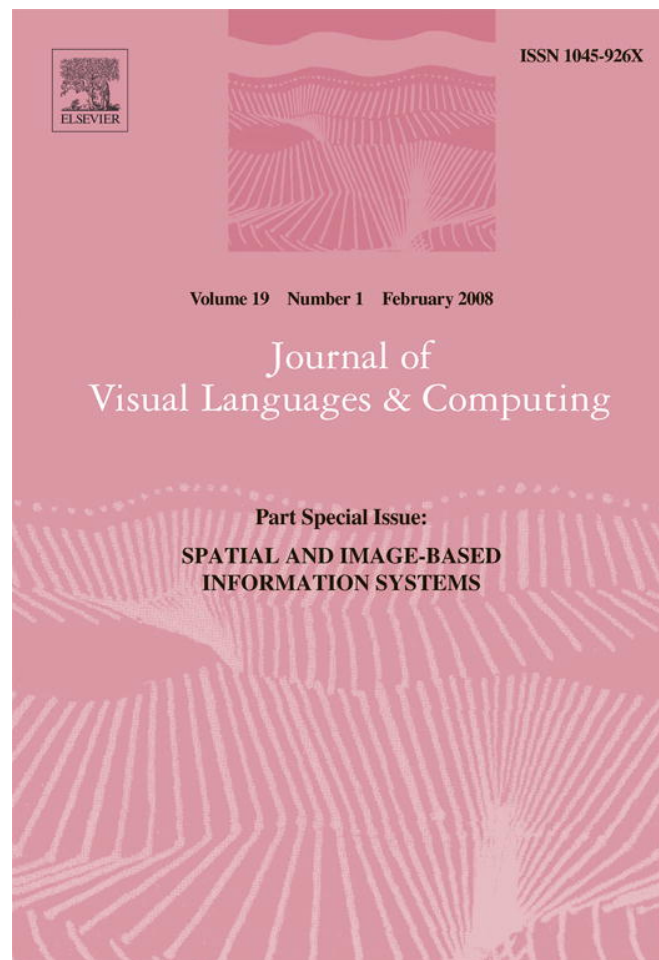


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# Qualitative spatial reasoning about relative point position

Reinhard Moratz<sup>a,\*</sup>, Marco Ragni<sup>b</sup>

<sup>a</sup>*Department of Mathematics and Informatics, University of Bremen, Bibliothekstr. 1, 28359 Bremen, Germany*

<sup>b</sup>*Institute for Informatics, University of Freiburg, Georges-Köhler-Allee 52, D-79110 Freiburg, Germany*

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## Abstract

Qualitative spatial reasoning (QSR) abstracts metrical details of the physical world. The two main directions in QSR are topological reasoning about regions and reasoning about orientations of point configurations. Orientations can refer to a global reference system, e.g. cardinal directions or instead only to relative orientation, e.g. egocentric views. Reasoning about relative orientations poses additional difficulties compared to reasoning about orientations in an absolute reference frame.

Qualitative knowledge about relative orientation can be naturally expressed in the form of ternary point calculi. Designing such calculi requires compromising between desired mathematical properties and the power to describe and model concrete “real-world” problems. Research has shown that using basic notions such as granularity leads to imprecise reasoning and as a consequence to underdetermined knowledge which is difficult to handle efficiently.

Concrete problems need a combination of qualitative knowledge of orientation and qualitative knowledge of distance. We present a calculus based on ternary relations where we introduce a qualitative distance measurement based on two of the three points. Its main advantage is that it utilizes finer distinctions than previously published calculi. Furthermore, it permits differentiations which are useful in realistic application scenarios such as robot navigation that cannot be directly dealt with in coarser calculi.

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*Keywords:* Qualitative spatial reasoning; Cognitive modelling; Robot navigation

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\*Corresponding author. Tel.: +49 179 5221512.

*E-mail addresses:* [moratz@tzi.de](mailto:moratz@tzi.de), [moratz@informatik.uni-bremen.de](mailto:moratz@informatik.uni-bremen.de) (R. Moratz), [ragni@informatik.uni-freiburg.de](mailto:ragni@informatik.uni-freiburg.de) (M. Ragni).

## 1. Introduction

Qualitative spatial reasoning (QSR) abstracts metrical details of the physical world and enables computers to make predictions about spatial relations even when precise quantitative information is unavailable [1]. From a practical viewpoint QSR is an abstraction that summarizes similar quantitative states into one qualitative characterization. A complementary view from the cognitive perspective is that the qualitative method *compares* features within the object domain rather than by *measuring* them in terms of some artificial external scale [2]. This is the reason why qualitative descriptions are quite natural for humans.

The two main directions in QSR are topological reasoning about regions [3–5] and positional reasoning about point configurations, like reasoning about orientation and distance [2,6–8]. More recent approaches in QSR that model orientations are a cyclic relation algebra [9] and the dipol calculus [10].

What is the motivation for considering calculi of relative orientation and relative distance? First, for robot navigation, the notion of path is central [11] and requires the representation of orientation and distance information [12,13]. Therefore we would like to provide this field with a calculus which has only a small number of relations but is fine-grained enough for solving reasoning problems. The reason for using only a small number of relations is the fact that we intent to have a small representation complexity. Second, we want to provide a calculus which can be used for cognitive modelling as well. For this reason this calculus is based on results of psycholinguistic research on reference systems [14].

An additional aim of this paper is to show with a small sample application how to apply QSR-based knowledge integration to robotics—to present a calculus which can be directly used for communicating with robots and is expressive enough to be helpful in tasks concerning spatial reasoning. For these reasoning steps to be successful, we need a minimum number of base relations. By connecting these two fields explicitly, we hope to be able to achieve synergy effects, i.e. by reducing the representation complexity—substituting quantitative descriptions such as exact position data through qualitative descriptions that are fine-grained enough to discern important informations—we aim at providing the field with a calculus which can be used to model strategic planning about scenarios, and to eliminate inconsistent scenarios easily.

In a first step, we initiate the use of ternary calculi and investigate the consequences of adding a relative distance measurement. Then, in a second step, we present complexity results concerning constraint satisfaction problems and investigate the notion of composition. The different calculi are additionally motivated by problems from robotics.

Positional calculi are influenced by results of psycholinguistic research [14] in the field of reference systems. The results point to three different options to give a qualitative description of spatial arrangements of objects labelled by Levinson [15] as *intrinsic*, *relative*, and *absolute*.

We can find examples of all three options of reference systems in the QSR literature. For instance, an intrinsic reference system was used in the dipol calculus [10,16], a relative reference system in QSR was introduced by Freksa [2], and finally Frank's cardinal direction calculus corresponds to an absolute reference system [17,18]. More details about human reference systems and their relations to QSR can be found in Section 3.4.

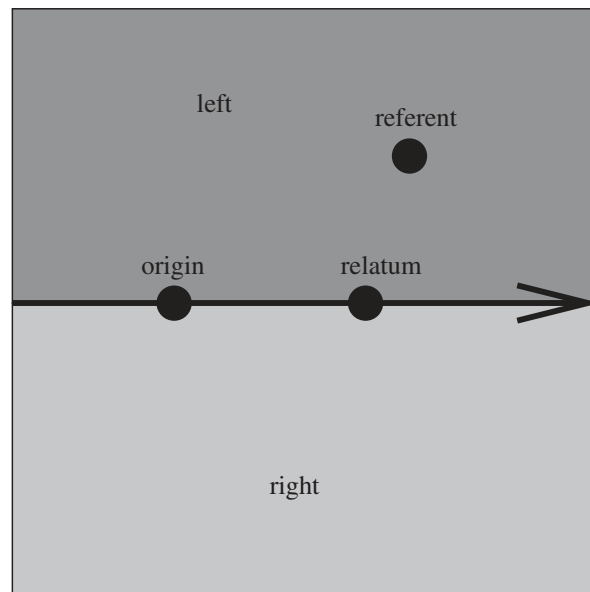


Fig. 1. The left/right-dichotomy in a relative reference system.

