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QUALITATIVE ANALYSIS OF BASIC NOTIONS IN PARAMETRIC CONVEX PROGRAMMING, II

(Parameters in the objective function)

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A short survey of recent results in the field of parametric convex programming from the qualitative point of view can be found in [4].

In this paper the same notions as those introduced in [4], i.e. the notions of the solvability set, the stability set of the first kind and the stability set of the second kind, are defined and analyzed qualitatively for the problem

$$\text{(II)} \qquad \min \sum_{a=1}^{m} \lambda_a \Phi_a(x) ,$$

subject to

$$\mathbf{M} = \{x \in \mathrm{R}^n / g_r(x) \leqq 0, \ r = 1, 2, \ldots, l\} ,$$

where $\Phi_a(x)$, $a = 1, 2, \ldots, m$; $g_r(x)$, $r = 1, 2, \ldots, l$ are convex functions possessing continuous first order partial derivatives on R^n (the vector space of all ordered n-tuples of real numbers) and λ_a, $a = 1, 2, \ldots, m$ are arbitrary nonnegative real numbers. The restriction set $\mathbf{M}$ is supposed to be nonempty and fixed.

1. CHARACTERIZATION OF THE SOLVABILITY SET

Definition 1. *The solvability set of problem* (II) *denoted by* $\mathbf{B}$, *is defined by*

$$(1) \qquad \mathbf{B} = \{\lambda \in {}'\mathrm{R}^m_+ / \min_{x \in \mathbf{M}} \sum_{a=1}^{m} \lambda_a \Phi_a(x) \text{ exists}\} ,$$

where ${}'\mathrm{R}^m_+$ *denotes the nonnegative orthant of the* ${}'\mathrm{R}^m$ *vector space of parameters.*

Lemma 1. *If the set* $\mathbf{B}$ *is defined by* (1), *then it is a cone with vertex at* $\lambda = 0$.

Proof. It is clear that $\lambda = 0$ is a point in $\boldsymbol{B}$. Let us assume that $\bar{\lambda} \in \boldsymbol{B}$, $\bar{\lambda} \neq 0$, then there exists $\bar{x} \in \boldsymbol{M}$ such that

$$\sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(\bar{x}) = \min_{x \in \boldsymbol{M}} \sum_{a=1}^{m} \lambda_a \Phi_a(x)$$

and therefore, for all $0 < t < \infty$ we have

$$\sum_{a=1}^{m} t\bar{\lambda}_a \Phi_a(\bar{x}) = \min_{x \in \boldsymbol{M}} \sum_{a=1}^{m} t\bar{\lambda}_a \Phi_a(x),$$

i.e. $\lambda^* \in \boldsymbol{B}$, where $\lambda^* = t\bar{\lambda}$, $0 < t < \infty$ and hence the result.

Lemma 2. *If problem* (II) *is solvable for* λ^1, λ^2 $(\lambda^1 \neq \lambda^2)$, *then it is solvable for all* $\lambda = \mu_1\lambda^1 + \mu_2\lambda^2$, $\mu_1 + \mu_2 = 1$ $(\mu_1 \geqq 0, \mu_2 \geqq 0)$ *iff for the problem*

(II)' $$\min_{\in x \boldsymbol{M}} [\mu_1 \mathsf{H}_1(x) + \mu_2 \mathsf{H}_2(x)], \quad \mu_1 + \mu_2 = 1\,(\mu_1 \geqq 0, \mu_2 \geqq 0),$$

where

$$\mathsf{H}_i(x) = \sum_{a=1}^{m} \lambda_a^i \Phi_a(x), \quad i = 1, 2$$

the solvability set $\boldsymbol{B}^{\sim}$ *is convex in* $'\mathrm{R}^2$, *where*

$$\boldsymbol{B}^{\sim} = \{(\mu_1, \mu_2) \in {'\mathrm{R}^2} / \min_{x \in \boldsymbol{M}} [\mu_1 \mathsf{H}_1(x) + \mu_2 \mathsf{H}_2(x)] \text{ exists}, \mu_1 + \mu_2 = 1\,(\mu_1 \geqq 0, \mu_2 \geqq 0)\}.$$

Proof. i) Suppose that if problem (II) is solvable for λ^1, λ^2 $(\lambda^1 \neq \lambda^2)$, then it is solvable for all $\lambda = \mu_1\lambda^1 + \mu_2\lambda^2$, $\mu_1 + \mu_2 = 1$ $(\mu_1 \geqq 0,\ \mu_2 \geqq 0)$ and let $(\mu_1^*, \mu_2^*) \in \boldsymbol{B}^{\sim}$; then there exists $x^* \in \boldsymbol{M}$ such that

(2) $$\sum_{a=1}^{m} (\mu_1^*\lambda_a^1 + \mu_2^*\lambda_a^2) \Phi_a(x^*) \leqq \sum_{a=1}^{m} (\mu_1^*\lambda_a^1 + \mu_2^*\lambda_a^2) \Phi_a(x), \quad \forall x \in \boldsymbol{M}.$$

Further let $(\hat{\mu}_1, \hat{\mu}_2) \in \boldsymbol{B}^{\sim}$, then there exists $\hat{x} \in \boldsymbol{M}$ such that

(3) $$\sum_{a=1}^{m} (\hat{\mu}_1\lambda_a^1 + \hat{\mu}_2\lambda_a^2) \Phi_a(\hat{x}) \leqq \sum_{a=1}^{m} (\hat{\mu}_1\lambda_a^1 + \hat{\mu}_2\lambda_a^2) \Phi_a(x), \quad \forall x \in \boldsymbol{M},$$

where $\mu_1^*; \mu_2^*; \hat{\mu}_1; \hat{\mu}_2 \geqq 0$, $\mu_1^* + \mu_2^* = 1$, $\hat{\mu}_1 + \hat{\mu}_2 = 1$. Let us denote $\gamma_1 = \mu_1^*\lambda^1 + \mu_2^*\lambda^2$, $\gamma_2 = \hat{\mu}_1\lambda^1 + \hat{\mu}_2\lambda^2$. From (2), (3) it follows that problem (II) is solvable for γ_1, γ_2 and by the assumptions of the lemma it is solvable for all $\gamma = (1 - \omega)\gamma_1 + \omega\gamma_2$, $0 \leqq \omega \leqq 1$, and hence $(1 - \omega)(\mu_1^*, \mu_2^*) + \omega(\hat{\mu}_1, \hat{\mu}_2) \in \boldsymbol{B}^{\sim}$, $0 \leqq \omega \leqq 1$, i.e. the set $\boldsymbol{B}^{\sim}$ is convex.

ii) Assume that the set $\boldsymbol{B}^{\sim}$ is convex and let $(\mu_1^*, \mu_2^*) \in \boldsymbol{B}^{\sim}$, $(\hat{\mu}_1, \hat{\mu}_2) \in \boldsymbol{B}^{\sim}$, then it follows that $(1 - \omega)(\mu_1^*, \mu_2^*) + \omega(\hat{\mu}_1, \hat{\mu}_2) \in \boldsymbol{B}^{\sim}$, $0 \leqq \omega \leqq 1$, therefore, if $\gamma_1 \in \boldsymbol{B}$, $\gamma_2 \in \boldsymbol{B}$, $\gamma_1 \neq \gamma_2$, then $(1 - \omega)\gamma_1 + \omega\gamma_2 \in \boldsymbol{B}$, $0 \leqq \omega \leqq 1$, where γ_1, γ_2 are defined in i).

