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Research Article

Quadratic-Quartic Functional Equations in RN-Spaces

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We obtain the general solution and the stability result for the following functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary t-norms $f(2x+y)+f(2x-y) = 4[f(x+y)+f(x-y)]+2[f(2x)-4f(x)]-6f(y)$.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms. Let $(G_1,\cdot)$ be a group and let $(G_2,*,d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality $d(h(x\cdot y), h(x)*h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, under what condition does there exists a homomorphism near an approximate homomorphism? The concept of stability for functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Hyers [2] gave a first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x+y)-f(x)-f(y)\| \le \delta \tag{1.1}$$

for all $x, y \in E$ and some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x)-T(x)\| \le \delta \tag{1.2}$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then T is $\mathbb{R}$-linear. In 1978, Rassias [3] provided a generalization of Hyers' theorem which allows the Cauchy difference to be unbounded. In 1991, Gajda [4] answered the question for the case $p > 1$, which was raised by Rassias. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (see [5–12]). The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.3}$$

is related to a symmetric biadditive mapping. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic functional equation (1.3) is said to be a quadratic mapping. It is well known that a mapping f between real vector spaces is quadratic if and only if there exits a unique symmetric biadditive mapping B such that $f(x) = B(x,x)$ for all x (see [5, 13]). The biadditive mapping B is given by

$$B(x,y) = \frac{1}{4}\left(f(x+y) - f(x-y)\right). \tag{1.4}$$

The Hyers-Ulam-Rassias stability problem for the quadratic functional equation (1.3) was proved by Skof for mappings $f : A \to B$, where A is a normed space and B is a Banach space (see [14]). Cholewa [15] noticed that the theorem of Skof is still true if relevant domain A is replaced an abelian group. In [16], Czerwik proved the Hyers-Ulam-Rassias stability of the functional equation (1.3). Grabiec [17] has generalized the results mentioned above.

In [18], Park and Bae considered the following quartic functional equation

$$f(x+2y) + f(x-2y) = 4\left[f(x+y) + f(x-y) + 6f(y)\right] - 6f(x). \tag{1.5}$$

In fact, they proved that a mapping f between two real vector spaces X and Y is a solution of (1.5) if and only if there exists a unique symmetric multiadditive mapping $M : X^4 \to Y$ such that $f(x) = M(x,x,x,x)$ for all x. It is easy to show that the function $f(x) = x^4$ satisfies the functional equation (1.5), which is called a quartic functional equation (see also [19]). In addition, Kim [20] has obtained the Hyers-Ulam-Rassias stability for a mixed type of quartic and quadratic functional equation.

The Hyers-Ulam-Rassias stability of different functional equations in random normed and fuzzy normed spaces has been recently studied in [21–26]. It should be noticed that in all these papers the triangle inequality is expressed by using the strongest triangular norm T_M.

The aim of this paper is to investigate the stability of the additive-quadratic functional equation in random normed spaces (in the sense of Sherstnev) under arbitrary continuous t-norms.

In this sequel, we adopt the usual terminology, notations, and conventions of the theory of random normed spaces, as in [22, 23, 27–29]. Throughout this paper, Δ^+ is the space of distribution functions, that is, the space of all mappings $F : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. Also, D^+ is a subset of Δ^+ consisting of all functions $F \in \Delta^+$ for which $l^- F(+\infty) = 1$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases} \tag{1.6}$$

Definition 1.1 (see [28]). A mapping $T : [0,1] \times [0,1] \to [0,1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a,1) = a$ for all $a \in [0,1]$;