Qualitative position calculi can be viewed as computational models for projective relations in relative reference systems. To model projective relations (like “left”, “right”, “front”, “behind”) in relative reference systems, all objects are mapped onto the plane  $\mathcal{D}$ . The mapping of an object  $O$  onto the plane  $\mathcal{D}$  is called  $p_{\mathcal{D}}(O)$ . The center  $\mu$  of this area can be used as point-like representation  $O'$  of the object  $O$ :  $O' = \mu(p_{\mathcal{D}}(O))$ . Using this abstraction, we will henceforth consider only point-like objects in the 2D-plane.

Fig. 1 shows a simple model for the left/right-dichotomy in a relative reference system, which is given by *origin* and *relatum* (corresponding to Levinson's terminology [15]). In this figure *origin* and *relatum* define the reference axis. The reference axis naturally partitions the surrounding space in a left/right-dichotomy. The spatial relation between the reference system and the *referent* is then described by naming the part of the partition in which the referent lies. In the configuration depicted in Fig. 1 the referent lies to the *left*<sup>1</sup> of the *relatum* as viewed from the *origin*.

This scheme ignores configurations in which the referent is positioned on the reference axis. Freksa [2] used a partition that splits these configurations into three sets, corresponding to the *relatum*: the referent is either behind, at the same position or in front of the *relatum*. Ligozat [19] subdivided the arrangements with the referent in front of the *relatum* in those cases where the referent is between the *relatum* and the *origin*, at the same position as the *origin*, or behind the *origin*. We then obtain the partition shown in Fig. 2. Ligozat calls this the flip-flop calculus. For a compact notation, we use abbreviations for the relation symbols.

For  $A$ ,  $B$ , and  $C$  as *origin*, *relatum*, and *referent*, Fig. 3 shows point configurations and their qualitative descriptions, respectively. Isli and Moratz [8] introduced two additional configurations in which the *origin* and the *relatum* have exactly the same location. In one of the configurations the referent has a different location, this relation is called **dou** (for double point). The configuration with all three points at the same location is called **tri**

<sup>1</sup>The natural language terms used here are meant to improve the readability of the paper. For issues of using QSR representations for modelling natural language expressions please refer to [14].

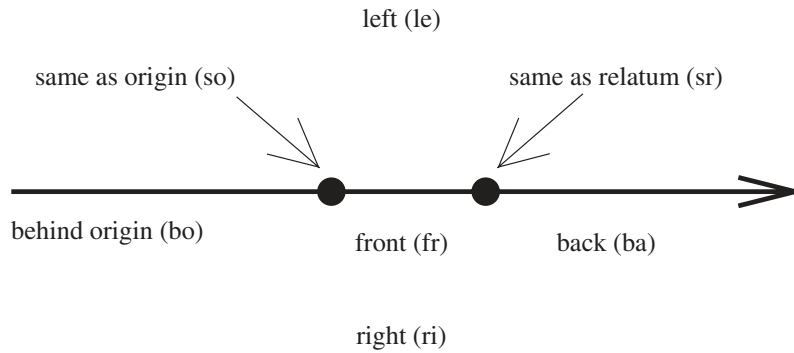


Fig. 2. Adding relations for referents on the reference axis.

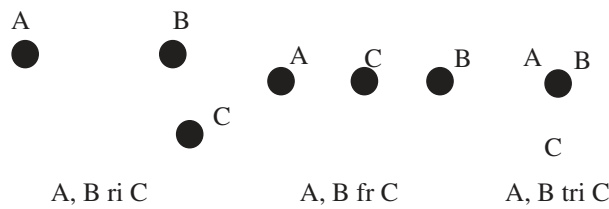


Fig. 3. Examples of point configurations and their expressions in the flip-flop calculus. We use an infix notation where the reference system consisting of origin and relatum is in front of the relation symbol and the referent is behind the relation symbol.

(for triple point). A system of qualitative relations which describes all the configurations of the domain and does not overlap is called jointly exhaustive and pairwise disjoint (JEPD). Such a calculus was formulated in the scheme of a relation algebra [20] by Scivos and Nebel [21].

The simple flip-flop calculus models “front” and “back” only as linear acceptance regions. Vorwerg et al. [22] showed empirically that a cognitively adequate model for projective regions needs acceptance regions for “front” and “back”, which have a similar extent as “left” and “right”. Freksa’s single cross calculus [2] has this feature (see Fig. 4). The front region consists of “left/front” and “right/front”, the left region consists of “left/front” and “left/back”. The intersection of both regions models the left/front relation.

This paper is organized as follows: in Section 2, we review some basic concepts of ternary relation calculi in general and introduce the *ternary point configuration calculus* (TPCC) and investigate some questions concerning relative distance measurement. Then, in the Section 3, we investigate the structure of such calculi and we show that these calculi are not ternary relation algebras [9]. A natural but not trivial result of closing any relation algebra with a “sensible” qualitative distance measure leads to a non-closedness of the composition operator. Furthermore, we present some complexity results concerning constraint solution problems. In Section 4, we investigate spatial reasoning in a knowledge integration scenario. Finally, Section 5 summarizes the results of the paper and gives a short overview of some questions that are left open in this paper.

## 2. The TPCC

The calculus we present is derived from the single cross calculus but makes finer distinctions. These finer distinctions are motivated by robot application domains like the

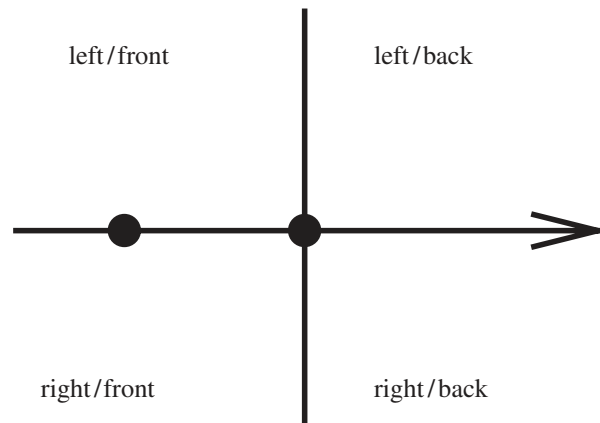


Fig. 4. The single cross calculus.

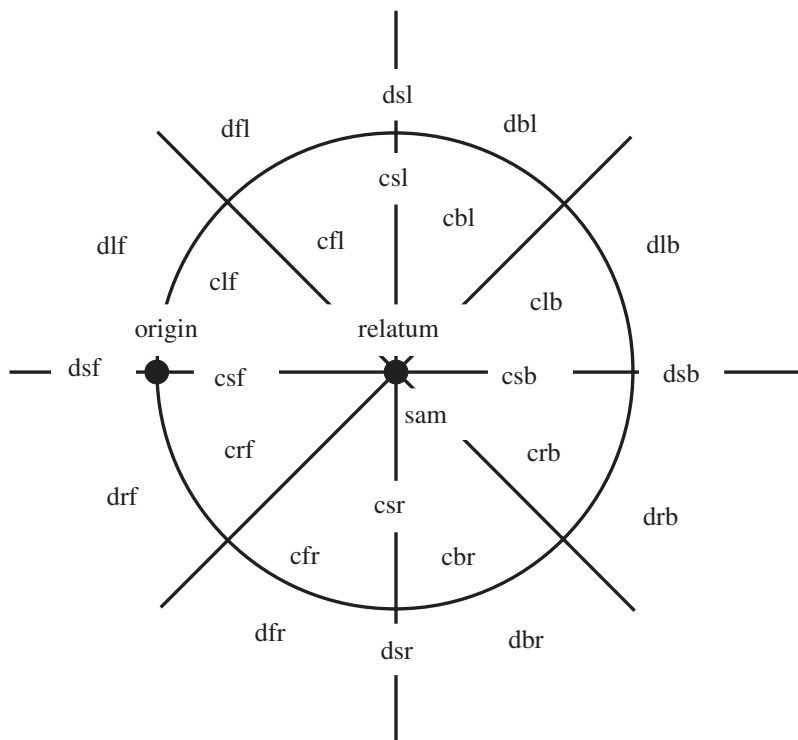


Fig. 5. The reference system used by the TPCC.

one presented at the end of this paper. The consequences of these finer distinctions are compared with other calculi in Section 3.4. The partition of the calculus is shown in Fig. 5.

