Quadrature Mirror Filter Banks, M-Band Extensions and Perfect-Reconstruction Techniques

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Abstract

In this paper, quadrature mirror filters (QMF) are reviewed. After a brief introduction to multirate building blocks, the two-band QMF bank is discussed. Various distortions caused by the structure, and methods to eliminate these distortions are outlined. Perfect-reconstruction structures for the two-band case are reviewed, and the results are extended to the case of arbitrary number of channels. The relation between perfect-reconstruction QMF banks, and the concept of *losslessness* in transfer-matrices is indicated. New lattice structures are presented, which perform the perfect reconstruction, sometimes even under coefficient quantization.

INTRODUCTION

QUADRATURE MIRROR FILTER (QMF) banks have been of great interest during the past decade, ever since their introduction by Croisier, Esteban and Galand [1], [2]. These find applications in situations where a discrete-time signal $x(n)$ is to be split into a number of consecutive bands in the frequency domain, so that each subband signal $x_k(n)$ can be processed in an independent manner. Typical 'processing' includes undersampling the subband signals, encoding them and transmitting over a channel, or merely storing the coded signals. Eventually, at some point in the process, the subband signals should somehow be recombined so that the original signal is properly reconstructed. Typical applications of such signal-splitting include subband coders for speech signals [2], [3], [29], digital transmultiplexers [26] used in FDM/TDM conversion, and frequency domain speech-scramblers [30].

In this paper we look at the QMF problem purely as a signal-processing problem. The exact signal characteristics and the nature of the application are not given any emphasis. This blissful freedom from specific aspects of the external world enables us to concentrate on the basic science of the problem from the viewpoint of signal-reconstruction. Our purpose here is to outline the signal-processing issues and mathematics involved. We address issues such as this: what kinds of errors are involved in the implementation of a QMF bank? Which of these errors can be reduced to acceptable levels? Which of these errors can be *completely* eliminated (at finite cost)? Is it possible to completely eliminate *all* errors that result due to the transmission of a signal through a QMF analysis/synthesis bank? If so, what constraints does this impose on the filters that take part in the filter-bank?

The QMF problem, basically simple looking, is in fact a fundamental *signal reconstruction* problem, and opens up several intriguing possibilities when addressed from a theoretical viewpoint. It is related, surprisingly, to the theory of lattice structures and orthogonal matrix-functions, even though these relations are not explicit from the problem statement. Our attempt here is to tie up several apparently unrelated results in a unified manner, so as to place in evidence the enormous scope and implicit beauty of this problem.

THE TWO CHANNEL QMF BANK

The QMF bank is a multirate digital filter bank. The term *multirate* signifies that the sampling rate is not constant throughout the system [3]; there are decimators in the system which down-sample a sequence, and there are interpolators which perform up-sampling. Since decimators and interpolators are the building blocks of any multirate digital system, let us briefly review their characteristics. Details and proofs can be found in the text by Crochiere and Rabiner [3].

Decimators

An M-fold decimator is shown in Fig. 1(a). Its input is a sequence $x(n)$, and the output sequence $y(n)$ is a compressed version of $x(n)$. More specifically, the output is obtained by retaining only those samples of $x(n)$ which occur at times which are multiples of the integer M. The input-output relation is $y(n) = x(Mn)$. The figure demonstrates the decimation operation for $M = 2$ and also shows that a decimator is a time-varying device, even though it is linear. Accordingly, it cannot be represented by a *transfer function;* we should seek other means of describing the decimator in the transform-domain. Since a decimator causes a compression in the time-domain, we expect a 'stretching' in the frequency domain. For example, with $M = 2$, the quantity $X(e^{j\omega/2})$ is a stretched version of $X(e^{j\omega})$. However, since $X(e^{j\omega/2})$ has a period of 4π rather than 2π, it is not a valid transform of a sequence. It can be verified that $Y(e^{j\omega})$ in fact has two terms. The first is $X(e^{j\omega/2})$, and the second term is $X(-e^{j\omega/2})$ which is a shifted version of the first term (shifted by an amount 2π). More formally, the input-output relation for a two-fold decimator can be written in the transform domain as [3]

$$Y(e^{j\omega}) = 0.5[X(e^{j\omega/2}) + X(-e^{j\omega/2})]. \qquad (1)$$

Note that $Y(e^{j\omega})$ given as above does have a period of 2π. This is demonstrated in Fig. 1(b), (c) where $x(n)$ is assumed to be a lowpass type of signal. If the transform of $x(n)$ is not bandlimited to $-\pi/2 \le \omega \le \pi/2$, there is an overlap of the two terms in (1) as shown by the shaded area in Fig. 1(c). This overlap is the *aliasing* effect, caused by undersampling. There is *no way* we can get back the original signal $x(n)$ from $y(n)$, once aliasing has taken place. In order to convince ourself that the scale factor of 0.5 is required in the expression (1), let us imagine, as an example, that $x(n)$ is the unit pulse $\delta(n)$. Then $X(z)$ is unity for all values of the argument z, hence from (1) we have $Y(e^{j\omega}) = 0.5(1 + 1) = 1.0$ for all ω, which is consistent with the fact that $y(n) = \delta(n)$ in this case.

For an M-fold decimator we have $y(n) = x(Mn)$, and the transform-domain relation is precisely an extension of (1). Instead of two terms, we now have M terms; the first term is merely a stretched version of $X(e^{j\omega})$ (by a factor of M), namely $X(e^{j\omega/M})$. The remaining terms are uniformly shifted versions of the first term (the amount of shift being integer multiples of 2π). Thus, it can be shown that

$$Y(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X(e^{j(\omega-2\pi k)/M}), \qquad Y(z) = \frac{1}{M}\sum_{k=0}^{M-1} X(z^{1/M}W^k) \qquad (2)$$

where $W = e^{-2\pi j/M}$ and $z = e^{j\omega}$. The scale factor of $1/M$ in (2) can be understood in a manner analogous to the factor 0.5 in (1).

What happens if we decimate a *highpass* signal, say by

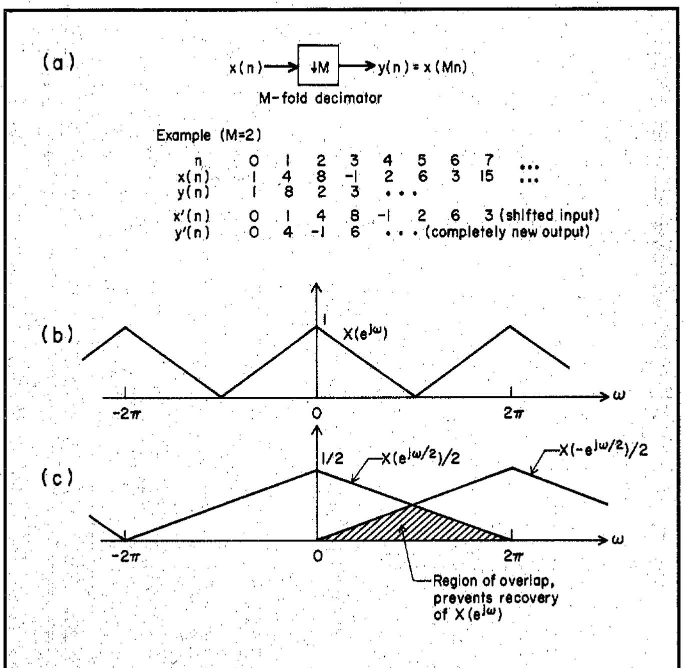

Figure 1. A *decimator* is a system which takes in a sequence $x(n)$ and produces a time-compressed sequence $y(n) = x(Mn)$. The compression ratio M is an integer. This operation is denoted by the downgoing arrow (indicative of down-sampling). The figure demonstrates the operation of a two-fold decimator. In general the output $y(n)$ of an M-fold decimator is obtained by retaining only the samples of the input sequence occurring at times that are integer-multiples of M. A decimator is a *time-varying* system (even though linear). The figure demonstrates this, by showing that a shifted version of the input does not produce a shifted version of the output. The effect of *compression* in the time domain is an *expansion* (or stretching) in the frequency domain. The quantity $\frac{1}{2}X(e^{j\omega/2})$ in the figure represents this, while $\frac{1}{2}X(-e^{j\omega/2})$ represents *aliasing* caused by downsampling. For arbitrary M, equation (2) gives the transform of the decimated sequence.

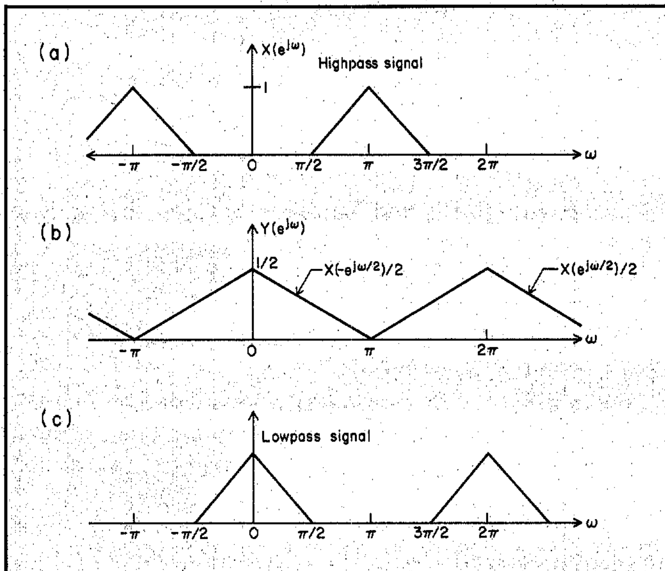

Figure 2. What happens when a highpass signal gets decimated? Fig. 2(a) represents the transform $X(e^{j\omega})$ (assumed to be real for simplicity) of a highpass signal. If this signal is decimated two-fold, the resulting transform has two components, proportional to $X(e^{j\omega/2})$ and $X(-e^{j\omega/2})$, as shown in Fig. 2(b). Accordingly, the decimated signal looks like a regular *lowpass* signal. In fact if a lowpass signal with transform as in Fig. 2(c) were decimated by a factor of two, the result would again have appearance as in Fig. 2(b). This means that given the transform in Fig. 2(b), it is impossible to tell whether is came from Fig. 2(a) or 2(c).

a factor of two? Since it is *not* bandlimited to $-\pi/2 \le \omega \le \pi/2$, the first impression is that there will be aliasing. Let us refer to Fig. 2. Assuming the highpass signal to be bandlimited to $\pi/2 \le \omega \le 3\pi/2$ (Fig. 2(a)), the decimator output $Y(e^{j\omega})$ is as shown in Fig. 2(b). Thus, the decimated version of the highpass signal looks like a lowpass signal. Notice that if a signal having a *lowpass* spectrum as in Fig. 2(c) were decimated, we would obtain precisely the spectrum of Fig. 2(b) again. In other words, simply by looking at Fig. 2(b), it is not possible to tell whether it came from Fig. 2(a) or Fig. 2(c). This is a kind of aliasing; theoretically speaking, we *have* undersampled a highpass signal. However, since there is no overlap between the two curves in Fig. 2(b), we can, in principle, reconstruct the signal $x(n)$ from $y(n)$, as long as we have the additional information as to whether $x(n)$ is a lowpass or highpass signal.

In practice, before a signal is passed through a decimator, it is first bandlimited (by using a bandpass filter, of which lowpass filtering is a special case), so as to reduce the effects of aliasing. Such filters are called *decimation filters.*

Interpolators

An M-fold interpolator, schematically shown in Fig. 3, inserts $M - 1$ zeros between adjacent samples. Its input output relation is given in Fig. 3. The effect of this stretching in the time domain is a *compression* in the frequency domain, as demonstrated in Fig. 3(b). Since $Y(e^{j\omega})$ has $M - 1$ replica (or *images*) of the basic prototype spectrum, the interpolator is said to cause an imaging effect (which is the dual of the aliasing effect of a decimator). The transform domain relation is simple, given by

$$Y(e^{j\omega}) = X(e^{j\omega M}), \quad Y(z) = X(z^M). \tag{3}$$

It can be verified through simple examples, that an interpolator is also a time-varying, linear system.

