Quadratic-phase wave packet transform

M. Younus Bhat¹, Aamir H. Dar², Didar Urynbassarova ^{3,4} and Altyn Urynbassarova^{5,6}

¹ Department of Mathematical Sciences, Islamic University of Science and Technology Awantipora, Pulwama, Jammu and Kashmir 192122, India. E-mail: gyounusg@gmail.com

 ² Department of Mathematical Sciences, Islamic University of Science and Technology Awantipora, Pulwama, Jammu and Kashmir 192122, India. E-mail: ahdkul740@gmail.com
 ³ School of Information and Electronics, Beijing Institute of Technology, Beijing 100081, China ⁴ Beijing Key Laboratory of Fractional Signals and Systems, Beijing 100081, China
 ⁵ Department of Technology and Ecology, School of Society, Technology and Ecology, Narxoz University, Almaty 050000, Kazakhstan

⁶ Department of Information Security, Faculty of Information Technology, Eurasian National University, Nur-Sultan 010000, Kazakhstan

ABSTRACT. The quadratic phase Fourier transform(QPFT) has gained much popularity in recent years because of its applications in image and signal processing. However, the QPFT is inadequate for localizing the quadratic-phase spectrum which is required in some applications. In this paper, the quadratic-phase wave packet transform (QP-WPT) is proposed to address this problem, based on the wave packet transform (WPT) and QPFT. Firstly, we propose the definition of the QP-WPT and gave its relation with windowed Fourier transform (WFT). Secondly, several notable inequalities and important properties of newly defined QP-WPT, such as boundedness, reconstruction formula, Moyal's formula, Reproducing kernel are derived. Finally, we formulate several classes of uncertainty inequalities such as Leib's uncertainty principle, logarithmic uncertainty inequality and the Heisenberg uncertainty inequality.

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1. Introduction

The Fourier transform (FT) is an important tool in optical communication and signal processing [1]. However, owing to its global kernel the FT is incapable of obtaining information about local properties of the signal. But, the actual signals are often non-stationary or time-variable, so to overcome this problem, the short time Fourier transform (STFT) is employed that uses a time window of fixed length applied at regular intervals so that we can obtain a portion of the signal considered to be stationary [2]. The resulting time-varying spectral depiction is critical for non-stationary signal analysis, but in this case it comes at fixed spectral and temporal resolution. The wavelet analysis(WT) [3, 4] provides an attractive and pinch-hitting tool to the STFT by using an optical multichannel correlator with a bank of WT filters, can provide a better illustration of the signal instead of the STFT. Nonetheless, in the high frequency region WT has poor frequency resolution. To solve this defect the wave packet transform (WPT) was proposed by combining the merits of STFT and WT [5, 6]. WPT is a linear transform which uses the Weyl operator and the wave packages.

In recent years, researchers have successfully applied wave packet transform (WPT) in the fields of wireless communication, denoising, and image compression [7]-[14]. Wave packet transform (WPT) is used widely in signal processing as it has some better morality than wavelet transform (WT) [15, 16]. Moreover, it can realize multilevel decomposition and analyze the high frequency decomposition that is not achieved in traditional discrete WT. The frequency subbands of signal are selected via wave packet decomposition, that improves the time-frequency resolution capability of the signal. However, the WPT is defined as the FT of the signal windowed with the wavelet, so the results obtained by WPT will not be optimal in dealing with chirp signals whose energy is not well concentrated in FT domain.

A superlative generalized version of the Fourier transform (FT) called quadratic-phase Fourier transform (QPFT) has been introduced by Castro et al[17, 18]. This novel transform has overthrown all the the applicable signal processing tools as it provides a unified analysis of both transient and non-transient signals in an easy and insightful fashion. The QPFT is actually a generalization of several well known transforms like Fourier, fractional Fourier and linear canonical transforms whose kernel is in the exponential form. Due to its extra degrees of freedom, the quadratic phase Fourier transform (QFT) has marked its importance in treatment of problems demanding several controllable parameters arising in diverse branches of science and engineering, including harmonic analysis, sampling, image processing, and so on [19]-[25].

Recently Prasad and Sharma [26] introduced the quadratic phase Fourier wavelet transform (QPFWT), which is generalization of classical continuous wavelet transform, [27]-[33] continuous fractional wavelet transform [34, 35, 36] as well as generalization of linear canonical wavelet transform[36, 37]. QPFWT intertwine the advantages of the quadraticphase Fourier and wavelet transforms into a novel integral transform which assimilates their individual properties. However, the transform neither relies on the complete kernel of the QPFT nor exhibits any existing convolution structure in the quadratic-phase Fourier transform. So Shah and Lone [20] introduced Quadratic-phase wavelet in different approach which is completely reliant upon convolution associated with QPFT.

As one of the generalization of the classical WPT, the fractional WPT(Fr-WPT) and linear canonical wave packet transform (LC-WPT) have been introduced to improve the performance in concentration [38, 39, 40]. They have attained a much more attention of the signal processing community and optics. But to the best of our knowledge theory about quadratic phase wave packet transform (QP-WPT) have never been proposed up to date, therefore it is worthwhile to study the theory of QP-WPT based on the wave packet transform (WPT) and QPFT which can be productive for signal processing theory and applications. Therefore, the cynosure of this paper is to rigorously study the quadratic phase wave packet transform (QP-WPT).

The highlights of the paper are pointed out below:

- To introduce a novel integral transform coined as the quadratic phase wave packet transform.
- To establish relationship between quadratic phase wave packet transform with Fourier transform(FT) and windowed Fourier transform (WFT).
- To study several notable inequalities and important properties of newly defined QP-WPT, such as boundedness, reconstruction formula, Moyal's formula, Reproducing kernel.
- To formulate several classes of uncertainty inequalities, such as Leib-type, the logarithmic uncertainty inequalities and the Heisenberg-type uncertainty inequalities

associated with the quadratic phase wave packet transform.

The paper is organised as follows. In Section 2, we provide some preliminary results required in subsequent sections. In Section 3, we provide the definition of quadratic phase wave packet transform (QP-WPT). Then we investigated several basic properties of the QP-WPT which are important for signal representation in signal processing. In Section 4, we develop a series of uncertainty inequalities such as Leib's uncertainty principle, the logarithmic uncertainty inequality and the Heisenberg-type inequality associated with the quadratic phase wave packet transform. Finally, a conclusion is extracted in Section 5.

2. Preliminaries

In this section we recall some basic concepts and notations, which will be useful in our study on quadratic-phase wave packet transfrom(QPWPT).