The letters f, b, l, r, s, d, c stand for front, back, left, right, straight, distant and close, respectively. The use of the TPCC relations in natural language applications is shown in [14]. These authors use the TPCC relations for natural human robot interaction by linguistic spatial references using projective predicates. Certain aspects of modelling linguistic predicates by QSR relations are discussed in Section 3.4.

The configuration in which the referent is at the same position as the relatum is called **sam** (for “same location”). The two special configurations in which the origin and the relatum have the same location are **dou** and **tri** and are also base relations of this calculus. This system of qualitative spatial relations and the inference rules described in the next

section is called TPCC. When assigning a precise, formal definition of the relations, we describe the corresponding geometric configurations on the basis of a Cartesian coordinate system represented by  $\mathbb{R}^2$ . First we define for  $A = (x_A, y_A)$ ,  $B = (x_B, y_B)$ , and  $C = (x_C, y_C)$  the special cases:

$$A, B \text{ dou } C := x_A = x_B \wedge y_A = y_B \wedge (x_C \neq x_A \vee y_C \neq y_A),$$

$$A, B \text{ tri } C := x_A = x_B = x_C \wedge y_A = y_B = y_C.$$

For the cases with  $A \neq B$ , we define a relative radius  $r_{A,B,C}$  and a relative angle  $\phi_{A,B,C}$ :

$$r_{A,B,C} := \frac{\sqrt{(x_C - x_B)^2 + (y_C - y_B)^2}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}},$$

$$\phi_{A,B,C} := \tan^{-1} \frac{y_C - y_B}{x_C - x_B} - \tan^{-1} \frac{y_B - y_A}{x_B - x_A}.$$

With this we are able to define a partition on the Euclidean plane:

Semirelation	Angle
sb	$\phi_{A,B,C} = 0$
lb	$0 < \phi_{A,B,C} \leq \pi/4$
bl	$\pi/4 < \phi_{A,B,C} < \pi/2$
sl	$\phi_{A,B,C} = \pi/2$
fl	$\pi/2 < \phi_{A,B,C} < 3/4 \pi$
lf	$3/4\pi \leq \phi_{A,B,C} < \pi$
sf	$\phi_{A,B,C} = \pi$
rf	$\pi < \phi_{A,B,C} \leq 5/4 \pi$
fr	$5/4 \pi < \phi_{A,B,C} < 3/2 \pi$
sr	$\phi_{A,B,C} = 3/2 \pi$
br	$3/2 \pi < \phi_{A,B,C} < 7/4 \pi$
rb	$7/4 \pi \leq \phi_{A,B,C} < 2 \pi$

How can we now introduce different granularities? By setting  $c := 0 < r_{A,B,C} < 1$  and  $d := 1 \leq r_{A,B,C}$ , we can partition the plane in the two qualitative distance measures *close* and *distant*. For instance, the relation  $csb := c \wedge sb$  is defined by  $0 < r_{A,B,C} < 1 \wedge \phi_{A,B,C} = 0$ . From this it follows that we can define all relations of Fig. 5. Furthermore, by applying this technique, we can easily construct a set of **JEPD** relations of any granularity.

From this definition it follows that this new set of 24 relations is a partition of the Euclidean plane.

There are cases in which we only have rough spatial knowledge or in which we are at the border of a segment of the partition and cannot decide safely due to measurement errors. In such cases we use sets of the above defined relations to denote disjunctions of relations. Fig. 6 shows a situation where it is not sensible to decide visually between the alternatives  $A, B \text{ clb } C$  and  $A, B \text{ cbl } C$ . Such a configuration is described by the relation  $A, B \text{ (cbl, clb) } C$ .

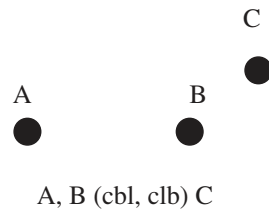


Fig. 6. Coarser spatial knowledge.

### 3. The structure of a ternary representation algebra

The well-known concept of relation algebra has been widely investigated and there exist many well-known techniques [23,24] for handling this kind of structure. However, there are some algebras (Double-Cross and even RCC-8 for non-regular sets) which are natural from a human viewpoint, but whose structural properties cannot be captured by the notion of relation algebra.

In order to “capture” these algebras, we propose and generalize the concept of *representation algebra* [25]. The purpose for using this structural type is merely that the most interesting question is whether a given set of constraints (over a set of relations) is satisfiable in a given model. The reason for this statement lies in the fact (due to [26]) that all other reasoning problems (for constraint systems) can be reduced in polynomial time to satisfiability. The (strong) composition is used as a reasoning “tool” for deriving facts or for reducing the search tree, but it does not imply a greater expressiveness of the structure. This leads to the idea that the two concepts of reasoning and representing *should not* be part of the same structure, instead, as common in AI, there is a Knowledge Base (a model), where the facts are expressed in a *knowledge representation language* [27] and a set of inference rules, which provides the framework for reasoning. It is obvious that any representation structure needs at least the expressiveness of propositional logic to be useful. In other words, any sort of algebra which is used for representation needs to be at least a Boolean algebra. For these reasons we next investigate various structures. We start with the concept of relational algebra. We then present the concept of a representation algebra, which up to now is defined for binary relations. Then, finally, we discuss a weak representation algebra, which is the generalization of the representation algebra for non-binary algebras.

#### 3.1. Generalization of binary representation algebras

We begin by recalling the definition of ternary relations.

**Definition 1** (*Ternary relation*). Let  $U$  be a set. A subset  $R$  of the Cartesian product  $U \times U \times U$  is called a ternary relation on  $M$ .

**Definition 2** (*Logical characterization of operators*). Let  $a, b, c \in U$  and  $R, S$  be relations of  $U$ :

$$\begin{aligned}
 R \sqcup S &= \{(a, b, c) | (a, b, c) \in R \vee (a, b, c) \in S\}, \\
 R \sqcap S &= \{(a, b, c) | (a, b, c) \in R \wedge (a, b, c) \in S\}, \\
 \overline{R} &= \{(a, b, c) | (a, b, c) \notin R\}.
 \end{aligned}$$



This is an obvious generalization of the binary operators  $\sqcup, \sqcap, \bar{\cdot}$ . The binary operators  $\circ$  and the unary operator  $^T$  [24] cannot be so easily generalized because there are not only six possible generalizations of the “converse” operator but also more than one possible generalization of the operator of composition. In the previous section we defined relations between triples of points on the 2D-plane. Now we define a set of unary and binary operations that allows deducing new relations about point sets from given relations about these points. Unary operations (transformations) use relations about three points to deduce a relation which holds for a permuted sequence of the same points. Binary operations (compositions) deduce information from two relations which have two points in common (the set consists of four points). The result is then a relation of one of the common points with the two other points.

As we have three arguments, we have  $3! = 6$  possible ways of arranging the arguments for a transformation. Following Zimmermann and Freksa [7] we use the following terminology and symbols to refer to these permutations of the arguments (a,b : c):

Term	Symbol	Arguments
Identical	ID	a,b : c
Inversion	INV	b,a : c
Short cut	SC	a,c : b
Inverse short cut	SCI	c,a : b
Homing	HM	b,c : a
Inverse homing	HMI	c,b : a

The transformation table for the TPCC is in Fig. 8. For any permutation  $p$  there exists an inverse permutation  $p'$  such that  $R R \subseteq p'(p(R))$  for any relation  $R$ .

