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Quantification in Some Non-normal Modal Logics

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Abstract

This paper offers a semantic study in multi-relational semantics of quantified **N**-Monotonic modal logics with varying domains with and without the identity symbol. We identify conditions on frames to characterise Barcan and Ghilardi schemata and present some related completeness results. The characterisation of Barcan schemata in multi-relational frames with varying domains shows the independence of **BF** and **CBF** from well-known propositional modal schemata, an independence that does not hold with constant domains. This fact was firstly suggested for classical modal systems by Stolpe (2003), but unfortunately that work used only models and not frames.

1 Introduction

A number of significant contributions in the last four decades show that non-normal modal logics (NMLs) can be fruitfully employed in several fields. One well-known domain is epistemic logic, where NMLs are a solution to alleviate the so-called omniscience problem that affects stronger (normal) modal systems (Fagin et al., 1995). Deontic logic is another field where NMLs have been traditionally proposed to avoid many drawbacks of standard deontic logic (i.e., deontic **KD**), which does not tolerate deontic conflicts and gives rise to a number of paradoxes (Goble, 2005; Jones and Carmo, 2002). Other important applications are those systems capturing different aspects of the concepts of action and agency: The modal logic of agency (Elgesem, 1997; Governatori and Rotolo, 2005; Segerberg, 1992), concurrent propositional dynamic logic (Goldblatt, 1992), game logic (Parikh, 1985), and coalition logic (Pauly, 2002), among others, are all examples where some modal operators are axiomatized in logics weaker than **K**.

Quantified modal logic has also a long and distinguished tradition (Garson, 2001; Fitting and Mendelsohn, 1998), which is still lively and technically productive (see, among others, Corsi, 2002; Brauner and Ghilardi, 2007; Gabbay et al., 2009; Goldblatt, 2011). Nevertheless, almost all efforts have so far been devoted to the analysis of the normal case: Besides a few significant exceptions (Arló-Costa and Pacuit, 2006; Arló-Costa, 2002; Stolpe, 2003; Waagbø, 1992), which are based on neighbourhood semantics, the study of quantification

in non-normal modal logics is still neglected. Despite that, quantified non-normal modal logics (QNML henceforth) exhibit a different behaviour with respect to normal modal logics: For instance the Barcan and the Converse Barcan schemata (i) are not characterised by decreasing and increasing domains (ii) are tightly connected to the validity of propositional modal axiom schemata.

This paper provides a semantic analysis of quantification in the system MN, i.e., a class of non-normal modal logics called **N**-Monotonic (Calardo and Rotolo, 2014; Chellas, 1980). This class is characterised by the inference rules **RM** and **RN** or, equivalently, by adding to E (i.e., the minimal classical modal logic closed under logical equivalence) the axiom schemata **M** and **N**[1]:

$$\textbf{RM} := \ \vdash A \to B \ \Rightarrow \ \vdash \Box A \to \Box B \qquad\qquad \textbf{RE} := \ \vdash A \leftrightarrow B \ \Rightarrow \ \vdash \Box A \leftrightarrow \Box B$$
$$\textbf{RN} := \ \vdash A \ \Rightarrow \ \vdash \Box A \qquad\qquad\qquad\qquad \textbf{M} := \ \Box(A \wedge B) \to (\Box A \wedge \Box B)$$
$$\textbf{N} := \ \Box \top$$

However, instead of working in the standard setting of neighborhood semantics (Chellas, 1980; Hansen, 2003; Montague, 1970; Scott, 1970; Segerberg, 1971), we shall focus on multi-relational semantics. From the structural point of view, this setting looks like a direct generalization of mono-relational (Kripke) semantics for normal modal logics. A multi-relational model is a structure $\langle W, \mathcal{R}, V \rangle$ where W is a set of possible worlds, $\mathcal{R}$ is a (possibly infinite) set of binary relations on W, and V is the usual evaluation function of Kripke semantics. Starting from the works of Jennings and Schotch (1981a,b) and Goble (2001, 2003, 2004a), two different versions of it have been later identified in the literature (cf. Calardo and Rotolo, 2014; Governatori and Rotolo, 2005), depending on how to evaluate $\Box$-formulae.

Here we shall only present a *weak* formulation of multi-relational semantics, namely the one that simply extends the case of Kripke semantics (Goble, 2001, 2004b; Jennings and Schotch, 1981a,b), since it requires that the Kripke-style evaluation clause for $\Box$-formulae is satisfied for at at least one relation in the set of relations of the model. In other words:

$$w \vDash \Box \phi \ \text{iff} \ \exists R_j \in \mathcal{R} : \forall w'(\text{if } wR_jw' \text{ then } w' \vDash \phi) \tag{1}$$

It is plain that this evaluation clause directly generalizes the one for Kripke semantics,[2] since it simply requires that the Kripke-style evaluation clause is satisfied for at least one relation in the set of relations in the model.[3]

The aim of this paper is to study quantification within this semantic setting. We consider frames with varying domains and formal languages with and without identity. Thus, we address research issues not yet explored in the literature, while keeping the intuition behind Kripke semantics although working with logics that are stronger than classical systems (E and above), yet strictly weaker than K. On the other hand, working with varying domains (i.e., with sets of individual existing in possible worlds that can vary from world to world) technically amounts to studying the most general case of quantified modal logics, which is

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[1] In Section 3 the reader can find two tables and a figure classifying the main systems and schemata.

[2] For this reason, throughout the paper we will use interchangeably the expressions "Kripke frame/model", "relational frame/model", and "1-relational frame/model".

[3] The *strong* version, on the other hand, is as follows (Governatori and Rotolo, 2005):

$$w \vDash \Box \phi \ \text{iff} \ \exists R_j \in \mathcal{R} : \forall w'(wR_jw' \text{ iff } w' \vDash \phi) \tag{2}$$

Things here are the same as in (1), except that ϕ is required to be true in w' *if and only if* w' is related with w by any relation R_j. The technical motivation of this second choice is that the resulting semantics is appropriate for any (non-normal) modal logics including the classical ones, i.e., the weakest ones that only consist of **RE**, $\vdash A \equiv B/\vdash \Box A \equiv \Box B$. On the contrary, the version with (1) validates, among others, **RM**, i.e., $\vdash A \to B/\vdash \Box A \to \Box B$, which opens the door to stronger logics below system **K**. However, multi-relational models based on the clause (2) make the semantics different from Kripke's, despite the structural similarity of how worlds are connected in frames.

mostly neglected in the literature on modal predicate logics weaker than K. Finally, the role of the identity symbol in QNML is still entirely overlooked: Indeed, identity plays a central technical role in ensuring completeness results.

Paper outline   The layout of the paper is as follows.

Section 2 introduces the state of the art and some results concerning quantified non-normal modal logics and neighborhood frames, as well as a first technical introduction to Barcan formulae and the problems related to such schemata.

Section 3 provides some preliminary material and useful auxiliary results from (Calardo and Rotolo, 2014);

Section 4 presents multi-relational first-order frames. We choose to analyse frames with varying domains, which correspond to the most general case and allow for offering a finer distinction between actual individuals and *possibilia*, namely, between the individuals that exist in each possible world and those that are only possible from that viewpoint, though not actually existing there (Garson, 2001; Fitting and Mendelsohn, 1998). Also, this choice technically amounts to studying the most general case of quantified modal logics.