2.1. Fourier transform. We use the following definition of Fourier transform [1] on $L^1(\mathbb{R})$ space

$$\mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-it\xi} dt, \quad \forall \xi \in \mathbb{R}.$$
(2.1)

2.2. Continuous wavelet transform. Wavelet transform presents an attractive alternative to the STFT by using a time-frequency window that changes with frequency, which can effectively provide resolution of varying granularity. The continuous wavelet transform (CWT) of a signal $f(t) \in L^2(\mathbb{R})$ is defined as [29, 30]

$$CWT_f(\beta,\alpha) = \frac{1}{\sqrt{\alpha}} \int_{\mathbb{R}} f(t)\psi^*\left(\frac{t-\beta}{\alpha}\right) dt,$$
(2.2)

where * denotes the complex conjugate and t is time, β is the translation parameter, α is the scaling parameter and $\psi(t)$ is the transforming function, called mother wavelet. Here $\alpha > 0$ and ψ is normalized such that the $\|\psi\| = 1$ in $L^2(\mathbb{R})$ space.

2.3. Windowed Fourier transform. The windowed Fourier transform of $f(t) \in L^2(\mathbb{R})$ with respect to the windowed function $\phi \in L^2(\mathbb{R})$ is defined as [41]

$$\mathcal{G}_{\phi}[f](w,\beta) = \int_{\mathbb{R}} f(t)\phi^*(t-\beta)e^{-i\xi t}dt$$
(2.3)

and the inverse of the function $f(t) \in L^2(\mathbb{R})$ is defined by [24]

$$f(t) = \frac{b}{2\pi \langle \phi, \psi \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{G}_{\phi}[f](\xi, \beta) e^{i\xi t} \psi(t-\beta) d\xi d\beta, \qquad (2.4)$$

where, $\psi \in L^2(\mathbb{R})$.

2.4. Wave packet transform. The wave packet transform (WPT) combines elements of STFT and CWT, and it can be viewed as [7, 8]

$$WPT_f(\xi,\beta,\alpha) = \frac{1}{\sqrt{2\pi\alpha}} \int_{\mathbb{R}} f(t) \overline{\psi_\alpha (t-\beta)} e^{-i\xi t} dt$$
(2.5)

where $\psi_{\alpha}(t-\beta) = \psi\left(\frac{t-\beta}{\alpha}\right)$.

The WPT is the Fourier transform of a signal windowed with a wavelet that is dilated by α and translated by β .

Lemma 2.1. [39] Let $\psi \in L^{p}(\mathbb{R})$, $p \in [1, \infty)$. Then, $\|\psi_{\alpha}(t-\beta)\|_{L^{p}(\mathbb{R})} = \alpha^{(1/p-1/2)} \|\psi\|_{L^{p}(\mathbb{R})}$,

2.5. Quadratic-phase Fourier transform.

In this subsection we introduce the Quadratic-phase Fourier transform which is a neoteric addition to the classical integral transforms and we also gave its inversion formula and some other classical results which are already present in literature.

Definition 2.1. Given a parameter $\mu = (a, b, c, d, e)$, the QPFT of any signal f is defined by [26]

$$\mathcal{Q}_{\mu}[f](\xi) = \int f(t) K_{\mu}(t,\xi) dt, \qquad (2.6)$$

where $K_{\mu}(t,\xi)$ is the quadratic-phase Fourier kernel, given by

$$K_{\mu}(t,\xi) = \sqrt{\frac{b}{2\pi i}} e^{(at^2 + bt\xi + c\xi + dt + e\xi)}$$
(2.7)

with $a, b, c, d, e \in \mathbb{R}$, $b \neq 0$.

Theorem 2.1. The inversion formula of the quadratic-phase Fourier transform is given by [26]

$$f(t) = \int \mathcal{Q}_{\mu}[f](\xi) \overline{K_{\mu}(t,\xi)} d\xi.$$
(2.8)

Using the inversion theorem, we can get the Parseval's relation given by [26]

$$\langle f, g \rangle = \langle \mathcal{Q}_{\mu}[f], \mathcal{Q}_{\mu}[g] \rangle$$
 (2.9)

and Plancherel identity is given by

$$\int |\mathcal{Q}_{\mu}[f](\xi)|^2 d\xi = \int |f(t)|^2 dt.$$
(2.10)

Theorem 2.2. [20, 21]Let $f, g \in L^2(\mathbb{R})$ and $\alpha, \beta, \tau \in \mathbb{R}$ then

- $\mathcal{Q}_{\mu}[\alpha f + \beta g](\xi) = \alpha \mathbb{Q}_{\mu}[f](w) + \beta \mathbb{Q}_{\mu}[g](w).$ $\mathcal{Q}_{\mu}[f(t-\tau)](\xi) = exp\{-i(a\tau^{2} + b\tau\xi + d\tau)\}\mathcal{Q}_{\mu}[e^{-2ia\tau t}f(t)](\xi).$ $\mathcal{Q}_{\mu}[f(-t)](\xi) = \mathcal{Q}_{\mu'}[f(t)](-\xi), \quad \mu' = (a, b, c, -d, -e).$
- $\mathcal{Q}_{\mu}[e^{i\alpha t}f(t)](\xi) = exp\{i(\alpha^2 + 2\alpha b\xi + \alpha eb)\frac{1}{b}\}\mathcal{Q}_{\mu}[f](w + \frac{a}{b}).$
- $\mathcal{Q}_{\mu}[\overline{f(t)}](w) = \overline{\mathcal{Q}_{-\mu}[f(t)](w)}.$

Theorem 2.3 (Convolution[26]). If $f, q \in L^2(\mathbf{R})$ then

$$\mathcal{Q}_{\mu}[f *_{\mu} g](\xi) = \sqrt{\frac{2\pi i}{b}} e^{-i(c\xi^{2} + e\xi)} \mathcal{Q}_{\mu}[f](\xi) \mathcal{Q}_{\mu}[e^{-ia(\cdot)^{2} - id(\cdot)}g](\xi).$$
(2.11)

Where

$$(f *_{\mu} g)(t) = \int_{\mathbf{R}} f(x)g(t-x)e^{-ia(t^2-z^2)-id(t-z)}dz.$$
 (2.12)

2.6. Quadratic-phase wavelet transform.

The generalization of the classical continuous wavelet transform, continuous fractional wavelet transform, as well as generalization of linear canonical wavelet transform, is the quadratic phase wavelet transform (QPWT).