In practice, an interpolator is followed by a filter called the interpolation filter, which eliminates the images in Fig. 3 so that the result is a simple bandpass signal (or lowpass signal, as desired).

It is interesting to see what happens when a decimator and an interpolator are cascaded (Fig. 4(a)). The decimator causes stretching and aliasing, whereas the interpolator causes compression, all in the frequency domain. The end result is shown in Fig. 4(c). If the spectrum of $x_k(n)$ is bandlimited to $-\pi/M \le \omega \le \pi/M$, then Fig. 4(c) is an imaged version of Fig. 4(b) with no stretching or compression.

Notice in passing, that if the decimator and interpolator in the cascade are *interchanged,* the result is just an identity system; the decimator simply gets rid of the zeros thrown in by the interpolator.

The Two-Channel Quadrature Mirror Filter (QMF) Bank

The two-channel QMF bank, shown in Fig. 5, is one of the earliest and most commonly employed structures [1], [2]. The analysis bank is composed of a lowpass filter $H_0(z)$ and a highpass filter $H_1(z)$, which split the incoming sequence $x(n)$ into two frequency bands. The lowpass signal $x_0(n)$ and the highpass signal $x_1(n)$ are then decimated by factors of two. The decimated signals are typically encoded [31], and transmitted. At the receiver end, the signals are decoded, and passed through the interpolators. The decimator-interpolator cascade causes aliasing and imaging as discussed in Fig. 4 earlier. The purpose of the synthesis filters $F_0(z)$ and $F_1(z)$ is to eliminate the images. $F_0(z)$ is a lowpass filter so that the highpass image of the interpolated lowpass signal $x_0(n)$ is suppressed. Similarly, $F_1(z)$ is highpass so that the *lowpass image* of the interpolated highpass signal is eliminated. As a result, the signals $v_0(n)$ and $v_1(n)$ are good approximations of $x_0(n)$ and $x_1(n)$, and the reconstructed signal $\hat{X}(z)$ (hopefully) resembles $X(z)$ closely. Note in passing that the name quadrature mirror filter derives from the fact that, the response of $H_1(z)$ is the *mirror-image* of the response of $H_0(z)$, with respect to frequency $\pi/2$ (which is a *quarter* of the sampling frequency).

Now what makes the QMF problem nontrivial and fascinating? In order to avoid aliasing (due to decimation), the responses of $H_0(z)$ and $H_1(z)$ must be disjoint as in Fig. 6(a). On the other hand, in order that no frequency range shall be 'left out' by the analysis bank, the responses should be *overlapping* as in Fig. 6(b). The only obvious solution to this dilemma is to make the responses very sharp (approximating the ideal response in Fig. 6(c)), but

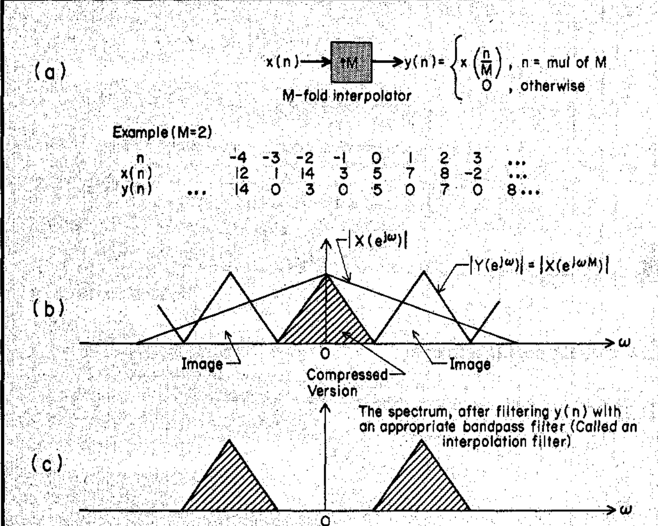

Figure 3. An *M*-fold *interpolator* merely inserts $M - 1$ zero-valued samples between adjacent samples of the input sequence $x(n)$. An upgoing arrow with M written on the side indicates this (Fig. 3(a)). The effect of this stretching in the time domain is a *compression* in the frequency domain as shown. Thus, $M - 1$ images of the original spectrum are caused by this operation. The term *imaging* is often used [3] to describe this, and is the dual of the aliasing effect caused by a decimator. Interpolators are linear, time-varying operators. An interpolation filter is an ordinary digital filter, which rejects the images in Fig. 3(b) and retains a single, desired image. For example, a bandpass filter can be used on the interpolated signal, to produce the spectrum of Fig. 3(c).

it is well known that sharp cutoff filters require very high order, are highly sensitive to quantization, and often cause instability problems (if IIR).

The philosophy adopted [1], [2], [5] in the QMF bank in order to overcome this problem is to *permit* aliasing at the output of the decimator, by designing the analysis filters as in Fig. 6(b), and then choosing the synthesis filters $F_0(z)$ and $F_1(z)$ such that the imaging produced by the interpolators cancels the aliasing. In fact *exact cancelation* is possible. This observation relieves the designer of a very stringent analysis-filter design problem.

Based on the relations (2) and (3) for a decimator and an interpolator respectively, it is possible to express $\hat{X}(z)$ in Fig. 5 as

$$\hat{X}(z) = \frac{1}{2}[H_0(z)F_0(z) + H_1(z)F_1(z)]X(z) + \frac{1}{2}[H_0(-z)F_0(z) + H_1(-z)F_1(z)]X(-z) \tag{4}$$

Because of the second term in (4) involving $X(-z)$, we cannot write down an expression for $\hat{X}(z)/X(z)$ that is independent of $X(z)$ itself. This is not surprising, since the QMF bank is not time-invariant (as the decimators and interpolators are time-variant). The second term in (4) represents the effects of aliasing and imaging. This term can be made to disappear simply by choosing the synthesis filters to be

$$F_0(z) = H_1(-z), \qquad F_1(z) = -H_0(-z) \tag{5}$$

Once the aliasing is so canceled, the QMF bank becomes a (linear and) time-invariant system with transfer function

$$T(z) = \frac{\hat{X}(z)}{X(z)} = \frac{1}{2}[H_0(z)H_1(-z) - H_1(z)H_0(-z)] \tag{6}$$

Ideally, we would like $T(z)$ to be a delay, i.e., $T(z) = z^{-n_0}$, so that the reconstructed signal is a delayed version of $x(n)$. Since $T(z)$ is in general not a delay, it represents a distortion and is called the *distortion function* or the *overall* transfer function. The quantity $|T(e^{j\omega})|$ is the amplitude distortion and $arg[T(e^{j\omega})]$ is the phase distortion. If $T(z)$ is an allpass function, i.e., if $|T(e^{j\omega})| = constant$ for all ω, then there is no amplitude distortion. If $T(z)$ is a linear-phase FIR function, then $arg[T(e^{j\omega})] = K\omega$, and there is no phase distortion. Barnwell [29] has shown that there are several interesting ways in which the filters $H_0(z)$, $H_1(z)$, $F_0(z)$ and $F_1(z)$ can be related, so as to obtain an appropriate functional form for $T(z)$.

It is typical to choose $H_1(z) = H_0(-z)$, so that we have a lowpass/highpass pair. Then

$$T(z) = \frac{1}{2}[H_0^2(z) - H_1^2(z)] = \frac{1}{2}[H_0^2(z) - H_0^2(-z)] \tag{7}$$

which represents the distortion function. Suppose $H_0(z)$ and $H_1(z)$ are linear phase FIR filters, then $T(z)$ given by (7) clearly has linear phase, and phase distortion is easily eliminated. In summary, the choice of transfer functions according to

$$H_1(z) = H_0(-z), \qquad F_0(z) = H_0(z), \qquad F_1(z) = -H_1(z) \tag{8}$$

leads to complete elimination of aliasing; if $H_0(z)$ has linear phase, then phase distortion is also eliminated. Assuming $H_0(z)$ to be a linear phase lowpass FIR filter of order $N - 1$, we can write $H_0(e^{j\omega}) = e^{-j\omega(N-1)/2}H_{0,a}(e^{j\omega})$ where $H_{0,a}(e^{j\omega})$ is the (real-valued) *amplitude response* [32]. With this, $T(e^{j\omega})$ takes on a nice form:

$$T(e^{j\omega}) = \frac{e^{-j\omega(N-1)}}{2}[|H_0(e^{j\omega})|^2 - (-1)^{N-1}|H_1(e^{j\omega})|^2] \tag{9}$$

If $N - 1$ is even, then referring to Fig. 6(b), at the frequency $\omega = \pi/2$, $T(e^{j\omega})$ given by (9) is zero! This implies severe amplitude distortion. Accordingly, with the choice of filters as in (8), we must always pick the order $N - 1$ of the linear phase FIR filter $H_0(z)$ to be odd‡. Equation (9) then yields

$$|T(e^{j\omega})| = \frac{1}{2}[|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2] \tag{10}$$

which represents the residual amplitude distortion. Since $T(z)$ has linear phase, phase-distortion is absent.

Now comes the bad news: if two *linear phase* transfer functions $H_0(z)$ and $H_1(z)$ are such that $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2$ is constant for all ω, then $H_0(z)$ and $H_1(z)$ *must* be trivial transfer functions [24] with frequency responses of the form $|H_0(e^{j\omega})|^2 = \cos^2(K\omega)$ and $|H_1(e^{j\omega})|^2 = \sin^2(K\omega)$. In

‡If an even order is *required* for some other reasons, there is a trick which can be employed to avoid the above distortion; see [5].

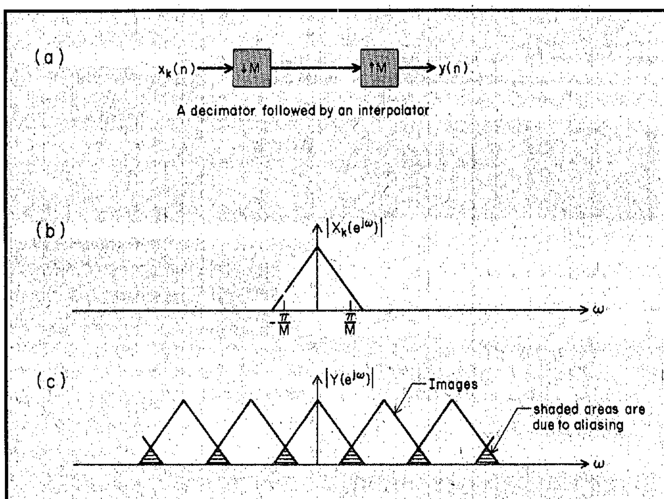

Figure 4. When a decimator is followed by an interpolator, aliasing and imaging effects are simultaneously present; the stretching and aliasing caused by the decimator are followed up by the imaging process caused by the interpolator. It is not possible to recover $x_k(n)$ from $y(n)$, unless the shaded areas in Fig. 4(c) are absent.

other words, for the choice of filters as in (8), there *does not* exist a non-trivial linear-phase transfer function $H_0(z)$ such that phase distortion and amplitude distortion are simultaneously eliminated!

To complement the above discussion, let us now see how *amplitude distortion* can be completely eliminated, when one agrees to tolerate phase distortion. There are many ways of doing this [11], [27], [29], [43]; let us look at one. In the world of IIR digital filters, there exists a beautiful subset of transfer functions which can be implemented as a sum of two allpass functions [33], [34], [39], [40], [43]. If such transfer functions are used in the analysis bank, this automatically forces $T(z)$ to be allpass. Without getting into the theoretical details, let us state the essence compactly here.