(d) $T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Typical examples of continuous t-norms are $T_P(a,b) = ab$, $T_M(a,b) = \min(a,b)$ and $T_L(a,b) = \max(a+b-1,0)$ (the Lukasiewicz t-norm). Recall (see [30, 31]) that if T is a t-norm and $\{x_n\}$ is a given sequence of numbers in $[0,1]$, then $T_{i=1}^{n} x_i$ is defined recurrently by $T_{i=1}^{1} x_i = x_1$ and $T_{i=1}^{n} x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$. $T_{i=n}^{\infty} x_i$ is defined as $T_{i=1}^{\infty} x_{n+i-1}$. It is known [31] that for the Lukasiewicz t-norm, the following implication holds:

$$\lim_{n\to\infty} (T_L)_{i=1}^{\infty} x_{n+i-1} = 1 \Longleftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty. \tag{1.7}$$

Definition 1.2 (see [29]). A *random normed space* (briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm, and μ is a mapping from X into D^+ such that the following conditions hold:

(RN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;

(RN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all $x \in X$, $\alpha \neq 0$;

(RN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x,y \in X$ and $t,s \geq 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X,μ,T_M), where

$$\mu_x(t) = \frac{t}{t + \|x\|} \tag{1.8}$$

for all $t > 0$, and T_M is the minimum t-norm. This space is called the induced random normed space.

Definition 1.3. Let (X,μ,T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x}(\epsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $\mu_{x_n - x_m}(\epsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) An RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4 (see [28]). *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty}\mu_{x_n}(t) = \mu_x(t)$ almost everywhere.*

Recently, Gordji et al. establish the stability of cubic, quadratic and additive-quadratic functional equations in RN-spaces (see [32, 33]).

In this paper, we deal with the following functional equation:

$$f(2x+y) + f(2x-y) = 4[f(x+y) + f(x-y)] + 2[f(2x) - 4f(x)] - 6f(y) \tag{1.9}$$

on RN-spaces. It is easy to see that the function $f(x) = ax^4 + bx^2$ is a solution of (1.9).

In Section 2, we investigate the general solution of the functional equation (1.9) when f is a mapping between vector spaces and in Section 3, we establish the stability of the functional equation (1.9) in RN-spaces.

2. General Solution

We need the following lemma for solution of (1.9). Throughout this section, X and Y are vector spaces.

Lemma 2.1. *If a mapping $f : X \to Y$ satisfies* (1.9) *for all $x, y \in X$, then f is quadratic-quartic.*

Proof. We show that the mappings $g : X \to Y$ defined by $g(x) := f(2x) - 16f(x)$ and $h : X \to Y$ defined by $h(x) := f(2x) - 4f(x)$ are quadratic and quartic, respectively.

Letting $x = y = 0$ in (1.9), we have $f(0) = 0$. Putting $x = 0$ in (1.9), we get $f(-y) = f(y)$. Thus the mapping f is even. Replacing y by $2y$ in (1.9), we get

$$f(2x+2y) + f(2x-2y) = 4[f(x+2y) + f(x-2y)] + 2[f(2x) - 4f(x)] - 6f(2y) \tag{2.1}$$

for all $x, y \in X$. Interchanging x with y in (1.9), we obtain

$$f(2y+x) + f(2y-x) = 4[f(y+x) + f(y-x)] + 2[f(2y) - 4f(y)] - 6f(x) \tag{2.2}$$

for all $x, y \in X$. Since f is even, by (2.2), one gets

$$f(x+2y) + f(x-2y) = 4[f(x+y) + f(x-y)] + 2[f(2y) - 4f(y)] - 6f(x) \tag{2.3}$$

for all $x, y \in X$. It follows from (2.1) and (2.3) that

$$\begin{aligned} &[f(2(x+y)) - 16f(x+y)] + [f(2(x-y)) - 16f(x-y)] \\ &\quad = 2[f(2x) - 16f(x)] + 2[f(2y) - 16f(y)] \end{aligned} \tag{2.4}$$

for all $x, y \in X$. This means that

$$g(x+y) + g(x-y) = 2g(x) + 2g(y) \tag{2.5}$$

for all $x, y \in X$. Therefore, the mapping $g : X \to Y$ is quadratic.