For a signal $f(t) \in L^2(\mathbb{R}^2)$, the continuous quadratic-phase wavelet transform of f with respect to an analyzing wavelet $\psi \in L^2(\mathbb{R})$ and the parameter set $\mu = (a, b, c, d, e)$ is defined by [26]

$$CQPWT_f(\beta,\alpha) = \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} f(t)\overline{\psi^{\mu}_{\beta,\alpha}(t)}dt, \qquad (2.13)$$

where the family $\psi^{\mu}_{\beta,\alpha}(t)$ is called quadratic-phase wavelet (QPW) and is given by

$$\psi^{\mu}_{\beta,\alpha}(t) = \frac{1}{\sqrt{\alpha}} \psi\left(\frac{t-\beta}{\alpha}\right) e^{-ia(t^2-\beta^2)-id(t-\beta)}.$$
(2.14)

Lemma 2.2. [26] If $\psi \in L^2(\mathbb{R})$, Then $\psi^{\mu}_{\beta,\alpha} \in L^2(\mathbb{R})$ with $\|\psi^{\mu}_{\beta,\alpha}\|^2 = \|\psi\|^2$.

Now we are ready to introduce a novel integral transform the quadratic phase wave packet transform.

3. Quadratic-phase wavelet packet transform(QP-WPT)

In this section, we propose a definition of QP-WPT based on the idea of WPT by adding extra dimension to kernel and wavelet. We replace the wave packet transform kernel by the QPFT kernel and the wavelet by the quadratic-phase wavelet (QPW).

Definition 3.1 (QP-WPT). The QP-WPT transform of a function $f \in L^2(\mathbb{R})$ with respect to wavelet function ψ is defined as

$$W_{f}^{\mu}(\xi,\beta,\alpha) = \int_{\mathbb{R}} f(t) \overline{\psi_{\beta,\alpha}^{\mu}(t)} K_{\mu}(t,\xi) dt$$
$$= \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} e^{i(at^{2}+bt\xi+c\xi^{2}+dt+e\xi)} f(t) \overline{\psi_{\beta,\alpha}^{\mu}(t)} dt,$$
(3.1)

where $\psi^{\mu}_{\beta,\alpha}(t) = \psi_{\alpha}(t-\beta) e^{-ia(t^2-\beta^2)-id(t-\beta)}$ and $\psi_{\alpha}(t) = \frac{1}{\alpha}\psi\left(\frac{t}{\alpha}\right)$.

Remark 3.1. By varying the parameter $\mu = (a, b, c, d, e)$ Definition 3.1 embodies certain existing time-frequency transforms and also give birth to some novel time-frequency tools which are yet to be reported in the open literature which are listed below:

• For $\mu = (a/2b, -1/b, c/2b, 0, 0)$, Definition 3.1 boils down to the novel linear canonical wave packet transform

$$W_f^{\mu}(\xi,\beta,\alpha) = \int_{\mathbb{R}} f(t) K_{\mu}(t,\xi) \overline{\psi\left(\frac{t-\beta}{\alpha}\right)} e^{i\frac{\alpha}{2b}(t^2-\beta^2)} dt$$

• For $\mu = (\cot \theta, -\csc \theta, \cot \theta, 0, 0), \ \theta \neq n\pi$ Definition 3.1 reduces to the novel fractional wave packet transform

$$W_f^{\mu}(\xi,\beta,\alpha) = \int_{\mathbb{R}} f(t) K_{\mu}(t,\xi) \overline{\psi\left(\frac{t-\beta}{\alpha}\right)} e^{i \cot \theta (t^2 - \beta^2)} dt$$

• For $\mu = (1, b, 0, 1, 0), b \neq 0$, we can obtain the novel Fresnel wave packet transform

$$W_f^{\mu}(\xi,\beta,\alpha) = \int_{\mathbb{R}} f(t) K_{\mu}(t,\xi) \overline{\psi\left(\frac{t-\beta}{\alpha}\right)} e^{i(t^2-\beta^2)+id(t-\beta)} dt$$

• For $\mu = (0, -1, 1, 0, 0)$, Definition 3.1 reduces to the classical wave packet transform

$$W_f^{\mu}(\xi,\beta,\alpha) = \int_{\mathbb{R}} f(t) K_{\mu}(t,\xi) \overline{\psi\left(\frac{t-\beta}{\alpha}\right)} dt$$

Theorem 3.2. Let $W_f^{\mu}(\xi, \beta, \alpha)$ and $Q_{\mu}[f]$ be the QP-WPT and QPFT of a function $f \in L^2(\mathbb{R})$, respectively and let $\psi_{\beta,\alpha}^{\mu}$ be the QPW, then we have

$$W_{f}^{\mu}(\xi,\beta,\alpha) = \sqrt{\alpha} \int_{\mathbb{R}} K_{\mu}(w,\beta) e^{-i[c(\alpha w)^{2} + e(\alpha w) - 2cw\xi]} \mathcal{Q}_{\mu}[e^{ia(.)^{2} + id(.)}f(t)](w+\xi) \mathcal{Q}_{\mu}[e^{-ia(.)^{2} - id(.)}\psi(.)](\alpha w)dw$$
(3.2)

Proof. Let us denote

$$f_{\xi,\mu} = \sqrt{\frac{b}{2\pi i}} e^{i(at^2 + bt\xi + c\xi^2 + dt + e\xi)} f(t).$$

On taking QPFT on both sides of above equation, we have

From [26], we have

$$Q_{\mu}[\psi^{\mu}_{\beta,\alpha}](w) = \sqrt{\alpha}e^{i(a\beta^{2}+b\beta w+cw^{2}+d\beta+ew)-ic(\alpha w)^{2}-ie(\alpha w)} \times Q_{\mu}[e^{-ia(.)^{2}-id(.)}\psi(.)](\alpha w).$$

The QP-WPT is represented in terms of inner product of $f_{\xi,\mu}$ and $\psi^{\mu}_{\beta,\alpha}$ and by Parseval theorem of QPFT, we have

$$\begin{split} W_{f}^{\mu}(\xi,\beta,\alpha) &= \langle f_{\xi,\mu},\psi_{\beta,\alpha}^{\mu} \rangle \\ &= \langle \mathcal{Q}_{\mu}[f_{\xi,\mu}], Q_{\mu}[\psi_{\beta,\alpha}^{\mu}] \rangle \\ &= \sqrt{\frac{\alpha b}{2\pi i}} \int_{\mathbb{R}} e^{i(a\beta^{2}+b\beta w+cw^{2}+d\beta+ew-c(\alpha w)^{2}-e(\alpha w)-2cw\xi)} \\ &\quad \times \mathcal{Q}_{\mu}[e^{i(at^{2}+dt)}f(t)](w+\xi)Q_{\mu}[e^{-ia(.)^{2}-id(.)}\psi(.)](\alpha w)dw. \end{split}$$

Now using 2.7, we get the desired proof.