Remark 1. If problem (II)′ is solvable for $\mu_1 = 0$; $\mu_2 = 1$, then

$$\min_{x \in M} [\mu_1 H_1(x) + \mu_2 H_2(x)] = \min_{x \in M} H_2(x) = \min_{x \in M} \sum_{a=1}^{m} \lambda_a^2 \Phi_a(x),$$

and if it solvable for $\mu_1 = 1$; $\mu_2 = 0$, then

$$\min_{x \in M} [\mu_1 H_1(x) + \mu_2 H_2(x)] = \min_{x \in M} H_1(x) = \min_{x \in M} \sum_{a=1}^{m} \lambda^1 \Phi_a(x).$$

Lemma 3. *If* $f_1(x)$; $f_2(x)$ *are convex functions on* M *such that* $f_1(x) \geqq 0$; $f_2(x) \geqq 0$ *for all* $x \in M$, *then*

$$\max [f_1(x), f_2(x)] \leqq f_1(x) + f_2(x), \quad \forall x \in M,$$

and the functions $\max [f_1(x), f_2(x)]$; $f_1(x) + f_2(x)$ *are convex on* M, *where* M *is defined in problem* (II).

Proof. Let

$$A_1 = \{x \in M / f_1(x) \geqq f_2(x)\},$$
$$A_2 = \{x \in M / f_1(x) \leqq f_2(x)\},$$

then

$$\max [f_1(x), f_2(x)] = f_1(x) \leqq f_1(x) + f_2(x), \quad \forall x \in A_1,$$
$$\max [f_1(x), f_2(x)] = f_2(x) \leqq f_1(x) + f_2(x), \quad \forall x \in A_2,$$

which implies that $\max [f_1(x), f_2(x)] \leqq f_1(x) + f_2(x)$, $\forall x \in M$. The convexity of the function $\max [f_1(x), f_2(x)]$ follows from the fact that

$$\max \{f_1[(1 - \omega) x^1 + \omega x^2], f_2[(1 - \omega) x^1 + \omega x^2]\} \leqq$$
$$\leqq \max \{[(1 - \omega) f_1(x^1) + \omega f_1(x^2)], [(1 - \omega) f_2(x^1) + \omega f_2(x^2)]\} \leqq$$
$$\leqq \max [(1 - \omega) f_1(x^1), (1 - \omega) f_2(x^1)] + \max [\omega f_1(x^2) + \omega f_2(x^2] =$$
$$= (1 - \omega) \max [f_1(x^1), f_2(x^1)] + \omega \max [f_1(x^2), f_2(x^2)]$$

for all $0 \leqq \omega \leqq 1$.
The convexity of $f_1(x) + f_2(x)$ is clear [3], [5].

Lemma 4. *If* $f_1(x)$, $f_2(x)$ *are strictly convex and closed functions on* M [6] *and* $\min_{x \in M} [f_i(x)]$, $i = 1, 2$ *exists, then both the sets* $A_1(k)$, $A_2(k)$ *defined by*

$$A_1(k) = \{x \in M / f_1(x) \leqq k\}, \tag{4}$$

$$A_2(k) = \{x \in M / f_2(x) \leqq k\} \tag{5}$$

are bounded for all $k \in {}'R$ *and such that* $A_1(k) \neq \emptyset$, $A_2(k) \neq \emptyset$.

Proof. Let $\min_{x \in \mathbf{M}} f_i(x) = f_i(x^i) = k_i$, $i = 1, 2$, where $x^1 \in \mathbf{M}$, $x^2 \in \mathbf{M}$.

Then the sets $A_1(k_1)$, $A_2(k_2)$ given by

$$A_1(k_1) = \{x \in \mathbf{M}/f_1(x) \leqq k_1\} ;$$
$$A_2(k_2) = \{x \in \mathbf{M}/f_2(x) \leqq k_2\}$$

are clearly bounded since $A_1(k_1) = x^1$, $A_2(k_2) = x^2$ (which follows from the strict convexity of the functions $f_1(x)$, $f_2(x)$ on $\mathbf{M}$). Therefore, a lemma given in [6] (this lemma states: "The nonvoid level sets $\mathbf{S}(\alpha) = \{x \in R^n/f(x) \leqq \alpha\}$ of a closed convex function f are either all bounded or all unbounded") implies directly the results.

Remark 2. The nonvoid level sets [6] $\{x \in \mathbf{M}/f(x) \leqq k, k \in {}'R\}$ are bounded iff the nonvoid level sets $\{x \in \mathbf{M}/f(x) + a \leqq k, k \in {}'R\}$ are bounded for any constant $a \in R$.

Lemma 5. *If the assumptions of Lemma* 4 *are satisfied, then the sets* $\Gamma(k)$ *defined by*

$$\Gamma(k) = \{x \in \mathbf{M}/f_1(x) + f_2(x) \leqq k\} \tag{6}$$

are bounded for all $k \in {}'R$ *such that* $\Gamma(k) \neq \emptyset$.

Proof. From the assumptions it follows that there exist constants $a_i \in R$, $i = 1, 2$ with $a_i > |\min_{x \in \mathbf{M}} f_i(x)|$, $i = 1, 2$ such that

$$f_i(x) + a_i \geqq 0 , \quad i = 1, 2 \quad \text{for all} \quad x \in \mathbf{M} .$$

From Lemma 3 we have

$$\max \{[f_1(x) + a_1], [f_2(x) + a_2]\} \leqq f_1(x) + f_2(x) + a_1 + a_2$$

and therefore

$$\{x \in \mathbf{M}/f_1(x) + f_2(x) + a_1 + a_2 \leqq k\} \subset$$
$$\subset \{x \in \mathbf{M}/\max \{[f_1(x) + a_1], [f_2(x) + a_2]\} \leqq k\} .$$

It is clear from (4), (5) that

$$\{x \in \mathbf{M}/\max [f_1(x), f_2(x)] \leqq k\} = A_1(k) \cap A_2 (k) \tag{7}$$

and hence the result follows from Lemma 4, Remark 2.