Let $H_0(z)$ be a lowpass IIR function with numerator polynomial $P(z)$ and denominator polynomial $D(z)$ of orders N, i.e.,

$$H_0(z) = P(z)/D(z) = \sum_{n=0}^{N} p_n z^{-n} \Big/ \sum_{n=0}^{N} d_n z^{-n}. \quad (11)$$

A typical magnitude response of $H_0(z)$ is indicated in Fig. 8(a). Let us define the highpass function $H_1(z)$ to be such that $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$, *for all* ω. Such a pair $[H_0(z), H_1(z)]$ is called a power-complementary pair. Given $H_0(z)$ such that $|H_0(e^{j\omega})| \leq 1$, we can *always* find such $H_1(z)$ by defining it to be $H_1(z) = Q(z)/D(z)$ where $Q(z)$ is a spectral factor of

$$|Q(e^{j\omega})|^2 = |D(e^{j\omega})|^2 - |P(e^{j\omega})|^2. \quad (12)$$

Usually, the zeros of $H_0(z)$ are on the unit circle, hence $P(z)$ is a symmetric polynomial, i.e., $p_n = p_{N-n}$. Moreover, it is

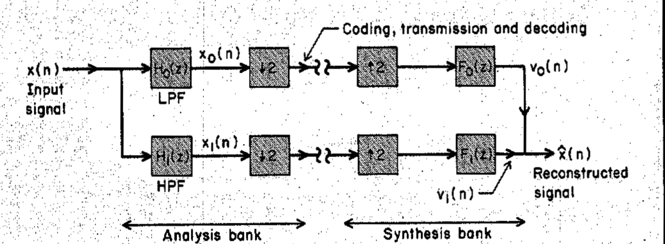

Figure 5. The two-channel Quadrature Mirror Filter (QMF) bank. Here, $H_0(z)$ and $H_1(z)$ represent lowpass and highpass filters respectively (called the *analysis filters*). The filtered signals $x_0(n)$ and $x_1(n)$ are decimated by a factor of two and transmitted, with possible encoding. At the receiver, the signals are interpolated (after decoding if necessary), and filtered by $F_0(z)$ and $F_1(z)$ (called the *synthesis filters*), before recombination. The purpose of $H_0(z)$ and $H_1(z)$ is to make the two frequency-bands as independent as possible, while the purpose of $F_0(z)$ and $F_1(z)$ is to eliminate the images caused by the decimator-interpolator system. The reconstructed signal $\hat{x}(n)$ is related to $x(n)$ by equation (4), where the term involving $X(-z)$ represents the effects of aliasing and imaging. The first term in this equation is a filtered version of $X(z)$. Ideally, we would like the second term to be zero, and the first term to be of the form $cz^{-n_0}X(z)$, so that $\hat{x}(n)$ is a (delayed) replica of $x(n)$.

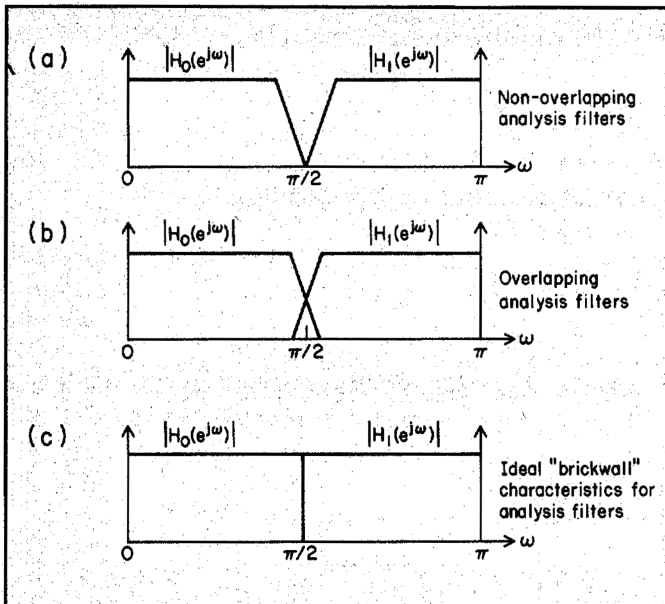

Figure 6. In the two-channel QMF structure, it is a non-trivial task to design $H_0(z)$ and $H_1(z)$. If the responses are disjoint, as in Fig. 6(a), then there is practically a spectral 'hole' which is undesirable. If they overlap as in Fig. 6(b), then there is aliasing due to subsequent decimation, because the filters have bandwidths exceeding $\pi/2$. Since a brickwall response as in Fig. 6(c) is not practicable, the response shown in Fig. 6(b) is usually chosen. The aliasing caused by decimation is then *canceled* by judicious choice of the synthesis filters $F_0(z)$ and $F_1(z)$. If these filters are chosen as in equation (5) then aliasing is *completely* canceled (no matter how $H_0(z)$ and $H_1(z)$ are chosen) and the QMF system becomes a (linear and) *time-invariant* system, characterized by the transfer function $T(z)$ in (6). $T(z)$ is called the *distortion* transfer function or the *overall* transfer function.

very common for $H_1(z)$ to have all zeros on the unit circle too (Fig. 8(a)). For example, if $H_0(z)$ is a digital Butterworth, Chebyshev or elliptic filter, this is always the case; accordingly, $Q(z)$ is a symmetric or antisymmetric polynomial. It is antisymmetric (i.e., $q_n = -q_{N-n}$) if $Q(z)$ has a zero at $\omega = 0$, which usually is the case when N is odd. Now, there is a result [34], [39] which says that if two transfer functions $H_0(z) = P(z)/D(z)$ and $H_1(z) = Q(z)/D(z)$ with symmetric $P(z)$ and antisymmetric $Q(z)$ are such that $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1$, *for all* ω, then they must be of the form

$$H_0(z) = (A_0(z) + A_1(z))/2, \qquad H_1(z) = (A_0(z) - A_1(z))/2 \quad (13)$$

where $A_0(z)$ and $A_1(z)$ are allpass functions. In other words, we can implement the analysis bank as in Fig. 8(b). Assume that in addition to all the above conditions, $H_1(z)$ also satisfies $H_1(z) = H_0(-z)$. You can ensure this by designing $H_0(z)$ such that the response $|H_0(e^{j\omega})|^2$ has a symmetry around $\omega = \pi/2$, i.e., $\omega_p + \omega_S = \pi$ and $\varepsilon_1 = \varepsilon_2$ as shown in Fig. 8(c). Simply by employing such $H_0(z)$ and $H_1(z)$ in the QMF bank of Fig. 5, and picking $F_0(z)$ and $F_1(z)$ as in (5) so as to cancel aliasing, we end up with the

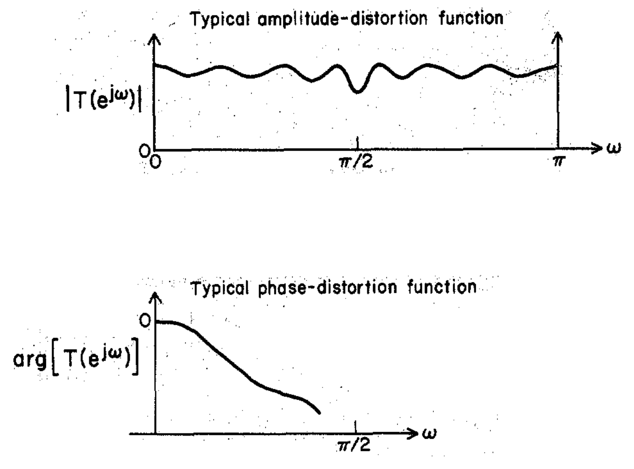

Figure 7. Once aliasing-error has been eliminated by choice of synthesis filters as in equation (5), the QMF bank suffers from two errors, viz., the amplitude distortion $|T(e^{j\omega})|$ and the phase distortion $arg|T(e^{j\omega})|$, which can be evaluated from equation (7) when the filters are related by equation (8). There is no amplitude distortion if $|T(e^{j\omega})|$ is constant for all ω; there is no phase distortion if $arg|T(e^{j\omega})|$ is of the form $K\omega$ for constant K. If $H_0(z)$ is a linear-phase FIR filter, then $T(z)$ in (7) has linear-phase too. Phase distortion can therefore be completely eliminated. The remaining amplitude distortion is given by (10) with $(N-1)$ odd; see text.

distortion function $T(z) = A_0(z)A_1(z)/2$ which is allpass! In other words, amplitude distortion is completely eliminated (and so is aliasing, of course). The phase response of $T(z)$ leads to some phase distortion. A discussion on *how to design* transfer functions $H_0(z)$ and $H_1(z)$ satisfying (13) and the condition $H_1(z) = H_0(-z)$, can be found in [11], [43].

It is worth pointing out some of the good features of implementations based on allpass decompositions, such as (13). The allpass filters $A_0(z)$ and $A_1(z)$ can be implemented using the Gray-and-Markel lattice structures [38] which come in various convenient forms. All of these forms can be made free from limit cycles [44] (which are parasitic oscillations under zero-input, caused by quantizer-nonlinearities in feedback loops of IIR filters). Moreover, instability problems due to coefficient quantization are absent in these lattice structures. It is also known that implementations based on the decomposition of (13) have low passband sensitivity [34] (even though this is not very crucial for QMF applications).

The relations (13) imply that $A_0(z) = H_0(z) + H_1(z)$, and $A_1(z) = H_0(z) - H_1(z)$. Since in addition $H_1(z) = H_0(-z)$, you can verify that $A_0(z)$ in fact has the form $a_0(z^2)$ and $A_1(z)$ has the form $z^{-1}a_1(z^2)$. Accordingly, the QMF bank can be implemented as in Fig. 8(d). If $a_0(z)$ and $a_1(z)$ are implemented using a cascade of the *one-multiplier* Gray-Markel lattice structure [38], then the entire QMF bank (i.e., all four transfer functions $H_0(z)$, $H_1(z)$, $F_0(z)$ and $F_1(z)$) can be implemented with a total of about N multipliers, where N is the order of $H_0(z)$! In other words, the structure of Fig. 8(d) is dramatically efficient.

The Perfect-Reconstruction Two Channel QMF Bank

So we see that either phase distortion or amplitude distortion can be completely eliminated, according to choice. The remaining distortion can be either minimized using computer-aided techniques, or *equalized* by cascading with a filter.

