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QUADRIC HYPERSURFACES OF FINITE TYPE

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Introduction. A submanifold M of the Euclidean m-space E^m is said to be of *finite type* (see [C1] for details) if each component of its position vector field X can be written as a finite sum of eigenfunctions of the Laplacian Δ of M, i.e., if

$$X = X_0 + X_1 + \ldots + X_k$$

where X_0 is a constant vector and $\Delta X_t = \lambda_t X_t$ for $t = 1, \ldots, k$. If in particular all eigenvalues $\lambda_1, \ldots, \lambda_k$ are mutually different, then M is said to be of k-*type*. If we define a polynomial P by

$$P(T) = \prod_{t=1}^{k} (T - \lambda_t),$$

then $P(\Delta)(X - X_0) = 0$. If M is compact, then the converse also holds, i.e., if there exists a constant vector X_0 and a nontrivial polynomial P such that $P(\Delta)(X - X_0) = 0$, then M is of finite type [C1].

The class of finite type submanifolds is very large, including minimal submanifolds of E^m, minimal submanifolds of a hypersphere, parallel submanifolds, compact homogeneous submanifolds equivariantly immersed in a Euclidean space, and also isoparametric hypersurfaces of a hypersphere. On the other hand, very few hypersurfaces of finite type in a Euclidean space are known, other than minimal hypersurfaces (which are of 1-type). Therefore the following problem seems to be quite interesting.

PROBLEM. Classify all finite type hypersurfaces in E^m.

For $m = 2$, this problem was solved completely. In fact, it is known that

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circles and straight lines are the only curves of finite type in E^2 (see [C1] and [CDVV] for details). For $m = 3$, the first result in this respect given in [C2], states that circular cylinders are the only tubes in E^3 which are of finite type. In [CDVV] it is shown that a ruled surface in E^3 is of finite type if and only if it is a plane, a circular cylinder or a helicoid. In [G], it is shown that a cone in E^m is of finite type if and only if it is minimal. In [D], some ruled submanifolds of finite type are classified.

If M' is an algebraic hypersurface with singularities in E^n, then M' is said to be of finite type if $M' - \{\text{singularities}\}$ is of finite type.

Combining the notion of algebraic hypersurfaces and the notion of submanifolds of finite type, the first two authors proved in [CD] that the only quadric surfaces of finite type in E^3 are the circular cylinders and the spheres. In this article, we shall completely classify quadric hypersurfaces of finite type.

2. Quadric hypersurfaces. A subset M of an n-dimensional Euclidean space E^n is called a *quadric hypersurface* if it is the set of points $(x_1, \dots, x_n)$ satisfying the following equation of the second degree:

$$\sum_{i,k=1}^{n} a_{ik} x_i x_k + \sum_{i=1}^{n} b_i x_i + c = 0 \,, \tag{2.1}$$

where a_{ik}, b_i, c are all real numbers. We can assume without loss of generality that the matrix $A = (a_{ik})$ is symmetric and A is not a zero matrix. By applying a coordinate transformation in E^n if necessary, we may assume that (2.1) takes one of the following canonical forms:

$$\sum_{i=1}^{r} a_i x_i^2 + 1 = 0 \,, \tag{I}$$

$$\sum_{i=1}^{r} a_i x_i^2 + 2x_{r+1} = 0 \,, \tag{II}$$

$$\sum_{i=1}^{r} a_i x_i^2 = 0 \tag{III}$$

where $(a_1, \dots, a_r, 0, \dots, 0)$ (with $n - r$ zeros) is proportional to the eigenvalues of the matrix A. In general, we have $1 \leq r \leq n$. In the cases where $r = n$ in (I) and (III) and $r + 1 = n$ in (II) the hypersurface is called a *properly* $(n-1)$*-dimensional quadric hypersurface*, and in other cases, a *quadric cylindrical hypersurface*. In cases (I) and (III), the quadric cylindrical hypersurface is the product of an $(n - r)$-dimensional linear subspace E^{n-r} and a properly $(r - 1)$-dimensional quadric hypersurface. In case (II), the quadric cylindrical hypersurface is the product of an $(n-r-1)$-dimensional linear subspace and a properly r-dimensional quadric hypersurface.

Let $S^p(r)$ denote the hypersphere in E^{p+1} with radius r and centered at the origin. Denote by $M_{p,q}$ the product of spheres

$$S^p\left(\sqrt{\frac{p}{p+q}}\right) \times S^q\left(\sqrt{\frac{q}{p+q}}\right) \subset S^{p+q+1}(1) \subset E^{p+q+2}\,.$$

We denote by $C_{p,q}$ the $(p+q+1)$-dimensional cone in E^{p+q+2} with vertex at the origin shaped on $M_{p,q}$. It is easy to see that $C_{p,0}$ and $C_{0,q}$ are hyperplanes in E^{p+2} and E^{q+2}, respectively, and $C_{p,q}$ with $p > 0$, $q > 0$ are algebraic hypersurfaces of degree 2.