Section 5 analyses alternative semantic characterisations for the Converse Barcan schema (**CBF**). After comparing our results to the standard ones in Kripke semantics, we shall see different ways to generalise the concept of *increasing inner domains*.

Section 6 provides Henkin-style completeness theorems for several systems, namely, the smallest *free* quantified non-normal **N**-monotonic logic $Q^\circ_=.NM$ and some extensions, including $Q^\circ_=.NM \oplus \textbf{CBF}$.

Section 7 offers a critical analysis and technical considerations on the Barcan schema **BF**. This schema raises some interesting questions in the context of non-normal modal logics and in multi-relational semantics.

Section 8 discusses the technical role of the identity relation. As we shall argue, the choice of a language with identity does not only increase the expressive power of the logic, but facilitates the construction of canonical models for **CBF**-frames, a potentially difficult goal to be achieved without it.

To the best of our knowledge, this is the first study on quantification in multi-relational semantics, the second one investigating the case of varying domains in non-normal modal logics, and the first that provides a frame characterization of Barcan schemata with varying domains with a language with identity.

2   Neighbourhood Semantics for QNML

Neighbourhood semantics—and the equivalent strong version of multi-relational semantics (cf. Governatori and Rotolo, 2005)—is in our opinion the best tool to deal with classical systems and above. For this reason, it has always played a central role within non-normal logics. Below we shall summarise the most influential results within this area. This approach will be useful to carry out a comparison between neighbourhood structures and multi-relational ones.

2.1   Syntax of Quantified Modal Logics

Let us extend the standard language $\mathcal{L}$ of propositional modal logic (Blackburn et al., 2001) with the universal quantifier $\forall$, a countable set of individual constants $\text{Const}:=\{a, b, c, \dots\}$, a set of individual variables, $\text{VAR}:= \{x, y, z, \dots\}$, the identity predicate $=$, and a set of n-ary predicate symbols (where $\omega > n \geq 1$). A *term* is either a variable, or an individual constant and $t_1, t_2, \dots$ are meta-variables for terms.

   Well formed formuale (wff) are defined as usual:

(a) $\perp$ is wff;

(b) If P^n is an n-ary predicate symbol and $t_1, \ldots, t_n$ are terms, then $P^n(t_1, \ldots, t_n)$ is a wff;

(c) If A and B are wff, then $A \to B$, $\Box A$, and $\forall x A$ are wff;

(d) Nothing else is a wff.

Both boolean operators, and the existential quantifier $\exists$ are defined as usual: $\exists x A \leftrightarrow \neg \forall x \neg A$. Moreover $A(t/s)$ is the formula obtained by replacing in $A(s)$ all the free occurrences of s with t (cf. Corsi, 2002, 1484).

2.2 Neighbourhood Models

Neighbourhood semantics for quantified modal logics was introduced a long time ago (see Gabbay, 1976). Nevertheless, it received very little attention until the beginning of the 1990s (cf. Waagbø, 1992). The study of QNML became then the subject of a few works (cf., for instance, Arló-Costa, 2002; Arló-Costa and Pacuit, 2006; Stolpe, 2003). While Waagbø (1992), Arló-Costa (2002), and Arló-Costa and Pacuit (2006) study the case of structures with constant domains, Stolpe (2003) developed a preliminary investigation of varying domains. In this section we summarise their main results.

Let us consider the case of constant domains.

Definition 2.1 (Constant domain neighbourhood frames and models) *A constant domain neighbourhood frame $\mathcal{F}$ is a structure $\langle W, \mathcal{N}, D \rangle$ where*

- W is a non-empty set of possible worlds;

- $\mathcal{N}$ is a function from w to 2^{2^W};

- D is a non-empty set of individuals (the domain of the frame).

For any $w \in W$, a w-assignment σ is a function $\sigma : Var(\mathcal{L}) \mapsto D$.
An x-variant τ of a w-assignment σ is a w-assignment which may differ from σ for the value assigned to x.

A constant domain neighbourhood model $\mathcal{M}$ is a structure $\langle W, \mathcal{N}, D, I \rangle$ where $\langle W, \mathcal{N}, D \rangle$ is a constant domain neighbourhood frame and I is an interpretation function such that, for any assignment σ and world w:

- $I_w^\sigma(x) \in D$ (global interpretation of variables/terms);

- $\forall w, v \in W, I_w^\sigma(x) = I_v^\sigma(x)$ (rigidity of variables/terms);

- $I_w^\sigma(P(x_1, \ldots, x_n)) \subseteq D^n$.

Notice that the notion of truth set has to take into account that the truth or falsity of open formulae depends on particular interpretations.

Definition 2.2 *Let $\mathcal{M}$ be a model with interpretation I, σ an assignment, w any world, and A any formula. The truth set of A with respect to to $\mathcal{M}$ and I^σ, $\|A\|_I^\sigma$ is thus defined:*[4]

$$\|A\|_I^\sigma := \{w \in W : \mathcal{M} \vDash_w^\sigma A\}.$$

The valuation conditions are as follows:

- $\mathcal{M} \vDash_w^\sigma P(x_1, \ldots, x_n)$ iff $\langle I_w^\sigma(x_1), \ldots, I_w^\sigma(x_n) \rangle \in I_w(P)$;

- Standard valuation conditions for negation and boolean connectives;

- $\mathcal{M} \vDash_w^\sigma \Box A$ iff $\|A\|_I^\sigma \in \mathcal{N}(w)$;

- $\mathcal{M} \vDash_w^\sigma \forall x A(x)$ iff, for every x-variant τ of σ, $\mathcal{M} \vDash_w^\tau A(x)$.

Let us consider the Barcan and the Converse Barcan schemata

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[4]When clear form the context, we also omit the reference to the model. The truth set of a closed formula does not depend on any interpretation and assignment.

BF:= $\forall x \Box A \to \Box \forall x A$ **CBF:=** $\Box \forall x A \to \forall x \Box A.$

The choice of constant domains (i.e., that the resulting modal logics are extensions of standard First Order Logic, FOL henceforth), does not correspond to the validity of **BF** and **CBF**:

Theorem 2.3 *(Arló-Costa and Pacuit, 2006) The class of all constant domain neighbourhood frames is sound and complete for* FOL $\oplus$ E.

BF and **CBF** were characterised in (Waagbø, 1992), but such results have been later made more precise in (Arló-Costa, 2002; Arló-Costa and Pacuit, 2006).

Let us consider the following frame properties (Arló-Costa, 2002; Arló-Costa and Pacuit, 2006):

Definition 2.4 (Frame properties) *A frame is* consistent *iff* $\forall w \in W : \mathcal{N}(w) \neq \varnothing$ *and* $\{\varnothing\} \notin \mathcal{N}(w)$.