To prove that $h : X \to Y$ is quartic, we have to show that

$$h(x+2y) + h(x-2y) = 4[h(x+y) + h(x-y) + 6h(y)] - 6h(x) \tag{2.6}$$

for all $x, y \in X$. Since f is even, the mapping h is even. Now if we interchange x with y in the last equation, we get

$$h(2x+y) + h(2x-y) = 4[h(x+y) + h(x-y) + 6h(x)] - 6h(y) \tag{2.7}$$

for all $x, y \in X$. Thus, it is enough to prove that h satisfies (2.7). Replacing x and y by $2x$ and $2y$ in (1.9), respectively, we obtain

$$f(2(2x+y)) + f(2(2x-y)) = 4[f(2(x+y)) + f(2(x-y))] + 2[f(4x) - 4f(2x)] - 6f(2y) \tag{2.8}$$

for all $x, y \in X$. Since $g(2x) = 4g(x)$ for all $x \in X$,

$$f(4x) = 20f(2x) - 64f(x) \tag{2.9}$$

for all $x \in X$. By (2.8) and (2.9), we get

$$f(2(2x+y)) + f(2(2x-y)) = 4[f(2(x+y)) + f(2(x-y))] + 32[f(2x) - 4f(x)] - 6f(2y) \tag{2.10}$$

for all $x, y \in X$. By multiplying both sides of (1.9) by 4, we get

$$4[f(2x+y) + f(2x-y)] = 16[f(x+y) + f(x-y)] + 8[f(2x) - 4f(x)] - 24f(y) \tag{2.11}$$

for all $x, y \in X$. If we subtract the last equation from (2.10), we obtain

$$\begin{aligned} h(2x+y) + h(2x-y) &= [f(2(2x+y)) - 4f(2x+y)] + [f(2(2x-y)) - 4f(2x-y)] \\ &= 4[f(2(x+y)) - 4f(x+y)] + 4[f(2(x-y)) - 4f(x-y)] \\ &\quad + 24[f(2x) - 4f(x)] - 6[f(2y) - 4f(y)] \\ &= 4[h(x+y) + h(x-y) + 6h(x)] - 6h(y) \end{aligned} \tag{2.12}$$

for all $x, y \in X$.

Therefore, the mapping $h : X \to Y$ is quartic. This completes the proof of the lemma. □

Theorem 2.2. *A mapping $f : X \to Y$ satisfies* (1.9) *for all $x, y \in X$ if and only if there exist a unique symmetric multiadditive mapping $M : X^4 \to Y$ and a unique symmetric bi-additive mapping $B : X \times X \to Y$ such that*

$$f(x) = M(x,x,x,x) + B(x,x) \tag{2.13}$$

for all $x \in X$.

Proof. Let f satisfy (1.9) and assume that $g, h : X \to Y$ are mappings defined by

$$g(x) := f(2x) - 16f(x), \qquad h(x) := f(2x) - 4f(x) \tag{2.14}$$

for all $x \in X$. By Lemma 2.1, we obtain that the mappings g and h are quadratic and quartic, respectively, and

$$f(x) = \frac{1}{12}h(x) - \frac{1}{12}g(x) \tag{2.15}$$

for all $x \in X$.

Therefore, there exist a unique symmetric multiadditive mapping $M : X^4 \to Y$ and a unique symmetric bi-additive mapping $B : X \times X \to Y$ such that $(1/12)h(x) = M(x,x,x,x)$ and $(-1/12)g(x) = B(x,x)$ for all $x \in X$ [5, 18]. So

$$f(x) = M(x,x,x,x) + B(x,x) \tag{2.16}$$

for all $x \in X$. The proof of the converse is obvious. □

3. Stability

Throughout this section, assume that X is a real linear space and (Y, μ, T) is a complete RN-space.