Further, the definition of the QP-WPT in 3.1 can be rewritten as

$$W_{f}^{\mu}(\xi,\beta,\alpha) = \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} e^{i(at^{2}+bt\xi+c\xi^{2}+dt+e\xi)+ia(t^{2}-\beta^{2})+id(t-\beta)} \times f(t)\overline{\psi_{\alpha}(t-\beta)}dt$$
$$= \int_{\mathbb{R}} f(t)\overline{\psi_{\xi,\beta,\alpha}}^{\mu}dt, \qquad (3.3)$$

where

$$\psi^{\mu}_{\xi,\beta,\alpha,}(t) = \overline{\left(\sqrt{\frac{b}{2\pi i}}\right)} e^{-i(at^2 + bt\xi + c\xi^2 + dt + e\xi) - ia(t^2 - \beta^2) - id(t - \beta)} \psi_{\alpha}(t - \beta).$$
(3.4)

Proposition 3.1 (Relation with WFT).

$$W_{f}^{\mu}(\xi,\beta,\alpha) = \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} e^{i(at^{2}+bt\xi+c\xi^{2}+dt+e\xi)+ia(t^{2}-\beta^{2})+id(t-\beta)} f(t)\overline{\psi_{\alpha}(t-\beta)}dt$$

$$= e^{i(c\xi^{2}+e\xi-a\beta^{2}-d\beta)} \int_{\mathbb{R}} \sqrt{\frac{b}{2\pi i}} e^{i(2at^{2}+bt\xi+2dt)} f(t)\overline{\psi_{\alpha}(t-\beta)}dt$$

$$= e^{i(c\xi^{2}+e\xi-a\beta^{2}-d\beta)} \int_{\mathbb{R}} \sqrt{\frac{b}{2\pi i}} e^{i(2at^{2}+2dt)} f(t)\overline{\psi_{\alpha}(t-\beta)}e^{ibt\xi}dt$$

$$= e^{i(c\xi^{2}+e\xi-a\beta^{2}-d\beta)} \mathcal{G}_{\psi^{\mu}}[h](b\xi,\beta)$$
(3.5)

where $h(t) = \sqrt{\frac{b}{2\pi i}}e^{i(2at^2+2dt)}f(t)$

3.1. Basic properties of the QP-WPT. In this subsection we prove some notable inequalities associated with the QP-WPT. Moreover, we also investigate some basic properties of the QP-WPT which are important for signal representation in signal processing.

Lemma 3.1. Let $\psi_{\alpha} \in L^{p}(\mathbb{R})$ and $f \in L^{q}(\mathbb{R})$ and $p, q \in [1, \infty)$ with $\frac{1}{p} + \frac{1}{q}$, then

$$|W_{f}^{\mu}(\xi,\beta,\alpha)| \leq \alpha^{1/p-1/2} \sqrt{\frac{b}{2\pi}} \|\psi\|_{L^{p}(\mathbb{R})} \|f\|_{L^{q}(\mathbb{R})}.$$
(3.6)

Proof. From (3.3), we have

$$\begin{aligned} |W_{f}^{\mu}(\xi,\beta,\alpha)| &= \left| \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} e^{i(at^{2}+bt\xi+c\xi^{2}+dt+e\xi)+ia(t^{2}-\beta^{2})+id(t-\beta)} f(t)\overline{\psi_{\alpha}(t-\beta)}dt \right| \\ &= \left| \sqrt{\frac{b}{2\pi}} \left| \int_{\mathbb{R}} e^{i(at^{2}+bt\xi+c\xi^{2}+dt+e\xi)+ia(t^{2}-\beta^{2})+id(t-\beta)} f(t)\overline{\psi_{\alpha}(t-\beta)}dt \right| \\ &\leq \sqrt{\frac{b}{2\pi}} \left| \int_{\mathbb{R}} f(t)\overline{\psi_{\alpha}(t-\beta)}dt \right|. \end{aligned}$$

By Lemma 2.1 and Holder's inequality, above yields

$$|W_{f}^{\mu}(\xi,\beta,\alpha)| \leq \alpha^{1/p-1/2} \sqrt{\frac{b}{2\pi}} \|\psi\|_{L^{p}(\mathbb{R})} \|f\|_{L^{q}(\mathbb{R})}$$

which completes the proof.

Theorem 3.3 (Boundedness). For ψ , $f \in L^2(\mathbb{R})$, the QP-WPT is bounded on $L^2(\mathbb{R})$. Proof. By taking p = q = 2 in Lemma 3.1, we have:

$$|W_{f}^{\mu}(\xi,\beta,\alpha)| \leq \sqrt{\frac{b}{2\pi}} \|\psi\|_{L^{2}(\mathbb{R})} \|f\|_{L^{2}(\mathbb{R})}$$

which shows that the QP-WPT is bounded on $L^2(\mathbb{R})$.

Theorem 3.4. Let $\psi \in L^p(\mathbb{R})$ and $f \in L^1(\mathbb{R}) \cap L^1(\mathbb{R})$. Then we have

$$\|W_{f}^{\mu}(\xi,\beta,\alpha)\|_{L^{p}(\mathbb{R})} \leq \alpha^{(1/p-1/2)} \sqrt{\frac{b}{2\pi}} \|\psi\|_{L^{p}(\mathbb{R})} \|f\|_{L^{1}(\mathbb{R})}.$$
(3.7)

Proof. By applying the Minkowski's inequality to (3.3), we obtain $\|W_f^{\mu}(\xi,\beta,\alpha)\|_{L^p(\mathbb{R})}$

$$= \left(\int_{\mathbb{R}} \left| \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} e^{i(at^{2}+bt\xi+c\xi^{2}+dt+e\xi)+ia(t^{2}-\beta^{2})+id(t-\beta)} f(t)\overline{\psi_{\alpha}(t-\beta)} dt \right|^{p} d\beta \right)^{1/p}$$

$$\leq \sqrt{\frac{b}{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| f(t)\overline{\psi_{\alpha}(t-\beta)} \right|^{p} d\beta \right)^{1/p} dt.$$
here

Setting $t - \beta = y$, we have

$$\begin{split} \|W_{f}^{\mu}(\xi,\beta,\alpha)\|_{L^{p}(\mathbb{R})} &= \sqrt{\frac{b}{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| f(t) \overline{\psi_{\alpha}(y)} \right|^{p} dy \right)^{1/p} dt \\ &\leq \sqrt{\frac{b}{2\pi}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| \overline{\psi_{\alpha}(y)} \right|^{p} dy \right)^{1/p} |f(t)| dt \\ &\leq \sqrt{\frac{b}{2\pi}} \|\psi_{\alpha}\|_{L^{p}(\mathbb{R})} \|f\|_{L^{1}(\mathbb{R})} \\ &\leq \alpha^{(1/p-1/2)} \sqrt{\frac{b}{2\pi}} \|\psi\|_{L^{p}(\mathbb{R})} \|f\|_{L^{1}(\mathbb{R})}. \end{split}$$

Which completes the proof.