A frame is closed under $\leq \kappa$ intersections *(where κ is a cardinal) iff*

$$\forall w \in W, \ \forall X = \{\mathcal{X}_i | i \in I\} \ \ where \ |I| \leq \kappa, \ \bigcap_{i \in I} \mathcal{X}_i \in \mathcal{N}(w).$$

A frame is trivial *iff* $|D| = 1$, *otherwise it is* non-trivial.
A frame is supplemented *iff* $\forall w \in W, X \cap Y \in \mathcal{N}(w) \Rightarrow X \in \mathcal{N}(w)$ *and* $Y \in \mathcal{N}(w)$.

Theorem 2.5 (Arló-Costa (2002)) **BF** *is valid in the class of frames that are either (i) trivial, or (ii) closed under finite intersection, if D is finite, or (iii) closed under $\leq \kappa$ intersections, if D is infinite and $|D| = \kappa$.*

Theorem 2.6 (Waagbø (1992); Arló-Costa and Pacuit (2006)) **CBF** *is valid in the class of frames that are either supplemented or trivial.*

Provided that frames are not trivial, Theorem 2.6 establishes a strong relationship between **M** and **CBF**, since supplementation characterises **M**. Hence, for constant non-trivial domain neighbourhood frames **CBF** is valid whenever **M** is. Also, under the same non-triviality assumption, since the closure under $\leq \kappa$ intersections implies the closure under intersection, it is not possible to falsify **C** when **BF** is valid. However, from the above theorem we can build a countermodel for **BF** given **C** (Waagbø, 1992), although this is possible only for infinite frames (Arló-Costa, 2002).

An open problem of this semantics is that the system FOL$\oplus$E$\oplus$**CBF** is strongly complete with respect to the class of frames that are either trivial, or supplemented: Arló-Costa (2011) conjectured that **M** is thus derivable by adding in the logic a schema expressing non-triviality but no result is available.

Let us now move to the case of varying domains, which was explored in QNML only by Stolpe (2003). The peculiarity of Stolpe (2003)'s analysis is that it works only with models and not with frames. Models are standardly defined as follows:

Definition 2.7 (Varying domain neighbourhood models) *A* varying domain neighbourhood frame $\mathcal{M}$ *is a structure* $\langle W, \mathcal{N}, D, \Sigma, I \rangle$ *where*

- *W, $\mathcal{N}$, D, and I are like in Definition 2.1 and*

- *Σ is a function assigning to each world $w \in W$ a set D_w of elements of D.*

The valuation condition for $\forall$-formulae is now as follows:

$$\mathcal{M} \vDash_w^\sigma \forall x A(x) \text{ iff } \mathcal{M} \vDash_w^\tau A(x) \text{ for every } x\text{-variant } \tau \text{ of } \sigma \text{ such that } \tau(x) \in D_w.$$

Stolpe (2003) defines two classes of varying neighbourhood models that characterize **BF** and **CBF**:

Theorem 2.8 (BF and CUPI models) *A varying neighbourhood model $\mathcal{M}$ = $\langle W, \mathcal{N}, D, I, \Sigma \rangle$ is a CUPI model iff for any world $w \in W$, if $\|P(x)\|_I^\sigma \in \mathcal{N}(w)$ for every σ such that $\sigma(x) \in D_w$, then $\|\forall x P(x)\|_I \in \mathcal{N}(w)$.*

*   **BF** is valid in the class of CUPI models.*

Theorem 2.9 (CBF and CUPO models) *A varying neighbourhood model $\mathcal{M}$ = $\langle W, \mathcal{N}, D, I, \Sigma \rangle$ is a CUPO model iff for any world $w \in W$, if $\|\forall x P(x)\|_I \in \mathcal{N}(w)$, then $\|P(x)\|_I^\sigma \in \mathcal{N}(w)$ for every σ such that $\sigma(x) \in D_w$.*

*   **CBF** is valid in the class of CUPO models.*

CUPI and CUPO models impose properties that trivially reflect the evaluation of **BF** and **CBF**. The main limit of this approach is that it does not appeal to frames. Thus, Arló-Costa (2011) rightly argues that Stolpe (2003) leaves open many questions, including the general characterization of **BF** and **CBF**. In this paper, we address this very problem in quantified **N**-Monotonic logics with varying domains.

3   Preliminaries: Multi-relational Semantics for the Propositional Case

In this section, we present some basic notions and results by Calardo and Rotolo (2014) for non-normal propositional modal logics and multi-relational semantics. The alphabet of the language includes as usual a countable set of propositional letters, modal and Boolean operators.

Inference Rules:

$\quad$ **RE**:= $\vdash A \leftrightarrow B \;\Rightarrow\; \vdash \Box A \leftrightarrow \Box B$

$\quad$ **RM**:= $\vdash A \rightarrow B \;\Rightarrow\; \vdash \Box A \rightarrow \Box B$

$\quad$ **RN**:= $\vdash A \;\Rightarrow\; \vdash \Box A$

$\quad$ **RR**:= $\vdash A \wedge B \rightarrow C \;\Rightarrow\; \vdash \Box A \wedge \Box B \rightarrow \Box C$

$\quad$ **RK**:= $\vdash A_1 \wedge \ldots \wedge A_n \rightarrow B \;\Rightarrow\; \vdash \Box A_1 \wedge \ldots \wedge \Box A_n \rightarrow \Box B \;\; n \geq 0$

Axiom Schemata:

$\quad$ **EFQ**:= $A \wedge \neg A \rightarrow B$

$\quad$ **M**:= $\Box(A \wedge B) \rightarrow (\Box A \wedge \Box B)$

$\quad$ **C**:= $(\Box A \wedge \Box B) \rightarrow \Box(A \wedge B)$

$\quad$ **K**:= $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$

$\quad$ **N**:= $\Box \top$

$\quad$ **CON**:= $\neg \Box \bot$

$\quad$ **D**:= $\Box A \rightarrow \neg \Box \neg A$

$\quad$ **T**:= $\Box A \rightarrow A$

$\quad$ **4**:= $\Box A \rightarrow \Box \Box A$

$\quad$ **B**:= $A \rightarrow \Box \Diamond A$

$\quad$ **DEX**:= $\Box A \wedge \Box \neg A \rightarrow \Box B$.

There are different systems of propositional modal logics built to model various situations. In the following table we list some simple systems which may be considered as a *base* for more complex systems (for further details, see Chellas, 1980).

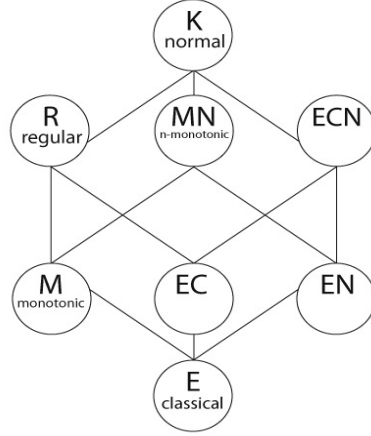|		Rules	Axioms
E	classical	**RE**	
M	monotonic	**RM**	E $\oplus$ **M**
MN	**N**-monotonic	**RM** $\oplus$ **RN**	E $\oplus$ **M** $\oplus$ **N**
R	regular	**RR**	E $\oplus$ **M** $\oplus$ **C**
K	normal	**RK**	E $\oplus$ **K** $\oplus$ **N**
			E $\oplus$ **M** $\oplus$ **C** $\oplus$ **N**

Figure 1: The lattice of non-normal propositional systems (cf. Chellas, 1980, 237).