Permutation	Inverse
ID	ID
INV	INV
SC	SC
SCI	HM
HM	SCI
HMI	HMI

The TPCC is not closed under transformations This means that results of a transformation can constitute proper subsets of the base relations. Since we need many sets of relations as a result of transformed relations, we introduce here an iconic notation of these relations, which makes the presentation more compact.

The segments corresponding to a relation are presented as filled segments (see Fig. 7 for the correspondence between icon segments and atomic relations). Unions of relations then simply have several segments that are filled. The reference axis and the dividing lines between left, right, front, and back are also presented in the icon to make the visual identification of the relation symbols easier. The iconic representation is easier to translate

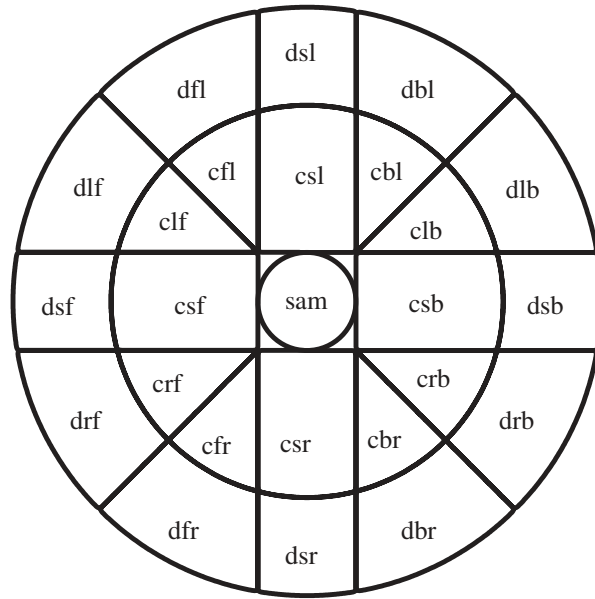


Fig. 7. Iconic representation for TPCC relations.

ID	
INV	
SC	
SCI	
HM	dou
HMI	dou

Fig. 8. Permutation table for TPCC relations.

into its semantic content (the denoted spatial point configuration) compared with a representation that uses the textual relation symbol (Fig. 8). Furthermore, unions can be expressed in a compact way.

In order to reduce the size of the table, the trivial cases for **dou** and **tri** are omitted. Symmetric cases can be derived by using a reflection operation (reflection on an axis). The results of  $Sc(\mathbf{dsf})$  and  $Sci(\mathbf{dsf})$  also include **dou** as a result.

As mentioned above the TPCC is not closed under transformations since results of a transformation can constitute proper subsets of the base relations. For a given calculus one can try to build the closure with respect to a set of operations by iteratively adding the operation results (e.g. potential subsets of the original base relations) to a new set of base relations until a fix point is reached. This construction was performed for Freksa's single cross calculus [2] (see Section 1) by Scivos and Nebel [28]. The resulting calculus is an extension of Freksa's original Double-Cross calculus [2]. The acceptance regions of the extended Double-Cross calculus are depicted on Fig. 9.

With ternary relations, one can think of different ways of composing them. However there are only a few ways to compose them in such a way that we can use them to enforce

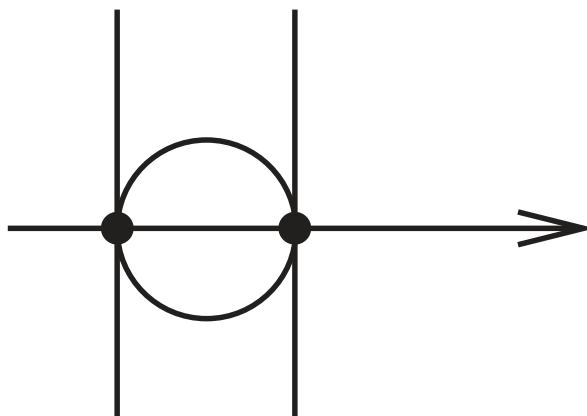


Fig. 9. Acceptance regions of the extended Double-Cross calculus.

local consistency [28]. By trying to generalize the path-consistency algorithm [29], we want to enforce 4-consistency [9]. We use the following (strong) composition operation:

$$\forall A, B, D : A, B(r_1 \diamond r_2)D \leftrightarrow \exists C : A, B(r_1)C \wedge B, C(r_2)D.$$

It can be concluded from the granularity, that the TPCC is not closed under strong composition. This is an inherent problem, but there are other algebras, as well, such as RCC-5 for non-regular sets which are not closed under composition. As a consequence the 4-consistency cannot be directly enforced. Nonetheless, a weak composition operation  $r_1 \diamond r_2$  for two relations  $r_1$  and  $r_2$  can be defined. This is the most specific relation such that

$$\forall A, B, D : A, B(r_1 \diamond r_2)D \leftarrow \exists C : A, B(r_1)C \wedge B, C(r_2)D.$$

While using the weak composition, we cannot enforce 4-consistency, as we have mentioned, though we still get useful inferences. We use this weak composition for inferences in the application scenario in Section 4.

The table for weak composition of TPCC relations is shown in Fig. 10. The first operand determines the row, the second operand the column. Again, the table omits entries which can be found by reflection in order to reduce the size of the table. And the trivial cases for **dou** and **tri** are again omitted.

We now show that the composition is not closed. In [21] it was shown that calculations with relations can be done by calculating with complex numbers. For this reason a relation  $R \subset (\mathbb{T} \setminus \text{sum} \setminus \text{eq})$  can be identified with the region  $\text{Reg}R := \{z = x + iy \in \mathbb{C} \mid ((0, 0), (1, 0), (x, y)) \in R\}$ . Examples are  $\text{csb} = ]1, 2[$  and  $\text{dsb} = ]2, \infty[$ . In [21] it was proven that there is an easy description for all composition using this formalism. In our case we get

$$R_1 \diamond R_2 = \{z_1 + (1 - z_2) - z_1(1 - z_2) \mid z_1 \in \text{Reg}(R_1) \text{ and } z_2 \in \text{Reg}(R_2)\}.$$

Hence,

$$\text{Reg}(\text{csb} \diamond \text{csb}) = \{z_1 + (1 - z_2) - z_1(1 - z_2) \mid z_1 \in \text{Reg}(]1, 2[) \text{ and } z_2 \in \text{Reg}(]1, 2[)\}.$$

In other words, the range of the function  $z_1 + (1 - z_2) - z_1(1 - z_2)$  on the domain  $]1, 2[ \times ]1, 2[$  is  $]1, 3[$ . Thus we see that the composition of  $\text{csb} \diamond \text{csb}$ , which is a subset of  $\{\text{csb}, \text{dsb}\}$  is actually a proper subset, because  $\text{csb} \diamond \text{csb} = \{\text{csb}, \text{dsb}\} \upharpoonright_{]2, 3[} \subsetneq \text{csb} \diamond \text{csb}$ .

In a paper by Isli and Cohn [9] the notion ternary relation algebra was introduced for the first time. We set  $T_1 := \text{SC}$  and  $T_2 = \text{SCR}$ :