For example, once phase distortion has been eliminated, amplitude distortion can be minimized by use of nonlinear optimization software [23]. Usually, an objective function is formulated which is a sum of the stopband error of $H_0(z)$ and the amplitude distortion error $\int[|T(e^{j\omega})|^2 - 1]^2 d\omega$; the coefficients of $H_0(z)$ are found such that this objective function is minimized. The remaining filters $H_1(z)$, $F_0(z)$ and $F_1(z)$ are found from (8). An

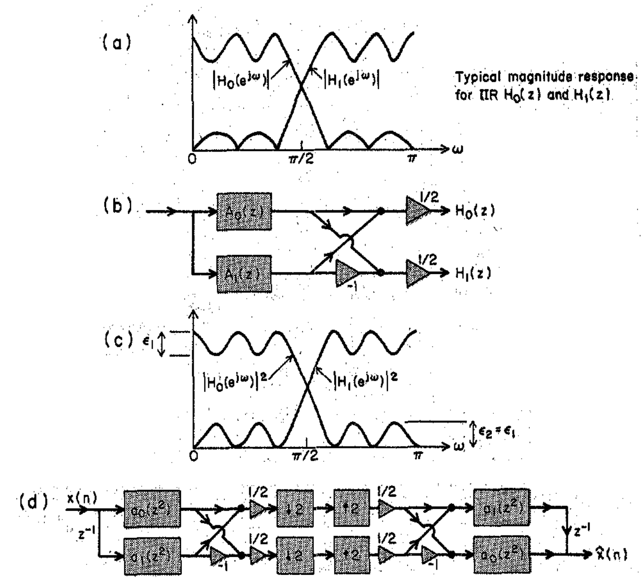

Figure 8. Once aliasing is canceled in Fig. 5, it is possible to *completely* eliminate amplitude distortion rather than phase distortion. For this, let $H_0(z) = P(z)/D(z)$ and $H_1(z) = Q(z)/D(z)$ be IIR transfer functions, with typical magnitude responses as in Fig. 8(a). All zeros of $P(z)$ and $Q(z)$ are on the unit circle, and moreover $Q(z)$ has a zero at $\omega = 0$. Hence, $P(z)$ is a symmetric polynomial (i.e., $p_n = p_{N-n}$) and $Q(z)$ is antisymmetric (i.e., $q_n = -q_{N-n}$). Such pairs of transfer functions $[H_0(z), H(z)]$ can be implemented [34] as a sum and difference of two allpass functions $A_0(z)$ and $A_1(z)$ (Fig. 8(b)), as long as they form a power complementary pair (See text). In addition, if $H_1(z) = H_0(-z)$ (which is required in (8)) the response $|H_0(e^{j\omega})|^2$ exhibits symmetry with respect to $\pi/2$. This in turn forces $A_0(z)$ to be of the form $a_0(z^2)$ and $A_1(z)$ to be of the form $z^{-1}a_1(z^2)$ whence [43] the entire QMF bank can be drawn as in Fig. 8(d). This QMF structure which involves four transfer functions $H_0(z)$, $H_1(z)$, $F_0(z)$ and $F_1(z)$ each of order N, can be implemented with a *total* of about N multipliers, and in addition has very low passband sensitivity. The all-pass functions can be implemented such that, under quantized environment, there are no instability problems, and no limit cycles [38], [44].

alternative to this optimization would be to cascade a linear-phase FIR filter to the output $\hat{x}(n)$ and equalize the amplitude distortion.

Now comes the crucial question: can we *simultaneously eliminate* both amplitude and phase distortions in a two-channel QMF bank? In an interesting article, Smith and Barnwell have shown [6] that the answer is *yes*. Such a QMF bank is said to have *perfect-reconstruction* property, because $\hat{x}(n)$ is a replica of $x(n)$ except for a delay. We can accomplish this by exploiting a property of linear-phase FIR halfband filters. Assuming that the required order of $H_0(z)$ is $N - 1$, let us first design a linear-phase FIR half-band filter $G(z)$ of order $2(N - 1)$ with amplitude response $G_a(e^{j\omega})$ as in Fig. 9(a); the response exhibits a symmetry with respect to $\pi/2$ (i.e., the passband ripple is equal to the stop band ripple, and $\omega_p + \omega_S = \pi$). From $G(z)$ we can construct a new halfband filter $G_+(z) = G(z) + \delta z^{-(N-1)}$ so that the amplitude response of $G_+(z)$ is nonnegative (as shown in Fig. 9(b)). (The term 'amplitude response' has been defined earlier; see, for example, discussion preceding equation (9)). We can therefore find a spectral factor[‡] $H_0(z)$ of $G_+(z)$ having only real coefficients. Because of symmetry around $\pi/2$ we have,

$$G_+(z) + (-1)^{N-1}G_+(-z) = dz^{-(N-1)} \qquad (14)$$

[‡]i.e., a function $H_0(z)$ such that $G_+(z) = z^{-(N-1)}H_0(z^{-1})H_0(z)$.

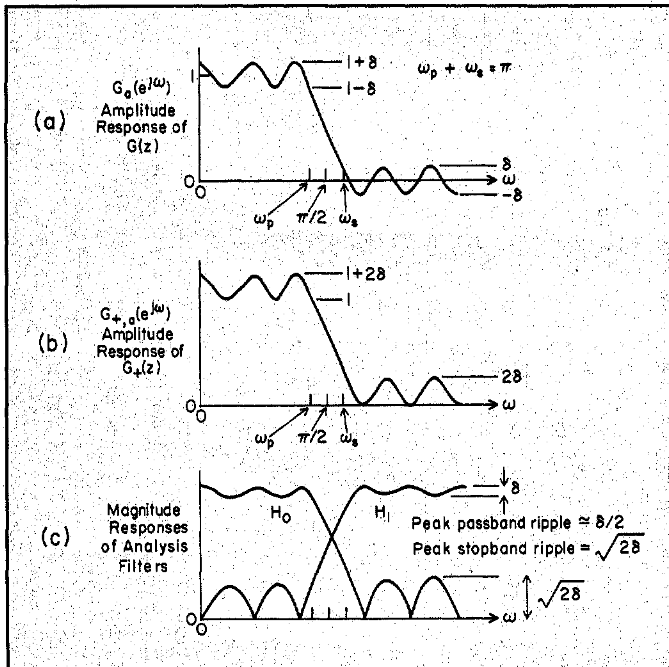

Figure 9. If the QMF bank is such that all three distortions (viz., aliasing, amplitude distortion and phase distortion) are eliminated, then we have $\hat{x}(n) = cx(n - n_0)$. Such a system is said to have *Perfect Reconstruction Property* (PRP for short). For the two-channel QMF bank, a system with PRP can be constructed by designing $H_0(z)$ to be a spectral factor of an appropriately conditioned FIR half-band filter, and then choosing the remaining transfer functions as in (16); see text.

where d is a constant, hence

$$H_0(z^{-1})H_0(z) + H_0(-z^{-1})H_0(-z) = d\,. \qquad (15)$$

Our objective is to pick $H_0(z)$ and $H_1(z)$ such that, after aliasing has been canceled (by the choice of (5)), the distortion $T(z)$ in (6) is a delay. Since $H_0(z)$ satisfies (15), we can accomplish our goal simply by choosing $N - 1$ to be odd and $H_1(z) = z^{-(N-1)}H_0(-z^{-1})$. This is therefore a neat solution to the perfect reconstruction problem! In summary, let $H_0(z)$ be a spectral factor of a linear-phase FIR half band filter $G_+(z)$ having a positive amplitude response. Choose the remaining filters according to

$$H_1(z) = z^{-(N-1)}H_0(-z^{-1}), \qquad F_0(z) = z^{-(N-1)}H_0(z^{-1}),$$
$$F_1(z) = z^{-(N-1)}H_1(z^{-1}). \qquad (16)$$

Then we have perfect reconstruction in the QMF bank of Fig. 5, and $\hat{x}(n) = cx(n - N + 1)$, where c is a constant. The expressions for $F_0(z)$ and $F_1(z)$ in (16) come from the alias-cancelation condition (5). It is important to notice that the stopband attenuation and stopband edge ω_S of $H_0(z)$ can be adjusted to any desired value, simply by

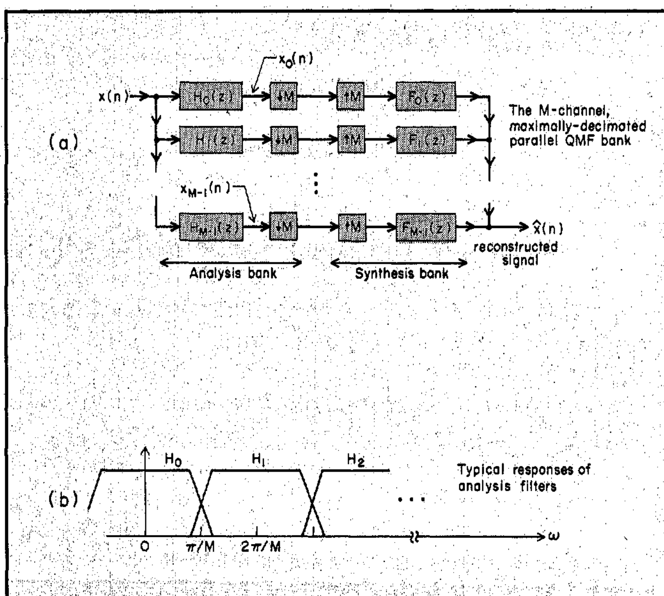

Figure 10. An extension of the two-channel QMF bank is the *M*-channel QMF bank. The analysis bank has *M* filters $H_0(z)$, $H_1(z)$, ... $H_{M-1}(z)$ which split the signal into *M* frequency bands. These subband signals $x_k(n)$ are *maximally* decimated, i.e., decimated by the factor *M*, and transmitted (after possible encoding). At the receiver end, each signal is decoded, interpolated and recombined through the set of *M* synthesis filters $F_0(z)$, $F_1(z)$, ... $F_{M-1}(z)$. The analysis filters should produce as much isolation between bands as possible, while at the same time not creating spectral gaps. Usually, their spectra overlap (Fig. 10(b)), and hence aliasing is inevitable. The choice of $F_k(z)$ should be such that the *imaging* produced by the interpolators tends to *cancel* aliasing as accurately as possible. The complicated relation between the signals $x(n)$ and $\hat{x}(n)$ makes the design problem more challenging and interesting, as compared to the two-channel problem. The adjective *parallel* in Fig. 10(a) is meant to distinguish it from a tree-structured bank [3], [6].

designing the halfband filter $G(z)$ appropriately, by using the McClellan-Parks algorithm [35].

THE M-CHANNEL QMF BANK

Figure 10 shows the M-channel QMF bank for arbitrary M. The analysis filters $H_0(z), H_1(z), \ldots H_{M-1}(z)$ split the signal $x(n)$ into M frequency bands, and each subband signal $x_k(n)$ is decimated by a factor of M. This is called a maximally decimated structure because, the decimation factors are equal to M, which is the number of bands. The signals $x_k(n)$ are encoded and transmitted. At the synthesizer end, they are decoded, interpolated, and filtered by the synthesis filters $F_0(z), F_1(z), \ldots, F_{M-1}(z)$. The signal $\hat{x}(n)$ is a reconstructed version of $x(n)$, and we wish it to be 'close' to $x(n)$ in some sense.

In analogy with the two-channel QMF problem, $\hat{x}(n)$ suffers from three types of errors. First, there is aliasing. Figure 10(b) shows the typical analysis filter responses‡. Since there is an overlap between adjacent filters, the signals are not strictly bandlimited to a sufficient extent. This causes aliasing. And then there are amplitude and phase distortions.

In absence of the decimators and interpolators, the relation between $\hat{X}(z)$ and $X(z)$ is nice and simple: $\hat{X}(z) = X(z)\sum_{k=0}^{M-1} H_k(z)F_k(z)$. The decimators and interpolators cause aliasing and imaging; by applying the relations in (2) and (3) we can arrive at the following key expression for $\hat{X}(z)$:

$$\hat{X}(z) = \frac{1}{\boldsymbol{M}} \sum_{n=0}^{M-1} X(z\boldsymbol{W}^n) \sum_{k=0}^{M-1} H_k(z\boldsymbol{W}^n)F_k(z) \qquad (17a)$$

‡The cutoff frequency of the lowpass filter $H_0(z)$ is not a quarter of 2π unless $M = 2$. Thus, the name *quadrature* mirror filter is a misnomer, when $M > 2$. However, this has become more or less standard, and there is no reason to change the name as long as we remember its origin.