The lattice depicted in Figure 1 illustrates the inclusion relations in non-normal systems.

In this paper we shall work with predicate extensions of **N**-monotonic modal ones. For this class of logics, multi-relational semantics is as follows.

Definition 3.1 *A multi-relational frame is a structure $\mathcal{F} := \langle W, \mathcal{R} \rangle$ where W is a non empty set and $\mathcal{R}$ is a set at most countable of accessibility relations.*

As usual, a model is obtained by adding an evaluation function V to any frame.

Truth conditions. The truth conditions for all Boolean operations are standard. Given a multi-relational a model $\mathcal{M} := \langle \mathcal{F}, V \rangle$ and a world w from W, the clauses to evaluate modal formulae are a direct generalisation of the standard Kripke approach: $\vDash_w^V \Box A$ if and only if there exists an R_i in $\mathcal{R}$ such that for all $v \in W$ $(wR_i v \Rightarrow \vDash_v^V A)$.

Notation and abbreviations. Given a relation R_i and a world w, by the symbol $R_i(w)$ we refer to the set of all the worlds R_i-accessible from w, i.e.: $R_i(w) := \{x : wR_i x\}$.
Given a model $\mathcal{M} := \langle W, \mathcal{R}, V \rangle$ and a formula A, we define the *truth set* of A, in symbols $\|A\|_V$, as the set of all the worlds of the model in which A is true, i.e.: $\|A\|_V := \{w : \vDash_w^V A\}$.

Finally, let us mentions a few relevant results from Calardo and Rotolo (2014) that work on the weak formulation of multi-relational semantics based on the clause (1).

Lemma 3.2 (Schema N) *The schema $\mathbf{N} := \Box\top$ is valid in the class of all multi-relational frames.*

Lemma 3.3 (Frame characterisation of C) *For any multi-relational frame $\mathcal{F}$ the following holds: $\mathcal{F} \vDash \Box A \wedge \Box B \rightarrow \Box(A \wedge B)$ iff for any world w, for any relation R_i, R_k there exists a relation R_j such that $R_j(w) \subseteq R_k(w) \cap R_i(w)$.*

Theorem 3.4 (N-Monotonic logics - Soundness) *Let $\mathsf{MN}_\vdash := \{A \mid \mathsf{CPC} \oplus \mathbf{RE} \oplus \mathbf{M} \oplus \mathbf{N} \vdash A\}$—where CPC stands for Classical Propositional Calculus—and $\mathsf{MN}_\vDash := \{A \mid \vDash A\}$. Then $\mathsf{MN}_\vdash \subseteq \mathsf{MN}_\vDash$.*

Theorem 3.5 (Completeness of MN) *The logic MN is complete with respect to the class of multi-relational frames.*

4 Quantification in N-Monotonic Modal Logics

We have seen that a few works offer a semantic analysis of quantified non-normal modal systems using neighbourhood semantics. However, as far as we are concerned, nothing has been done so far regarding multi-relational semantics in a first order modal framework.

Let us define multi-relational structures for any quantified modal logic.

Definition 4.1 (Multi-relational frames) *A multi-relational frame is a tuple $\mathcal{F} :=$ $\langle W, \mathcal{R}, D, U \rangle$ where:*
- *W is a non empty set of worlds*
- *$\mathcal{R}$ is a (possibly infinite) set of binary relations over W*
- *D is a function associating to each world $w \in W$ a set D_w of individuals (the inner domain of w)*
- *U is a function associating to each world $w \in W$ a set U_w of individuals (the outer domain of w) such that for any $w \in W$, $U_w \neq \varnothing$ and $D_w \subseteq U_w$ and if wRv for some R, then $U_w \subseteq U_v$.*

The original definition given by Kripke (1963) states that for all worlds w, $U_w = \bigcup_{v \in W} D_v$, setting a unique outer domain for the whole frame. However, we decided to follow the more general approach proposed by Corsi (2002):

> The fact that $U_w \subseteq U_v$, if wRv, does not prevent D_w from being disjoint from D_v. Kripke (1963) stipulates that for all $v \in W$, $U_v = \bigcup_{w \in W} D_w$. We generalise Kripke's original semantics by allowing $U_w \subseteq U_v$, if wRv, and $\bigcup_{w \in W} U_w \supseteq \bigcup_{w \in W} D_w$. $\bigcup_{w \in W} U_w$ may contain individuals that never happen to come into existence. (Corsi, 2002, 1485)

Models, assignments, and the concepts of *satisfaction, truth, validity* are defined in the standard way.

Definition 4.2 (Multi-relational models) *A multi-relational model is a tuple $\mathcal{M} :=$ $\langle W, \mathcal{R}, D, U, I \rangle$ where $\langle W, \mathcal{R}, D, U \rangle$ is a multi-relational frame and I is a function $I : \mathcal{L} \times W \mapsto \bigcup_{w \in W} U_w$ such that:*
- *$I_w(P^n) \subseteq (U_w)^n$*
- *$I_w(c) \in U_w$*
- *$I_w(=) = \{\langle d, d \rangle : d \in U_w\}$.*

Definition 4.3 (Assignments) *For any $w \in W$, a w-assignment σ is a function $\sigma : Var(\mathcal{L}) \mapsto U_w$.*
An x-variant τ of a w-assignment σ is a w-assignment which may differ from σ for the value assigned to x.

Notice that within the semantics framework proposed by Kripke (1963), since the outer domains are constant, any w-assignment σ is also a v-assignment for any couple of worlds. However, it should be noticed that, here, the fact that $U_w \subseteq U_v$, if wRv for some R, still guarantees that if two worlds w, v are related by some R, then any w-assignment is also a v-assignment, as all the variables of the language are still mapped on individuals without gaps.

Definition 4.4 (σ-interpretation) *Given a w-assignment σ, define*

(a) *$I_w^\sigma(c) = I_w(c)$, and*

(b) *$I_w^\sigma(x) = \sigma(x)$.*

Definition 4.5 (Truth conditions) *Let $\mathcal{M} := \langle W, \mathcal{R}, D, U, I \rangle$ be any multi-relational model, σ any assignment, and $w \in W$. Truth evaluation clauses are as follows:*
- *$\mathcal{M} \vDash_w^\sigma P^n(t_1, \ldots, t_n)$ iff $\langle I_w^\sigma(t_1), \ldots, I_w^\sigma(t_n) \rangle \in I_w(P^n)$*
- *$\mathcal{M} \nvDash_w^\sigma \bot$*
- *$\mathcal{M} \vDash_w^\sigma \forall x A$ iff for every x-variant τ of σ such that $\tau(x) \in D_w$, $\mathcal{M} \vDash_w^\tau A(x)$*

- $\mathcal{M} \vDash^\sigma_w \Box A$ *iff* $\exists R_i \in \mathcal{R} \; \forall v \in W (wR_i v \Rightarrow \mathcal{M} \vDash^\sigma_v A)$.