The purpose of this article is to prove the following classification theorem.

THEOREM. *A quadric hypersurface M in E^{n+1} is of finite type* (*even locally*) *if and only if it is one of the following hypersurfaces:*

(a) *hypersphere,*

(b) *one of the algebraic cones $C_{p,n-p-1}$, $0 < p < n-1$,*

(c) *the product of a linear subspace E^l and a hypersphere of E^{n-l+1} $(0 < l < n)$,*

(d) *the product of a linear subspace E^l and one of the algebraic cones $C_{p,n-l-p-1}$ $(0 < p < n-l-1)$.*

3. Properly n-dimensional quadric hypersurfaces. Let M be a hypersurface in E^{n+1}. Consider a parametrization

$$X(u_1,\dots,u_n) = (u_1,\dots,u_n,v) \tag{3.1}$$

where

$$v = v(u_1,\dots,u_n)\,. \tag{3.2}$$

Denote $\partial_i v (= \partial v/\partial u_i)$ by v_i. Then we have

$$g_{ij} = \delta_{ij} + v_i v_j\,, \qquad g^{ij} = \delta_{ij} - \frac{v_i v_j}{g} \tag{3.3}$$

where

$$g = \det(g_{ij}) = 1 + \sum_{i=1}^{n} v_i^2\,, \tag{3.4}$$

and $g_{ij} = \langle \partial_i X, \partial_j X\rangle$. The Laplacian Δ of M is given by

$$\Delta = -\sum_{i,j}\left(\frac{\partial_i g}{2g} g^{ij} + \partial_i g^{ij}\right)\partial_j - \sum_{i,j} g^{ij}\partial_i\partial_j\,. \tag{3.5}$$

If M is a properly n-dimensional quadric hypersurface, then either M is an algebraic cone of degree 2 or M is of one of the following two kinds:

$$v^2 = \sum_{i=1}^{n} b_i u_i^2 + c\,, \qquad b_1 \dots b_n c \neq 0\,, \tag{I}$$

$$v = \frac{1}{2}\sum_{i=1}^{n} b_i u_i^2\,, \qquad b_1 \dots b_n \neq 0\,. \tag{II}$$

In the following two sections, we study properly n-dimensional quadric hypersurfaces of kinds (I) and (II), separately.

4. Proper quadric hypersurfaces of kind (I). In this section we assume M is a properly n-dimensional quadric hypersurface of kind (I). We may consider the following parametrization:

$$X = (u_1, \dots, u_n, v)\,, \quad v^2 = a_1 u_1^2 + \dots + a_n u_n^2 + c\,, \quad a_1 \dots a_n c \neq 0\,. \tag{4.1}$$

In this case, we have

$$v_i = \partial_i v = a_i u_i / v\,. \tag{4.2}$$

Thus, (3.3) and (3.4) imply

$$g_{ij} = \delta_{ij} + \frac{a_i a_j u_i u_j}{W}\,, \qquad g^{ij} = \delta_{ij} - \frac{a_i a_j u_i u_j}{gW}\,, \tag{4.3}$$

$$g = 1 + \frac{1}{W}\sum_i (a_i u_i)^2\,, \qquad \frac{1}{g} = 1 - \frac{1}{gW}\sum_i (a_i u_i)^2\,, \tag{4.4}$$

where

$$W = v^2 = a_1 u_1^2 + \dots + a_n u_n^2 + c\,. \tag{4.5}$$

From (4.4) we find

$$\partial_i g = \frac{2}{W}(a_i u_i (1 + a_i - g))\,, \tag{4.6}$$

$$\widetilde{g} := gW = c + \sum_i (1 + a_i) a_i u_i^2\,. \tag{4.7}$$

We put

$$\begin{aligned} A_k &= \frac{1}{2W}\Big\{(gW - a_k^2 u_k^2)\partial_k g - a_k u_k \sum_{t \neq k} a_t u_t \partial_t g\Big\} \\ &= \frac{1}{2} g \sum_t g^{tk} \partial_t g\,. \end{aligned} \tag{4.8}$$

Then from (4.3) and a straightforward computation, we have

$$-\sum_t \partial_t g^{tk} = \frac{a_k u_k}{gW}\sum_{t \neq k} a_t + \frac{2A_k}{g^2}\,. \tag{4.9}$$

From (3.5), (4.8) we obtain

$$\Delta = \frac{1}{g^2}\sum_i A_i\partial_i + \frac{1}{gW}\sum_j \Big(\sum_{t\neq k} a_t\Big) a_j u_j \partial_j - \sum_{i,j} g^{ij}\partial_i\partial_j\,. \tag{4.10}$$