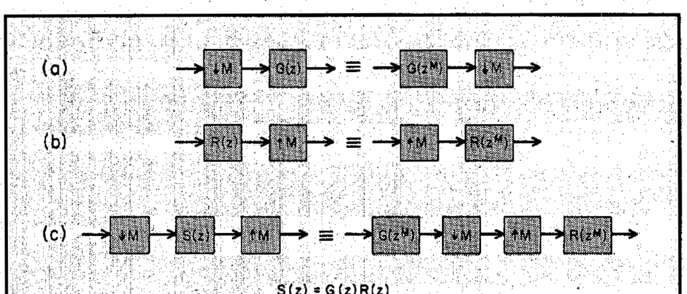

Figure 11. The rules for moving a transfer function around a decimator or interpolator are simple; the figure shows some useful identities. Figure 11(a) says that a transfer function $G(z)$ following an M-fold decimator is equivalent to a transfer function $G(z^M)$ preceding an M-fold decimator. Figure 11(b) is the corresponding identity for interpolators. If a transfer function $S(z)$ is sandwiched between a decimator and an interpolator, then the structure can be rearranged as shown in Fig. 11(c), where $G(z)R(z)$ is any *arbitrary* factorization of $S(z)$.

If the boldfaced quantities in (17a) are dropped, it is equivalent to dropping the decimators and interpolators. The term corresponding to $n = 0$ represents the genuine output which would result if the decimators and interpolators were absent. The terms corresponding to $1 \le n \le M - 1$ are unwanted aliasing terms. If we wish to cancel aliasing completely, we should choose $F_k(z)$ such that these terms are equal to zero for every possible input signal $x(n)$. Assuming that aliasing *is* somehow canceled, we have the relation

$$\frac{\hat{X}(z)}{X(z)} = T(z) = \frac{1}{M} \sum_{k=0}^{M-1} H_k(z)F_k(z) \qquad (17b)$$

where $T(z)$ represents the overall transfer function, or the *distortion function*. If $H_k(z)$ and $F_k(z)$ are such that $T(z)$ is allpass, then there is no amplitude distortion; on the other hand, if you force $T(z)$ to be a linear phase FIR function, there is no phase distortion. Finally, if $T(z)$ is a pure delay, (and if aliasing has already been canceled) then we have a perfect-reconstruction QMF bank. For the two-channel case, we saw that perfect-reconstruction is a tricky issue, the solution being not really obvious. For the M-band case, it is even more challenging. Our purpose here is to indicate a few steps towards the solution to the perfect-reconstruction problem for arbitrary M. A scheme shall eventually be presented, which is an attractive candidate for study and research.

Equation (17a) gives rise to M equations, which can be looked upon as conditions for perfect reconstruction. These in turn can be written in matrix form:

$$\mathbf{H}(z)\mathbf{f}(z) = \mathbf{v} \qquad (18)$$

where $\mathbf{f}(z) = [F_0(z)\ F_1(z) \ldots F_{M-1}(z)]^T$, $\mathbf{v} = [cz^{-n_0}\ 0\ 0 \ldots 0]^T$, and $\mathbf{H}(z)$ has elements $H_{n,k} = H_k(zW^n)$. The matrix $\mathbf{H}(z)$ is known as the alias-cancelation matrix (AC-matrix for short). In principle, inversion of this matrix leads to a solution for the synthesis filters $F_k(z)$ in terms of the analysis filters $H_k(z)$. However, there are problems associated with such an approach: first, the resulting synthesis filters are unlikely to be stable for a given set of $H_k(z)$. It is a tricky issue to try to restrict $H_k(z)$ to be such that the above approach gives rise to stable synthesis filters. Second, even if they are stable, they tend to have very high orders. Finally, since $\mathbf{H}(z)$ is a function of z rather than a constant, its inversion is difficult. Accordingly, there is considerable interest in finding other procedures to attack the perfect reconstruction problem.

A Couple of Useful Identities

In Fig. 1(a), imagine that we wish to pass the decimated signal $y(n)$ through a delay z^{-1}. This is equivalent to delaying the signal $x(n)$ by M units and then decimating the result. Extending this idea further, we can show that a transfer function $G(z)$ following a decimator (Fig. 11(a)) can be equivalently moved to the *left* of the decimator, as long as each z is replaced with z^M. In an analogous manner, a transfer function $R(z)$ preceding an interpolator can be moved to the right by replacing z with z^M, as shown in

Fig. 11(b). These identities are *exact,* and can be proved based on the input-output relations of decimators and interpolators.

Now consider Fig. 11(c), where $S(z)$ is a transfer function sandwiched between a decimator and an interpolator. Let $S(z) = G(z)R(z)$ be an arbitrary factorization. Then we can move $G(z)$ and $R(z)$ around to obtain the equivalent diagram shown in the figure. Such manipulations are very useful in obtaining a quick understanding of certain important issues in the QMF problem.

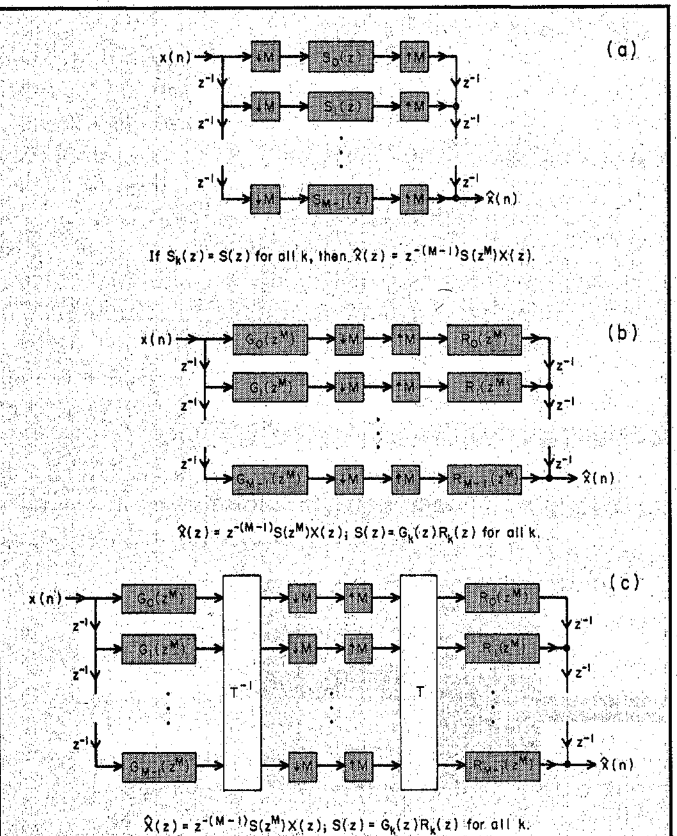

Figure 12. Since the relation between $\hat{X}(z)$ and $X(z)$ in the M-channel maximally decimated parallel QMF bank is complicated, it is difficult to see how to cancel aliasing *completely.* (Perfect reconstruction is even more difficult). However, certain fundamental observations enable one to obtain complete and correct answers in a simple manner. In Fig. 12(a), it can be shown that $\hat{X}(z)$ is free of aliasing if and only if $S_k(z)$ are independent of channel number k, i.e., $S_k(z) = S(z)$ for all k. Under this condition, if $S(z)$ is factorized as $S(z) = G_k(z)R_k(z)$, for $k = 0, 1 \ldots M-1$ in M arbitrary ways, and the structure rearranged (with the help of the identities in Fig. 11), we get Fig. 12(b), which represents a general alias-free system. If an arbitrary nonsingular matrix $\mathbf{T}$ is inserted as shown in Fig. 12(c), the system still satisfies $\hat{X}(z) = z^{-(M-1)}S(z^M)X(z)$, and is free of aliasing. In particular, if $\mathbf{T}$ is chosen as the DFT matrix, the analysis and synthesis banks are uniform-DFT banks!

A General M-Band Alias-Free System

For a minute, let us switch our minds to a completely different picture, viz. Fig. 12. Figure 12(a) shows a set of transfer functions $S_0(z), S_1(z), \ldots S_{M-1}(z)$ sandwiched between M-fold decimators and interpolators. The decimators are preceded by a chain of delays, whereas the interpolators are followed by a chain of delays. Since decimation is not preceded by filtering, there is in general severe aliasing in the structure, for arbitrary $x(n)$. However, it may be possible to pick the functions $S_k(z)$ such that the aliasing is *somehow* canceled by the imaging effect of the interpolators. Let us probe deeper into this possibility.

First, suppose that $S_k(z)$ are not there (i.e., $S_k(z) = 1$ for all k.) Then the decimators and interpolators do not do any harm to the signal $x(n)$; we can formally prove in this case [37], that $\hat{x}(n) = x(n - M + 1)$. Next suppose that $S_k(z)$ are present, but the following relation holds:

$$S_k(z) = S(z), \quad \text{for all } k \tag{19}$$

i.e., $S_k(z)$ is independent of k, but otherwise arbitrary. By making use of the identity (b) in Fig. 11, we can move $S(z)$ past the interpolators, and then past the delay chain, to show that $\hat{X}(z) = z^{-(M-1)}S(z^M)X(z)$. Thus, the system is represented by a transfer function $T(z) = z^{-(M-1)}S(z^M)$, and is therefore (linear and) time-invariant. In particular, aliasing has been completely canceled. More formally, it is possible to prove that, in Fig. 12(a), $\hat{x}(n)$ is free from aliasing if and *only if* (19) holds.

Let us now assume that (19) holds, and factorize $S(z)$ in M arbitrary ways:

$$S(z) = G_k(z)R_k(z) \tag{20}$$

By making use of the identities in Fig. 11, we can then redraw Fig. 12(a) as in Fig. 12(b). If we now insert an *arbitrary* nonsingular constant matrix $\mathbf{T}$ and its inverse into the structure as shown in Fig. 12(c), an observer observing $\hat{x}(n)$ will not even know that we have inserted these! In other words, for a given $x(n)$, $\hat{x}(n)$ in Fig. 12(a) is exactly same as $\hat{x}(n)$ in Fig. 12(c), no matter what $\mathbf{T}$ is. Accordingly, as long as $G_k(z)R_k(z)$ is independent of k, $\hat{x}(n)$ in Fig. 12(c) is entirely free of aliasing for any $\mathbf{T}$. The effective analysis and synthesis filters in Fig. 12(c) can be verified to be

$$H_k(z) = \sum_{l=0}^{M-1} z^{-l}[\mathbf{T}^{-1}]_{kl}G_l(z^M),$$

$$F_k(z) = \sum_{l=0}^{M-1} z^{-(M-1-l)}\mathbf{T}_{lk}R_l(z^M). \tag{21}$$

Based on this observation, how do we construct some useful QMF banks? As a possible example, let us imagine that $\mathbf{T}$ is related to the DFT matrix, i.e., $\mathbf{T}_{mn} = W^{mn}/M$ where $W = e^{-2\pi j/M}$. In particular, this means,

$$H_0(z) = [G_0(z^M) + z^{-1}G_1(z^M) + z^{-2}G_2(z^M) + \ldots + z^{-(M-1)}G_{M-1}(z^M)] \tag{22}$$

and $H_k(z) = H_0(zW^k)$. In other words, we have the following situation: suppose somebody gives us a lowpass