Clearly, since the operators $\exists$ and $\Diamond$ are defined ones, the evaluation clauses are simply:
$\mathcal{M} \vDash^\sigma_w \exists x A$ iff for some x-variant τ of σ such that $\tau(x) \in D_w$ $\mathcal{M} \vDash^\tau_w A(x)$ and
$\mathcal{M} \vDash^\sigma_w \Diamond A$ iff $\forall R_i \in \mathcal{R}$ there is a $v \in W (wR_i v \& \mathcal{M} \vDash^\sigma_v A)$.

Satisfaction, Truth, Validity. A model $\mathcal{M}$ *satisfies* a set of formulae Δ iff for some world w and some w-assignment σ, $\mathcal{M} \vDash^\sigma_w A$ for all $A \in \Delta$. A formula A is *true in a world w* of a model $\mathcal{M}$, $\mathcal{M} \vDash_w A$, iff for any w-assignment σ, $\mathcal{M} \vDash^\sigma_w A$. A formula A is *true in a model $\mathcal{M}$*, $\mathcal{M} \vDash A$, iff for all w, $\mathcal{M} \vDash_w A$. A formula A is *valid* on a frame $\mathcal{F}$, $\mathcal{F} \vDash A$, iff for any model $\mathcal{M}$ on $\mathcal{F}$, $\mathcal{M} \vDash A$. Given a class of frames $\mathbb{F}$, a formula A is $\mathbb{F}$-*valid*, $\mathbb{F} \vDash A$, iff for any frame $\mathcal{F} \in \mathbb{F}$, $\mathcal{F} \vDash A$. $\mathcal{M}$ is a model for a logic L iff $\mathcal{M} \vDash A$ for all $A \in \mathsf{L}$.

Abbreviations. Given a frame $\mathcal{F}$ and a model $\mathcal{M}$ on $\mathcal{F}$, for any formula A,
Satisfaction: $\|A\|^\sigma_I := \{w \mid \mathcal{M} \vDash^\sigma_w A\}$
Truth: $\|A\|_I := \{w \mid \mathcal{M} \vDash^\sigma_w A, \text{ for any assignment } \sigma\}$
Validity: $\|A\| := \{w \mid \mathcal{F} \vDash^\sigma_w A \text{ for any assignment } \sigma \text{ and any interpretation } I\}$.

Definition 4.6 *Given a multi-relational model $\mathcal{M} = \langle W, \mathcal{R}, D, U, I \rangle$, an individual constant c is said to be a* rigid designator *iff for any w, v in W, if there is a relation R_i such that $(wR_i v)$, then $I_w(c) = I_v(c)$.*

Lemma 4.7 *Given a multi-relational model $\mathcal{M} = \langle W, \mathcal{R}, D, U, I \rangle$ and a w-assignment σ, if an individual constant c is a rigid designator, then $\vDash^\sigma_w A(c/x)$ iff $\vDash^\tau_w A(x)$ for any w-assignment τ which is an x-variant of σ such that $\tau(x) = I_w(c)$. (Cf. (Corsi, 2002, Lemma 1.1).)*

PROOF. The proof is given by induction on the length of a formula A. We shall provide a proof only for the modal case, since the others are quite standard. Suppose A has the form $\Box B(x)$. If $\vDash^\sigma_w \Box B(c/x)$ then there is a relation R_i such that for any world v, if $wR_i v$, then $\vDash^\sigma_v B(c/x)$ and by induction hypothesis we have $\vDash^\tau_v B(x)$ where τ is an x-variant of σ such that $\tau(x) = I_v(c)$. Since c is a rigid designator by hypothesis, $I_v(c) = I_w(c)$, τ is a w-assignment and hence $\vDash^\tau_w \Box B(x)$.

If $\vDash^\tau_w \Box B(x)$ then there is a relation R_i such that for any world v, if $wR_i v$, then $\vDash^\tau_v B(x)$ and by induction hypothesis we have $\vDash^\sigma_v B(c/x)$ where τ is an x-variant of σ such that $\tau(x) = I_v(c)$. Since c is a rigid designator by hypothesis, $I_v(c) = I_w(c)$, σ is a w-assignment and hence $\vDash^\sigma_w \Box B(c/x)$. ∎

We assume all individual constants to be rigid designators.

Lemma 4.8 *Given a multi-relational frame $\mathcal{F}$, a model $\mathcal{M} := \langle W, \mathcal{R}, D, U, I \rangle$ on it and a world w, if σ and τ are two w-assignments which coincide on any free variable occurring in a formula A, then it holds that $\mathcal{M} \vDash^\sigma_w A$ iff $\mathcal{M} \vDash^\tau_w A$.*

PROOF. The proof is given by induction on the length of a formula A. Suppose A has the form $P^n(t_1, \ldots, t_n)$, then $\vDash^\sigma_w P^n(t_1, \ldots, t_n)$ if and only if $\langle I^\sigma_w(t_1), \ldots, I^\sigma_w(t_n) \rangle \in I_w(P^n)$. If t_i is an individual constant c, then $I^\sigma_w(c) = I_w(c) = I^\tau_w(c)$. Otherwise, if t_i is a variable x, $I^\sigma_w(x) = \sigma(x) = \tau(x)$ by hypothesis, hence $I^\sigma_w(x) = I^\tau_w(x)$. The other steps are straightforward. ∎

5 The Converse Barcan and the Ghilardi schemata

(Stolpe, 2003) is certainly one of the most significant studies of quantified non-normal modal logics. Stolpe's aim is to find out "[...] what semantical restrictions must be imposed on a minimal *model* in order to validate the Barcan and the converse Barcan formulae" (Stolpe, 2003, 559, emphasis added). However, as Arló Costa pointed out:

[...] unfortunately Stolpe does not appeal to frames in his semantics (he only uses models). So, it is obvious that there are many open questions not considered in Stolpe's paper. For example it would be nice to get frame conditions characterizing the Barcan and the Converse Barcan in this setting. (Arló-Costa, 2011, 21)

Our aim is to attack this research question in the multi-relational setting. In particular, in this section we shall propose frame conditions for **CBF** in multi-relational semantics.

Standard results in Kripke Semantics (1-relational frames) state that both **BF** and **CBF** are valid in constant domain frames (i.e., when D_w and U_w coincide for each w). Within first-order Kripke frames with varying domains, however, these schemata cease to be valid. Let us recall below the proof of this well known result.