Theorem 1. *If* $f_1(x), f_2(x)$ *are strictly convex and closed functions on* $\mathbf{M}$ [6] *and* $\min_{x \in \mathbf{M}} f_i(x)$, $i = 1, 2$ *exists, then*

$$\min_{x \in \mathbf{M}} [f_1(x) + f_2(x)] \quad \textit{exists.}$$

Proof. Let us define the sets denoted by $\mathbf{C}$, $\mathbf{D}$ as follows:

$$\mathbf{C} = \{k \in {}'\mathrm{R}/\Lambda_1(k) \cap \Lambda_2(k) \neq \emptyset\} \quad \text{(see (7))},$$
$$\mathbf{D} = \{k \in {}'\mathrm{R}/\Gamma(k) \neq \emptyset\} \quad \text{(see (6))}.$$

It is clear (see [4]) that $\mathbf{C} \neq \emptyset$, $\mathbf{D} \neq \emptyset$. It follows from Lemma 3 that $\mathbf{D} \subset \mathbf{C}$. From the assumptions and from Lemma 1, Lemma 2 it follows that the sets $\mathbf{C}$; $\mathbf{D}$ are convex, closed and unbounded subsets of the real line and $\mathbf{C}$ has the form $\mathbf{C} = [k_0, \infty)$ where $k_0 = \min_{x \in \mathbf{M}} \{\max_{x \in \mathbf{M}} [\mathrm{f}_1(x), \mathrm{f}_2(x)]\}$. Hence $\min_{x \in \mathbf{M}} [\mathrm{f}_1(x) + \mathrm{f}_2(x)]$ exists.

Corollary 1. *If all the assumptions of Theorem 1 are satisfied, then all problems of the form*

$$\min_{x \in \mathbf{M}} [\mu_1 \mathrm{f}_1(x) + \mu_2 \mathrm{f}_2(x)], \quad \mu_1 \geqq 0, \quad \mu_2 \geqq 0$$

are solvable.

Remark 3. It should be noted that Theorem 1 can be proved under the assumptions that the functions $\mathrm{f}_1(x)$, $\mathrm{f}_2(x)$ are closed, convex on $\mathbf{M}$ and $\min \mathrm{f}_i(x)$, $i = 1, 2$ exists such that both the sets

$$\mathbf{m}^1_{\mathrm{opt}} = \{x^* \in \mathbf{M}/\mathrm{f}_1(x^*) = \min_{x \in \mathbf{M}} \mathrm{f}_1(x)\},$$
$$\mathbf{m}^2_{\mathrm{opt}} = \{x^* \in \mathbf{M}/\mathrm{f}_2(x^*) = \min_{x \in \mathbf{M}} \mathrm{f}_2(x)\}$$

are bounded (see the proof of Lemma 4).

Theorem 2. *If the set* $\mathbf{U}$ *is defined by*

$$\mathbf{U} = \{\lambda \in \mathbf{B}/\mathbf{m}_{\mathrm{opt}}(\lambda) \text{ is bounded}\}, \tag{8}$$

where $\mathbf{B}$ *is given by* (1), *and*

$$\mathbf{m}_{\mathrm{opt}}(\lambda) = \{\tilde{x} \in \mathbf{M}/\sum_{a=1}^{m} \lambda_a \Phi_a(\tilde{x}) = \min_{x \in \mathbf{M}} \sum_{a=1}^{m} \lambda_a \Phi_a(x)\}, \tag{9}$$

then $\mathbf{U}$ *is a convex set.*

Proof. Let $\lambda^1 \in U$, $\lambda^2 \in U(\lambda^1 \neq \lambda^2)$, then

$$\min_{x \in \mathbf{M}} \sum_{a=1}^{m} \lambda^1_a \Phi_a(x) = \min_{x \in \mathbf{M}} \mathrm{H}_1(x) \quad \text{exists},$$

and

$$\min_{x \in \mathbf{M}} \sum_{a=1}^{m} \lambda^2_a \Phi_a(x) = \min_{x \in \mathbf{M}} \mathrm{H}_2(x) \quad \text{exists}.$$

Since the functions $H_1(x)$, $H_2(x)$ are continuous and convex on R^n, they are convex and closed on $\boldsymbol{M}$ (since lower semicontinuity is equivalent to closedness over R^n) and hence from Corollary 1, Remark 3 it follows that

$$\min_{x \in \boldsymbol{M}} [\mu_1 H_1(x) + \mu_2 H_2(x)] \quad \text{exists}, \quad \mu_1 + \mu_2 = 1, \quad \mu_1 \geqq 0, \quad \mu_2 \geqq 0,$$

i.e. $$\min_{x \in \boldsymbol{M}} \sum_{a=1}^{m} (\mu_1 \lambda_a^1 + \mu_2 \lambda_a^2) \Phi_a(x) \quad \text{exists}, \quad \mu_1 + \mu_2 = 1, \quad \mu_1 \geqq 0, \quad \mu_2 \geqq 0,$$

and hence $\mu_1 \lambda^1 + \mu_2 \lambda^2 \in \boldsymbol{U}$ for all $\mu_1 + \mu_2 = 1$, $\mu_1 \geqq 0$, $\mu_2 \geqq 0$, therefore $\boldsymbol{U}$ is convex.

Remark 4. If $\boldsymbol{B} = \boldsymbol{U}$, then the solvability set of problem (II) $\boldsymbol{B}$ is convex.

Corollary 2. *If the set $\boldsymbol{M}$ is bounded, then (8) implies that $\boldsymbol{B} = \boldsymbol{U}$ and therefore $\boldsymbol{B}$ is convex by Remark 4.*

Corollary 3. *If the functions $\Phi_a(x)$, $a = 1, 2, \ldots, m$ are strictly convex on $\boldsymbol{M}$, then (8) implies $\boldsymbol{B} = \boldsymbol{U}$, and therefore $\boldsymbol{B}$ is convex by Remark 4.*

Lemma 6. *If for problem (II) $\boldsymbol{m}_{\text{opt}}(\lambda)$ is defined by (9), then it is convex and closed in R^n.*

Proof. If $\boldsymbol{m}_{\text{opt}}(\lambda)$ is a one-point set, or the empty set, or the whole R^n-space, the result is clear. Suppose that x^1, x^2 are two points in $\boldsymbol{m}_{\text{opt}}(\lambda)$, then the convexity of the set $\boldsymbol{M}$ and the functions $\Phi_a(x)$, $a = 1, 2, \ldots, m$, yields

$$\sum_{a=1}^{m} \lambda_a \Phi_a[(1 - \omega) x^1 + \omega x^2] \leqq (1 - \omega) \sum_{a=1}^{m} \lambda_a \Phi_a(x^1) + \omega \sum_{a=1}^{m} \lambda_a \Phi_a(x^2) =$$
$$= \min_{x \in \boldsymbol{M}} \sum_{a=1}^{m} \lambda_a \Phi_a(x), \quad 0 \leqq \omega \leqq 1$$

and hence $(1 - \omega) x^1 + \omega x^2 \in \boldsymbol{m}_{\text{opt}}(\lambda)$ for all $0 \leqq \omega \leqq 1$, i.e. the set $\boldsymbol{m}_{\text{opt}}(\lambda)$ is convex. Assume that $\tilde{x}_n \in \boldsymbol{m}_{\text{opt}}(\lambda)$, $n = 1, 2, \ldots$ is a sequence of points which converges to $\tilde{x}$. Then

$$\sum_{a=1}^{m} \lambda_a \Phi_a(\tilde{x}_n) = \min_{x \in \boldsymbol{M}} [\sum_{a=1}^{m} \lambda_a \Phi_a(x)],$$

$$\lim_{n \to \infty} \sum_{a=1}^{m} \lambda_a \Phi_a(\tilde{x}_n) = \min_{x \in \boldsymbol{M}} [\sum_{a=1}^{m} \lambda_a \Phi_a(x)].$$

From the finiteness of the sum and the continuity of the functions $\Phi_a(x)$, $a = 1, 2, \ldots$ $\ldots, m$, we have

$$\sum_{a=1}^{m} \lambda_a \Phi_a(\lim_{n \to \infty} \tilde{x}_n) = \sum_{a=1}^{m} \lambda_a \Phi_a(\tilde{x}) = \min_{x \in \boldsymbol{M}} [\sum_{a=1}^{m} \lambda_a \Phi_a(x)].$$

Hence $\tilde{x} \in \boldsymbol{m}_{\text{opt}}(\lambda)$ and the set $\boldsymbol{m}_{\text{opt}}(\lambda)$ is therefore closed.

