QUADRATURE OF THE NORMAL CURVE

By

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There are three formulas for the calculation of areas under the normal probability curve, only two of which seem to be generally recognized in American statistical circles. Herewith is presented an outline of the mathematical development of these three formulas and a determination of the bounds of practical utility of each.

The well-known equation for the normal curve,

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

may be expanded into the series

$$y = \frac{N}{\sigma\sqrt{2\pi}}\left[1 - \left(\frac{x}{\sigma\sqrt{2}}\right)^2 + \frac{1}{|\underline{2}}\left(\frac{x}{\sigma\sqrt{2\pi}}\right)^4 - \frac{1}{|\underline{3}}\left(\frac{x}{\sigma\sqrt{2\pi}}\right)^6 + \cdots\right]$$

(Ref. 3)

by means of Maclaurin's Theorem. (See any good calculus text-book.) (7) This expansion is readily accomplished by making the substitution

$$t = \frac{x}{\sigma\sqrt{2}}$$

so that

$$e^{-\frac{x^2}{2\sigma^2}} = e^{-t^2}$$

and

$$f(t) = e^{-t^2}$$

The process of successive differentiation is quite lengthy, since every other term differentiated becomes zero and therefore 2n terms in the Maclaurin series are required to produce n terms in the new series. After the expansion has been carried to five or six terms, a regular law of formation becomes evident from inspection of the new series

$$e^{-t^2} = 1 - t^2 + \frac{1}{|\underline{2}} t^4 - \frac{1}{|\underline{3}} t^6 + \frac{1}{|\underline{4}} t^8 - \cdots$$

viz.:

$$\text{nth term} = \frac{1}{\lfloor n-1} t^{2n-2}$$

After making the reconversion $t = \frac{x}{\sigma\sqrt{2}}$ and substituting the value of $e^{-\frac{x^2}{2\sigma^2}}$ in the original equation for the normal curve, we have

$$y = \frac{N}{\sigma\sqrt{2\pi}}\left[1 - \left(\frac{x}{\sigma\sqrt{2}}\right)^2 + \frac{1}{\lfloor 2}\left(\frac{x}{\sigma\sqrt{2}}\right)^4 - \frac{1}{\lfloor 3}\left(\frac{x}{\sigma\sqrt{2}}\right)^6 + \cdots\right]$$

(Ref. 1, 7)

as previously indicated. This series is uniformly convergent and may therefore be integrated term by term.

The area under all or any portion of the normal curve is calculated from the integral of the equation for the curve:

$$\int y = \frac{N}{\sigma\sqrt{2\pi}}\int e^{-\frac{x^2}{2\sigma^2}}dx.$$

To simplify the procedure, let $x = \sigma \cdot \sqrt{2} \cdot t$.

Then $dx = \sigma \cdot \sqrt{2} \cdot dt$

and $$\int y = \frac{N}{\sigma\sqrt{2\pi}}\int e^{-t^2} \cdot \sigma\sqrt{2} \cdot dt = \frac{N}{\sqrt{\pi}}\int e^{-t^2}dt$$

$$= \frac{1}{\sqrt{\pi}}\int e^{-t^2}dt \qquad (\text{when } N=1).$$

The value of the definite integral representing the area between the ordinate at the mean and the general ordinate whose abscissa is t is

$$\int_0^t y = \frac{1}{\sqrt{\pi}}\int_0^t e^{-t^2}dt.$$

Then, integrating the expanded series for e^{-t^2} (above) term by term, we have:

$$\frac{1}{\sqrt{\pi}}\int_0^t\left[1 - t^2 + \frac{t^4}{\lfloor 2} - \frac{t^6}{\lfloor 3} + \frac{t^8}{\lfloor 4} - \cdots\right]dt =$$

$$\frac{t}{\sqrt{\pi}}\left[1 - \frac{t^2}{2} + \frac{t^4}{4\lfloor 2} - \frac{t^6}{6\lfloor 3} + \frac{t^8}{8\lfloor 4} - \frac{t^{10}}{10\lfloor 5} + \frac{t^{12}}{12\lfloor 6} - \cdots\right].$$