We put

$$c_{ij} = gg^{ij}\,. \tag{4.11}$$

From (4.3), (4.4) and (4.11) we have

$$c_{ij} = \delta_{ij} + \frac{1}{W}\Big(\delta_{ij}\sum_t a_t^2u_t^2 - a_ia_ju_iu_j\Big)\,. \tag{4.12}$$

For later use, we note that from (4.8), (4.12) we have

$$\sum_{i,j} c_{ij}(\partial_i g)(\partial_j g) = 2\sum_j A_j\partial_j g\,. \tag{4.13}$$

Also note from (4.7) that

$$\widetilde{g} = gW \text{ is a polynomial in } u_1,\ldots,u_n\,. \tag{4.14}$$

Lemma 1. *We have*

$$\Delta^t u_k = g^{1-3t}A_k\alpha_t\Big(\sum_i A_i\partial_i g\Big)^{t-1} + g^{2-3t}P_{k,t}(u_1,\ldots,u_n,1/W)$$

where $P_{k,t}$ is a polynomial in $n+1$ variables and α_t is given by

$$\alpha_t = (4-3t)(6t-5)\alpha_{t-1}\,, \qquad \alpha_1 = 1\,. \tag{4.15}$$

Proof. The proof goes by induction. For $t = 1$, the formula follows from (4.10). Suppose the lemma is true for $t-1$. Then it follows from (4.10), (4.11) and (4.13) that

$$\begin{aligned}
\Delta^t u_k &= g^{1-3t}\sum_j A_jA_k\alpha_{t-1}\Big(\sum_i A_i\partial_i g\Big)^{t-2}(4-3t)\partial_j g \\
&\quad - g^{1-3t}\sum_{i,j} c_{ij}A_k\alpha_{t-1}\Big(\sum_l A_l\partial_l g\Big)^{t-2}(4-3t)(3-3t)\partial_j g\partial_i g \\
&\quad + g^{2-3t}P_{k,t}(u_1,\ldots,u_n,1/W) \\
&= g^{1-3t}A_k\alpha_t\Big(\sum_i A_i\partial_i g\Big)^{t-1} + g^{2-3t}P_{k,t}(u_1,\ldots,u_n,1/W)\,,
\end{aligned}$$

which proves the lemma.

Now, suppose that M is of k-type. Then there exist real numbers $c_1,\ldots,c_k$ such that

$$\Delta^{k+1}X + c_1\Delta^k X + \ldots + c_k\Delta X = 0\,, \tag{4.16}$$

$$\Delta^{k+1}u_i + c_1\Delta^k u_i + \ldots + c_k\Delta u_i = 0\,, \qquad i = 1,\ldots,n\,. \tag{4.17}$$

From Lemma 1 and (4.17) we get

$$\Big(\sum_i A_i\partial_i g\Big)^{k+1} = gP(u_1,\dots,u_n,1/W)\,, \tag{4.18}$$

where P is a polynomial in $n+1$ variables. We put

$$G(u_1,\dots,u_n) = W^5\sum_i A_i\partial_i g\,. \tag{4.19}$$

Then G is a polynomial in $u_1,\dots,u_n$. Since W is a polynomial in $u_1,\dots,u_n$, there is a natural number N and a polynomial R in n variables such that

$$W^N P(u_1,\dots,u_n,1/W) = R(u_1,\dots,u_n)\,. \tag{4.20}$$

From (4.7), (4.18)–(4.20), we have

$$W^{N+1}G^{k+1} = \widetilde{g}W^{5k+5}R\,. \tag{4.21}$$

For any fixed j, $1\le j\le n$, we put $u_i=0$ for $i\ne j$ in (4.21) to obtain

$$\begin{aligned}(c+a_ju_j^2)^{N+k+2}&2^{k+1}(a_j^2cu_j)^{2k+2}\\ &= (c+a_j(a_j+1)u_j^2)(c+a_ju_j^2)^{5k+5}R(0,\dots,0,u_j,0,\dots,0)\,.\end{aligned} \tag{4.22}$$

Since $a_1\dots a_nc\ne 0$, this implies $a_j=-1$. Because this is true for any j, M is a hypersphere.