Remark 5. If $\Phi_a(x)$, $a = 1, 2, \ldots, m$ are strictly convex functions on $\mathbf{M}$ and $\min_{x \in \mathbf{M}} \Phi_a(x)$, $a = 1, 2, \ldots, m$ exists, then the solvability set of problem (II) $\mathbf{B}$ is given by $\mathbf{B} = {}'\mathrm{R}^m_+$.

Theorem 3. *If the solvability fuction of problem* (II) *denoted by* $\xi(\lambda)$ *is defined by*

$$\xi(\lambda) = \min_{x \in \mathbf{M}} \left[\sum_{a=1}^{m} \lambda_a \Phi_a(x) \right], \tag{10}$$

then it is concave on $\mathbf{U}$, *where* $\mathbf{U}$ *is given by* (8).

Proof. If λ^1, λ^2 are any two points in $\mathbf{U}$, then by Theorem 2, $(1 - \omega)\lambda^1 + \omega\lambda^2 \in \mathbf{U}$ for all $0 \leqq \omega \leqq 1$, and therefore

$$\xi[(1 - \omega)\lambda^1 + \omega\lambda^2] = \min_{x \in \mathbf{M}} \sum_{a=1}^{m} [(1 - \omega)\lambda^1_a + \omega\lambda^2_a] \Phi_a(x) \geqq$$

$$\geqq (1 - \omega) \min_{x \in \mathbf{M}} \sum_{a=1}^{m} \lambda^1_a \Phi_a(x) + \omega \min_{x \in \mathbf{M}} \sum_{a=1}^{m} \lambda^2_a \Phi_a(x) =$$

$$= (1 - \omega)\xi(\lambda^1) + \omega\,\xi(\lambda^2), \quad 0 \leqq \omega \leqq 1.$$

Hence the function $\xi(\lambda)$ is concave on the set $\mathbf{U}$.

Corollary 4. *If the functions* $\Phi_a(x)$, $a = 1, 2, \ldots, m$ *are strictly convex on* $\mathbf{M}$, *or if the set* $\mathbf{M}$ *is bounded, then the solvability function* $\xi(\lambda)$ *is concave on* $\mathbf{B}$ (*see Corollaries* 2 *and* 3).

2. CHARACTERIZATION OF THE STABILITY SET OF THE FIRST KIND

Definition 2. *Suppose that* $\bar{\lambda} \in \mathbf{B}$ *with a corresponding optimal point* $\bar{x}$, *then the stability set of the first kind of problem* (II) *corresponding to* $\bar{x}$ *denoted by* $\mathbf{S}(\bar{x})$ *is defined by*

$$\mathbf{S}(\bar{x}) = \{\lambda \in \mathbf{B} / \sum_{a=1}^{m} \lambda_a \Phi_a(\bar{x}) = \min_{x \in \mathbf{M}} [\sum_{a=1}^{m} \lambda_a \Phi_a(x)]\}. \tag{11}$$

Lemma 7. *If the set* $\mathbf{S}(\bar{x})$ *is defined by* (11), *then it is a cone in* $'\mathrm{R}^m$ *with vertex at* $\lambda = 0$.

Proof. It is clear that $0 \in \mathbf{S}(\bar{x})$. Suppose that $\lambda^* \in \mathbf{S}(\bar{x})$, $\lambda^* \neq 0$, then $\sum_{a=1}^{m} \lambda^*_a \Phi_a(\bar{x}) = \min_{x \in \mathbf{M}} [\sum_{a=1}^{m} \lambda^*_a \Phi_a(x)]$ and therefore $\sum_{a=1}^{m} t\lambda^*_a \Phi_a(\bar{x}) = \min_{x \in \mathbf{M}} [\sum_{a=1}^{m} t\lambda^*_a \Phi_a(x)]$ for all $t > 0$, i.e. $t\lambda^* \in \mathbf{S}(\bar{x})$ for all $t > 0$. Hence the result.

Theorem 4. *If the functions $g_r(x)$, $r = 1, 2, \ldots, l$ (see problem (II)) satisfy any one of the constraint qualifications [1], [3] (for example Slater), then the set $\mathbf{S}(\bar{x})$ is convex and closed in $'\mathrm{R}^m$.*

Proof. If $\mathbf{S}(\bar{x})$ is a one-point set, or the empty set, or the whole nonnegative orthant of the $'\mathrm{R}^m$ space, it is convex and closed. Suppose that $\lambda^1 \in \mathbf{S}(\bar{x})$, $\lambda^2 \in \mathbf{S}(\bar{x})$, $\lambda^1 \neq \lambda^2$, then there exist $u^1 \in \mathrm{R}^l$, $u^2 \in \mathrm{R}^l$ such that $(\bar{x}, u^1)$ and $(\bar{x}, u^2)$ solve the Kuhn-Tucker problem [1], [3], i.e.

$$\sum_{a=1}^{m} \lambda_a^1 \frac{\partial \Phi_a}{\partial x_\alpha}(\bar{x}) + \sum_{r \notin \mathrm{I}_1} u_r^1 \frac{\partial g_r}{\partial x_\alpha}(\bar{x}) = 0, \quad \alpha = 1, 2, \ldots, n,$$

$$g_r(\bar{x}) \leqq 0; \quad u_r^1 g_r(\bar{x}) = 0, \quad r = 1, 2, \ldots, l,$$

$$u_r^1 = 0, \quad r \in \mathrm{I}_1 \subset \{1, 2, \ldots, l\}, \quad u_r^1 \geqq 0, \quad r \in \{1, 2, \ldots, l\} - \mathrm{I}_1,$$

and

$$\sum_{a=1}^{m} \lambda_a^2 \frac{\partial \Phi_a}{\partial x_\alpha}(\bar{x}) + \sum_{r \notin \mathrm{I}_2} u_r^2 \frac{\partial g_r}{\partial x_\alpha}(\bar{x}) = 0, \quad \alpha = 1, 2, \ldots, n,$$

$$g_r(\bar{x}) \leqq 0; \quad u_r^2 g_r(\bar{x}) = 0, \quad r = 1, 2, \ldots, l,$$

$$u_r^2 = 0, \quad r \in \mathrm{I}_2 \subset \{1, 2, \ldots, l\}, \quad u_r^2 \geqq 0, \quad r \in \{1, 2, \ldots, l\} - \mathrm{I}_2.$$