Theorem 3.1. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ ($\rho(x,y)$ is denoted by $\rho_{x,y}$) with the property:*

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x,y}(t) \tag{3.1}$$

for all $x, y \in X$ and all $t > 0$. If

$$\lim_{n\to\infty} T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{2n+i+1}t\right)\right), T\left(\rho_{2^{n+i-1}x,2\cdot 2^{n+i-1}x}\left(\frac{2^{2n+i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{2n+i}t}{3}\right)\right)\right) = 1,$$
$$\lim_{n\to\infty} \rho_{2^n x,2^n y}\left(2^{2n}t\right) = 1 \tag{3.2}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quadratic mapping $Q_1 : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-Q_1(x)}(t) \geq T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(2^{i+1}t\right),\ T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^i t}{4}\right),\ \rho_{0,2^{i-1}x}\left(\frac{2^i t}{3}\right)\right)\right)\right) \tag{3.3}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = x$ in (3.1), we obtain

$$\mu_{f(3x)-6f(2x)+15f(x)}(t) \geq \rho_{x,x}(t) \tag{3.4}$$

for all $x \in X$ and all $t > 0$. Letting $y = 2x$ in (3.1), we get

$$\mu_{f(4x)-4f(3x)+4f(2x)+8f(x)-4f(-x)}(t) \geq \rho_{x,2x}(t) \tag{3.5}$$

for all $x \in X$ and all $t > 0$. Putting $x = 0$ in (3.1), we obtain

$$\mu_{3f(y)-3f(-y)}(t) \geq \rho_{0,y}(t) \tag{3.6}$$

for all $y \in X$ and all $t > 0$. Replacing y by x in (3.6), we see that

$$\mu_{3f(x)-3f(-x)}(t) \geq \rho_{0,x}(t) \tag{3.7}$$

for all $x \in X$ and all $t > 0$. It follows from (3.5) and (3.7) that

$$\mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}(t) \geq T\left(\rho_{x,2x}\left(\frac{t}{2}\right), \rho_{0,x}\left(\frac{2t}{3}\right)\right) \tag{3.8}$$

for all $x \in X$ and all $t > 0$. If we add (3.4) to (3.8), then we have

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq T\left(\rho_{x,x}(2t), T\left(\rho_{x,2x}\left(\frac{t}{4}\right), \rho_{0,x}\left(\frac{t}{3}\right)\right)\right). \tag{3.9}$$

Let

$$\varphi_{x,x}(t) = T\left(\rho_{x,x}(2t), T\left(\rho_{x,2x}\left(\frac{t}{4}\right), \rho_{0,x}\left(\frac{t}{3}\right)\right)\right) \tag{3.10}$$

for all $x \in X$ and all $t > 0$. Then we get

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq \varphi_{x,x}(t) \tag{3.11}$$

for all $x \in X$ and all $t > 0$. Let $g : X \to Y$ be a mapping defined by $g(x) := f(2x) - 16f(x)$. Then we conclude that

$$\mu_{g(2x)-4g(x)}(t) \geq \psi_{x,x}(t) \tag{3.12}$$

for all $x \in X$ and all $t > 0$. Thus we have

$$\mu_{g(2x)/2^2-g(x)}(t) \geq \psi_{x,x}\left(2^2 t\right) \tag{3.13}$$

for all $x \in X$ and all $t > 0$. Hence

$$\mu_{g(2^{k+1}x)/2^{2(k+1)}-g(2^k x)/2^{2k}}(t) \geq \psi_{2^k x,2^k x}\left(2^{2(k+1)} t\right) \tag{3.14}$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. This means that

$$\mu_{g(2^{k+1}x)/2^{2(k+1)}-g(2^k x)/2^{2k}}\left(\frac{t}{2^{k+1}}\right) \geq \psi_{2^k x,2^k x}\left(2^{k+1} t\right) \tag{3.15}$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. By the triangle inequality, from $1 > 1/2 + 1/2^2 + \cdots + 1/2^n$, it follows that