Lemma 5.1 CBF *is not valid in the class of multi-relational frames with varying inner domains.*

PROOF. The proof is trivial. Let $\mathcal{M} = \langle W, \mathcal{R}, D, U, I \rangle$ be a 1-relational model such that: $W := \{w, v\}$, $\mathcal{R} := \{R\}$, $R := \{\langle w, v \rangle\}$, $D_w := \{a, b\}$, $D_v := \{a\}$, $U_w = U_v = \{a, b\}$, $I_w^\sigma(P^1) = \varnothing$ for all σ and $I_v^\sigma(P^1) = D_v$, for $\sigma(x) = a$ (see Figure 2). Then $\|\forall x(P)\|_I^\sigma = D_v$ and there is a relation, i.e., R, such that for any world t, wRt (if and) only if $\vDash_t^\sigma \forall x P$. Hence $\vDash_w^\sigma \square \forall x P$. Moreover $\vDash_w^\sigma \exists x \Diamond \neg P$, as $\vDash_v^\tau \neg P(x)$ for $\tau(x) = b$. ∎
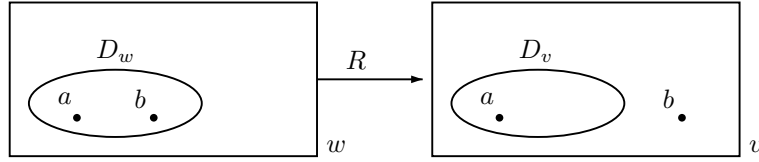

Figure 2: A frame to build a countermodel for **CBF**. Indeed set $I_w^\sigma(P^1) = \varnothing$ for all σ and $I_v^\sigma(P^1) = D_v$, for $\sigma(x) = a$. Hence $v \vDash_v^\sigma \forall x A$.

Let us first consider the definition of *local seriality*, which us needed to study Barcan schemata:

Definition 5.2 (Local Seriality) *Given a multi-relational frame $\mathcal{F}$, a world w is* locally serial *iff for any relation R_i from $\mathcal{F}$ there is a world v such that wR_iv.*

In what follows we shall present two different characterisation results connecting **CBF** to the concept of *increasing domains*. The first one (Lemma 5.3) is certainly the less intuitive, but it is needed to provide a further, more appealing, characterisation (Lemma 5.11). In fact, the lemma below provides necessary and sufficient conditions to build counter-models for **CBF**. After this preliminary result, we shall discuss the semantic role of **CBF** in further detail.

Lemma 5.3 (CBF Characterisation result) *For any multi-relational frame $\mathcal{F} := \langle W, \mathcal{R}, D, U \rangle$, $\mathcal{F} \vDash \square \forall x A \to \forall x \square A$ iff $\forall w \in W$, if w is locally serial, then for any a such that $a \in D_w$, for any relation R_j, there is some relation R_i such that $R_i(w) \subseteq R_j(w)$ and for all t, $(wR_it \Rightarrow a \in D_t)$.*

PROOF. According to the statement of the Lemma above, any counter-model for **CBF** should be based on a frame $\mathcal{F}$ with the following three conditions: There is a world $w \in W$, and a w-assignment σ such that:

(a) BOTH w is locally serial, i.e., for all k, $R_k(w) \neq \varnothing$

(b) AND there exists a w-assignment τ, which is an x-variant of σ such that $\tau(x) \in D_w$, there is a relation R_j

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b.1 BOTH $\exists t(wR_j t \And \tau(x) \notin D_t)$, i.e., the inner domains are not increasing

b.2 AND for all the other relations R_i, IF $R_i(w) \subseteq R_j(w)$, THEN there exists some world t_i such that $(wR_i t_i \And \tau(x) \notin D_{t_i})$.

We will now show that these conditions are necessary and sufficient to build a countermodel for **CBF**. The idea is to provide an interpretation to make the class $R_j(w)$ to be exactly the truth set of $\forall x A$ for some formula A, so that the antecedent of **CBF** holds. Then, we need an individual which *actually exists* in w and satisfies $\neg A(x)$ in some R_k-accessible state for all k; this would guarantee that the consequent of **CBF** is false.

Indeed, let P be some unary predicate and let I be an interpretation defined in the following way: For any ϑ and any world z, $z \in \|P(x)\|_\vartheta^I$ if and only if $wR_j z$ and ϑ is an x-variant of τ such that $\vartheta(x) \in D_z$. Thus, $R_j(w) = \|\forall x P(x)\|_\tau^I$ and hence $\vDash_w^\tau \square \forall x P$.

Consider now any world z such that $z \notin R_j(w)$; We defined I in such a way that for any z-assignment ϑ, $\nvDash_z^\vartheta P(x)$ and thus this holds also for τ, $\nvDash_z^\tau P(x)$. Moreover our frame conditions guarantee that for any relation R_i, if $R_i(w) \subseteq R_j(w)$, then there is some $z \in R_i(w)$ such that $\tau(x) \notin D_z$ and, following our construction of I, $\nvDash_z^\tau P(x)$. Since $\tau(x) \in D_w$ by definition, it holds that $\vDash_w^\tau \exists x \lozenge \neg P(x)$, thus falsifying **CBF**.

For the opposite direction, it is easy to see that if **CBF** does not hold on a model, its frame fulfills the conditions stated above. Assume that for some frame $\mathcal{F}$, $\mathcal{F} \nVdash$ **CBF**. Then there are an assignment σ, a valuation I and a world w such that (a) $\vDash_w^\sigma \square \forall x A$ and (b) $\vDash_w^\sigma \exists x \lozenge \neg A$ for some formula A. From (b) it follows that $\vDash_w^\tau \lozenge \neg A(x)$ for some $\tau(x) \in D_w$, thus there exists some set $\{z_1, z_2 \ldots\}$ such that $wR_k z_k$ for any $R_k \in \mathcal{R}$ and $\nvDash_{z_k}^\tau A$. From (a) it follows that for some relation R_i, $R_i(w) \subseteq \|\forall x A\|_I^\sigma$, where $\|\forall x A\|_I^\sigma := \{t \mid \vDash_t^\tau A(x)$ for any τ such that $\tau(x) \in D_t\}$. Now, consider any relation R_k; if $R_k(w) \subseteq R_i(w)$, then by (b), $R_k \subseteq \|\forall x A\|_I^\sigma$ and by (a) there is some $z \in R_k(w)$ such that $z \notin \|A(x)\|_I^\tau$ for some $\tau(x) \in D_w$, thus we must conclude that $\tau(x) \notin D_z$. ∎

This lemma leads to a result that is very close to that for Kripke semantics, as we shall see in what follows.

Some remarks on CBF. An immediate, yet interesting result is that the schemata **M** and **CBF** are independent, in contrast with Theorem 2.6, which establishes that, for *constant* non-trivial domain neighbourhood frames **CBF** is valid whenever **M** is. Indeed by the following lemma, it holds that **M** is a valid schema, whereas it is possible to build a countermodel for **CBF** (see Lemma 5.1).

Lemma 5.4 *The schema* **M** *is valid in the class of all multi-relational frames.*

It is hence possible to derive the following:

Corollary 5.5 *The validity of* **M** *does not imply that of* **CBF**.

Lemma 5.3 offers some interesting insights about the class of **CBF**-frames. First of all, it states an important property about individuals and their behaviour with respect to different accessibility relations. It is well known that **CBF** imposes *increasing inner domains* on Kripke frames:

Definition 5.6 (Increasing inner domains - Kripke semantics) *A relational frame* $\mathcal{F}$ *has* increasing inner domains *iff for all worlds* w, v, *if* wRv *then* $D_w \subseteq D_v$.