Hence we deduce that for all $0 \leqq \omega \leqq 1$,

$$\sum_{a=1}^{m} [(1 - \omega)\lambda_a^1 + \omega\lambda_a^2] \frac{\partial \Phi_a}{\partial x_\alpha}(\bar{x}) + \sum_{r \notin (\mathrm{I}_1 \cap \mathrm{I}_2)} u_r^* \frac{\partial g_r}{\partial x_\alpha}(\bar{x}) = 0, \quad \alpha = 1, 2, \ldots, n,$$

$$g_r(\bar{x}) \leqq 0; \quad u_r^* g_r(\bar{x}) = 0, \quad r = 1, 2, \ldots, l,$$

$$u_r^* = 0, \quad r \in \mathrm{I}_1 \cap \mathrm{I}_2, \quad u_r^* \geqq 0, \quad r \in \{1, 2, \ldots, l\} - (\mathrm{I}_1 \cap \mathrm{I}_2),$$

where

$$\begin{aligned} u_r^* &= (1 - \omega) u_r^1, && r \in [\{1, 2, \ldots, l\} - \mathrm{I}_1] \cap \mathrm{I}_2, \\ &= \omega u_r^2, && r \in \mathrm{I}_1 \cap [\{1, 2, \ldots, l\} - \mathrm{I}_2], \\ &= (1 - \omega) u_r^1 + \omega u_r^2, && r \in [\{1, 2, \ldots, l\} - \mathrm{I}_1] \cap [\{1, 2, \ldots, l\} - \mathrm{I}_2], \\ &= 0, && r \in \mathrm{I}_1 \cap \mathrm{I}_2. \end{aligned}$$

Therefore it follows from the sufficient optimality theorem of Kuhn-Tucker [1], [3] that $(1 - \omega)\lambda^1 + \omega\lambda^2 \in \mathbf{S}(\bar{x})$ for all $0 \leqq \omega \leqq 1$. Hence the set $\mathbf{S}(\bar{x})$ is convex in $'\mathrm{R}^m$. Assume that $\hat{\lambda}$ is a boundary point of $\mathbf{S}(\bar{x})$, then for any interior point λ^0 of $\mathbf{S}(\bar{x})$

the open line segment $(\lambda^0, \hat{\lambda})$ lies in $\mathbf{S}(\bar{x})$ due to the convexity of $\mathbf{S}(\bar{x})$. For any $\lambda \in (\lambda^0, \hat{\lambda})$ we have

$$\sum_{a=1}^{m} \lambda_a \Phi_a(\bar{x}) \leqq \sum_{a=1}^{m} \lambda_a \Phi_a(x), \quad \forall x \in \mathbf{M}$$

and therefore

$$\lim_{\lambda \to \hat{\lambda}} \sum_{a=1}^{m} \lambda_a \Phi_a(\bar{x}) \leqq \lim_{\lambda \to \hat{\lambda}} \sum_{a=1}^{m} \lambda_a \Phi_a(x), \quad \forall x \in \mathbf{M}.$$

From the finiteness of the sum and the continuity of the functions $\Phi_a(x)$, $a = 1, 2, \ldots$ $\ldots, m$ on $\mathbf{M}$ it follows that

$$\sum_{a=1}^{m} \lim_{\lambda \to \hat{\lambda}} [\lambda_a \Phi_a(\bar{x})] \leqq \sum_{a=1}^{m} \lim_{\lambda \to \hat{\lambda}} [\lambda_a \Phi_a(x)], \quad \forall x \in \mathbf{M}.$$

The limiting process concerns the path directed from λ^0 to $\hat{\lambda}$ as a straight line, and since λ^0 is an arbitrary point in int $\mathbf{S}(\bar{x})$, this path is considered to be arbitrary, and therefore

$$\sum_{a=1}^{m} \hat{\lambda}_a \Phi_a(\bar{x}) \leqq \sum_{a=1}^{m} \hat{\lambda}_a \Phi_a(x), \quad \forall x \in \mathbf{M}.$$

Hence $\hat{\lambda} \in \mathbf{S}(\bar{x})$, and therefore the set $\mathbf{S}(\bar{x})$ is closed.

Theorem 5. *If* int $[\mathbf{S}(x^1) \cap \mathbf{S}(x^2)] \neq \emptyset$, *then* $\mathbf{S}(x^1) = \mathbf{S}(x^2)$, *where* $\mathbf{S}(x^1)$, $\mathbf{S}(x^2)$ *are the stability sets of the first kind of problem* (II) *corresponding to* x^1, x^2 *respectively* $(x^1 \neq x^2)$.

Proof. Let $\lambda^0 \in \text{int}\,[\mathbf{S}(x^1) \cap \mathbf{S}(x^2)]$, then

(12) $$\sum_{a=1}^{m} \lambda_a^0 \Phi_a(x^1) = \sum_{a=1}^{m} \lambda_a^0 \Phi_a(x^2).$$

Assume that $\lambda^1 \in \mathbf{S}(x^1)$, $\lambda^1 \neq \lambda^0$, then there exists $0 < \omega < 1$ such that $\lambda^* =$ $= (1 - \omega)\lambda^1 + \omega\lambda^0 \in \mathbf{S}(x^2)$, and therefore

$$\sum_{a=1}^{m} \lambda_a^* \Phi_a(x^2) \leqq \sum_{a=1}^{m} \lambda_a^* \Phi_a(x^1), \quad \text{i.e.}$$

$$(1 - \omega) \sum_{a=1}^{m} \lambda_a^1 \Phi_a(x^2) + \omega \sum_{a=1}^{m} \lambda_a^0 \Phi_a(x^2) \leqq (1 - \omega) \sum_{a=1}^{m} \lambda_a^1 \Phi_a(x^1) + \omega \sum_{a=1}^{m} \lambda_a^0 \Phi_a(x^1).$$

Using (12) we get

$$\sum_{a=1}^{m} \lambda_a^1 \Phi_a(x^2) \leqq \sum_{a=1}^{m} \lambda_a^1 \Phi_a(x^1) \leqq \sum_{a=1}^{m} \lambda_a^1 \Phi_a(x), \quad \forall x \in \mathbf{M},$$

therefore $\lambda^1 \in \mathbf{S}(x^2)$, and hence $\mathbf{S}(x^1) \subseteq \mathbf{S}(x^2)$. Similarly it can be shown that $\mathbf{S}(x^2) \subseteq$ $\subseteq \mathbf{S}(x^1)$. Hence $\mathbf{S}(x^1) = \mathbf{S}(x^2)$.

In order to have an analytic description for the set $\mathbf{S}(\bar{x})$ defined by (11), let us proceed in the following way: We order the functions $g_r(x)$, $r = 1, 2, \ldots, l$ in such a way that

$$r \in \{1, 2, \ldots, s\} \quad \text{if} \quad g_r(\bar{x}) = 0 \,, \quad s \geqq 1 \,,$$
$$r \in \{s + 1, \ldots, l\} \quad \text{if} \quad g_r(\bar{x}) < 0 \,.$$

Consider the system of equations

$$(13) \qquad \sum_{a=1}^{m} \lambda_a \frac{\partial \Phi_a}{\partial x_\alpha}(\bar{x}) + \sum_{r=1}^{s} u_r \frac{\partial g_r}{\partial x_\alpha}(\bar{x}) = 0 \,, \quad \alpha = 1, 2, \ldots, n \,.$$

It represents n linear homogeneous equations in $m + s$ unknowns λ_a, $a = 1, 2, \ldots, m$ and u_r, $r = 1, 2, \ldots, s$, which can be solved explicitly.