5. Proper quadric hypersurfaces of kind (II). For such hypersurfaces we consider a parametrization

$$X = (u_1,\dots,u_n,v)\,,\qquad v=\frac12\sum_i b_iu_i^2\,,\qquad b_1\dots b_n\ne 0\,. \tag{5.1}$$

From (3.3)–(3.5) we may find

$$g_{ij} = \delta_{ij}+b_ib_ju_iu_j\,,\qquad g^{ij} = \delta_{ij}-\frac{b_ib_ju_iu_j}{g}\,, \tag{5.2}$$

$$g = \det(g_{ij}) = 1+\sum_i b_i^2u_i^2\,, \tag{5.3}$$

$$\begin{aligned}\Delta = \frac{1}{g^2}\sum_j\Big\{b_j+\sum_i(b_j-b_i)b_i^2u_i^2\Big\}&b_ju_j\partial_j\\ &-\sum_{i,j}g^{ij}\partial_i\partial_j+\frac1g\sum_j\Big(\sum_{i\ne j}b_i\Big)b_ju_j\partial_j\,.\end{aligned} \tag{5.4}$$

Lemma 2. *We have*

$$g^2\Delta g = Q(u_1,\dots,u_n)+gT(u_1,\dots,u_n)\,, \tag{5.5}$$

$$||\nabla g||^2 = \frac2g Q(u_1,\dots,u_n)\,, \tag{5.6}$$

where Q and T are some polynomials in $u_1,\dots,u_n$ and ∇g is the gradient of g.

Proof. From (5.3) and (5.4) we find

$$\Delta g=\frac{2}{g^2}\sum_j b_j^2u_j\Big\{(b_j+\sum_i(b_j-b_i)b_i^2u_i^2)b_ju_j+g\Big(\sum_{i\neq j}b_i\Big)b_ju_j\Big\}-2\sum_j b_j^2g^{jj}\,.$$

Thus, if we put

$$Q=2\sum_j b_j^3u_j^2\Big\{b_j+\sum_i(b_j-b_i)b_i^2u_i^2\Big\}\,, \tag{5.7}$$

$$T=2\sum_j b_j^3u_j^2\Big(\sum_{i\neq j}b_i\Big)-2g\sum_i g^{ii}b_i^2\,, \tag{5.8}$$

then we obtain (5.5). It is obvious that Q and T are polynomials in $u_1,\dots,u_n$. (5.6) follows from the definition of the norm of ∇g, (5.2), (5.3) and (5.7).

LEMMA 3. *We have*

$$\Delta^t u_j=g^{1-3t}Q^{t-1}b_ju_j\Big\{b_j+\sum_i(b_j-b_i)b_i^2u_i^2\Big\}\alpha_t+g^{2-3t}\widetilde{P}_{j,t}$$

where $\widetilde{P}_{j,t}$ is a polynomial in $u_1,\dots,u_n$ and α_t is given by (4.15).

Proof. The proof goes by induction. For $t=1$ the formula follows easily from (5.4). Assume it is true for $t-1$. Then we have

$$\begin{aligned}\Delta^t u_j&=\Delta\Big\{g^{4-3t}Q^{t-2}b_ju_j\Big(b_j+\sum_i(b_j-b_i)b_i^2u_i^2\Big)\alpha_{t-1}+g^{5-3t}\widetilde{P}_{j,t-1}\Big\}\\&=g^{1-3t}Q^{t-2}b_ju_j\Big(b_j+\sum_i(b_j-b_i)b_i^2u_i^2\Big)\alpha_{t-1}\\&\qquad\times\{(4-3t)g^2\Delta g-(4-3t)(3-3t)g||\nabla g||^2\}+g^{2-3t}\widehat{P}_{j,t}\,.\end{aligned}$$

where $\widehat{P}_{j,t}$ is a polynomial in $u_1,\dots,u_n$. Thus, Lemma 2 implies the assertion.

If M is of k-type, then again there exist real numbers $c_1,\dots,c_k$ such that

$$\Delta^{k+1}u_j+c_1\Delta^k u_j+\ldots+c_k\Delta u_j=0\,,\qquad j=1,\dots,n\,.$$

From Lemma 3 and (5.7) we obtain

$$Q^{k+1}=gP(u_1,\dots,u_n)$$

where P is a polynomial in $u_1,\dots,u_n$. Since $b_1\dots b_n\neq 0$, $g=1+\sum b_i^2u_i^2$ is irreducible. Moreover, because $Q/g=\frac{1}{2}||\nabla g||^2$ is not a polynomial in $u_1,\dots,u_n$, we obtain a contradiction. Thus, there exist no proper quadric hypersurfaces of kind (II) which are of finite type.

6. Proof of Theorem. If M is a properly n-dimensional quadric hypersurface of finite type in E^{n+1}, then either M is an algebraic conic hypersurface of degree 2 or, according to §§3–5, M is a hypersphere. If M is an algebraic conic hypersurface of degree 2, then because M is of finite type, M is a minimal cone [G]. Thus, by a result of [H], M is one of the algebraic cones $C_{p,n-p-1}$, $0 < p < n-1$.

If M is a quadric cylindrical hypersurface of finite type in E^{n+1}, then M is the product of a linear subspace E^l and a proper quadric hypersurface, say N. Since M is of finite type, N is also of finite type. Thus, N is either a hypersphere or an algebraic cone $C_{p,n-l-p-1}$ for some suitable p.

The converse is easy to verify.

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