Theorem 3.5 (Reconstruction theorem). Every signal $f \in L^2(\mathbb{R})$, can be reconstructed from QP-WPT by the formula

$$f(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} W_f^{\mu}(\xi, \beta, \alpha) \psi_{\xi, \beta, \alpha}^{\mu}(t) d\xi d\beta.$$
(3.8)
(3.9)

Proof. Let $h(t), \psi, \phi \in L^2(\mathbb{R})$. Assuming $\langle \phi, \psi \rangle \neq 0$ and ψ_{α} as a windowed function then by the inverse of the WFT (2.4), we have

$$h(t) = \frac{b}{2\pi \langle \phi, \psi \rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathcal{G}_{\psi^{\mu}}[h](b\xi, \beta) e^{-ibt\xi} \psi_{\alpha}(t-\beta) d\xi d\beta.$$

By virtue of (3.5), we have from above equation

$$\sqrt{\frac{b}{2\pi i}}e^{i(2at^{2}+2dt)}f(t) = \frac{b}{2\pi\langle\phi,\psi\rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(c\xi^{2}+e\xi-a\beta^{2}-d\beta)}e^{-ibt\xi} \\
\times\psi_{\alpha}(t-\beta)W_{f}^{\mu}(\xi,\beta,\alpha)d\xi d\beta \\
f(t) = \sqrt{\frac{bi}{2\pi}}\frac{1}{\langle\phi,\psi\rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(c\xi^{2}+e\xi-a\beta^{2}-d\beta+2dt+bt\xi+2at^{2})} \\
\times\psi_{\alpha}(t-\beta)W_{f}^{\mu}(\xi,\beta,\alpha)d\xi d\beta \\
= \frac{1}{\langle\phi,\psi\rangle} \int_{\mathbb{R}} \int_{\mathbb{R}} \psi_{\xi,\beta,\alpha}^{\mu}(t)W_{f}^{\mu}(\xi,\beta,\alpha)d\xi d\beta.$$
(3.10)

For perfect reconstruction take $\langle \phi, \psi \rangle = 1$, above equation yields

$$f(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} W_f^{\mu}(\xi, \beta, \alpha) \psi_{\xi, \beta, \alpha}^{\mu}(t) d\xi d\beta.$$

Which completes the proof.

Theorem 3.6 (Moyal's Formula). Let $W_f^{\mu}(\xi, \beta, \alpha)$ and $W_g^{\mu}(\xi, \beta, \alpha)$ be the QP-WPT with respect to the wavelets ψ and ϕ respectively, then

$$\langle W_f^{\mu}(\xi,\beta,\alpha), W_g^{\mu}(\xi,\beta,\alpha) \rangle_{L^2(\mathbb{R}^2)} = \overline{\langle \psi,\phi \rangle}_{L^2(\mathbb{R})} \langle f,g \rangle_{L^2(\mathbb{R})}$$
(3.11)

 $\langle W_f^{\mu}(\xi,\beta,\alpha), W_g^{\mu}(\xi,\beta,\alpha) \rangle$

$$\begin{split} &= \int_{\mathbb{R}^2} W_f^{\mu}(\xi,\beta,\alpha) \overline{W_g^{\mu}(\xi,\beta,\alpha)} d\xi d\beta \\ &= \int_{\mathbb{R}^2} \left\{ \int_{\mathbb{R}} f(t) \overline{\psi_{\beta,\alpha}^{\mu}(t)} K_{\mu}(t,\xi) dt \\ &\quad \times \int_{\mathbb{R}} \overline{g(t')} \phi_{\beta,\alpha}^{\mu}(t') \overline{K_{\mu}(t',\xi)} dt' \right\} d\xi d\beta \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(t) \overline{g(t')} \phi_{\beta,\alpha}^{\mu}(t') \overline{\psi_{\beta,\alpha}^{\mu}(t)} \int_{\mathbb{R}} K_{\mu}(t,\xi) \overline{K_{\mu}(t',\xi)} d\xi dt dt' d\beta \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(t) \overline{g(t')} \phi_{\alpha}(t'-\beta) \overline{\psi_{\alpha}(t-\beta)} \frac{b}{2\pi} \int_{\mathbb{R}} e^{ib\xi(t-t')} d\xi dt dt' d\beta \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}} f(t) \overline{g(t')} \phi_{\alpha}(t'-\beta) \overline{\psi_{\alpha}(t-\beta)} \delta(t-t') dt' dt d\beta \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} f(t) \overline{g(t)} dt \int_{\mathbb{R}} \frac{1}{\alpha} \phi\left(\frac{t-\beta}{\alpha}\right) \overline{\psi}\left(\frac{t-\beta}{\alpha}\right) d\beta \\ &= \overline{\langle \psi, \phi \rangle}_{L^2(\mathbb{R})} \langle f, g \rangle_{L^2(\mathbb{R})}. \end{split}$$

Which completes the proof.

Consequences of the Theorem 3.6:

• If $\psi = \phi$, then

$$\langle W_f^{\mu}(\xi,\beta,\alpha), W_g^{\mu}(\xi,\beta,\alpha) \rangle_{L^2(\mathbb{R}^2)} = \|\psi\|_{L^2(\mathbb{R})}^2 \langle f,g \rangle_{L^2(\mathbb{R})}.$$
(3.12)

• If $\psi = \phi$, and f = g then

$$\langle W_f^{\mu}(\xi,\beta,\alpha), W_g^{\mu}(\xi,\beta,\alpha) \rangle_{L^2(\mathbb{R}^2)} = \|\psi\|_{L^2(\mathbb{R})}^2 \|f\|_{L^2(\mathbb{R})}^2.$$
(3.13)

• If $\psi = \phi = 1$, and f = g then

$$\langle W_f^{\mu}(\xi,\beta,\alpha), W_g^{\mu}(\xi,\beta,\alpha) \rangle_{L^2(\mathbb{R}^2)} = \|f\|_{L^2(\mathbb{R})}^2.$$
 (3.14)

Remark 3.7 (Energy conservation). Equation (3.14) yields the conservation of energy for the QP-WPT

$$\int_{\mathbb{R}^2} \left| W_f^{\mu}(\xi,\beta,\alpha) \right|^2 d\xi d\beta = \int_{\mathbb{R}} \left| f(t) \right|^2 dt.$$
(3.15)