Substituting in this series the value of $t = \frac{x}{\sigma\sqrt{2}}$ and keeping the expression $\frac{x}{\sigma}$ separate, we obtain:

$$\int_0^{\frac{x}{\sigma}} y = \frac{\frac{x}{\sigma}}{\sqrt{2\pi}} \left\{ 1 - \frac{\left(\frac{x}{\sigma}\right)^2}{2\cdot 3} + \frac{\left(\frac{x}{\sigma}\right)^4}{2^2\cdot 5\cdot \underline{|2}} - \frac{\left(\frac{x}{\sigma}\right)^6}{2^3\cdot 7\cdot \underline{|3}} + \frac{\left(\frac{x}{\sigma}\right)^8}{2^4\cdot 9\cdot \underline{|4}} - \cdots \right\}.$$

This series may be extended indefinitely, since the general term is seen to be

$$\frac{\left(\frac{x}{\sigma}\right)^{2n-2}}{2^{n-1}\cdot(2n-1)\underline{|n-1}}$$

It will be referred to hereinafter as Series A.

A published statement that Series A is divergent when $t > 1$ is erroneous (7). It is always convergent, regardless of the value of the deviate, but converges very slowly when t is not small, in which case it is better to use another series obtained by integrating by parts. (1, 7)

We may write

$$\int e^{-t^2} dt = \int \left[-\tfrac{1}{2t}\right]\left[-2t\,e^{-t^2}dt\right]$$

$$= \int \underbrace{\left[-\tfrac{1}{2t}\right]}_{\boxed{u}}\underbrace{\left[d\left(e^{-t^2}\right)\right]}_{\boxed{dv}}$$

Then

$$\int e^{-t^2} dt = -\frac{1}{2t}\left(e^{-t^2}\right) - \frac{1}{2}\int \frac{e^{-t^2}}{t^2}\,dt$$

(Formula) $\boxed{\int u\,dv} = \boxed{u\cdot v} - \boxed{\int v\,du}$

Integrating the integral expression in the last term above, i. e.,

$$-\frac{1}{2}\int \frac{e^{-t^2}}{t^2}\,dt$$

in like manner (by parts), another term appears, and the equation above becomes

$$\int e^{-t^2} dt = -\frac{e^{-t^2}}{2t} + \frac{e^{-t^2}}{4t^3} + \frac{3}{4}\int \frac{e^{-t^2}}{t^4}\,dt.$$

Continuing this process by breaking up the integral on the right into another term in the series plus a new integral, repeatedly, produces the infinite series:

$$\int e^{-t^2} dt = -\frac{e^{-t^2}}{2t} + \frac{e^{-t^2}}{4t^3} - \frac{3e^{-t^2}}{8t^5} + \frac{3.5e^{-t^2}}{16t^7} - \frac{3.5.7e^{-t^2}}{32t^9} + \cdots$$

$$= -\frac{e^{-t^2}}{2t}\left[1 - \frac{1}{2t^2} + \frac{1.3}{(2t^2)^2} - \frac{1.3.5}{(2t^2)^3} + \frac{1.3.5.7}{(2t^2)^4} - \cdots\right].$$

Now $$\int_0^t e^{-t^2} dt = \int_0^\infty e^{-t^2} dt - \int_t^\infty e^{-t^2} dt.$$

$$\boxed{\text{part}} = \boxed{\text{whole}} - \boxed{\text{part}}$$

But
$$\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}\ * \qquad \text{(Ref. 4)}$$

Therefore $$\int_0^t e^{-t^2} dt = \frac{\sqrt{\pi}}{2} - \int_t^\infty e^{-t^2} dt.$$