$$\mu_{g(2^n x)/2^{2n}-g(x)}(t) \geq T_{k=1}^{n}\left(\mu_{g(2^k x)/2^{2k}-g(2^{k-1}x)/2^{2(k-1)}}\left(\frac{t}{2^k}\right)\right) \geq T_{i=1}^{n}\left(\psi_{2^{i-1}x,2^{i-1}x}\left(2^i t\right)\right) \tag{3.16}$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{g(2^n x)/2^{2n}\}$, we replace x with $2^m x$ in (3.16) to obtain that

$$\mu_{g(2^{n+m}x)/2^{2(n+m)}-g(2^m x)/2^{2m}}(t) \geq T_{i=1}^{n}\left(\psi_{2^{i+m-1}x,2^{i+m-1}x}\left(2^{i+2m} t\right)\right). \tag{3.17}$$

Since the right-hand side of the inequality (3.17) tends to 1 as m and n tend to infinity, the sequence $\{g(2^n x)/2^{2n}\}$ is a Cauchy sequence. Thus we may define $Q_1(x) = \lim_{n\to\infty}(g(2^n x)/2^{2n})$ for all $x \in X$.

Now we show that Q_1 is a quadratic mapping. Replacing x, y with $2^n x$ and $2^n y$ in (3.1), respectively, we get

$$\begin{aligned}&\mu_{((g(2^n(2x+y))+g(2^n(2x-y))-4g(2^n(x+y))-4g(2^n(x-y))-2g(2^{n+1}x)+8g(2^n x)+6g(2^n y))/4^n)}(t)\\ &\quad\geq \rho_{(2^n x,2^n y)}\left(2^{2n} t\right).\end{aligned} \tag{3.18}$$

Taking the limit as $n \to \infty$, we find that Q_1 satisfies (1.9) for all $x, y \in X$. By Lemma 2.1, the mapping $Q_1 : X \to Y$ is quadratic.

Letting the limit as $n \to \infty$ in (3.16), we get (3.3) by (3.10).

Finally, to prove the uniqueness of the quadratic mapping Q_1 subject to (3.3), let us assume that there exists another quadratic mapping Q_1' which satisfies (3.3). Since $Q_1(2^n x) = 2^{2n}Q_1(x), Q_1'(2^n x) = 2^{2n}Q_1'(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, from (3.3), it follows that

$$\begin{aligned}
&\mu_{Q_1(x)-Q_1'(x)}(2t) \\
&\quad = \mu_{Q_1(2^n x)-Q_1'(2^n x)}\left(2^{2n+1}t\right) \\
&\quad \geq T\left(\mu_{Q_1(2^n x)-g(2^n x)}\left(2^{2n}t\right), \mu_{g(2^n x)-Q_1'(2^n x)}\left(2^{2n}t\right)\right) \\
&\quad \geq T\left(T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{2n+i+1}t\right), T\left(\rho_{2^{n+i-1}x,2\cdot 2^{n+i-1}x}\left(\frac{2^{2n+i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{2n+i}t}{3}\right)\right)\right)\right),\right. \\
&\qquad\qquad \left. T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{2n+i+1}t\right), T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(\frac{2^{2n+i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{2n+i}t}{3}\right)\right)\right)\right)\right)
\end{aligned} \tag{3.19}$$

for all $x \in X$ and all $t > 0$. Letting $n \to \infty$ in (3.19), we conclude that $Q_1 = Q_1'$, as desired. □

Theorem 3.2. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ ($\rho(x,y)$ is denoted by $\rho_{x,y}$) with the property:*