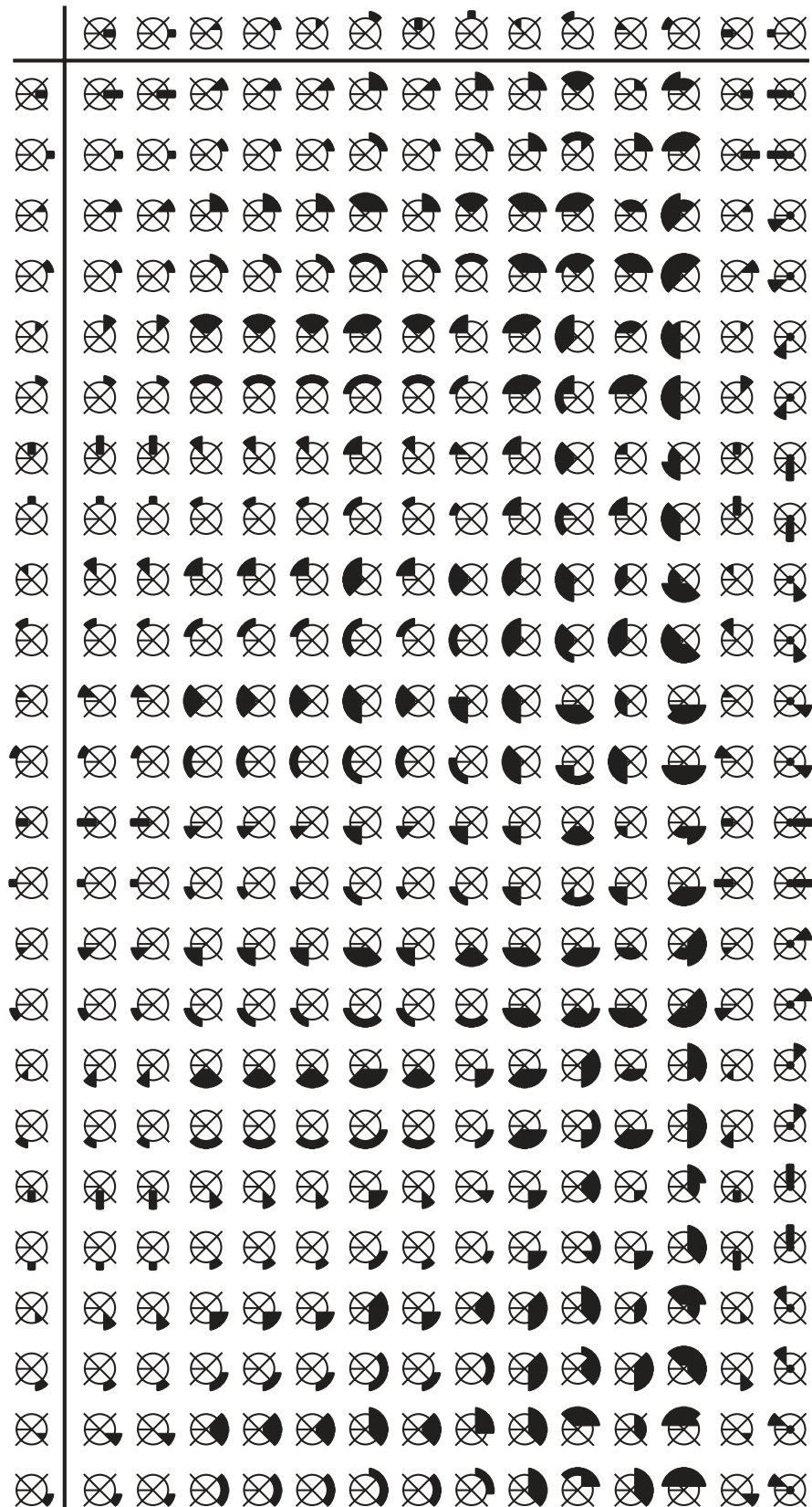


Fig. 10. Composition of TPCC relations.

**Definition 3** (*Ternary relation algebra*). A ternary relation algebra is a structure  $\mathcal{R}\mathcal{A} = (\mathcal{R}, \sqcup, \sqcap, \bar{\phantom{x}}, \top, \perp, id, \circ, {}^{T_1}, {}^{T_2})$  where  $(\mathcal{R}, \sqcup, \sqcap, \bar{\phantom{x}}, \top, \perp)$  is a Boolean algebra,  $\mathcal{R}$  is a set of ternary relations, and  $\circ$  is a binary operation,  ${}^{T_1}, {}^{T_2}$  are unary operations, where for all relations  $x, y, z \in \mathcal{R}$  the following holds:

$$(x \circ y) \circ z = x \circ (y \circ z), \tag{1}$$

$$(x \sqcup y) \circ z = (x \circ z) \sqcup (y \circ z), \tag{2}$$

$$x \circ id = id \circ x = x, \tag{3}$$

$$(x^{T_1})^{T_1} = x, \tag{4}$$

$$(x \sqcup y)^{T_1} = x^{T_1} \sqcup y^{T_1}, \tag{5}$$

$$(x \circ y)^{T_1} = y^{T_1} \circ x^{T_1}, \tag{6}$$

$$((x^{T_2})^{T_2})^{T_2} = x, \tag{7}$$

$$(x \sqcup y)^{T_2} = x^{T_2} \sqcup y^{T_2}, \tag{8}$$

$$x^{T_2} \circ \overline{(x \circ y)} \sqcap y = \perp. \tag{9}$$

It is not hard to see that Double-Cross or TPCC are not ternary relation algebras because those algebras are not closed under permutations or composition. For this reason we want to analyze the structure of these algebras and find out the properties we need. It must follow that if an algebra is not closed under transposition, the conditions (4), (7) cannot be generally satisfied.

A *weak representation algebra* is a representation algebra [25] that is not closed under transformation. More precisely:

**Definition 4** (*Weak representation algebra*). A *weak representation algebra* is a structure  $\mathcal{R}\mathcal{A} = (\mathcal{R}, \sqcup, \sqcap, \bar{\phantom{x}}, \top, \perp, id, {}^{T_1}, \dots, {}^{T_n})$ , where  ${}^{T_i}$  are transformations, over a universe  $U$  where the ternary relations are interpreted with the following properties:

- (1)  $\mathcal{R}$  is a relation partition on  $U \times U \times U$ .
- (2) A Boolean algebra for  $(\mathcal{R}, \sqcup, \sqcap, \bar{\phantom{x}}, \perp, \top)$  with the usual semantics.
- (3) Both structures are connected in a way that for each transformation  $T$  and the inverse  $T'$  they form a *normal Boolean algebra* with operators for

$$(x \sqcup y)^T = x^T \sqcup y^T,$$

$$x \subseteq (x^T)^{T'},$$

$$0^T = 0.$$

**Definition 5** (*Reasoning representation algebra*). A *reasoning representation algebra* is a structure  $(\mathcal{R}\mathcal{A}, \circ)$ , where  $\mathcal{R}\mathcal{A}$  is a representation algebra and  $\circ$  is an inference calculus with the following additional properties:

$$(x \sqcup y) \circ z = (x \circ z) \sqcup (y \circ z),$$

$$x \circ id = id \circ x = x.$$

The reasoning representation algebra  $(\mathcal{R}\mathcal{A}, \circ)$  is a *relation algebra* if it is closed under composition and under transformation. If  $\circ$  is the weak composition ( $\diamond$ )

it is called [28] constraint algebra. It is now possible to identify these new kinds of structures.

**Lemma 1.** *TPCC is a weak representation algebra, with weak composition it is a constraint algebra.*

### 3.2. Reasoning about ternary relations

The standard method for reasoning with relation algebras is to use Ladkin and Reinefeld's algorithm [30] which uses backtracking and employing the path-consistency algorithm as a forward checking method. This scheme was extended by Isli and Cohn [9] for ternary relation algebras. It can be easily applied to the flip-flop calculus, which was described in Section 2.

A prerequisite for using the standard constraint algorithms is to express the calculi in terms of relation algebras in the sense of Tarski [24]. But since the TPCC is neither closed under transformations nor under composition, we cannot use this scheme. However, simple path-based inferences can be performed using the following scheme. The last two relations of a path are composed. Then the reference system is incrementally moved towards the beginning of the path following backward chaining.

**Definition 6 (CSP).** *A constraint satisfaction problem (CSP) is characterized by*

- a set  $V$  of  $n$  variables  $\{v_1, \dots, v_n\}$ ,
- the possible values  $D_i$  of variables  $v_i$ ,
- constraints (sets of relations) over subsets of variables.

Sets of constraints on spatial formulae are called *TPCC CSP*. The variables are interpreted over pairs of the real numbers  $\mathbb{R}$  in the Euclidean plane. We call an interpretation a model of a *TPCC CSP* iff all constraints are satisfied. If such a model exists, we say that the *TPCC CSP* is satisfiable.