And since the value of the definite integral

$$\int_t^\infty e^{-t^2} dt = \left[-\frac{e^{-t^2}}{2t}\left\{1 - \frac{1}{2t^2} + \frac{1.3}{(2t^2)^2} - \frac{1.3.5}{(2t^2)^3} + \cdots\right\}\right]_t^\infty.$$

Then

$$\int_0^t e^{-t^2} dt = \frac{\sqrt{\pi}}{2} - \frac{e^{-t^2}}{2t}\left\{1 - \frac{1}{2t^2} + \frac{1.3}{(2t^2)^2} - \cdots\right\},$$

and, since $$\frac{N}{\sigma\sqrt{2\pi}} \int e^{-\frac{x^2}{2\sigma^2}} dx = \frac{1}{\sqrt{\pi}} \int e^{-t^2} dt,$$

(when $N = 1$ and $t = \frac{x}{\sigma\sqrt{2}}$)

* Good proof in The Encyclopedia Brittanica, American Edition, 1896, in article on Infinitesimal Calculus. Also (2).

$$\frac{1}{\sqrt{\pi}}\int_0^t e^{-t^2}dt = \frac{1}{2}\left[1 - \frac{e^{-t^2}}{t\sqrt{\pi}}\left\{1 - \frac{1}{2t^2} + \frac{3}{(2t^2)^2} - \frac{3\cdot5}{(2t^2)^3} + \frac{3\cdot5\cdot7}{(2t^2)^4} - \cdots\right\}\right]$$

Substituting $\frac{x}{\sigma\sqrt{2}}$ for t and keeping $\frac{x}{\sigma}$ separate, this series becomes:

$$\frac{1}{2} - \frac{e^{-\frac{x^2}{2\sigma^2}}}{\frac{x}{\sigma}\sqrt{2\pi}}\left\{1 - \frac{1}{\left(\frac{x}{\sigma}\right)^2} + \frac{3}{\left(\frac{x}{\sigma}\right)^4} - \frac{3\cdot5}{\left(\frac{x}{\sigma}\right)^6} + \frac{3\cdot5\cdot7}{\left(\frac{x}{\sigma}\right)^8} - \cdots\right\}$$

This series will be referred to as Series B. It is asymptotic or semi-convergent, (5) (8), a type of series which is frequently obtained by integrating by parts (6). Series B is divergent for values of $\frac{x}{\sigma}$ below unity. Weld (7) states that this series converges rapidly when $\frac{x}{\sigma} > \sqrt{2}$, but does not mention its peculiar asymptotic nature whereby it converges until a minimum term is reached and then diverges. As Townsend (6) indicates, the best approximation of the sum of an asymptotic series is obtained if the series is terminated with the term having the smallest absolute value. This minimum term is the second term for $\frac{x}{\sigma} = \sqrt{2}$ and while the error due to dropping the succeeding terms is less than the last term retained, still this second term has too large a value to permit any very accurate calculation of the area (as will be shown later). However, the accuracy increases as $\frac{x}{\sigma}$ takes on larger values, since it then takes longer for convergence to the minimum term and this minimum term also grows smaller.

Brunt (1) advocates the use of another series, developed by Schlömilch, which he states is better when $\frac{x}{\sigma}$ is large. This

Schlömilch series, hereinafter referred to as Series S, is as follows, in terms of $\frac{x}{\sigma}$:

$$\text{Area}\Big]_0^{\frac{x}{\sigma}} = \frac{1}{2}\Bigg[1 - \frac{2e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}}{\frac{x}{\sigma}\sqrt{2\pi}}\Bigg\{1 - \frac{1}{\left(\frac{x}{\sigma}\right)^2+2} + \frac{1}{\left\{\left(\frac{x}{\sigma}\right)^2+2\right\}\left\{\left(\frac{x}{\sigma}\right)^2+4\right\}} -$$

