QUADRICS ASSOCIATED WITH A SIMPLEX IN n-SPACE

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1. Introduction

It is well known that the projections of a pair of points from the vertices of a triangle onto the opposite sides lie on a conic and that when the points are the centroid and orthocentre of the triangle, this conic is a circle. Analogously the projections of the centroid and orthocentre of a simplex from its vertices onto the opposite $(n-1)$-dimensional faces, if the simplex is orthocentric, lie on a hypersphere [2, 5]. Further the projections of two points onto the edges of a general simplex from the opposite faces lie on quadric [1]; and when the points are the centroid and orthocentre respectively and the simplex is orthocentric, this quadric is a hypersphere [2]. The results as regards projections onto $(n-1)$-dimensional and 1-dimensional faces being thus known, it remains to see what results hold in the case of intermediary faces. And in this note we prove that a similar result holds for projections onto intermediary faces as well.

2.

Let S be any simplex in n-space. Let it be the simplex of reference with vertex-vectors $\boldsymbol{e}_0, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$. Let $\boldsymbol{u} = \sum_{i=0}^n u_i \boldsymbol{e}_i = (u_0, u_1, \cdots, u_n)$ and $\boldsymbol{v} = \sum_{i=0}^n v_i \boldsymbol{e}_i = (v_0, v_1, \cdots, v_n)$ be the co-ordinate vectors of two arbitrary points U and V disjoint from S.

The projections of the points $\boldsymbol{u}$ and $\boldsymbol{v}$ onto the r-dimensional face determined by the $(r+1)$ vertices $\boldsymbol{e}_{i_0}, \boldsymbol{e}_{i_1}, \cdots, \boldsymbol{e}_{i_r}$ from the opposite faces of the simplex are the points $\sum_{j=0}^r u_{i_j} \boldsymbol{e}_{i_j}$ and $\sum_{j=0}^r u_{i_j} \boldsymbol{e}_{i_j}$. It is easily verified that these points lie on the quadric:

$$\left(\sum_{i=0}^{n} x_i|u_i\right)\left(\sum_{i=0}^{n} x_i|v_i\right) - (r+1)\left(\sum_{i=0}^{n} x^2|u_i v_i\right) = 0.$$

Thus we have

THEOREM 1. *The projections of a pair of points onto the r-dimensional faces of a simplex from the opposite faces lie on a quadric*, $(r = 0, 1, 2, \cdots, (n-1))$.

The point $\boldsymbol{u}$ and the prime $L \equiv \sum_{i=0}^n x_i|u_i = 0$ are pole and polar for the simplex of reference [3, 4] and similarly V and $M \equiv \sum_{i=0}^{n} x_i|v_i = 0$ are pole and polar for S. The quadric $F \equiv \sum_{i=0}^{n} x^2|u_i v_i = 0$ is such that S is self-polar for it and the polar of either of the points U, V with regard to it is the polar for S of the other. There is one and only one quadric with this property when the two points disjoint from S are given. We shall designate it as the polar quadric of S relative to the two points. The $(n-1)$ dimensional quadric $LM-rF = 0$ we shall denote as $Q_r(U, V)$ or $Q_r(\boldsymbol{u}, \boldsymbol{v})$ and refer to it as the rth associated quadric of the points U and V for the simplex S. It is clear that the projections of U and V onto the r-dimensional faces from the opposite faces of S lie on the $(r+1)$th associated quadric $Q_{r+1}(U, V)$, $r = 0, 1, 2, \cdots, (n-1)$. The first associated quadric is easily seen to circumscribe the simplex of reference.

When $r = 0$, the projections of U and V onto the faces of S give only $(n+1)$ points viz. the vertices of the simplex, and then the associated quadric $Q_1(U, V)$ is only one of the many possible quadrics through these $(n+1)$ points. When $r = (n-1)$, projections of U and V give $2(n+1)$ points which are not sufficient to determine a quadric except in space of dimensions less than 3, since the minimum number of points required to determine a quadric in n-space is $(n+3)n/2$. Hence $Q_n(U, V)$, unless $n < 3$, is only one of the possible quadrics through the projections of U and V onto the $(n-1)$-dimensional faces of S. But when $r = 1, 2, \cdots, (n-2)$, the number of points which projections of U and V give is $2\binom{n+1}{r+1} \geqq (n+3)n/2$, so that $Q_r(U, V)$ is the definite quadric which passes through the projections of U and V onto the r-dimensional faces of S from the opposite faces.