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x,y}(t) \tag{3.20}$$

for all $x, y \in X$ and all $t > 0$. If

$$\begin{aligned}
&\lim_{n\to\infty} T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{4n+3i+1}t\right),\right.\right. \\
&\qquad\qquad \left.\left. T\left(\rho_{2^{n+i-1}x,2\cdot 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{3}\right)\right)\right)\right) = 1, \\
&\lim_{n\to\infty} \rho_{2^n x,2^n y}\left(2^{4n}t\right) = 1
\end{aligned} \tag{3.21}$$

for all $x, y \in X$ and all $t > 0$, then there exists a unique quartic mapping $Q_2 : X \to Y$ such that

$$\mu_{f(2x)-4f(x)-Q_2(x)}(t) \geq T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(2^{3i+1}t\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^{3i}t}{4}\right), \rho_{0,2^{i-1}x}\left(\frac{2^{3i}t}{3}\right)\right)\right)\right) \tag{3.22}$$

for all $x \in X$ and all $t > 0$.

Proof. Putting $y = x$ in (3.20), we obtain

$$\mu_{f(3x)-6f(2x)+15f(x)}(t) \geq \rho_{x,x}(t) \tag{3.23}$$

for all $x \in X$ and all $t > 0$. Letting $y = 2x$ in (3.20), we get

$$\mu_{f(4x)-4f(3x)+4f(2x)+8f(x)-4f(-x)}(t) \geq \rho_{x,2x}(t) \tag{3.24}$$

for all $x \in X$ and all $t > 0$. Putting $x = 0$ in (3.20), we obtain

$$\mu_{3f(y)-3f(-y)}(t) \geq \rho_{0,y}(t) \tag{3.25}$$

for all $y \in X$ and all $t > 0$. Replacing y by x in (3.25), we get

$$\mu_{3f(x)-3f(-x)}(t) \geq \rho_{0,x}(t) \tag{3.26}$$

for all $x \in X$ and all $t > 0$. It follows from (3.5) and (3.26) that

$$\mu_{f(4x)-4f(3x)+4f(2x)+4f(x)}(t) \geq T\left(\rho_{x,2x}\left(\frac{t}{2}\right), \rho_{0,x}\left(\frac{2t}{3}\right)\right) \tag{3.27}$$

for all $x \in X$ and all $t > 0$. If we add (3.23) to (3.27), then we have

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq T\left(\rho_{x,x}(2t), T\left(\rho_{x,2x}\left(\frac{t}{4}\right), \rho_{0,x}\left(\frac{t}{3}\right)\right)\right). \tag{3.28}$$

Let

$$\varphi_{x,x}(t) = T\left(\rho_{x,x}(2t), T\left(\rho_{x,2x}\left(\frac{t}{4}\right), \rho_{0,x}\left(\frac{t}{3}\right)\right)\right) \tag{3.29}$$

for all $x \in X$ and all $t > 0$. Then we get

$$\mu_{f(4x)-20f(2x)+64f(x)}(t) \geq \varphi_{x,x}(t) \tag{3.30}$$

for all $x \in X$ and all $t > 0$. Let $h : X \to Y$ be a mapping defined by $h(x) := f(2x) - 4f(x)$. Then we conclude that

$$\mu_{h(2x)-16h(x)}(t) \geq \varphi_{x,x}(t) \tag{3.31}$$

for all $x \in X$ and all $t > 0$. Thus we have

$$\mu_{h(2x)/2^4-h(x)}(t) \geq \varphi_{x,x}\left(2^4 t\right) \tag{3.32}$$

for all $x \in X$ and all $t > 0$. Hence

$$\mu_{h(2^{k+1}x)/2^{4(k+1)}-h(2^k x)/2^{4k}}(t) \geq \varphi_{2^k x, 2^k x}\left(2^{4(k+1)}t\right) \tag{3.33}$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. This means that

$$\mu_{h(2^{k+1}x)/2^{4(k+1)}-h(2^k x)/2^{4k}}\left(\frac{t}{2^{k+1}}\right) \geq \varphi_{2^k x, 2^k x}\left(2^{3(k+1)}t\right) \tag{3.34}$$

for all $x \in X$, all $t > 0$ and all $k \in \mathbb{N}$. By the triangle inequality, from $1 > 1/2 + 1/2^2 + \cdots + 1/2^n$, it follows that