Theorem 5.7 (CBF characterisation in Kripke frames) CBF *characterises the class of Kripke frames with increasing inner domains.*

Consider any Kripke frame $\mathcal{F} := \langle W, R, D, U \rangle$ (which is nothing but an 1-relational frame with varying domains). From the condition stated above in Lemma 5.3 we have that $\mathcal{F} \vDash \square \forall x A \to \forall x \square A$ iff $\forall w \in W$, for any world z, if wRz then for any w-assignment σ such that $\sigma(x) \in D_w$, $\sigma(x) \in D_z$, i.e., $D_w \subseteq D_z$. Thus the conditions imposed by **CBF** are the usual ones: **CBF** is valid in those frames whose connected worlds have increasing inner domains.

The situation within **N**-monotonic logics is rather different, yet connected. The first question that comes to mind is how to generalise the concept of *increasing inner domains*. There are a couple of alternatives at hand, which look very close to the definition given in Kripke semantics. We might either define increasing domains in a very strong sense, by asking that *all* relations enjoy such property:

Definition 5.8 (General increasing inner domains) *A multi-relational frame $\mathcal{F}$ has increasing inner domains iff for any couple of worlds w, v for* any *relation R_i, if wR_iv then $D_w \subseteq D_v$.*

Alternatively, we might keep such property lighter, asking for only one relation to fulfill it:

Definition 5.9 (Restricted increasing inner domains) *A multi-relational frame $\mathcal{F}$ has increasing inner domains iff for any couple of worlds w, v there is at least one relation R_i, such that if wR_iv then $D_w \subseteq D_v$.*

Generally, in the multi-relational semantics scenario, **CBF** does not capture any of these properties. Indeed Lemma 5.3 states that if a world w is locally serial, i.e., if there is access to other worlds under *any* relation, then each *actual* individual (i.e., any individual belonging to the inner domain of w) keeps being actual in a subset of every relation. So it says, somehow, that individuals are bound to "survive" under alternative relations, although not necessarily altogether. If Mary and Alex are both alive now (actual) the presence of **CBF** guarantees that for any relation, there is a subset of alternative possible worlds in which Mary keeps being actual, and another one in which Alex is still actual.

Logically, if we consider the big union of all the inner domains connected to w by some relation i, the inner domain of w is a subset of it, i.e.:

Lemma 5.10 *If the schema **CBF** is valid on a given frame, then on any world w, if w is locally serial, i.e., for any i, $R_i(w) \neq \varnothing$, then for any relation R_i*

$$D_w \subseteq \bigcup \{D_v \mid v \in R_i(w)\}.$$

PROOF. Assume that the property stated in this Lemma does not hold. Then there are a locally serial world w and a relation R_i such that for some individual $d \in D_w$, for all $v \in R_i(w)$, $d \notin D_v$, i.e., $D_w \nsubseteq \bigcup \{D_v \mid v \in R_i(w)\}$. Consider an assignment σ on any world belonging to $R_i(w)$ such that $\sigma(x) = d$, then for all $v \in R_i(w)$, $\vDash^\sigma_v \neg \exists x (x = d)$ and then $\vDash^\sigma_w \Box \forall x (x \neq d)$. By **CBF** we get $\vDash^\sigma_w \forall x \Box (x \neq d)$, i.e., for any w-assignment τ such that $\tau(x) \in D_w$, $\vDash^\tau_w \Box (x \neq d)$. Since $d \in D_w$, this holds true for σ as well, where $\sigma(x) = d$. Hence we get that the formula $\Box (d \neq d)$ holds true at w. This would impose that for some j, $R_j(w) = \varnothing$ which is contradictory with the assumptions. ∎

Hence, within the same set of alternative possible worlds, there is one world in which Mary lives and another in which Alex lives, although, again, they are not bound to necessarily coincide.

A natural question is then how to force individuals to survive altogether under some relation. Well, intuitively, at first sight, it would be enough to add partial closure under intersection, i.e., the semantic property stated in Section 3 (Lemma 3.3) and characterised by schema $\mathbf{C} := \Box A \wedge \Box B \to \Box (A \wedge B)$. However, this would generate a normal system (see the lattice in Figure 1). Thus, in the presence of Axiom **C**, the situation becomes closer to the Kripkean case. This said, Definition 5.9 seems to capture the concept of "increasing domains" better than the stronger alternative proposed. However, it turns out that the presence of **C** is not required to achieve the formerly stated property, which is actually (partially) granted by the presence of **CBF** alone:

Theorem 5.11 (CBF and Restricted Increasing Domains) *For any multi-relational frame $\mathcal{F} := \langle W, \mathcal{R}, D, U \rangle$, $\mathcal{F} \vDash \Box \forall x A \to \forall x \Box A$ iff $\forall w \in W$, if w is locally serial, then for all R_j, there is some relation R_i such that $R_i(w) \subseteq R_j(w)$ and for all worlds v in $R_i(w)$, $D_w \subseteq D_v$.*

PROOF. This is very straightforward. In fact, consider the formulation of Lemma 5.3, which states that if w is locally serial, then for any c such that $c \in D_w$, for any relation R_j, there is some relation R_i such that $R_i(w) \subseteq R_j(w)$ and for all t, $(wR_it \Rightarrow c \in D_t)$. Suppose the inner domain of w is the set $\{a, b\}$. Then, for any R_j there must be an R_i such that $R_i(w) \subseteq R_j(w)$ and a belongs to the inner domain of *all* the worlds from $R_i(w)$. However, since the statement of Lemma 5.3 refers to *all* relations and *all individuals*, we must consider also R_i and b. Hence there is an R_m such that $R_m(w) \subseteq R_i(w)$ and b belongs to the inner domain of any world from $R_m(w)$. Thus, all the worlds from $R_m(w)$ have an inner domain containing both a, and b and their inner domain is, therefore, a superset of D_w.

For the opposite direction, suppose by reductio that the property stated in the Lemma holds, whereas **CBF** does not. Let us recall the construction of a counter model for **CBF** as explained in Lemma 5.3. If **CBF** does not hold, there are an assignment σ, a valuation I and a world w such that (a) $\vDash^\sigma_w \Box\forall xA$ and (b) $\vDash^\sigma_w \exists x \Diamond \neg A$ for some formula A. From (b) it follows that $\vDash^\tau_w \Diamond\neg A(x)$ for some $\tau(x) \in D_w$, thus for any $R_k \in \mathcal{R}$, there is a world z_k such that wR_kz_k and $\nvDash^\tau_{z_k} A$. From (a) it follows that for some relation R_i, $R_i(w) \subseteq \|\forall xA\|^\sigma_I$, where $\|\forall xA\|^\sigma_I := \{t \mid \vDash^\tau_t A(x)$ for any τ such that $\tau(x) \in D_t\}$. Now consider the relation $R_i(w)$. The statement of this Lemma says that (c) there must be a relation R_j such that $R_j(w) \subseteq R_i(w)$ and for all worlds v in $R_i(w)$, $D_w \subseteq D_v$. Moreover, by (a), since $R_j(w) \subseteq R_i(w)$, for any $v \in R_j(w)$ it holds that $\vDash^\sigma_v \forall xA$. However by (b) for some $z \in R_j(w)$, $\nvDash^\tau_z A(x)$ and $\tau(x) \in D_w$. But by (c) $D_w \subseteq D_z$, hence we reach a contradiction. $\blacksquare$

By all means, these results are much more specific than those proposed by Stolpe (2003), as nothing is known about inner domains of related worlds. In fact, by translating in terms of multi-relational semantics Stolpe (2003)'s CUPO condition (see Theorem 2.9) it is not hard to prove a weaker correspondence result for **CBF** in multi-relational models.