**Theorem 1.** *The general satisfiability problem of TPCC is in PSPACE.*

**Proof.** The algebraic semantics of the relations implies that reasoning problems in the TPCC can be expressed as inequalities over polynomials of power 2 with integer coefficients. Systems of such quadratic inequalities can be solved using polynomial space [31].  $\square$

A natural question might now be: How hard is the problem at least, i.e. how difficult is it to solve the general satisfiability problem? Are there some tractable subclasses? Do the base relations and the universal relation form a tractable subclass? Or do the base relation and the universal relation alone already form an NP-hard problem? Unfortunately the proof of [28] cannot be repeated because there is nothing like an “equal” relation  $\{csb\}$  in our calculus. However, we are able to prove:

**Theorem 2.** *Satisfiability over  $\{csf, \top\}$  is NP-hard.*

Before we prove this, let us recall the definition of the Betweenness problem:

*Given:* A finite set  $M$ , a set  $C$  of ordered triples  $(a, b, c)$  of distinct elements of  $M$ .

**Question:** Is there a one-to-one function  $f : M \rightarrow \{1, 2, \dots, \|M\|\}$  such that for each  $(a, b, c) \in C$  we have either  $f(a) < f(b) < f(c)$  or  $f(c) < f(b) < f(a)$  ?

**Proof.** This will be proven by a reduction from the Betweenness problem. For a given instance of Betweenness  $\theta$  we construct a constraint set  $\theta'$  as follows: for each  $m \in M$  we add  $csf(x, m, y)$ , where  $x$  and  $y$  are two fixed elements not in  $M$ . This enforces that all  $m$  are on a line. Now, we add  $csf(a, c, b)$  for each tuple  $(a, b, c) \in M$ . It is easy to show that  $\theta$  is true if and only if  $\theta'$  is true and the transformation is of course a polynomial reduction.  $\square$

**Corollary 1.** *The general satisfiability problem of TPCC is NP-hard.*

### 3.3. Fix-point computation for ternary representation algebras

We want of course to be able to test the arc-consistency of a given set of constraints. For this reason we present here a ternary variant of the Mackworth algorithm with a runtime of  $O(n^4)$ .

A scenario consists of a set of objects  $O$ . Every three objects correspond to a relation, which is a proposition about an ordered triple  $C_{\langle X, Y, Z \rangle}$  with  $X, Y, Z \in O$ . In the following we may refer to this object triple as a node. A point configuration on the other hand is a (not ordered) set of three objects  $X, Y, Z \in O$ . Then, in a fully connected constraint network,  $3!$  nodes refer to the same point configuration.

The set of all possible triples of  $n$  objects is denoted by  $V_n^{(3)}$ . This is the number of nodes in a fully connected ternary constraint network about  $n$  objects. It clearly follows that the number of nodes is  $|V_n^{(3)}| = 3! \binom{n}{3} < n^3$ .

$|Rel_{TPCC}|$  constitutes the number of base relations of the TPCC. Then each node has only  $m \leq |Rel_{TPCC}|$  possible base relations that are claimed to be true for the given node constraint. The situation  $m = |Rel_{TPCC}|$  corresponds to a disjunction over all base relations (e.g. no information is given by this constraint).

To combine all information about the same point configuration given in all the  $3!$  corresponding nodes, a subprogram `enforceUnaryRules` uses the rules `INV`, `SC`, `SCI`, `HM`, and `HMI` to propagate knowledge from all  $3!$  permutations of the same three objects to a given node with a specific permutation of these objects.

The subprogram `enforceBinaryRule` returns `TRUE` if by using the composition table a constraint is replaced by a stricter constraint. For a comparison with other ternary constraint propagation algorithms, we refer to Dylla and Moratz [32].

The variable “change” is true if and only if for one node at least one base relation has been excluded. If this happens, then the *queue'* has been extended by a new element. For this reason we have in the worst case no more than  $n^3 |Rel_{TPCC}|$  nodes in the *queue* (for each  $C_{\langle x_i, y_i, z_i \rangle}$  with  $\langle x_i, y_i, z_i \rangle \in queue$ ). In the iteration of the loop each element leads to  $n$  calls of `CheckBinaryRule`. Therefore the complexity of this algorithm is  $O(n^4)$ .

**Algorithm 1.** A ternary variant of the Mackworth algorithm.

**procedure** *TernaryMackworth*( $\langle O, D, C \rangle$ )

1. *queue*  $\leftarrow \{ \langle X, Y, Z \rangle \mid C_{\langle X, Y, Z \rangle} \in C_{init} \}$
2. **while** *queue*  $\neq \emptyset$  **do**
3.     *initialize queue'*
4.     **for all**  $C_{\langle x_i, y_i, z_i \rangle}$  with  $\langle x_i, y_i, z_i \rangle \in queue$  **do**

```

5.      enforceUnaryRules( $C_{\langle x_i, y_i, z_i \rangle}$ )
6.      for all  $o_i \in O$  do
7.          if  $o_i \neq y \wedge o_i \neq z$  then
8.              changed  $\leftarrow$  enforceBinaryRule( $C_{\langle x, y, z \rangle}, C_{\langle y, z, o_i \rangle}$ )
9.          end if
10.         if changed then
11.              $queue' = queue' \cup \langle x, y, o_i \rangle$ 
12.         end if
13.     end for
14. end for
15.  $queue = queue'$ 
16. end while

```

### 3.4. Comparison between the different ternary point-based reasoning algebras

As mentioned in the introduction ternary point configuration calculi can model linguistic spatial location descriptions by human speakers. In this section we compare different calculi in more detail.

Previous research on reference systems for spatial descriptions has led to the identification of three different reference systems labelled by Levinson [15] as *intrinsic*, *relative*, and *absolute*.

In *intrinsic reference systems*, the relative position of one object (the *referent*) to another (the *relatum*) is described by referring to the relatum's intrinsic properties such as *front* or *back*. In such a situation, the speaker's or hearer's position are irrelevant for the identification of the object. However, the speaker's or hearer's front or back may also serve as origins in intrinsic reference systems. These reference systems can be modelled by binary orientation calculi that use oriented objects (e.g. oriented line segments) as basic elements [10,16].

Humans employing *relative reference systems* use the position of a third entity as origin instead of referring to inbuilt features of the relatum. Thus, a ball (= referent) may be situated to the left of a chair (= relatum) from the speaker's or the hearer's point of view (= origin). In Fig. 11 the human speaker might refer to an object by the expression "The object in front of the chair". Then the speaker would be the origin, the chair would be the relatum.<sup>2</sup>

In *absolute reference systems*, neither a third entity nor intrinsic features are used for reference. Instead, the earth's cardinal directions such as *north* and *south* serve as anchor directions. Calculi like Frank's cardinal direction calculus model these reference systems [17,18].

An established method in the multidisciplinary research field of Spatial Cognition is to start with psychological findings and theories and build initial computational models of human spatial concepts [33]. These computational models can then iteratively be improved based on experimental observations (in our context experiments about human–robot interaction). There would be no last, perfect model, but a sequence of iteratively improved models.

<sup>2</sup>A chair has a natural intrinsic direction (e.g. the direction of the backrest as anchor for "behind"). We focus here on the use of relative reference systems.



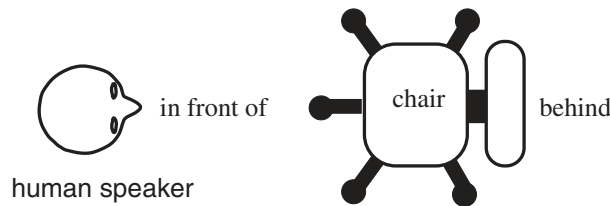


Fig. 11. Human relative reference by projective predicates.

When developing the TPCC our intention was to keep the set of distinctions which generate the base relations as simple as possible. The distinctions have to be capable of expressing projective location statements adequately. Preference was given to the simplest model which explains our empirical data [34]. This procedure enables meaningful research in the interdisciplinary and highly complex field of human–robot interaction without the prerequisite of sophisticated and expensive system parts, which may not even be required for enabling effective interaction.