Suppose that $\lambda_a^* \geqq 0$, $a = 1, 2, \ldots, m$; $u_r^* \geqq 0$, $r = 1, 2, \ldots, s$ solve the system (13), then it is clear that $(\bar{x}, \bar{u})$ solves the Kuhn-Tucker problem [1], [3], where $\bar{u}_r = u_r^*$, $r = 1, 2, \ldots, s$, $\bar{u}_r = 0$, $r = s + 1, \ldots, l$ and hence $\lambda^* \in \mathbf{S}(\bar{x})$. Let us define the set denoted by $\mathbf{p}(\lambda, u)$ as follows:

$$(14) \qquad \mathbf{p}(\lambda, u) = \{(\lambda, u) \in {}'\mathrm{R}_+^m \times \mathrm{R}_+^s / (\lambda, u) \text{ solves } (13)\} \,,$$

where ${}'\mathrm{R}_+^m$; R_+^s are the nonnegative orthants of the ${}'\mathrm{R}^m$ vector λ-space, and R^s vector u-space, respectively. Then

$$(15) \qquad \mathbf{S}(\bar{x}) = \{\lambda \in {}'\mathrm{R}^m / (\lambda, u) \in \mathbf{p}(\lambda, u)\} \,.$$

The representation of $\mathbf{S}(\bar{x})$ by (15) can be used to prove the convexity and closedness of the set $\mathbf{S}(\bar{x})$. If $g_r(\bar{x}) < 0$, $r = 1, 2, \ldots, l$, then it is easy to see that $\mathbf{S}(\bar{x})$ can be written in the form

$$\mathbf{S}(\bar{x}) = \left\{\lambda \in {}'\mathrm{R}_+^m / \sum_{a=1}^{m} \lambda_a \frac{\partial \Phi_a}{\partial x_\alpha}(\bar{x}) = 0 \,, \ \alpha = 1, 2, \ldots, n\right\}$$

and it is clear that this representation proves the convexity and the closedness of the set $\mathbf{S}(\bar{x})$.

It may happen that for some problems, the system (13) has only the trivial solution, and for such cases $\mathbf{S}(\bar{x})$ is a one-point set, namely $\mathbf{S}(\bar{x}) = \{0\}$.

3. CHARACTERIZATION OF THE STABILITY SET OF THE SECOND KIND

Definition 3. *Suppose that* $\bar{\lambda} \in \mathbf{B}$ *(see* (1)*) with a corresponding optimal point* $\bar{x}$ *and* $\Sigma(\bar{\lambda}, \mathrm{J})$ *denotes either the unique side of* $\mathbf{M}$ *from those given by* $\{x \in \mathrm{R}^n / g_r(x) = 0, r \in \mathrm{J};\ g_r(x) < 0,\ r \notin \mathrm{J}\}$ *which contains* $\bar{x}$, *or* int $\mathbf{M}$. Then *the stability set of the second kind of problem* (II) *corresponding to* $\Sigma(\bar{\lambda}, \mathrm{J})$ *denoted by* $\mathbf{Q}(\Sigma(\bar{\lambda}, \mathrm{J}))$, *is defined by*

$$(16) \qquad \mathbf{Q}(\Sigma(\bar{\lambda}, \mathrm{J})) = \{\lambda \in \mathbf{B} / \mathbf{m}_{\mathrm{opt}}(\lambda) \cap \Sigma(\bar{\lambda}, \mathrm{J}) \neq \emptyset\} \,,$$

where $\mathbf{m}_{\mathrm{opt}}(\lambda)$ *is defined by* (9).

(16) gives a definition for the stability set of the second kind corresponding to a side rather than to an index set as was done in [4], and this is due to the assumption that the set $\boldsymbol{M}$ is fixed and independent of parameters.

Let us adjoin to problem (II) the following problem

$$\text{(II)}' \qquad \min \left[\sum_{a=1}^{m} \lambda_a \, \Phi_a(x) \right],$$

subject to

$$'\boldsymbol{M} = \{x \in \mathrm{R}^n / g_r(x) \leqq 0 \, , \quad r \in \mathrm{J}\}$$

where J is the index set given in the definition of $\Sigma(\bar{\lambda}, \mathrm{J})$.

Lemma 8. *If $\bar{\lambda} \in \boldsymbol{B}$ with $\boldsymbol{m}_{\mathrm{opt}}(\bar{\lambda}) \subseteq \Sigma(\bar{\lambda}, \mathrm{J})$, Φ_k is strictly convex on R^n for at least one $k \in \{1, 2, \ldots, m\}$ for which $\bar{\lambda}_k > 0$, then*

$$\bar{x} \in \boldsymbol{m}_{\mathrm{opt}}(\bar{\lambda}) \Leftrightarrow \sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(\bar{x}) = \min_{x \in 'M} \left[\sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(x) \right],$$

where $'\boldsymbol{M}$ is the same as in problem (II)$'$ *and*

$$\boldsymbol{m}_{\mathrm{opt}}(\bar{\lambda}) = \{x^* \in \mathrm{R}^n / \sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(x^*) = \min_{x \in \boldsymbol{M}} \sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(x)\} \, .$$

Proof. i) Let $\bar{x} \in \boldsymbol{m}_{\mathrm{opt}}(\bar{\lambda})$, then $g_r(\bar{x}) = 0$, $r \in \mathrm{J}$, $g_r(\bar{x}) < 0$, $r \notin \mathrm{J}$ and hence $\bar{x} \in {}'\boldsymbol{M}$. Assume that there exists $x^* \in {}'\boldsymbol{M}$ such that $\sum_{a=1}^{m} \bar{\lambda}_a \, \Phi(\bar{x}) > \sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(x^*)$. It is easy to prove that there exists ω with $0 < \omega \leqq 1$ such that $\hat{x} = (1 - \omega) \, \bar{x} + \omega x^* \in \boldsymbol{M}$. From the convexity of the functions $\Phi_a(x)$, $a = 1, 2, \ldots, m$ we obtain

$$\begin{aligned} \sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(\hat{x}) &\leqq (1 - \omega) \sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(\bar{x}) + \omega \sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(x^*) < \\ &< (1 - \omega) \sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(\bar{x}) + \omega \sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(\bar{x}) = \\ &= \sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(\bar{x}) \end{aligned}$$

which contradicts our assumption, and hence

$$\sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(\bar{x}) \leqq \sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(x) \, , \quad \forall x \in {}'\boldsymbol{M} \, ,$$

i.e.

$$\sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(\bar{x}) = \min_{x \in '\boldsymbol{M}} \left[\sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(x) \right].$$

ii) Let

$$\sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(\bar{x}) = \min_{x \in '\boldsymbol{M}} \left[\sum_{a=1}^{m} \bar{\lambda}_a \, \Phi_a(x) \right].$$

If $\bar{x} \in \mathbf{M}$, the result is clear. Suppose that $\bar{x} \notin \mathbf{M}$ and let $x^0 \in \Sigma(\bar{\lambda}, \mathrm{J})$ be an optimal point corresponding to $\bar{\lambda}(x^0 \neq \bar{x})$ with $\sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x^0) = \min_{x \in \mathbf{M}} \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x)$.