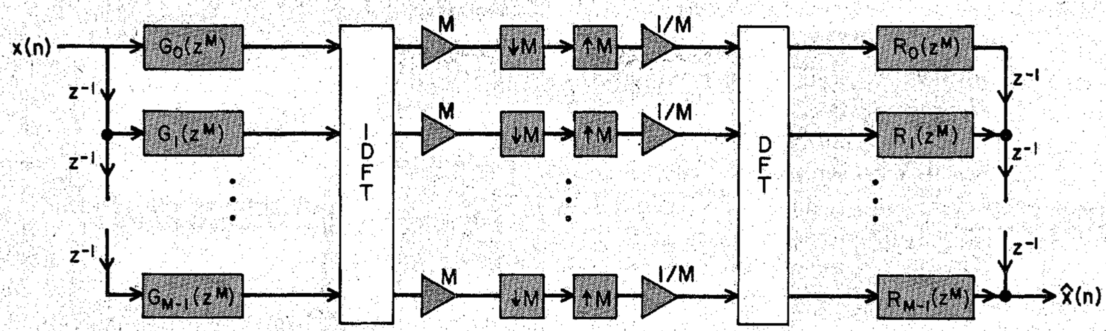

Figure 13. In this figure, the analysis and synthesis banks are of the uniform DFT type. The lowpass filter $H_0(z)$ is given by $G_0(z^M) + z^{-1}G_1(z^M) + \ldots + z^{-(M-1)}G_{M-1}(z^M)$ whereas $H_k(z) = H_0(zW^k)$. With $R_k(z)$ defined as in (25), aliasing is completely canceled, and the system is characterized by the transfer function $T(z) = z^{-(M-1)}\prod_{l=0}^{M-1} G_l(z^M)$. Such schemes for alias cancelation, though perfect, give rise to complicated synthesis filters (in terms of filter order). In practice, if aliasing due to nonadjacent channels is negligible, certain elegant and practical solutions are possible as shown by Rothweiler [9] and Chu [8]. Note that, both in this figure and in Fig. 12, if $R_k(z)$ are chosen to be equal to $1/G_k(z)$, we would have perfect reconstruction, and $\hat{x}(n) = x(n - M + 1)$. However, unless all the zeros of the numerator polynomials of $G_k(z)$ are strictly inside the unit circle (i.e., these numerators are *minimum-phase* polynomials) $R_k(z)$ become unstable. So, such perfect reconstruction structures are not very useful.

transfer function

$$H_0(z) = \sum_{n=0}^{\infty} h(n)z^{-n} \qquad (23)$$

We can write $H_0(z)$ in the form (22) simply by defining‡

$$G_l(z) = h(l) + h(l + M)z^{-1} + h(l + 2M)z^{-2} + \ldots \qquad (24)$$

That is, the impulse responses of $G_l(z)$ are decimated versions of the impulse response $h(n)$. Having done so, let us take the remaining analysis filters as $H_k(z) = H_0(zW^k)$. The frequency responses of $H_k(z)$ are uniformly shifted versions of the prototype $H_0(e^{j\omega})$, i.e., $H_k(e^{j\omega}) = H_0(e^{j(\omega-(2\pi k/M))})$. Such a set of analysis filters is very commonly used; the analysis bank is then called the uniform DFT bank. Now, what we learned from the exercise of Fig. 12 is that, in a QMF bank with such analysis filters, we can *completely eliminate* aliasing simply by choosing the synthesis filters $F_k(z)$ to be as in (21) with $T_{lk} = W^{lk}/M$, and with $R_k(z)$ such that the product $G_k(z)R_k(z)$ is independent of k. For example we could choose

$$R_k(z) = \prod_{\substack{l=0 \\ l\neq k}}^{M-1} G_l(z) \qquad (25)$$

or, as an alternative,

$$R_k(z) = 1/G_k(z)\,. \qquad (26)$$

With the choice (25) it is clear that the synthesis filters are stable as long as the analysis filters are stable; but the disadvantage is that the transfer functions $F_k(z)$ tend to have much higher orders than $H_k(z)$. The choice in (26) overcomes this problem and in addition leads to perfect reconstruction since $S(z) = 1$ here; but it does not give rise to stable synthesis filters, unless the numerators of the polyphase components $G_k(z)$ have minimum phase.

‡$G_l(z)$ are called *polyphase* components [3] of $H_0(z)$.

PERFECT RECONSTRUCTION M-CHANNEL QMF BANKS

For an M-band QMF bank with arbitrary M, we saw at least one technique for obtaining perfect reconstruction, namely Fig. 12(c) or Fig. 13 with $R_k(z)$ as in (26). As pointed out, such a scheme works under the constraint that $G_k(z)$ should have minimum-phase numerators. A different scheme is now outlined, which is free from such a requirement. Consider again the structure of Fig. 12(a), redrawn in Fig. 14(a), with $S_k(z) = 1$ for all k. This is then an ultrasimple QMF bank, with $H_k(z) = z^{-k}$ and $F_k(z) = z^{-(M-1-k)}$. We know from earlier discussions that $\hat{x}(n) = x(n - M + 1)$, i.e., the structure has perfect-reconstruction property. Let us now insert the matrices $\mathbf{R}$ and $\mathbf{R}^{\dagger}$ into the structure as shown in Fig. 14(b)‡, where $\mathbf{R}$ is a $M \times M$ unitary matrix (i.e., any matrix satisfying $\mathbf{R}^{\dagger}\mathbf{R} = c\mathbf{I}$), where c is a scalar. Evidently, this does not affect the output $\hat{x}(n)$ since the matrices $\mathbf{R}$ and $\mathbf{R}^{\dagger}$ simply cancel. Since $\mathbf{R}$ is memoryless, we can obviously move the matrices to obtain the perfect reconstruction system of Fig. 14(c). This works for *any* unitary $\mathbf{R}$; the only disadvantage is that the FIR filters $H_k(z)$ and $F_k(z)$ in Fig. 14(c) are of order $M - 1$. In order to accomplish higher orders, and hence sharper filters, let us extend this idea further. Thus, refer to Fig. 14(d) which is obtained from Fig. 14(a) by inserting the matrix *functions* $\mathbf{E}(z)$ and $\mathbf{E}^T(z^{-1})$. The matrix $\mathbf{E}(z)$ is unitary

‡ Superscript † stands for transposed conjugate and superscript T stands for transpose.

on the unit circle of the z-plane, i.e.,

$$\mathbf{E}^T(z^{-1})\mathbf{E}(z) = c\mathbf{I}, \quad \text{for } z = e^{j\omega}, \tag{27}$$

where c is a scalar constant. (Note that, assuming $\mathbf{E}(z)$ is real for real z, $\mathbf{E}^T(z^{-1})$ is precisely the transpose-conjugate of $\mathbf{E}(z)$ on the unit circle). If $\mathbf{E}(z)$ has complex coefficients, then $\mathbf{E}^T(z^{-1})$ should be replaced with $\mathbf{E}_*^T(z^{-1})$ where subscript '*' means coefficient conjugation. As a result, the output $\hat{x}(n)$ in Fig. 14(d) continues to be same as that in Fig. 14(a), except for a scaling constant. We can now invoke the identities in Fig. 11, and rearrange Fig. 14(d) as in Fig. 14(e), which is therefore a perfect reconstruction system! As such, unless $\mathbf{E}(z)$ is FIR, $\mathbf{E}^T(z^{-1})$ is unstable, so we assume $\mathbf{E}(z)$ is FIR. To avoid non causal operations, in practice, we insert a delay in front of $\mathbf{E}^T(z^{-1})$ so that there are no positive powers of z anywhere. From Fig. 14(e) you can deduce that the analysis and synthesis filters are effectively

$$H_k(z) = \sum_{l=0}^{M-1} z^{-l}E_{kl}(z^M) \tag{28a}$$

and

$$F_k(z) = z^{-\beta}H_k(z^{-1}) \tag{28b}$$

where β is a large enough positive integer to ensure that there are no positive powers of z in $F_k(z)$.

Now, if (27) holds everywhere on the unit circle, then it must be true for all z, by analytic continuation. Such matrices $\mathbf{E}(z)$ are said to be *paraunitary*‡. For our discussion, 'paraunitary' will therefore be used as a synonym to 'unitary on the unit circle'. We can thus state the following result: let $H_k(z)$ be FIR analysis filters with *polyphase components* $E_{kl}(z)$ such that the matrix $\mathbf{E}(z) = [E_{kl}(z)]$ is para-

‡ The concept of paraunitariness is well-known in classical, continuous-time network theory; scattering matrices that describe lossless multiports satisfy this property [19],[20].

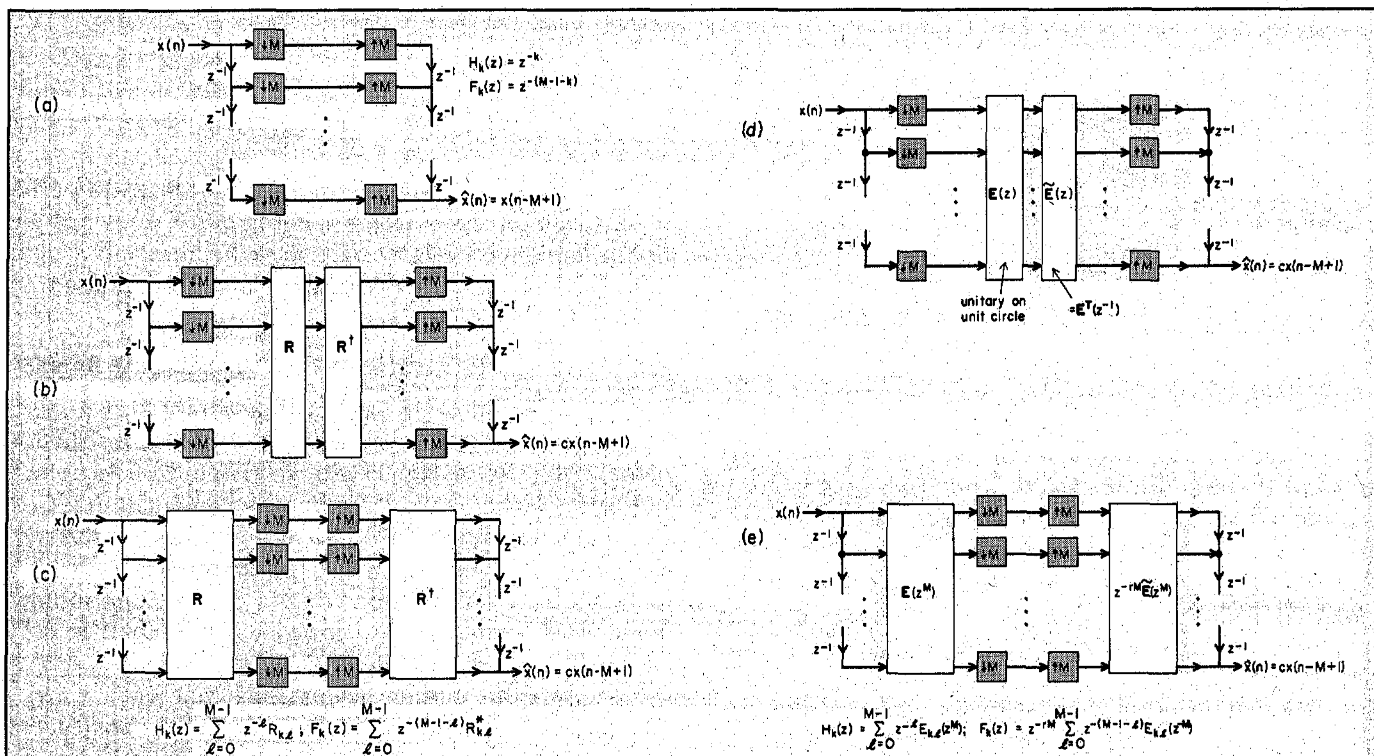

Figure 14. Figure 14(a) represents a QMF bank with very simple analysis and synthesis filters. This structure forms the basis for constructing more useful filter banks. It can be easily verified that, in Fig. 14(a), we have perfect reconstruction, i.e., $\hat{x}(n) = x(n - M + 1)$. Now consider Fig. 14(b) which is a modification of Fig. 14(a), with matrices $\mathbf{R}$ and $\mathbf{R}^\dagger$ introduced, where superscript dagger indicates transpose conjugation. If $\mathbf{R}$ is a unitary matrix (i.e., if it satisfies $\mathbf{R}^\dagger\mathbf{R} = c\mathbf{I}$, for some scalar c), then Fig. 14(b) is equivalent to Fig. 14(a), except for the scale factor c. Now the matrices $\mathbf{R}$ and $\mathbf{R}^\dagger$ (which are not functions of z) can be moved past the decimators and interpolators resulting in Fig. 14(c), which is therefore a perfect reconstruction QMF bank. The analysis and synthesis filters $H_k(z)$ and $F_k(z)$, can be written in terms of the elements of $\mathbf{R}$ as shown in the figure. In Fig. 14(c), the system is highly restricted because the filters $H_k(z)$ are constrained to be FIR with order $M - 1$. To obtain better and higher-order filters, we can push our imagination one step further. In Fig. 14(d), we have inserted the matrix *functions* $\mathbf{E}(z)$ and $\tilde{\mathbf{E}}(z)$, where the tilde accent is defined such that $\tilde{\mathbf{E}}(z) = \mathbf{E}^T(z^{-1})$. For steady state frequencies, (i.e., on the unit circle of the z-plane where $z = e^{j\omega}$) the tilde operation is same as transpose-conjugation, so Fig. 14(d) is an extension of the idea in Fig. 14(b). If $\mathbf{E}(z)$ is unitary on the unit circle of the z-plane, (i.e., if $\mathbf{E}^\dagger(e^{j\omega})\mathbf{E}(e^{j\omega}) = c\mathbf{I}$ for all ω, where c is a constant), the system is still a perfect reconstruction system [37]. By using the identities in Fig. 11, we can move these matrices to obtain the structure of Fig. 14(e), which satisfies the perfect reconstruction property! In practice, to ensure stability of $\mathbf{E}^T(z^{-1})$, the entries $E_{kl}(z)$ of the matrix $\mathbf{E}(z)$ are restricted to be FIR. Moreover, a delay element z^{-r} is thrown in front of $\mathbf{E}^T(z^{-1})$ so as to avoid noncausality.