In our human–robot interaction studies [34] we did not consider distance related concepts since the introduction of every new parameter increases the number of necessary human test subjects. However, current work of our students is consistent with the simple distance feature in the TPCC acceptance region partition. To sum up, the TPCC acceptance regions which are shown in Fig. 12 have been tested on human test subjects and are found to be a valid approximation to linguistic behavior observed in the real world. In contrast, the flip-flop calculus and the Double-Cross are less adequate for modelling projective predicates (see the table in Fig. 12). Their strength lies in their feature to be closed under transformations,<sup>3</sup> or even composition (flip-flop) (see the table in Fig. 13).

Point-based representations are more adequate for configurations in which the represented objects are small and are distant (e.g. they have a large distance to size ratio). In other configurations objects cannot be represented by a single point adequately. Point-based calculi then can represent these extended objects by representing the corners of their bounding boxes. However, for many domains it has advantages to use a region-based calculus to model extended objects directly. More details can be found in Section 5.

The main design decision for the TPCC relations was rather to make the distinctions necessary in a group of domains than to enforce closedness with respect to transformations or even composition. Equally sized, overlapping angular applicability areas of  $90^\circ$  each and a single distance distinction is a natural choice then for the areal acceptance regions of projective predicates in relative reference systems. The set of **JEPD** base relations of the TPCC is constructed to support these  $90^\circ$  sectors, iteratively rotated by  $45^\circ$ . The motivation for the linear acceptance regions (straight continuation and right angle) lies in our experiences with qualitative instruction maps for graphical human–robot interaction [35]. In these applications walls are represented by their endpoints (corners). Straight continuation and right angles are also important for representing idealized street networks [36].

However, the TPCC model is especially suited to *interpret* human spatial references. In cases in which the robot itself *generates* linguistic descriptions the acceptance regions do

<sup>3</sup>The extended Double-Cross is closed under the transformations, the original Double-Cross does not have this feature. With respect to projective predicates both variants are equal.

	left	behind	leftback
Flip Flop			
Double Cross			
TPCC			

Fig. 12. Relative reference by projective predicates for the different calculi.

	transformations	composition	linguistic adequacy
Flip Flop	closed	closed	weak
extended Double Cross	closed	not closed	moderate
TPCC	not closed	not closed	strong

Fig. 13. Main differences between the point-based calculi.

not have to overlap [37]. Then a set of base relations of  $45^\circ$  each (with an  $22.5$  offset to the reference direction) would be the simplest adequate model.

Also the addition of more distance distinctions has advantages for specific applications. But the very simple geometric features of the TPCC support the construction of a sound composition table. More complex calculi demand more effort for the construction of their composition tables.

#### 4. Spatial reasoning in a knowledge integration scenario

We will now demonstrate how to integrate local knowledge into survey knowledge with the presented TPCC. The context is a mobile robot able to perceive colors and to segment simple objects (see Fig. 14). Furthermore, the system is able to perform straight motion steps and turnings. Since the robot has no prior knowledge about the sizes of the objects and does not know whether the ground is perfectly flat, it cannot estimate their distances. Thus, the robot has only access to local orientation information relative to its position.

Users interact with our system by verbally issuing simple requests to the system. These requests to identify items in the system's perceptual range are detected with a Nuance Speech Recognizer.<sup>4</sup> These requests are then fed to a semantic analysis component (for a detailed description of the linguistic aspects please refer to [14]). To demonstrate the capabilities of our system we first give a detailed description how the system performs

<sup>4</sup>[www.nuance.com](http://www.nuance.com).



Fig. 14. Speech commands to an AIBO robot in a knowledge integration scenario.

knowledge integration on a specific example (see Fig. 14<sup>5</sup>). The task is to “move to the yellow cube behind the red disk”, i.e. one of the relations  $clb$ ,  $csb$ , or  $crb$  must hold for  $(R1, D, C1)$  or  $(R1, D, C2)$ . We will refer to the disjunction of the three relations as  $c?b$ . Fig. 15 shows the configuration from a bird’s eye view, using icons for the objects. The perceived constraints are also listed with respect to what is known about  $C1$  and  $C2$ .

The agent must move to deduce the desired knowledge. How a good action is selected is beyond the scope of this paper. Therefore we limit our description here to given actions. In our example the robot moves straight towards a position roughly in the center of the perceived scene.

In Fig. 16 it reaches a position ( $R2$ ) which provides new local perceptions. Relation 3 simply describes the agent’s move to a position where the red disk is orthogonal<sup>6</sup> to the moving direction of the robot which is still aligned with the directed line beginning with the starting point  $R1$ . Note that the robot only made straight movements which is because it still knows the direction of its starting point. The agent’s perception gives additional knowledge on  $C1$  and  $C2$  relative to  $(D, R2)$ . More relations are perceivable, but we concentrate on the relations relevant for this example. In order to make a composition, we have to transform relation 3 with the Sc transformation leading to relation 3’ (Fig. 16). Now 3’ can be composed with 5 leading to the fact that  $c?b$  is not valid for  $(R1, D, C2)$ . Composing 3’ and 4 (Fig. 17) shows that  $C1$  has a position behind or left of the red disk seen from  $R1$ . Since  $C2$  is not a candidate for being behind the red disk seen from the initial position of the robot,  $C1$  is the only cube which the instructor could have meant with his command.

The corresponding constraint net has a set of five objects and therefore needs 60 nodes (one for each object triple). The fixpoint computation needs 420 unary and 504 binary

<sup>5</sup>The distances on the image are smaller than in the real experiments for illustration purposes.

<sup>6</sup>For aspects of more adequate modelling of sensor-based information please refer to the discussion in the next section.

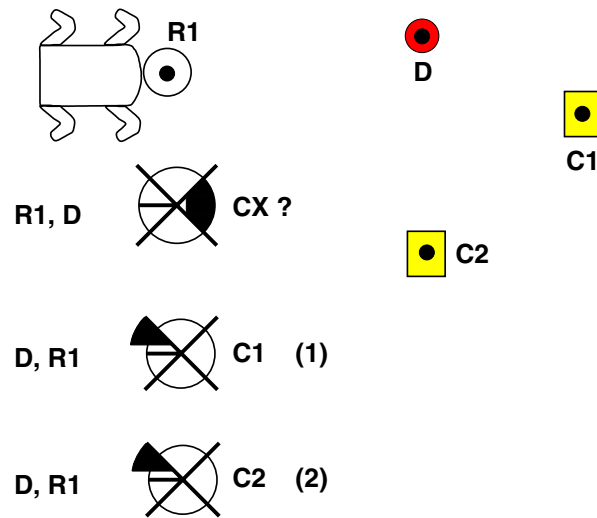


Fig. 15. The initial situation and the perceived relations.