3.

The quadrics $Q_r(U, V)$ form a pencil of quadrics through the intersection of the polar quadric $F = 0$ and the pair of primes $LM = 0$. So each associated quadric passes through two $(n-2)$-dimensional quadrics which are the intersections of the polar quadric of S relative to U and V by the polar primes of U and V for S. Hence

THEOREM 2. *Given a simplex and a pair of points disjoint from the simplex in n-space, there exists a pair of $(n-2)$-dimensional quadrics such that the projections of the two points onto the r-dimensional faces of the simplex from the opposite faces lie on an $(n-1)$-dimensional quadric which passes through them.*

The pair of points U and V also lie on a quadric which passes through the two $(n-2)$-dimensional quadrics, viz., the associated quadric $Q_{n+1}(U, V)$.

Further the polar of a point on the join of the points U and V with regard to any associated quadric is seen to be a prime through the intersection of $L = 0$ and $M = 0$ and so the pole of any prime through the intersection of $L = 0$ and $M = 0$ for any associated quadric is a point of the join of U and V. Hence

THEOREM 3. *Given a simplex and two points, the polars of the points for the simplex are primes such that their poles with respect to each associated quadric lie on the join of the two points.*

4.

Taking an arbitrary $(n-2)$-dimensional quadric Ω as the absolute quadric, the prime in which Ω lies being called absolute prime, any $(n-1)$-dimensional quadric which meets the absolute prime in Ω is called a *hypersphere* in n-space. A simplex is said to be *orthocentric* if there exists a hypersphere for which the simplex is self-polar. The centre of the hypersphere is then called the *orthocentre* of the simplex; and the hypersphere itself is then called the polar hypersphere of the simplex. The *centroid* of the simplex is defined to be the point the polar of which for the simplex is the absolute prime.

Suppose S is orthocentric. Then there is a hypersphere for which it is self-polar. Let H be its centre. Let G be the centroid of S. Then the polar quadric of S relative to G and H will be the same as the hypersphere for which S is self-polar, since the polar of G for S is the polar of H for the hypersphere. The polar of G for S is the absolute prime and the intersection of the polar quadric by it is the absolute quadric, the polar quadric being a hypersphere. The projections of G and H onto the r-dimensional faces of S from the opposite faces lie on the $(r+1)$th associated quadric of G and H. Since every associated quadric passes through two fixed $(n-2)$-dimensional quadrics which are the intersections of the polar quadric by the polar primes of the two points relative to S, it follows that each associated quadric of G and H meets the absolute prime in Ω. Hence each associated quadric is a hypersphere. Thus

THEOREM 4. *The projections of the centroid and orthocentre of an orthocentric simplex onto the r-dimensional faces from the opposite faces lie on a hypersphere. The n hyperspheres we obtain corresponding to* $r = 0, 1, 2, \cdots, (n-1)$ *form a coaxial system to which the polar hyperspheres also belongs. The centre of these hyperspheres lie on the line joining the centroid and orthocentre of the simplex; and the radical axis of the system is the polar of the centroid for the polar hypersphere.*

It may be observed that any simplex and a pair of points disjoint from the simplex being given, by a proper choice of the absolute quadric, we can have the simplex as orthocentric and the two points as its centroid and orthocentre respectively; so that the associated quadrics of the two points for the simplex are all hyperspheres belonging to a coaxial system. For, on choosing as the absolute quadric the intersection of the polar quadric of the simplex relative to the two points by the polar prime for the simplex of one of the points, at once we have that the polar quadric is a hypersphere and so the simplex is orthocentric and the two points are one the centroid and the other the orthocentre.