$$\frac{5}{\left\{\left(\frac{x}{\sigma}\right)^2+2\right\}\left\{\left(\frac{x}{\sigma}\right)^2+4\right\}\left\{\left(\frac{x}{\sigma}\right)^2+6\right\}} + \frac{9}{\left\{\left(\frac{x}{\sigma}\right)^2+2\right\}\left\{\left(\frac{x}{\sigma}\right)^2+4\right\}\left\{\left(\frac{x}{\sigma}\right)^2+6\right\}\left\{\left(\frac{x}{\sigma}\right)^2+8\right\}}$$

$$- \frac{129}{\left\{\left(\frac{x}{\sigma}\right)^2+2\right\}\cdots\left\{\left(\frac{x}{\sigma}\right)^2+10\right\}} + \frac{315^{\,*}}{\left\{\left(\frac{x}{\sigma}\right)^2+2\right\}\cdots\cdots\left\{\left(\frac{x}{\sigma}\right)^2+12\right\}}\cdots$$

Series S is readily developed from series B by transformation of successive terms in the B series to terms with the characteristic Schlömilch denominator. This is more easily accomplished when series B is in the "t" $\left(= \frac{x}{\sigma\sqrt{2}}\right)$ form.

To determine the limits of practical utility of each of these three series, actual calculations of areas were made at appropriate $\frac{x}{\sigma}$ intervals, with results shown in Table 1 and Figure 1.' All calculations were made "by hand" and carried to 10 or more decimal places.

The three series (formulas) were used in the following forms:

SERIES A:—

$$\text{Area}\Big]_0^{\frac{x}{\sigma}} = \frac{x}{\sigma}(.398\,942\,280\,3)\Bigg[1 - \frac{\left(\frac{x}{\sigma}\right)^2}{6} + \frac{\left(\frac{x}{\sigma}\right)^4}{40} - \frac{\left(\frac{x}{\sigma}\right)^6}{336}$$

$$+ \frac{\left(\frac{x}{\sigma}\right)^8}{3456} - \frac{\left(\frac{x}{\sigma}\right)^{10}}{42240} + \frac{\left(\frac{x}{\sigma}\right)^{12}}{599\,040} - \frac{\left(\frac{x}{\sigma}\right)^{14}}{9676800} + \frac{\left(\frac{x}{\sigma}\right)^{16}}{175\,472\,640}$$

$$\frac{\left(\frac{x}{\sigma}\right)^{18}}{353\,009\,6640} + \frac{\left(\frac{x}{\sigma}\right)^{20}}{780\,337\,15200} - \frac{\left(\frac{x}{\sigma}\right)^{22}}{188\,024\,094\,720\,0} + \cdots\Bigg]$$

* This term not given by Brunt (1), but calculated by present writer. Last term practicable to use, since next term also has plus sign.

SERIES B:—

$$\text{Area}\Big]_0^{\frac{x}{\sigma}} = \frac{1}{2} - \log^{-1}\Big[0 - \Big\{\log\frac{x}{\sigma} + \frac{\left(\frac{x}{\sigma}\right)^2}{2}(.434\,294\,481\,9) + .399\,089\,934\,2\Big\}\Big]\cdot\Big[1 - \frac{1}{\left(\frac{x}{\sigma}\right)^2} + \frac{3}{\left(\frac{x}{\sigma}\right)^4} - \frac{15}{\left(\frac{x}{\sigma}\right)^6} + \frac{105}{\left(\frac{x}{\sigma}\right)^8} - \frac{945}{\left(\frac{x}{\sigma}\right)^{10}} + \frac{10\,395}{\left(\frac{x}{\sigma}\right)^{12}} - \frac{135\,135}{\left(\frac{x}{\sigma}\right)^{14}} + \frac{202\,702\,5}{\left(\frac{x}{\sigma}\right)^{16}} - \frac{344\,594\,25}{\left(\frac{x}{\sigma}\right)^{18}} + \frac{654\,729\,075}{\left(\frac{x}{\sigma}\right)^{20}} - \frac{137\,493\,105\,75}{\left(\frac{x}{\sigma}\right)^{22}} + \cdots\Big]$$