$$\begin{aligned}\mu_{h(2^n x)/2^{4n}-h(x)}(t) &\geq T_{k=1}^{n}\left(\mu_{h(2^k x)/2^{4k}-h(2^{k-1}x)/2^{4(k-1)}}\left(\frac{t}{2^k}\right)\right) \\ &\geq T_{i=1}^{n}\left(\varphi_{2^{i-1}x, 2^{i-1}x}\left(2^{3i}t\right)\right)\end{aligned} \tag{3.35}$$

for all $x \in X$ and all $t > 0$. In order to prove the convergence of the sequence $\{h(2^n x)/2^{4n}\}$, we replace x with $2^m x$ in (3.35) to obtain that

$$\mu_{h(2^{n+m}x)/2^{4(n+m)}-h(2^m x)/2^{4m}}(t) \geq T_{i=1}^{n}\left(\varphi_{2^{i+m-1}x, 2^{i+m-1}x}\left(2^{3i+4m}t\right)\right). \tag{3.36}$$

Since the right-hand side of (3.36) tends to 1 as m and n tend to infinity, the sequence $\{h(2^n x)/2^{4n}\}$ is a Cauchy sequence. Thus we may define $Q_2(x) = \lim_{n\to\infty}(h(2^n x)/2^{4n})$ for all $x \in X$.

Now we show that Q_2 is a quartic mapping. Replacing x, y with $2^n x$ and $2^n y$ in (3.20), respectively, we get

$$\begin{aligned}&\mu_{(h(2^n(2x+y))+h(2^n(2x-y))-4h(2^n(x+y))-4h(2^n(x-y))-2h(2^{n+1}x)+8h(2^n x)+6h(2^n y))/16^n}(t) \\ &\quad \geq \rho_{2^n x, 2^n y}\left(2^{4n}t\right).\end{aligned} \tag{3.37}$$

Taking the limit as $n \to \infty$, we find that Q_2 satisfies (1.9) for all $x, y \in X$. By Lemma 2.1 we get that the mapping $Q_2 : X \to Y$ is quartic.

Letting the limit as $n \to \infty$ in (3.35), we get (3.22) by (3.29).

Finally, to prove the uniqueness of the quartic mapping Q_2 subject to (3.24), let us assume that there exists a quartic mapping Q_2' which satisfies (3.22). Since $Q_2(2^n x) = 2^{4n} Q_2(x)$ and $Q_2'(2^n x) = 2^{4n} Q_2'(x)$ for all $x \in X$ and all $n \in \mathbb{N}$, from (3.22), it follows that

$$\begin{aligned}
&\mu_{Q_2(x)-Q_2'(x)}(2t)\\
&\quad= \mu_{Q_2(2^n x)-Q_2'(2^n x)}\left(2^{4n+1}t\right)\\
&\quad\geq T\left(\mu_{Q_2(2^n x)-h(2^n x)}\left(2^{4n}t\right), \mu_{h(2^n x)-Q_2'(2^n x)}\left(2^{4n}t\right)\right),\\
&\quad\geq T\left(T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{4n+3i+1}t\right), T\left(\rho_{2^{n+i-1}x,2\cdot 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{3}\right)\right)\right)\right),\right.\\
&\qquad\left. T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{4n+3i+1}t\right) T\left(\rho_{2^{n+i-1}x,2\cdot 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{3}\right)\right)\right)\right)\right)
\end{aligned} \tag{3.38}$$

for all $x \in X$ and all $t > 0$. Letting $n \to \infty$ in (3.38), we get that $Q_2 = Q_2'$, as desired. □

Theorem 3.3. *Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\rho : X \times X \to D^+$ ($\rho(x, y)$ is denoted by $\rho_{x,y}$) with the property:*