Theorem 3.8 (Reproducing kernel). Let (ξ_0, β_0, α) be any point on the plane of (ξ, β, α) , the necessary and sufficient condition that the function $W_f^{\mu}(\xi, \beta, \alpha)$ is the QP-WPT of some function is that $W_f^{\mu}(\xi, \beta, \alpha)$ must satisfy the following reproducing kernel formula

$$W_f^{\mu}(\xi,\beta,\alpha) = \int_{\mathbb{R}} \int_{\mathbb{R}} W_f^{\mu}(\xi,\beta,\alpha) \mathbb{K}_{\psi^{\mu}}(\xi,\beta,\alpha:\xi_0,\beta_0,\alpha) d\xi d\beta$$
(3.16)

where $W_f^{\mu}(\xi_0, \beta_0, \alpha)$ is value of function $W_f^{\mu}(\xi, \beta, \alpha)$ at (ξ_0, β_0, α) , and $\mathbb{K}_{\psi^{\mu}}(\xi, \beta, \alpha : \xi_0, \beta_0, \alpha)$ is called the reproducing kernel given by

$$\mathbb{K}_{\psi^{\mu}}(\xi,\beta,\alpha:\xi_{0},\beta_{0},\alpha) = \langle \psi^{\mu}_{\xi,\beta,\alpha},\psi^{\mu}_{\xi_{0},\beta_{0},\alpha} \rangle$$
(3.17)

Proof. From (3.3) and (3.8), we have

$$\begin{split} W_{f}^{\mu}(\xi,\beta,\alpha) &= \int_{\mathbb{R}} f(t) \overline{\psi_{\xi,\beta,\alpha}^{\mu}}(t) dt \\ &= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} W_{f}^{\mu}(\xi,\beta,\alpha) \psi_{\xi,\beta,\alpha}^{\mu}(t) d\xi d\beta \right\} \overline{\psi_{\xi,\beta,\alpha}^{\mu}}(t) dt. \end{split}$$

Setting $(\xi, \beta, \alpha) = (\xi_0, \beta_0, \alpha)$, we have

$$\begin{split} W_{f}^{\mu}(\xi_{0},\beta_{0},\alpha) \\ &= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} W_{f}^{\mu}(\xi,\beta,\alpha) \psi_{\xi,\beta,\alpha}^{\mu}(t) d\xi d\beta \right\} \overline{\psi_{\xi_{0},\beta_{0},\alpha}^{\mu}}(t) dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} W_{f}^{\mu}(\xi,\beta,\alpha) \psi_{\xi,\beta,\alpha}^{\mu}(t) \overline{\psi_{\xi_{0},\beta_{0},\alpha}^{\mu}}(t) dt d\xi d\beta \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} W_{f}^{\mu}(\xi,\beta,\alpha) \left\{ \int_{\mathbb{R}} \psi_{\xi,\beta,\alpha}^{\mu}(t) \overline{\psi_{\xi_{0},\beta_{0},\alpha}^{\mu}}(t) dt \right\} d\xi d\beta \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} W_{f}^{\mu}(\xi,\beta,\alpha) \mathbb{K}_{\psi^{\mu}}(\xi,\beta,\alpha) \cdot \xi_{0},\beta_{0},\alpha) d\xi d\beta \end{split}$$

Which completes the proof.

4. Uncertainty Principle's for the QP-WPT

Uncertainty principle has applications in two main areas: harmonic analysis and signal analysis. This principle in harmonic analysis stems from the uncertainty principle in quantum mechanics, which tells that a particle's velocity and position cannot be measured with infinite precision. In signal analysis, it tells that if one observes a signal only for a finite time, then the knowledge about the frequencies consisted by the signal is lost. In this section, we first prove QP-WPT Lieb's uncertainty principle by considering the relationship between the WFT and QP-WPT. Then we will obtain a logarithmic uncertainty principle associated with the QP-WPT by using the relation fundamental between FT and QP-WPT. Finally, we will establish a generalization of the Heisenberg type uncertainty principle for the QP-WPT.

Theorem 4.1 (Leib's uncertainty principle). For $\psi, f \in L^2(\mathbb{R})$ and $2 \leq p < \infty$, the following inequality holds:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| W_f^{\mu}(\xi,\beta,\alpha) \right|^p d\xi d\beta \leq \frac{2}{p} (M_{\mu})^p \left(\|f\|_2 \|\psi\|_2 \right)^p \tag{4.1}$$

where $(M_{\mu}) = (2\pi)^{\frac{-1}{2}} |b|^{\frac{1}{2} - \frac{1}{p}}$

Proof. The Lieb's uncertainty principle for the windowed Fourier transform [42, 41] reads

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |\mathcal{G}_{\psi}[f](\xi,\beta)|^p \, d\xi d\beta \le \frac{2}{p} \left(\|f\|_2 \|\psi\|_2 \right)^p \tag{4.2}$$

for all $f, \psi \in L^2(\mathbb{R})$ and $2 \le p < \infty$.

For $f \in L^2(\mathbb{R})$ we have function $h(t) = \sqrt{\frac{b}{2\pi i}} e^{i(2at^2 + 2dt)} f(t) \in L^2(\mathbb{R})$, therefore we can replace f in (4.2)by h as:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| \mathcal{G}_{\psi_{\alpha}^{\mu}}[h](\xi,\beta) \right|^{p} d\xi d\beta \leq \frac{2}{p} \left(\|h\|_{2} \|\psi_{\alpha}^{\mu}\|_{2} \right)^{p} \\
= \frac{2}{p} \left(\left(\int_{\mathbb{R}} \left| \sqrt{\frac{b}{2\pi i}} e^{i(2at^{2}+2dt)} f(t) \right|^{2} dt \right)^{\frac{1}{2}} \|\psi_{\alpha}\|_{2} \right)^{p} . (4.3)$$

Substituting $\xi = b\xi$ in (4.3), we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |b| \left| \mathcal{G}_{\psi_{\alpha}^{\mu}}[h](b\xi,\beta) \right|^{p} d\xi d\beta \leq \frac{2}{p} \left(\frac{b}{2\pi} \right)^{\frac{p}{2}} \left(\left(\int_{\mathbb{R}} |f(t)|^{2} dt \right)^{\frac{1}{2}} \|\psi_{\alpha}\|_{2} \right)^{p}.$$
(4.4)

Using (3.5) in (4.4)

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| e^{-i(c\xi^2 + e\xi - a\beta^2 - d\beta)} W_f^{\mu}(\xi, \beta, \alpha) \right|^p d\xi d\beta \le \frac{2}{p|b|} \left(\frac{b}{2\pi} \right)^{\frac{p}{2}} \left(\left(\int_{\mathbb{R}} \left| f(t) \right|^2 dt \right)^{\frac{1}{2}} \|\psi_{\alpha}\|_2 \right)^p. \tag{4.5}$$

p

On further simplifying (4.5) and using lemma 2.2, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \left| W_{f}^{\mu}(\xi,\beta,\alpha) \right|^{p} d\xi d\beta \leq \frac{2}{p|b|} \left(\frac{b}{2\pi} \right)^{\frac{1}{2}} (\|f\|_{2} \|\psi\|_{2})^{p}$$
$$= \frac{2}{p} \left(\frac{1}{|b|^{\frac{1}{p}}} \right)^{p} \left(\frac{|b|^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}} \right)^{p} (\|f\|_{2} \|\psi\|_{2})^{p}$$

which completes the proof.