SERIES S:—

$$\text{Area}\Big]_0^{\frac{x}{\sigma}} = \frac{1}{2} - \log^{-1}\Big[0 - \Big\{\log\frac{x}{\sigma} + \frac{\left(\frac{x}{\sigma}\right)^2}{2}(.434\,294\,481\,9) + .399\,089\,934\,2\Big\}\Big]\cdot\Big[1 - \frac{1}{\left(\frac{x}{\sigma}\right)^2+2} + \frac{1}{\left\{\left(\frac{x}{\sigma}\right)^2+2\right\}\left\{\left(\frac{x}{\sigma}\right)^2+4\right\}} - \frac{5}{\left\{\left(\frac{x}{\sigma}\right)^2+2\right\}\left\{\left(\frac{x}{\sigma}\right)^2+4\right\}\left\{\left(\frac{x}{\sigma}\right)^2+6\right\}} + \frac{9}{\left\{\left(\frac{x}{\sigma}\right)^2+2\right\}\cdots\left\{\left(\frac{x}{\sigma}\right)^2+8\right\}} - \frac{129}{\left\{\left(\frac{x}{\sigma}\right)^2+2\right\}\cdots\left\{\left(\frac{x}{\sigma}\right)^2+10\right\}} + \frac{315}{\left\{\left(\frac{x}{\sigma}\right)^2+2\right\}\cdots\left\{\left(\frac{x}{\sigma}\right)^2+12\right\}} - .$$

Table I shows the numbers of terms required in each series to give areas accurate to 3, 4, 5, 7 and 9 decimal places, respectively, for values of $\frac{x}{\sigma}$ ranging from .25 to 5.00. Calculations were checked by Sheppard's Tables (4) and the accuarcy was determined on the principle that the error is less than the last term retained or the first term dropped *. Where x is used it indicates that the series cannot give the accuracy indicated.

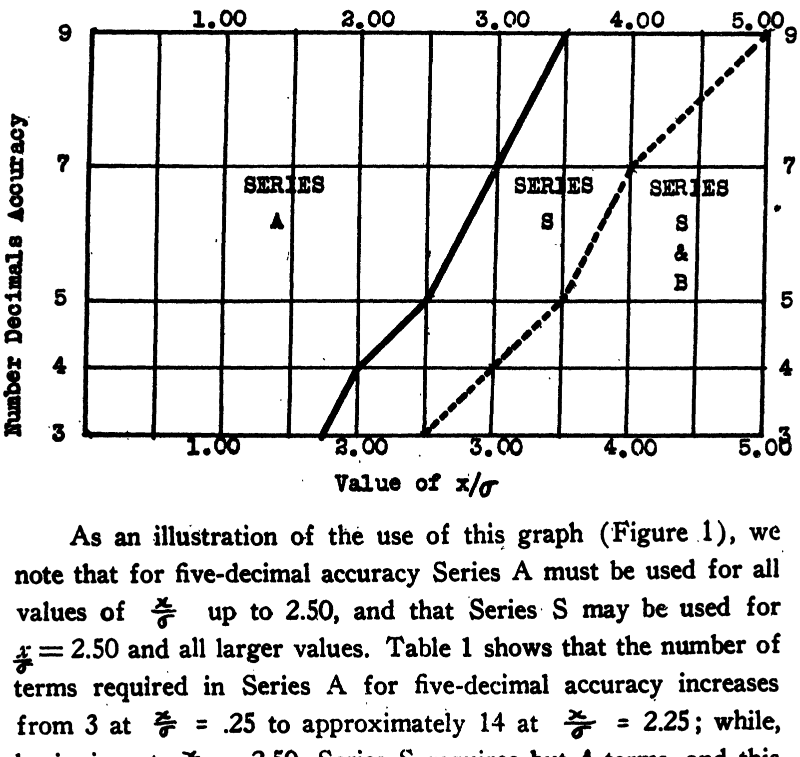
The graph, Figure 1, shows the approximate domain of

* This does not hold strictly true of Series S, since it is a modification of the true Series B.

utility of each series under conditions of accuracy ranging from 3 to 9 decimal places. Note that Series S covers a wider range than does Series B, including the entire domain of Series B. Hence we may conclude that, while it is essential as a basis for the derivation of Series S, Series B may be discarded for area calculations. Moreover, Series S is not only more valuable than Series B because of its wider range of utility but also because its more rapid convergence gives a desired degree of accuracy with fewer terms than are necessary with Series B.