unitary. If we pick the synthesis filters to be as in (28b) then aliasing is completely canceled, and there are no amplitude and phase distortions, i.e., the structure has perfect reconstruction property.

So! Perfect-reconstruction for QMF banks, *in principle*, is as simple as that! Now comes the important question: *how* do we construct an $M \times M$ matrix of FIR functions such that it is paraunitary? One procedure would be to construct $\mathbf{E}(z)$ as a product of simple unitary building blocks, as indicated in Fig. 15(a). In this figure, $\mathbf{K}_n$ are constant $M \times M$ orthogonal matrices (a unitary matrix with real entries is called an orthogonal matrix). The matrices $\Lambda_n(z)$ are diagonal matrices of delays, so that they are unitary on the unit circle. Since a product of unitary matrices is unitary, the cascade in Fig. 15(a) realizes the desired $\mathbf{E}(z)$. More general methods for synthesizing $\mathbf{E}(z)$ can be found in [41].

The Design Problem

What are the design issues now? In practice, we wish $H_k(z)$ to have good stopband attenuation, so that adjacent frequency bands are well isolated (this is important, even though aliasing is eventually canceled anyway, because, when the subband signals are encoded, maximum freedom from adjacent channels is desired [3]). So the filters $H_k(z)$ should have good stopband attenuation. We can now set up an optimization problem: find the orthogonal matrices in Fig. 15(a) such that the stopbands of the analysis filters have minimum possible energy. We need not worry about the passbands because the unitary nature of $\mathbf{E}(z)$ ensures that $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})^2 + \ldots + |H_{M-1}(e^{j\omega})|^2 = constant$ for all ω. Accordingly, if the stopbands are good, and if the passbands are defined to be disjoint, then the passbands are automatically good. Figures 15 and 16 provide further discussions and examples on this issue.

An $M \times M$ orthogonal matrix can be represented in terms of $\binom{M}{2}$ planar rotations. In Fig. 15, an example of the choice of building blocks is shown for the case of $M = 3$. A typical 3×3 orthogonal matrix, constructed as shown in Fig. 15(c), has $\binom{3}{2} = 3$ degrees of freedom which can be adjusted so as to obtain good stopband attenuation. The criss-cross nature of the orthogonal building block suggests the name *lattice structures* for these circuits. A meaningful objective function to be minimized would be the sum of the stopband energies of the analysis filters. Fig. 16 shows a design example for the case of $M = 3$, with $L - 1 = 31$. Figure 16(b) shows the optimized frequency response plots. The transfer functions $H_0(z)$, $H_1(z)$ and

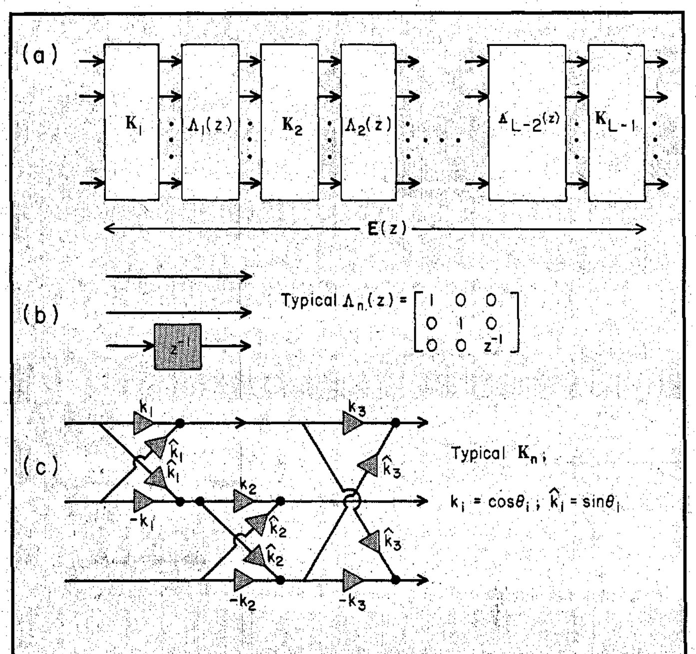

Figure 15. How do we design a M-input M-output FIR transfer matrix $\mathbf{E}(z)$ such that it is paraunitary? In the figure, $\mathbf{K}_n$ are constant orthogonal matrices whereas $\Lambda_n(z)$ are diagonal matrices, with delays along the diagonal. Accordingly, $\Lambda_n(z)$ is paraunitary, and the overall cascaded system is paraunitary. The figure shows typical building blocks for $M = 3$. The 3×3 orthogonal matrix $\mathbf{K}_n$ is merely a combination of three planar-rotation operators. In general an $M \times M$ orthogonal matrix is a combination of $\binom{M}{2}$ planar rotation operators [22]. The planar rotation angles can be optimized to maximize stop band attenuation of the analysis filters. Each building block $\mathbf{K}_n$ has the appearance of a generalized lattice, hence the above structure is called a cascaded FIR lattice [17], [37]. More general ways to synthesize $\mathbf{E}(z)$ can be found in [41].

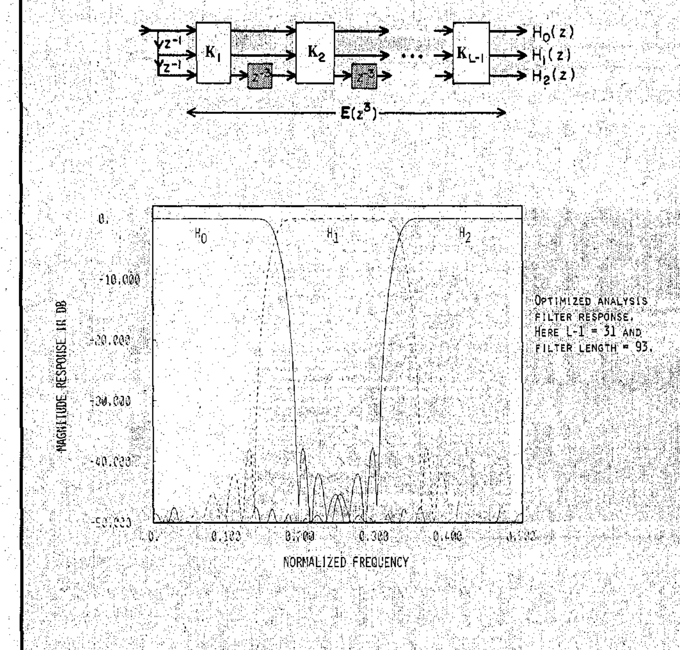

Figure 16. An example of a perfect reconstruction system. The coefficients of the 3×3 orthogonal matrices $\mathbf{K}_n$ are optimized to yield good stopband attenuation for the analysis filters. Since $\mathbf{E}(z)$ is paraunitary simply by virtue of the cascade of Fig. 16(a), the relation $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 + \ldots + |H_{M-1}(e^{j\omega})|^2 = constant$ holds for all ω; hence, it is sufficient to maximize the stopband attenuations of $H_k(z)$ and the passbands are *automatically* good. Fig. 16(b) shows the response of the analysis filters.

$H_2(z)$ have orders $N - 1 = 3(L - 2) + 2 = 92$. The optimization of the lattice coefficients was performed using software available in [23]‡.

In the case of $M = 2$, the lattice structure can be drawn in a particularly simple form, by considering a *denormalized* orthogonal matrix. This is indicated in Fig. 17. Notice that this lattice is different from the linear-prediction lattice structure because of the minus sign on one of the α's. This sign-difference is very crucial, and is what enables us to employ the structure for the QMF application.

This lattice structure has several unique properties [42]. First, the transfer functions $H_0(z)$ and $H_1(z)$ generated by the lattice satisfy the condition of equation (16); second, the synthesis-bank lattice, which is the transpose of the analysis bank has transfer functions $F_0(z)$ and $F_1(z)$ satisfying (16). Moreover, $H_0(z)$ is a spectral factor of a linear-phase halfband FIR filter. Finally, the condition of equation (15) is satisfied. These four properties hold, *regardless of* the values of α_k used in the structure! (The quantities α_k determine only the sharpness of cutoff and stopband attenuation of $H_0(z)$). In summary, if these lattice structures are used in the analysis and synthesis banks of the two-channel QMF circuit, then perfect reconstruction is *guaranteed* even if α_k are quantized to arbitrarily small number of bits (in a digital implementation). In other words, perfect-reconstruction is *structurally induced* by the lattice.

‡ This example was generated by Truong Q. Nguyen, at Caltech.

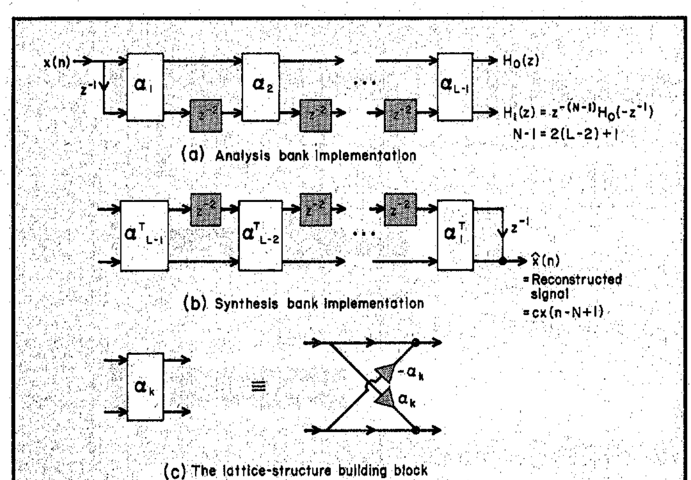

Figure 17. In the two-channel lattice structure, *regardless* of the values of α_k, the following property holds: $\hat{x}(n) = cx(n - N + 1)$ (perfect reconstruction). The lattice structure is therefore *robust* [42] and is attractive for implementation purposes. In particular, we have perfect reconstruction even if α_k are quantized to mere powers of two! (If the transfer functions $H_k(z)$ and $F_k(z)$ were implemented in direct-form, then coefficient quantization would have destroyed the perfect-reconstruction property). Notice that, the above lattice [17] is different from the *linear-prediction* lattice [36], because of the minus sign in the criss-cross.