Lemma 4.1 (Relation between QP-WPT and FT). We have from (3.3)

$$\begin{split} W_{f}^{\mu}(\xi,\beta,\alpha) &= \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} e^{i(at^{2}+bt\xi+c\xi^{2}+dt+e\xi)} f(t)\overline{\psi}_{\beta,\alpha}^{\mu}dt \\ &= \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} e^{i(at^{2}+bt\xi+c\xi^{2}+dt+e\xi)+ia(t^{2}-\beta^{2})+id(t-\beta)} f(t)\overline{\psi}_{\alpha}(t-\beta)dt \\ &= e^{i(c\xi^{2}+e\xi-a\beta^{2}-d\beta)} \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} e^{i(2at^{2}+bt\xi+2dt)} f(t)\overline{\psi}_{\alpha}(t-\beta)dt \\ &= e^{i(c\xi^{2}+e\xi-a\beta^{2}-d\beta)} \sqrt{\frac{b}{2\pi i}} \int_{\mathbb{R}} e^{i(2at^{2}+2dt)} e^{ib\xi t} f(t)\overline{\psi}_{\alpha}(t-\beta)dt \\ &= e^{i(c\xi^{2}+e\xi-a\beta^{2}-d\beta)} \sqrt{\frac{b}{i}} \mathcal{F}[g](b\xi) \end{split}$$

$$(4.6)$$

where

$$g(t) = e^{i(2at^2 + 2dt)} f(t)\psi_{\alpha}(t - \beta).$$
(4.7)

Theorem 4.2 (Logarithmic uncertainty principle). Let $\psi \in L^2(\mathbb{R})$ and $W^{\mu}_f(\xi, \beta, \alpha)$ be the QP-WPT of $f \in S(\mathbb{R})$ [Schwartz space]. Then, the following logarithmic inequality holds:

$$\begin{split} \|\psi\|^2 \int_{\mathbb{R}} \ln|t| |f(t)|^2 dt &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \ln|\xi| \left| W_f^{\mu}(\xi,\beta,\alpha) \right|^2 d\xi d\beta \\ &\geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|b| \right] \|f\|^2 \|\psi\|^2 \end{split}$$

Proof. For any $f \in \mathcal{S}(\mathbb{R})$ (Schwartz space in $L^2(\mathbb{R})$,) the logarithmic uncertainty principle for the classical Fourier transform reads[21]

$$\int_{\mathbb{R}} \ln|t| |f(t)|^2 dt + \int_{\mathbb{R}} \ln|\xi| |\mathcal{F}[f](\xi)|^2 d\xi \ge \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi\right] \int_{\mathbb{R}} |f(t)|^2 dt.$$
(4.8)

As $f \in \mathcal{S}(\mathbb{R})$, then it is evident that function g given in (4.7) belongs to the Schwartz space $\mathcal{S}(\mathbb{R})$. Therefore we can replace f in (4.8) by g as:

$$\int_{\mathbb{R}} \ln |t| |g(t)|^2 dt + \int_{\mathbb{R}} \ln |\xi| |\mathcal{F}[g](\xi)|^2 d\xi \ge \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln \pi \right] \int_{\mathbb{R}} |g(t)|^2 dt.$$
(4.9)

Changing ξ by $b\xi$, we obtain from (4.9)

$$\int_{\mathbb{R}} \ln |t| |g(t)|^2 dt + b \int_{\mathbb{R}} \ln |b\xi| |\mathcal{F}[g](b\xi)|^2 d\xi \ge \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln \pi\right] \int_{\mathbb{R}} |g(t)|^2 dt.$$
(4.10)

Applying Lemma 4.1 and (4.7) to (4.10), we obtain

$$\int_{\mathbb{R}} \ln|t| |f(t)\psi_{\alpha}(t-\beta)|^{2} dt + b \int_{\mathbb{R}} (\ln|b| + \ln|\xi|) \left| \sqrt{\frac{i}{b}} e^{-i(c\xi^{2} + e\xi - a\beta^{2} - d\beta)} W_{f}^{\mu}(\xi,\beta,\alpha) \right|^{2} d\xi$$

$$\geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi \right] \int_{\mathbb{R}} |f(t)\psi_{\alpha}(t-\beta)|^{2} dt.$$

$$(4.11)$$

On further simplifying (4.11), we get

$$\int_{\mathbb{R}} \ln |t| |f(t)\psi_{\alpha}(t-\beta)|^{2} dt + \int_{\mathbb{R}} \ln |b| \left| W_{f}^{\mu}(\xi,\beta,\alpha) \right|^{2} d\xi + \int_{\mathbb{R}} \ln |\xi| \left| W_{f}^{\mu}(\xi,\beta,\alpha) \right|^{2} d\xi \\
\geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln \pi \right] \int_{\mathbb{R}} |f(t)\psi_{\alpha}(t-\beta)|^{2} dt.$$
(4.12)

On integrating both sides of (4.12) with respect to β , we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \ln |t| |f(t)\psi_{\alpha}(t-\beta)|^{2} dt d\beta + \ln |b| \int_{\mathbb{R}} \int_{\mathbb{R}} \left| W_{f}^{\mu}(\xi,\beta,\alpha) \right|^{2} d\xi d\beta \\
+ \int_{\mathbb{R}} \int_{\mathbb{R}} \ln |\xi| \left| W_{f}^{\mu}(\xi,\beta,\alpha) \right|^{2} d\xi d\beta \\
\geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln \pi \right] \int_{\mathbb{R}} \int_{\mathbb{R}} \|f(t)\psi_{\alpha}(t-\beta)\|^{2} dt d\beta.$$
(4.13)

Now using (3.13) in (4.13), we get

$$\begin{split} \|\psi\|^{2} \int_{\mathbb{R}} \ln|t| |f(t)|^{2} dt &+ \int_{\mathbb{R}} \int_{\mathbb{R}} \ln|\xi| \left| W_{f}^{\mu}(\xi,\beta,\alpha) \right|^{2} d\xi d\beta \\ &\geq \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi \right] \|f\|^{2} \|\psi\|^{2} - \ln|b| \|f\|^{2} \|\psi\|^{2} \\ &= \left[\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|b| \right] \|f\|^{2} \|\psi\|^{2} \end{split}$$

which completes the proof.