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-2f(2x)+8f(x)+6f(y)}(t) \geq \rho_{x,y}(t) \tag{3.39}$$

for all $x, y \in X$ and all $t > 0$. If

$$\begin{aligned}
&\lim_{n\to\infty} T_{i=1}^{\infty}\left(T\left(\rho_{2^{n+i-1}x,2^{n+i-1}x}\left(2^{4n+3i+1}t\right),\right.\right.\\
&\qquad\left.\left. T\left(\rho_{2^{n+i-1}x,2\cdot 2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{4}\right), \rho_{0,2^{n+i-1}x}\left(\frac{2^{4n+3i}t}{3}\right)\right)\right)\right) = 1,\\
&\lim_{n\to\infty} \rho_{2^n x,2^n y}\left(2^{2n}t\right) = 1
\end{aligned} \tag{3.40}$$

for all $x, y \in X$ and all $t > 0$, then there exist a unique quadratic mapping $Q_1 : X \to Y$ and a unique quartic mapping $Q_2 : X \to Y$ such that

$$\begin{aligned}
&\mu_{f(x)-Q_1(x)-Q_2(x)}(t)\\
&\quad\geq T\left(T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(\frac{2^i t}{12}\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^i t}{4\cdot 24}\right), \rho_{0,2^{i-1}x}\left(\frac{2^i t}{3\cdot 24}\right)\right)\right)\right),\right.\\
&\qquad\left. T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(\frac{2^{3i}t}{24}\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^{3i}t}{4\cdot 24}\right), \rho_{0,2^{i-1}x}\left(\frac{2^{3i}t}{3\cdot 24}\right)\right)\right)\right)\right)
\end{aligned} \tag{3.41}$$

for all $x \in X$ and all $t > 0$.

Proof. By Theorems 3.1 and 3.2, there exist a quadratic mapping $Q_1' : X \to Y$ and a quartic mapping $Q_2' : X \to Y$ such that

$$\mu_{f(2x)-16f(x)-Q_1'(x)}(t) \geq T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(2^{i+1}t\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^i t}{4}\right), \rho_{0,2^{i-1}x}\left(\frac{2^i t}{3}\right)\right)\right)\right),$$
$$\mu_{f(2x)-4f(x)-Q_2'(x)}(t) \geq T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(2^{3i+1}t\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^{3i} t}{4}\right), \rho_{0,2^{i-1}x}\left(\frac{2^{3i} t}{3}\right)\right)\right)\right) \tag{3.42}$$

for all $x \in X$ and all $t > 0$. It follows from the last inequalities that

$$\begin{aligned}
&\mu_{f(x)+(1/12)Q_1'(x)-(1/12)Q_2'(x)}(t) \\
&\quad\geq T\left(\mu_{f(2x)-16f(x)-Q_1'(x)}\left(\frac{t}{24}\right), \mu_{f(2x)-4f(x)-Q_2'(x)}\left(\frac{t}{24}\right)\right) \\
&\quad\geq T\left(T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(\frac{2^i t}{12}\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^i t}{4\cdot 24}\right), \rho_{0,2^{i-1}x}\left(\frac{2^i t}{3\cdot 24}\right)\right)\right)\right),\right. \\
&\qquad\left. T_{i=1}^{\infty}\left(T\left(\rho_{2^{i-1}x,2^{i-1}x}\left(\frac{2^{3i} t}{24}\right), T\left(\rho_{2^{i-1}x,2\cdot 2^{i-1}x}\left(\frac{2^{3i} t}{4\cdot 24}\right), \rho_{0,2^{i-1}x}\left(\frac{2^{3i} t}{3\cdot 24}\right)\right)\right)\right)\right)
\end{aligned} \tag{3.43}$$

for all $x \in X$ and all $t > 0$. Hence we obtain (3.41) by letting $Q_1(x) = -(1/12)Q_1'(x)$ and $Q_2(x) = (1/12)Q_2'(x)$ for all $x \in X$. The uniqueness property of Q_1 and Q_2 is trivial. □

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