This unique nature of the lattice encourages us to design the two-channel perfect reconstruction analysis filter $H_0(z)$ in a different way rather than by spectral-factorizing a half-band filter (see Fig. 18). We simply formulate an objective function

$$P = \int_{\omega_S}^{\pi} |H_0(e^{j\omega})|^2 d\omega, \tag{29}$$

and find the set of parameters α_k that minimizes P. The lattice structure automatically ensures the rest; in particular the passband of $H_0(z)$ comes out to be good because, firstly, $|H_1(e^{j\omega})|$ is an image of $|H_0(e^{j\omega})|$ (because of (16)), and furthermore, $|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 =$ *constant* holds. In Fig. 19, we can see plots of $|H_0(e^{j\omega})|$ and $|H_1(e^{j\omega})|$ obtained by such an optimization‡. The reconstruction error is expected to be zero. Figure 19(b) shows a plot of $|T(e^{j\omega})|$ with the lattice coefficients quantized to 5 bits. Because of *structural* perfect-reconstruction property, the plot is constant for all ω. Instead of quantizing the lattice coefficients, if we quantize the direct-form coefficients, the resulting $|T(e^{j\omega})|$ is not completely flat, because the quantized $H_0(z)$ is then no longer a spectral factor of a half-band filter. The lattice structure in Fig. 17, therefore, seems to be desirable both from the viewpoint of designing $H_0(z)$ and from the viewpoint of implementing the analysis and synthesis banks.

It is worth pointing out yet another feature of the lattice structure, viz., the *modular* property. What we mean by this is the following: suppose we drop (i.e., simply eliminate) the lattice sections labeled α_{L-1} and α_{L-1}^T along with

‡ This design example was generated by Phuong-Quan Hoang at Caltech.

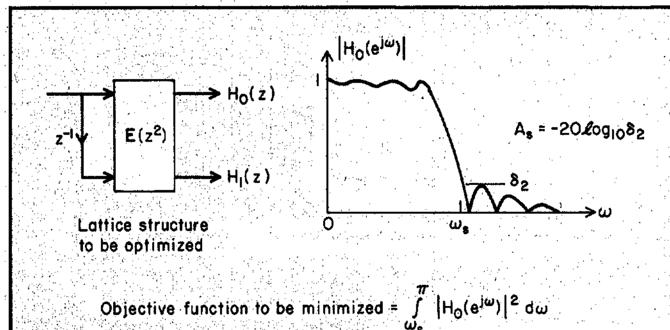

Figure 18. An optimization problem can be set up [42] so as to maximize the stopband attenuation provided by $H_0(z)$ in the two-channel lattice. Such an approach is different from the more standard approach, where a half-band FIR filter of order $2(N - 1)$ is first designed and then spectral-factorized. For high order filters with large stopband attenuations, such spectral factorization is numeri-cally inaccurate and difficult. The lattice-based design not only avoids such factorization, but in addition, produces a monotone-increasing peak stopband attenuation (often desired in QMF applications). Moreover, $H_0(z)$ resulting out of the lattice structure are automatically spectral factors of a half-band filter (regardless of values of α_k) [42].

the associated delays (z^{-2} elements). Then the resulting smaller lattice structure continues to have perfect reconstruction property. In other words, the effect of adding more sections or deleting sections is to change the attenuation characteristics of $H_0(z)$, but the signal $\hat{x}(n)$ continues to be a *perfect* replica of $x(n)$ except for a delay. Such a modular property is of course, not available with the direct-form structure.

CONCLUDING REMARKS

The purpose of this paper has been to outline some of the issues involved in the QMF-bank design problem. The relation of the QMF problem to the concept of losslessness (or unitariness) has been emphasized in this article. It is encouraging to know that perfect-reconstruction can be accomplished for arbitrary number of channels in a maximally decimated parallel QMF bank. Moreover, such reconstruction is possible even when the analysis filters $H_k(z)$ and synthesis filters $F_k(z)$ are all FIR, and of the same length N. Certain new lattice structures, fundamentally different from the linear-prediction lattice, are naturally placed in evidence when we attempt to do perfect reconstruction based on unitary building blocks. Moreover, for the case of two channels, these lattice structures give rise to transfer functions which have all the properties required for perfect reconstruction even when the lattice coefficients are quantized to arbitrarily small number of bits.

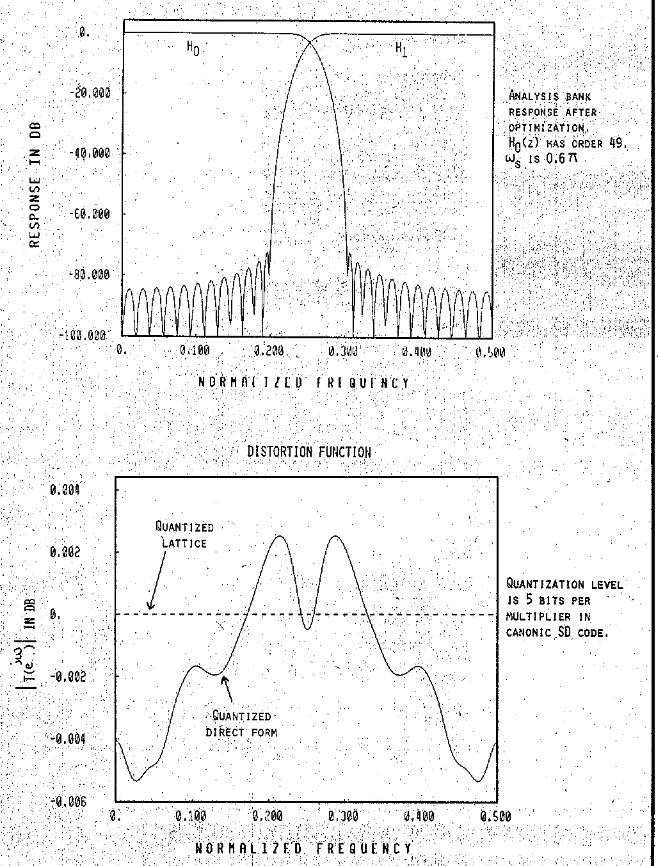

Figure 19. Example of an optimized lattice response. The attenuation provided by $H_0(z)$ exceeds 70 *dB*. Notice that the number of zeros on the unit circle (for $H_0(z)$) is equal to 25, which is the maximum permissible for the spectral factor of any half band filter of order 2 × 49. The distortion function $T(z)$ continues to be a pure delay even after coefficient quantization, provided the filters are implemented in lattice form rather than in direct form. Figure 19(b) shows the plot of $|T(e^{j\omega})|$ for an implementation with 5 bits per coefficient in canonic sign-digit code, both with the lattice structure and the direct-form structure.

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SUMMARY OF KEY RESULTS

The *M*-channel Quadrature Mirror Filter (QMF) bank in Fig. 10 is called the *maximally* decimated, parallel QMF bank. In order to avoid spectral gaps while splitting the signal $x(n)$ into M bands, the frequency responses of the *analysis filters* $H_k(z)$ are permitted to overlap. Consequently, there is aliasing at the output of the decimators. This aliasing can be canceled by the *imaging* effects of the interpolators, if the *synthesis filters* $F_k(z)$ are chosen appropriately. Some schemes [10], [11] for perfect cancelation of aliasing are shown in Fig. 12 and Fig. 13, and typically require high orders for $F_k(z)$. Approximate cancelation of aliasing can be achieved by use of suitable synthesis filters of low order [8], [9].

Once aliasing has been canceled, the reconstructed signal is given by $\hat{X}(z) = T(z)X(z)$ where $T(z)$ is the *overall* transfer function or the *distortion* transfer function. If $|T(e^{j\omega})|$ is constant independent of ω (i.e., if $T(z)$ is an allpass function) there is no amplitude distortion; if $arg[T(e^{j\omega})] = K\omega$ (i.e., if $T(z)$ is a linear-phase (FIR) function) then there is no phase distortion. In fact it has been possible in the past to thus eliminate either amplitude distortion *or* phase distortion *completely* [1], [5], [10], [11], [29]. Simultaneous elimination of *all* three distortions (i.e., aliasing, amplitude and phase distortions) is difficult but can be done. Such a QMF structure will be a *perfect-reconstruction structure* and satisfies $\hat{x}(n) = cx(n - n_0)$. If $E_{k,n}(z)$, $0 \leq n \leq M - 1$ represent the M polyphase components of the analysis filters $H_k(z)$, $0 \leq k \leq M - 1$, (see discussions around equations (23), (24) for meaning of polyphase components) and if the matrix function

$$\mathbf{E}(z) = [E_{k,n}(z)] \tag{A}$$

is unitary on the unit circle of the z-plane (Fig. 14), then it is possible to obtain perfect reconstruction simply by taking the synthesis filters to be

$$F_k(z) = z^{-\beta}H_k(z^{-1}) \tag{B}$$

where β is an integer large enough so that there are no positive powers of z in the expressions for $F_k(z)$'s.

Such perfect reconstruction is practicable provided $H_k(z)$ are FIR; notice that, the poles of $H_k(z^{-1})$ are the reciprocals of those of $H_k(z)$, hence if $H_k(z)$ are (stable and) IIR, $F_k(z)$ given by Eqn. (B) are necessarily unstable. When $H_k(z)$ are FIR, the above perfect-reconstruction scheme has FIR synthesis filters $F_k(z)$ having the *same order* as the analysis filters, and there is an ultrasimple *closed form* expression (Eqn. (B)) that gives $F_k(z)$ in terms of $H_k(z)$! No inversion of matrices and matrix-polynomials are involved in the design. An implementation of such a system is shown in Fig. 14(e), whereas Fig. 14(d) is a more efficient implementation with the polyphase filters $E_{k,n}(z)$ operating at the lowest possible rate.

A matrix which is unitary on the unit circle satisfies $\mathbf{E}^T(z^{-1})\mathbf{E}(z) = c\mathbf{I}$ for all $z = e^{j\omega}$, where c is a constant scalar. By analytic continuation, this implies $\mathbf{E}^T(z^{-1})\mathbf{E}(z) = c\mathbf{I}$ for *all* z. Such matrices will be called paraunitary or simply *lossless*. The term 'lossless' comes from the fact that if

$\mathbf{Y}(z) = \mathbf{E}(z)\mathbf{X}(z)$, then the energy in the vector sequence $\mathbf{y}(n)$ is equal to that in the vector sequence $\mathbf{x}(n)$. In the continuous-time world, such paraunitary systems are well-known; scattering matrices of lossless multiports are known to have this property [19], [20].

A second way to look at the perfect-reconstruction scheme is through the Alias Cancelation (AC-) matrix $\mathbf{H}(z)$ in Eqn. (18). If this matrix is unitary, then we can solve for the synthesis filter vector $\mathbf{f}(z)$ simply by taking $\mathbf{f}(z) = z^{-p}\mathbf{H}^T(z^{-1})\mathbf{v}$ (where p is large enough so that there are no positive powers of z in the expressions for $F_k(z)$). It turns out that, with this viewpoint, the same solutions viz. Eqn. (B) results and the (para)unitariness of $\mathbf{E}(z)$ is equivalent to (para)unitariness of $\mathbf{H}(z)$.

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ACOUSTICS, SPEECH AND SIGNAL PROCESSING SOCIETY

Call for Adcom Nominations

The nominating committee is currently preparing a slate of nominees for next year's Administrative Committee and would welcome suggestions from the society membership. If you would like to serve as a member of AdCom or know someone who you would like to have as a representative, please send this information to the nominating committee chairperson.

All suggestions will be carefully considered; however, a nominating petition with a minimum of 25 names of society membership will guarantee placement of the nominee on the slate of AdCom candidates to be presented to the membership for mail ballot. The deadline for the return of suggestions and petitions is August 14, 1987.

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