Theorem 4.3. For $\psi, f \in L^2(\mathbb{R})$ and $W^{\mu}_f(\xi, \alpha, \beta)$ be the QP-WPT of the signal f, then the following inequality holds:

$$\int_{\mathbb{R}} t^2 |f(t)|^2 dt \int_{\mathbb{R}} \int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta \ge \left(\frac{1}{2|b|} \|f\|^2 \|\psi\|\right)^2.$$
(4.14)

Proof. The classical Heisenberg-Pauli-Weyl inequality in the QPFT domain (see [21] Theorem 3.2) is given by

$$\int_{\mathbb{R}} t^2 |f(t)|^2 dt \int_{\mathbb{R}} \xi^2 |\mathcal{Q}_{\mu}[f](\xi)|^2 d\xi \ge \left(\frac{1}{2|b|} \int_{\mathbb{R}} |f(t)|^2 dt\right)^2.$$
(4.15)

Using the inverse transform for the QPFT into the LHS and Plancherel identity for QPFT (??) into the RHS of the (4.15), we have

$$\int_{\mathbb{R}} t^2 |Q_{\mu}^{-1}[Q_{\mu}[f](\xi)]|^2(t) dt \int_{\mathbb{R}} \xi^2 |\mathcal{Q}_{\mu}[f](\xi)|^2 d\xi \ge \left(\frac{1}{2|b|} \int_{\mathbb{R}} |Q_{\mu}[f](\xi)|^2 d\xi\right)^2.$$
(4.16)

For $f, \mathcal{Q}^{\mu}[f] \in L^2(\mathbb{R})$ we have $W_f^{\mu}(\xi, \beta, \alpha) \in L^2(\mathbb{R})$, so replacing $\mathcal{Q}^{\mu}[f]$ by $W_f^{\mu}(\xi, \beta, \alpha)$ in (4.16), we have

$$\int_{\mathbb{R}} t^2 |Q_{\mu}^{-1}[W_f^{\mu}(\xi,\beta,\alpha)]|^2 dt \int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi \ge \left(\frac{1}{2|b|} \int_{\mathbb{R}} |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi\right)^2.$$
(4.17)

Which implies

$$\left(\int_{\mathbb{R}} t^2 |Q_{\mu}^{-1}[W_f^{\mu}(\xi,\beta,\alpha)]|^2 dt\right)^{1/2} \left(\int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi\right)^{1/2} \ge \frac{1}{2|b|} \int_{\mathbb{R}} |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi.$$
(4.18)

Now integrating (4.19) both sides by β , we have

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} t^2 |Q_{\mu}^{-1}[W_f^{\mu}(\xi,\beta,\alpha)]|^2 dt \right)^{1/2} \left(\int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi \right)^{1/2} d\beta$$
$$\geq \frac{1}{2|b|} \int_{\mathbb{R}} \int_{\mathbb{R}} |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta$$
(4.19)

Now applying Cauchy-Schwartz inequality, (4.19) yields

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}}t^{2}|Q_{\mu}^{-1}[W_{f}^{\mu}(\xi,\beta,\alpha)]|^{2}dtd\beta\right)^{1/2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\xi^{2}|W_{f}^{\mu}(\xi,\beta,\alpha)|^{2}d\xid\beta\right)^{1/2}$$
$$\geq\frac{1}{2|b|}\int_{\mathbb{R}}\int_{\mathbb{R}}|W_{f}^{\mu}(\xi,\beta,\alpha)|^{2}d\xid\beta$$
(4.20)

Now, using (3.13) in (4.20), we obtain

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}}t^{2}|f(t)\psi(t-\beta)|^{2}dtd\beta\right)^{1/2}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\xi^{2}|W_{f}^{\mu}(\xi,\beta,\alpha)|^{2}d\xid\beta\right)^{1/2}$$

$$\geq\frac{1}{2|b|}\|f\|^{2}\|\psi\|^{2}.$$
(4.21)

On further simplifying (4.21), we get

$$\left(\int_{\mathbb{R}} t^2 |f(t)|^2 dt\right)^{1/2} \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta\right)^{1/2} \geq \frac{1}{2|b|} ||f||^2 ||\psi||.$$

$$(4.22)$$

Which implies

$$\int_{\mathbb{R}} t^2 |f(t)|^2 dt \int_{\mathbb{R}} \int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta \ge \left(\frac{1}{2|b|} \|f\|^2 \|\psi\|\right)^2$$

Which completes the proof.

Remark 4.4. By varying the parameter $\mu = (a, b, c, d, e)$ the Heisenberg-type inequality (4.14), embodies certain existing Heisenberg-type inequalities and also give birth to some novel Heisenberg-type inequalities which are yet to be reported in the open literature which are listed below:

• For $\mu = (a/2b, -1/b, c/2b, 0, 0)$, the Heisenberg-type inequality (4.14) boils down to the novel Heisenberg inequality for linear canonical wave packet transform(see Theorem 6.2 [39])

$$\int_{\mathbb{R}} t^2 |f(t)|^2 dt \int_{\mathbb{R}} \int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta \ge \left(\frac{|b|}{2} ||f||^2 ||\psi||\right)^2.$$

• For $\mu = (\cot \theta, -\csc \theta, \cot \theta, 0, 0), \ \theta \neq n\pi$, we can obtain the novel Heisenberg inequality for the fractional wave packet transform

$$\int_{\mathbb{R}} t^2 |f(t)|^2 dt \int_{\mathbb{R}} \int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta \ge \left(\frac{\sin\theta}{2} ||f||^2 ||\psi||\right)^2.$$

• For $\mu = (0, -1, 1, 0, 0)$, we can obtain the novel Heisenberg inequality for the the classical wave packet transform

$$\int_{\mathbb{R}} t^2 |f(t)|^2 dt \int_{\mathbb{R}} \int_{\mathbb{R}} \xi^2 |W_f^{\mu}(\xi,\beta,\alpha)|^2 d\xi d\beta \ge \left(\frac{1}{2} \|f\|^2 \|\psi\|\right)^2.$$

5. Conclusion

Based on quadratic phase Fourier transform (QPFT) and the classical wave packet transform (WPT) theory, we in this paper propose a novel integral transform coined as quadratic phase wave packet transform (QP-WPT) which rectifies the limitations of the WPT and QPFT. Overall, it not only combines the advantages of QPFT and WPT, but also preserves the properties of its conventional counterpart, and has better mathematical properties. Besides studying some notable inequalities and the fundamental properties including the Moyal's formula, inversion formula and a reproducing kernel, we also formulated several classes of uncertainty inequalities, such as Leib's uncertainty principle, the logarithmic uncertainty inequality and the Heisenberg inequality.

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