There exists a point $\tilde{x} = (1 - \omega)\bar{x} + \omega x^0 \in \mathbf{M}$, $0 < \omega \leqq 1$. Therefore, from the convexity of the functions $\Phi_a(x)$, $a = 1, 2, \ldots, m$; $a \neq k$ and the strict convexity of $\Phi_k(x)$, we obtain

$$\sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(\tilde{x}) < (1 - \omega) \sum_{a=1}^{m} \lambda_a \Phi_a(\bar{x}) + \omega \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x^0)$$

and by the assumption

$$\sum_{a=1}^{m} \lambda_a \Phi_a(x^0) < \sum_{a=1}^{m} \lambda_a \Phi_a(\tilde{x}) .$$

Therefore

$$\sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x^0) < \sum_{a=1}^{m} \bar{\lambda}_a \Phi_a(x) ,$$

which contradicts our assumption, and therefore $\bar{x} = x^0$ which follows from the strict convexity of $\Phi_k(x)$. Hence $\bar{x} \in \mathbf{m}_{\text{opt}}(\bar{\lambda})$.

Lemma 9. *If the functions $\Phi_a(x)$, $a = 1, 2, \ldots, m$ are strictly convex on $\mathbf{M}$ and $\Sigma(\lambda^1, \mathrm{J}_1)$; $\Sigma(\lambda^2, \mathrm{J}_2)$ are two distinct sides of $\mathbf{M}$ then*

$$\mathbf{Q}(\Sigma(\lambda^1, \mathrm{J}_1)) \cap \mathbf{Q}(\Sigma(\lambda^2, \mathrm{J}_2)) = \{0\} .$$

Proof. It is clear that $\lambda = 0$ belongs to all stability sets of the second kind corresponding to different sides of $\mathbf{M}$. Suppose that $\lambda^* \in \mathbf{Q}(\Sigma(\lambda^1, \mathrm{J}_1)) \cap \mathbf{Q}(\Sigma(\lambda^2, \mathrm{J}_2))$, $\lambda^* \neq 0$, then (16) yields

$$\mathbf{m}_{\text{opt}}(\lambda^*) \cap \Sigma(\lambda^1, \mathrm{J}_1) \neq \emptyset ,$$

$$\mathbf{m}_{\text{opt}}(\lambda^*) \cap \Sigma(\lambda^2, \mathrm{J}_2) \neq \emptyset .$$

This leads to a contradiction, since $\mathbf{m}_{\text{opt}}(\lambda^*)$ by the assumption consists only of a single point. Hence the result.

In order to have more properties concerning the stability set of the second kind, let us concentrate our attention to the problem

$$(\mathrm{II})_q \qquad \min \left[\sum_{i,j=1}^{n} \tfrac{1}{2} c_{ij} x_i x_j + \sum_{i=1}^{n} p_i x_i \right] ,$$

subject to the restriction set $\mathbf{M}$,

where $[c_{ij}]$, $i, j = 1, 2, \ldots, n$ is a real symmetric positive semidefinite matrix, p_i, $i = 1, 2, \ldots, n$ are arbitrary parameters and $\mathbf{M}$ is the same set as in problem (II).

Lemma 10. *If $\Sigma(\bar{p}, \mathrm{J}_L)$ denotes either a linear side of $\boldsymbol{M}$ or* int $\boldsymbol{M}$, *then the stability set of the second kind of problem $(\mathrm{II})_q$ corresponding to $\Sigma(\bar{p}, \mathrm{J}_L)$ denoted by $\mathbf{Q}_q(\Sigma(\bar{p}, \mathrm{J}_L))$ is convex in $'\mathrm{R}^n$ (the vector space of p_α, $\alpha = 1, 2, \ldots, n$).*

Proof. The proof will be done for the case of a linear side of $\boldsymbol{M}$, the proof for the case of int $\boldsymbol{M}$ being similar. Suppose that p^1, p^2 are two points in $\mathbf{Q}_q(\Sigma(\bar{p}, \mathrm{J}_L))$, then there exist u^1, u^2 in R^l such that (x^1, u^1) and (x^2, u^2) solve the Kuhn-Tucker problem [1], [3], where

$$x^1 \in \boldsymbol{m}_{\mathrm{opt}}(p^1) \cap \Sigma(\bar{p}, \mathrm{J}_L), \quad x^2 \in \boldsymbol{m}_{\mathrm{opt}}(p^2) \cap \Sigma(\bar{p}, \mathrm{J}_L),$$

$$\Sigma(\bar{p}, \mathrm{J}_L) = \{x \in \mathrm{R}^n / g_r(x) = 0, \quad r \in \mathrm{J}_L, \quad g_r(x) < 0, r \notin \mathrm{J}_L\},$$

and the functions $g_r(x)$, $r \in \mathrm{J}_L$ are linear over $\boldsymbol{M}$. Therefore,

$$\begin{aligned}
&\sum_{j=1}^{n} c_{\alpha j} x_j^1 + p_\alpha^1 + \sum_{r \in \mathrm{J}_L} u_j^1 \frac{\partial g_r}{\partial x_\alpha}(x^1) = 0, && \alpha = 1, 2, \ldots, n,\\
&g_r(x^1) = 0, \quad r \in \mathrm{J}_L, \quad g_r(x^1) < 0, && r \notin \mathrm{J}_L,\\
&u_r^1 \, g_r(x^1) = 0, && r = 1, 2, \ldots, l,\\
&u_r^1 = 0, \quad r \notin \mathrm{J}_L, \quad u_r^1 \geqq 0, && r \in \mathrm{J}_L
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j=1}^{n} c_{\alpha j} x_j^2 + p_\alpha^2 + \sum_{r \in \mathrm{J}_L} u_r^2 \frac{\partial g_r}{\partial x_\alpha}(x^2) = 0, && \alpha = 1, 2, \ldots, n,\\
&g_r(x^2) = 0, \quad r \in \mathrm{J}_L, \quad g_r(x^2) < 0, && r \notin \mathrm{J}_L,\\
&u_r^2 \, g_r(x^2) = 0, && r = 1, 2 \ldots, l,\\
&u_r^2 = 0, \quad r \notin \mathrm{J}_L, \quad u_r^2 \geqq 0, && r \in \mathrm{J}_L.
\end{aligned}$$