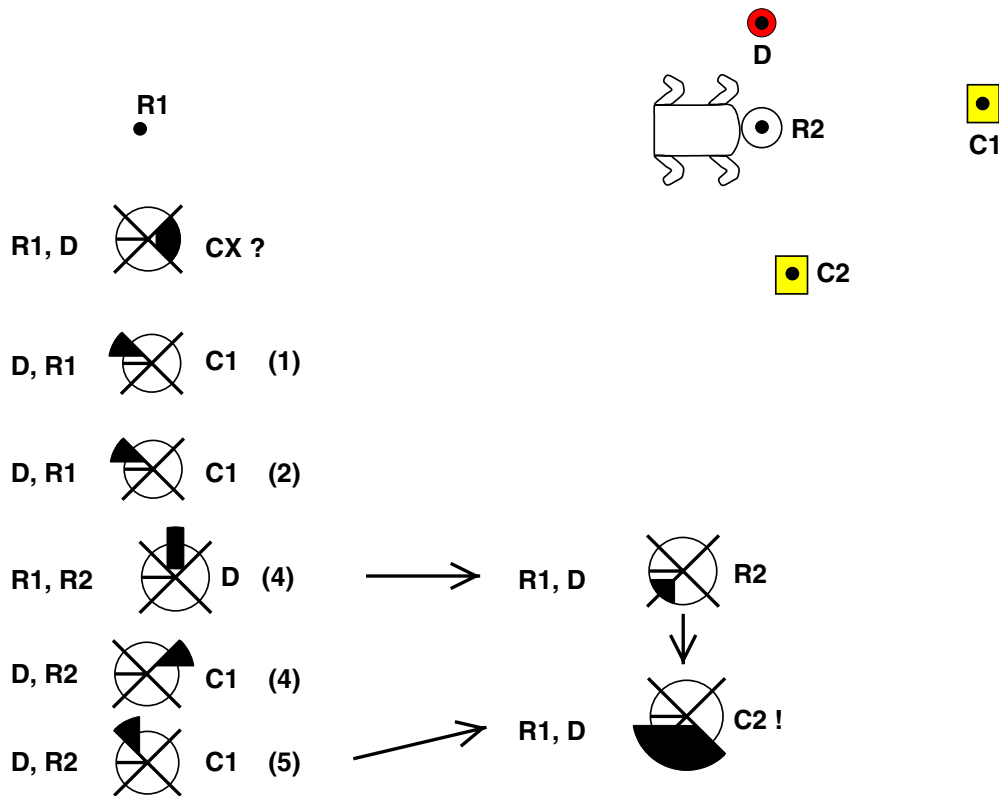


Fig. 16. Path-based integration of 3' with 5, resulting C2 being right of D.

update rule applications For this scenario we generated 11 random configurations and tested the TPCC reasoning. From the 11 cases two could already be solved by local perception. Six cases made successful use of constraint propagation, two cases were still ambiguous after constraint propagation, and one case could not be solved due to insufficient precision of the AIBO's odometry. This means that in eight out of 11 cases the TPCC relations were an adequate representation in our scenario.

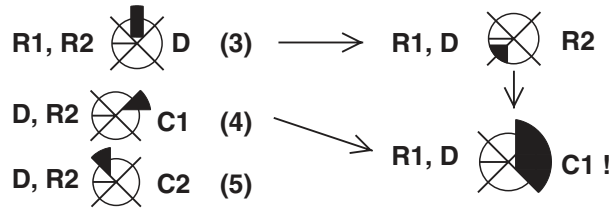


Fig. 17. Integration of 3' with 4 resulting in C1 being somewhere behind D.

## 5. Related work and outlook

Clementini et al. [6] suggested a framework for representing qualitative orientation and distance. They also derived a family of algorithms for composing two distance segments. Our TPCC can be directly expressed in their framework. We focused on a specific set of base relations and elaborated reasoning mechanisms which were originally designed for binary relation algebras. These constraint-based techniques are necessary to combine arbitrary knowledge sources by enforcing path consistency.

Isli and Cohn [9] developed a ternary relation algebra for reasoning about orientations. Their algebra has a tractable subset containing the base relations. However, Isli and Cohn's algebra is too coarse to directly represent linguistic projective relations like "left" or "front" [14].

The recent approach introduced by Klippel et al. [36] is an approach motivated by interdisciplinary investigations about modelling route information. Their wayfinding choreme theory proposes a formal treatment of conceptual route knowledge which is based on qualitative calculi and refined by behavioral experimental research. The wayfinding choreme representation uses a similar granularity level as the TPCC. The theory focuses on the modelling of chunking operations rather than constraint-based reasoning as performed by the TPCC.

The newly designed calculus GPCC [38] which is based on TPCC but uses more fine-grained base relations would be better suited to robotic applications in which a more precise modelling of sensor-based information is necessary. However, the formal status of GPCC compositions is still somewhat unclear. The composition tables derived so far are only approximations to minimal composition tables.

As mentioned in Section 3.4 the abstraction of extended objects by a point-like representative (e.g. centroid) is not adequate for many applications of orientation-based QSR. There are several region-based QSR calculi for orientation-based reasoning. Skiadopoulos and Koubarakis [39] developed a calculus for cardinal directions between regions. Since their calculus models absolute reference systems it cannot directly be compared to the TPCC. A calculus which can be viewed as a region-based counterpart to the TPCC was developed by Clementini and Billen [40]. They constructed a model for projective relations between regions. This model is based on a ternary point-based calculus which is very similar to the flip-flop calculus. Regions are modelled as point sets. Relations between the set of all points which belong to one object and the corresponding sets of two other objects can be represented by this point-based calculus. Thereby 34 projective base relations between the three regions can be constructed.

One can compare the application of Clementini's and Billen's calculus to linguistic projective predicates with the application of the TPCC presented in Section 3.4. For extended objects which are close to each other the "behind" region modelled in their

calculus has a very natural acceptance region since it is simply the “visually occluded” region behind the relatum. For more angular concepts like “left” finer grained angular resolutions might be beneficial which are supported by our TPCC. So both calculi seem to complement each other. There are also tables for the transformations and for the composition operation of the region-based calculus for projective relations [41]. Therefore one can apply the ternary variant of the Mackworth algorithm also this calculus. And our new result about the  $O(n^4)$ -complexity of this algorithm also holds for the region-based calculus.

In the field of robotics itself there were methods for representing uncertain position data in robotics using small acceptance areas with sharp boundaries even as early as the 1980s [42]. However, this approach to modelling was not followed up on, as probabilistic modelling became available [43]. In laboratory situations in which systematic errors could be excluded through calibration methods, as a result only statistically independent measuring and movement errors remain. Probabilistic approaches return very realistic estimates. There the particularly favorable property of independently sourced errors will be used: they can mutually partially compensate each other. One then obtains, in contrast with propositions relying on fixed regions of error, more precise estimations. However, these estimations are optimistic and are not adequate when a pessimistic estimation is necessary for a critical application in the real world. Propagation-based reasoning with fine-grained qualitative calculi derived from TPCC could be applied in these scenarios and produce more reliable results because they would be more pessimistic results.

Another even more important situation where probabilistic methods are not adequate is one in which generic spatial configurations need to be described. As already pointed out by Hernández [44], qualitative, symbolic spatial expressions can express underdetermined knowledge in a systematic way. Generic instructions can be given by linguistic, symbolic descriptions in a straightforward and intuitive way, for example by issuing a permanent or habitual instruction to a future semi-autonomous robot such as: “If there are shoes in front of a room door in a hotel, then polish the shoes”, “Whenever there is a risk of a collision with another robot or a person, avoid it by moving towards the right wall”. These generic spatial configurations can be quite well described and reasoned about using coarse, cognitively motivated calculi like our TPCC. However, these settings are still too technically demanding for the procedural modules of today’s autonomous robots, which is the reason why we gave a different sample application in the previous section.

Future investigations also aim to temporalize these calculi (comparable with Erwig’s and Schneider’s [45] temporalization of a topological calculus) in order to express dynamic problems and to bring deterministic planning into play.

## 6. Conclusion

In this paper we have presented the new TPCC for representing and reasoning about qualitative relative position information. The TPCC representation is based on results of psycholinguistic research on reference systems and is a combination of a relative position calculi with a qualitative distance measure. By this combination, it is possible to identify a system of 27 JEPD atomic relations between point triples on the real plane. An essential part of constraint-based reasoning methods are composition tables. Thus, we have provided these for TPCC. Theoretical investigations of TPCC has identified this calculus

as part of the weak representation algebra class, and that the general satisfiability problem lies in the same class as classical (deterministic) planning PSPACE.

Potential applications of the calculus are demonstrated with a robotics scenario. In the scenario, linguistic commands and coarse perceived configuration information have to be integrated by constraint propagation to get survey knowledge. In the conceptualization of a qualitative calculus, a compromise must be made between a fine resolution, necessary for robotic applications, versus the expenditure which must be made to develop a verifiable table of compositions. The accuracy of the calculus permits robotic applications, in particular in cognitively driven scenarios featuring linguistic communication and approximate visual perception.

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