FIGURE 1

DOMAINS OF PRACTICAL UTILITY OF THREE INFINITE SERIES IN CALCULATING AREA UNDER NORMAL PROBABILITY CURVE.



As an illustration of the use of this graph (Figure 1), we note that for five-decimal accuracy Series A must be used for all values of $\frac{x}{\sigma}$ up to 2.50, and that Series S may be used for $\frac{x}{\sigma} = 2.50$ and all larger values. Table 1 shows that the number of terms required in Series A for five-decimal accuracy increases from 3 at $\frac{x}{\sigma} = .25$ to approximately 14 at $\frac{x}{\sigma} = 2.25$; while, beginning at $\frac{x}{\sigma} = 2.50$, Series S requires but 4 terms, and this number diminishes to 1 term for $\frac{x}{\sigma} = 5.00$.

TABLE 1.

NUMBERS OF TERMS REQUIRED FOR VARYING DEGREES OF ACCURACY IN CALCULATION OF INCREASING PROPORTIONS OF AREA UNDER THE NORMAL PROBABILITY CURVE.

	3 decimals			4 decimals			5 decimals			7 decimals			9 decimals		
	A	B	S	A	B	S	A	B	S	A	B	S	A	B	S
.25	2	x	x	2	x	x	3	x	x	4	x	x	5	x	x
.50	3	x	x	3	x	x	4	x	x	5	x	x	6	x	x
.75	4	x	x	4	x	x	5	x	x	7	x	x	8	x	x
1.00	4	x	x	5	x	x	6	x	x	8	x	x	9	x	x
1.25	5	x	x	6	x	x	7	x	x	9	x	x	11	x	x
1.50	7	x	x	8	x	x	9	x	x	11	x	x	12	x	x
1.75	7	x	5	9	x	x	10	x	x	12	x	x	14*	x	x
2.00	9	x	3	10	x	4	12	x	x	14*	x	x	16*	x	x
2.25	11	x	3	12	x	3	14*	x	x	15*	x	x	18*	x	x
2.50	12	2	2	13*	x	2	15*	x	4	17*	x	x	20*	x	x
2.75	–	1	1	–	x	2	–	x	4	19*	x	x	23*	x	x
3.00	–	1	1	–	3	2	–	x	3	–	x	4	27*	x	x
3.50	–	1	1	–	1	1	–	3	2	–	x	4	–	x	7
4.00	–	1	1	–	1	1	–	1	1	–	5	3	–	x	6
5.00	–	1	1	–	1	1	–	1	1	–	1	1	–	3	2
	3 decimals			4 decimals			5 decimals			7 decimals			9 decimals		

* Estimated by graphic extrapolation.

Explanatory: Read table as follows: The number of terms required in Series A to calculate to 4 decimal places of accuracy the portion of the area under the normal curve lying between the ordinates at $\frac{x}{\sigma} = 0$ and $\frac{x}{\sigma} = 2.00$ is 10; with Series S it is 4; the calculation is impossible to this degree of accuracy with Series B.

Notes: x indicates impossible calculation. — indicates impracticable calculation.

CONCLUSIONS

All areas under the normal curve may be calculated by the use of Series A and Series S, the two being complementary.

Methods of developing Series A and Series B are outlined and it is indicated that Series S is derived from Series B.

The domain of practical utility for each series is shown in Figure 1. The numbers of terms required for various degrees of accuracy are shown in Table 1.

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