Hence it follows from the linearity of the functions $g_r(x)$, $r \in \mathrm{J}_L$ that for all $0 \leqq \omega \leqq 1$ we have

$$\begin{aligned}
&\sum_{j=1}^{n} c_{\alpha j} x_j^* + p_\alpha^* + \sum_{r \in \mathrm{J}_L} u_r^* \frac{\partial g_r}{\partial x_\alpha}(x^*) = 0, && \alpha = 1, 2, \ldots, n,\\
&g_r(x^*) = 0, \quad r \in \mathrm{J}_L, \quad g_r(x^*) < 0, && r \notin \mathrm{J}_L.\\
&u_r^* \, g_r(x^*) = 0, && r = 1, 2, \ldots, l,\\
&u_r^* = 0, \quad r \notin \mathrm{J}_L, \quad u_r^* \geqq 0, && r \in \mathrm{J}_L,\\
&x^* = (1 - \omega)\, x^1 + \omega x^2,\\
&p^* = (1 - \omega)\, p^1 + \omega p^2,\\
&u^* = (1 - \omega)\, u^1 + \omega u^2.
\end{aligned}$$

This together with the Kuhn-Tucker sufficient optimality theorem [1], [3] implies that

$$x^* \in \boldsymbol{m}_{\text{opt}}(p^*) \cap \Sigma(\bar{p}, \mathrm{J}_L)$$

for all $0 \leqq \omega \leqq 1$. Hence the set $\mathbf{Q}_q(\Sigma(\bar{p}, \mathrm{J}_L))$ is convex.

Remark 6. It is easy to prove that (see Lemma 9) if $[c_{ij}]$, $i; j = 1, 2, \ldots, n$ is a real symmetric positive definite matrix, then the nonempty stability sets of the second kind of problem (II) corresponding to certain sides of $\boldsymbol{M}$, int $\boldsymbol{M}$ are mutually disjoint and all together exhaust the solvability set of problem $(\mathrm{II})_q$.

Example. Consider the problem
Minimize

$$[x_1^2 + x_2^2 + p_1 x_1 + p_2 x_2],$$

subject to

$$\boldsymbol{M} = \{x \in \mathrm{R}^2 / x_1^2 + x_2^2 \leqq 1, \quad x_1 + x_2 \leqq 1\}.$$

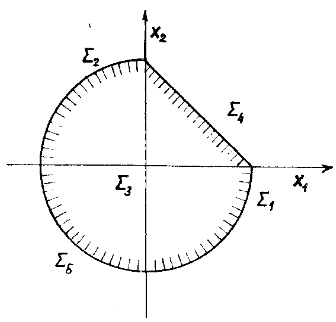

Fig. a. The set $\boldsymbol{M}$.

The set $\boldsymbol{M}$ is compact, and therefore $\mathbf{B} = {}'\mathrm{R}_2$. $\boldsymbol{M}$ consists of four distinct sides and int $\boldsymbol{M}$ (see Fig. a). Let $\mathbf{Q}_i$ denote the stability sets of the second kind corresponding to the sides Σ_i, $i = 1, 2, 4, 5$ while $\mathbf{Q}_3$ is the stability set of the second kind corresponding to $\Sigma_3 \equiv$ int $\boldsymbol{M}$. Then the sets $\mathbf{Q}_i$, $i = 1, 2, \ldots, 5$ are obtained in the form (see Fig. b)

$$\mathbf{Q}_1 = \{p \in {}'\mathrm{R}^2 / p_2 \leqq 0, \ p_2 - p_1 - 2 \geqq 0\},$$

$$\mathbf{Q}_2 = \{p \in {}'\mathrm{R}^2 / p_1 \leqq 0, \ p_1 - p_2 - 2 \geqq 0\},$$

$$\mathbf{Q}_3 = \{p \in {}'\mathrm{R}^2 / p_1^2 + p_2^2 < 4, \ p_1 + p_2 > -2\},$$

$$\mathbf{Q}_4 = \{p \in {}'\mathrm{R}^2 / p_1 + p_2 \leqq -2, -2 < p_2 - p_1 < 2\},$$

$$\mathbf{Q}_5 = \{p \in {}'\mathrm{R}^2 / p_1 > 0, \ p_2 > 0, \ p_1^2 + p_2^2 \geqq 4\} \cup$$
$$\cup \{p \in {}'\mathrm{R}^2 / p_1 < 0, \ p_2 > 0, \ p_1^2 + p_2^2 \geqq 4\} \cup$$
$$\cup \{p \in {}'\mathrm{R}^2 / p_1 > 0, \ p_2 < 0, \ p_1^2 + p_2^2 \geqq 4\}.$$

The set $\mathbf{B}$ is decomposed into the sets $\mathbf{Q}_i$, $i = 1, 2, 3, 4, 5$, and $\mathbf{Q}_i \cap \mathbf{Q}_j = \emptyset$, $i \neq j$, $i; j = 1, 2, 3, 4, 5$. The sets $\mathbf{Q}_i$, $i = 1, 2, 3, 4$ are convex. The convexity and the closedness of the sets $\mathbf{Q}_1$, $\mathbf{Q}_2$ follows from the fact that

$$\mathbf{Q}_1 = \mathbf{S}(1, 0),$$

$$\mathbf{Q}_2 = \mathbf{S}(0, 1),$$

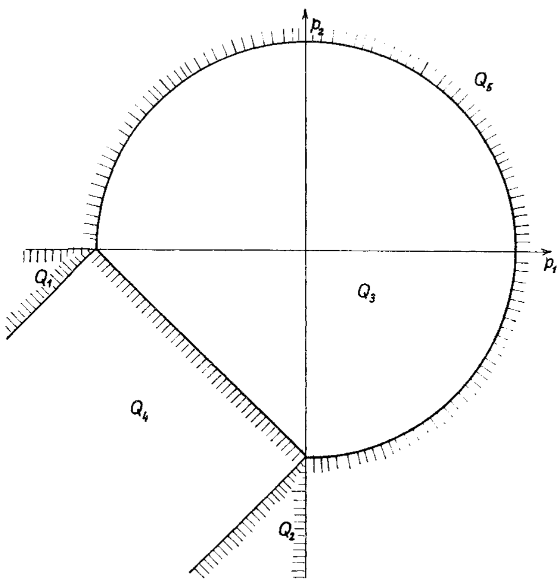

Fig. b. The nonempty stability sets of the second kind.

where $\mathbf{S}(1, 0)$, $\mathbf{S}(0, 1)$ are the stability sets of the first kind of our problem corresponding to the points $(1, 0)$, $(0, 1)$ respectively.

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Souhrn

KVALITATIVNÍ ANALÝZY ZÁKLADNÍCH POJMŮ PARAMETRICKÉHO KONVEXNÍHO PROGRAMOVÁNÍ, II

(Parametry v cílové funkci)

MOHAMED SAYED ALI OSMAN

V článku je podána kvalitativní analýza základních pojmů parametrického konvexního programování pro konvexní programy s parametry v cílové funkci. Jsou to pojmy množiny přípustných parametrů, množiny řešitelnosti a množin stability prvního a druhého druhu. Předpokládá se, že vyšetřované funkce mají spojité parciální derivace prvního řádu v R^n a že parametry nabývají libovolných reálných hodnot. Výsledky mohou být použity pro širokou třídu konvexních programů.

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