The $(r+1)$ vertices which determine an r-dimensional face form an r-simplex. When S is orthocentric the r-simplex also will be orthocentric; further the projections of the centroid and orthocentre of S onto the r-dimensional faces will be the centroid and orthocentre respectively of the r-simplex. For, as may be seen from the equations of the polar primes, the polar quadric and associated quadrics

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of U and V for S, the intersection of the polar primes of U and V by the r-dimensional face are the polars for the r-simplex of the projections of U and V onto the r-dimensional face; and the intersections of the polar and associated quadrics by the r-dimensional face are the polar and associated quadrics for the r-simplex of the projections of U and V.

5.

Suppose S is an orthocentric simplex in E_n. Let $\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_n$ be the position vectors of its vertices relative to the orthocentre as origin. These vectors, being dependent, are connected by a relation $\sum_{i=0}^{n} c_i \boldsymbol{a}_i = 0$. If q be the square of the radius of the polar hypersphere of S, then $q = \boldsymbol{a}_i \cdot \boldsymbol{a}_j$, $i \neq j$. The projection of the orthocentre onto the r-dimensional face determined by the vertices $\boldsymbol{a}_{i_0}, \boldsymbol{a}_{i_1}, \cdots, \boldsymbol{a}_{i_r}$ from the opposite face of S will be the point with position vector $\boldsymbol{p} = (\sum_{k=0}^{r} c_{i_k} \boldsymbol{a}_{i_k}) / (\sum_{k=0}^{r} c_{i_k})$ and the projection onto the opposite face will be the point with position vector $\boldsymbol{p}' = (\sum_{k=r+1}^{n} c_{i_k} \boldsymbol{a}_{i_k}) / (\sum_{k=r+1}^{n} c_{i_k})$. The two points $\boldsymbol{p}$ and $\boldsymbol{p}'$ are evidently collinear with the origin, that is, the orthocentre. Further $\boldsymbol{p} \cdot \boldsymbol{p}' = q$. It follows that the two points $\boldsymbol{p}$ and $\boldsymbol{p}'$ are inverses with regard to the polar hypersphere. Hence the hypersphere which contains the projections of the orthocentre on the r-dimensional faces and the hypersphere which contains the projections on the opposite faces, that is, on $(n-r-1)$-dimensional faces are inverses of each other with regard to the polar hypersphere. Hence

THEOREM 5. *The first n associated quadrics of the centroid and orthocentre for an orthocentric simplex in E_n are hyperspheres which are inverses in pairs with regard to the polar hypersphere.*

When n is odd, there is one such hypersphere, viz., the hypersphere which contains the projections of the centroid and orthocentre onto $((n-1)/2)$-dimensional faces, that is its own inverse. And this hypersphere therefore cuts the polar hypersphere orthogonally.

6.

When the two points U and V coincide, it is evident that the second associated quadric $Q_2(U, V)$ is a quadric which touches the edges of S. It may be now shown that $Q_{r+1}(U, V)$ for $r = 2, 3, \cdots, (n-1)$ also will be similarly a quadric touching all the r-dimensional faces of S. For, $Q_{r+1}(U, U)$ will be the quadric

$$\left(\sum_{i=0}^{n} x_i | u_i\right)^2 - (r+1)\left(\sum_{i=0}^{n} x_i^2 | u_i^2\right) = 0$$

And the polar with respect to $Qr+1(U, U)$ of the projection of U onto the r-dimensional face determined by $\boldsymbol{e}_{i_0}, \boldsymbol{e}_{i_1}, \cdots, \boldsymbol{e}_{i_r}$ will be the prime $\sum_{k=r+1}^{n} x_{i_k} | u_{i_k} = 0$

which evidently contains the r-dimensional face determined by $e_{i_0}, e_{i_1}, \cdots, e_{i_r}$. Thus

THEOREM 6. *Given a simplex and a point in n-space, the projections of the point on the r-dimensional faces of the simplex from the opposite faces lie on a quadric which touches the r-dimensional faces at these points.*

Choosing the intersections of the quadric $\sum_{i=0}^{n} x_i^2 | u_i^2 = 0$ by the prime $\sum_{i=0}^{n} x_i | u_i = 0$ as the absolute $(n-2)$-dimensional quadric, the simplex S becomes orthocentric with the centroid and orthocentre coinciding. And the associated quadrics $Q_r(U, V)$ all now become concentric hyperspheres each touching all the faces of the simplex of a definite dimension.

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