Lemma 5.12 (CUPO models) *For any multi-relational model $\mathcal{M} := \langle W, \mathcal{R}, D, U, I \rangle$, $\mathcal{M} \vDash \Box\forall xA \to \forall x \Box A$ if and only if for any world w the following holds: given a w-assignment σ, if there is a relation R_i such that $R_i(w) \subseteq \|\forall xA\|^\sigma_I$, then for all x-variant τ of σ such that $\tau(x) \in D_w$ there is some j such that $R_j(w) \subseteq \|A(x)\|^\tau_I$.*

PROOF. For the left arrow, suppose there are a model $\mathcal{M} := \langle W, \mathcal{R}, V \rangle$ and a σ such that $\vDash^\sigma_w \forall xA$. Hence, for some i, $R_i(w) \subseteq \|\forall xA\|^\sigma_w$. By hypothesis for all x-variant τ of σ such that $\tau(x) \in D_w$ there is some j such that $R_j(w) \subseteq \|A(x)\|^\tau_I$, thus ensuring $\vDash^\tau_w \Box A(x)$ and hence $\vDash^\sigma_w \forall x \Box A(x)$. The other version can be proved accordingly by contraposition. $\blacksquare$

The reader may easily notice that Stolpe's result does not say anything regarding the inner structure of frames, but it is strictly related to specific interpretations. On the other hand, a frame characterisation result is based on structural properties on domains of individuals (here also interplaying with relations), as analysed in the literature on quantified modal logics, where **CBF** is classically associated with the condition of increasing domains of individuals (Garson, 2001; Fitting and Mendelsohn, 1998; Corsi, 2002; Garson, 2005; Brauner and Ghilardi, 2007; Gabbay et al., 2009; Goldblatt, 2011).

Mirroring Kripke semantics, **CBF** and the Ghilardi schema $\exists x \Box A \to \Box\exists xA$ (**GF**) are semantically equivalent:

Lemma 5.13 **CBF** *and* **GF** *are semantically equivalent*

PROOF. To show this result it is enough to check that **GF** characterises the class of **CBF**-frames.

($\Rightarrow$) According to the statement of Lemma 5.3, any counter-model for **CBF** must be based on a frame $\mathcal{F}$ with the following three conditions:

there is a world $w \in W$, and a w-assignment σ such that:

(a) BOTH w is locally serial, i.e., for all k, $R_k(w) \neq \varnothing$

(b) AND there exists an individual a, $a \in D_w$, and there is a relation R_j

(b.1) BOTH $\exists t(wR_jt \ \& \ a \notin D_t)$, i.e., the inner domains are not increasing

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(b.2) AND for all the other relations R_i, IF $R_i(w) \subseteq R_j(w)$, THEN there exists some world t_i such that $(wR_it_i \ \& \ a \notin D_{t_i})$.

Let us sketch a valuation to build a counter-model for **GF**. Let A be a unary predicate, take any w-assignment τ and set I as follows:

(a) IF $\tau(x) \neq a$, then $\|A(x)\|_I^\tau = W$;

(b) OTHERWISE if $\tau(x) = a$, then $\|A(x)\|_I^\tau = W - R_i(w)$.

From (b) it follows that $R_i(w) \subseteq \|\neg A(x)\|_I^\tau$, hence $\vDash_w^\sigma \exists x \, \square \, \neg A(x)$.

Let us turn our attention to (a). Any world v which is not R_i-seen by w is such that for any assignment τ, $\vDash_v^\tau A(x)$ and hence $\vDash_v^\sigma \forall x A(x)$ for all σ. Concerning R_i the situation is the following: the fact that $\exists t(wR_jt \ \& \ a \notin D_t)$ guarantees that $\vDash_t^\sigma \forall x A(x)$, whereas the fact that for all the other relations R_i, IF $R_i(w) \subseteq R_j(w)$, THEN there exists some world t_i such that $(wR_it_i \ \& \ a \notin D_{t_i})$, guarantees the fact that there is always a world z in $R_j(w)$ such that $\vDash_z^\sigma \forall x A(x)$. This observation, together with the fact that w is locally serial by assumption, makes sure that $\vDash_w^\sigma \lozenge \forall x A(x)$, thus disproving an instance of **GF**. ∎

6 Completeness Results[5]

6.1 The System $\mathsf{Q}_=^\circ.\mathsf{MN}$

Here we present an axiomatic system extending MN with predicate logic, which is based on free quantified modal logic (see Corsi, 2002, 1498). The system $\mathsf{Q}_=^\circ.\mathsf{MN}$ (Free Quantified **N**-monotonic modal logic) contains the following axioms and inference rules:

- Propositional tautologies;
- $\mathbf{UI}^\circ := \forall y(\forall x A(x) \to A(y/x))$
- $\forall x \forall y A \leftrightarrow \forall y \forall x A$
- $A \to \forall x A$, x not free in A
- $\forall x(A \to B) \to (\forall x A \to \forall x B)$
- $\mathbf{I} := t = t$
- $(s = t) \to (A(s//x) \to A(t//x))$

- $\mathbf{ND} := \square A \wedge s \neq t \to \square(A \wedge s \neq t)$
- $\mathbf{NI} := \square A \wedge s = t \to \square(A \wedge s = t)$
- $\mathbf{M} := \square(A \wedge B) \to (\square A \wedge \square B)$
- $\mathbf{N} := \square \top$
- $\mathbf{MP} := A \to B, A/B$
- $\mathbf{RE} := A \leftrightarrow B/\square A \leftrightarrow \square B$
- $\mathbf{UG} := A/\forall x A$

Some remarks on $\mathsf{Q}_=^\circ.\mathsf{MN}$. Besides the propositional part, axiom schemata for the basic predicate part (not considering identity) are those originally proposed by Kripke (1963) (see Corsi, 2002) plus $\forall x \forall y A \leftrightarrow \forall y \forall x A$, which is conceptually harmless but needed to ensure completeness results (see Goldblatt, 2011). The language includes the identity symbol =, which makes the logic very expressive. Notice that the expression $A(t//s)$ denotes that some (all, none) free occurrences of s are replaced by t, whereas $A(s_1 \ldots, t/s_i, \ldots s,)$ stands for $A(s_1 \ldots, s_i \ldots s_n)(t/s_i)$. (see Corsi (2002)). In quantified modal logic with identity, logicians usually consider whether the following schemata are to be valid (for a philosophical discussion, see Kripke, 1980):

$$t = s \to \square(t = s) \qquad\qquad t \neq s \to \square(t \neq s)$$

Here, we consider a version of them that can also capture some *quite restricted* versions of **C**: indeed, **ND** and **NI** state that it is possible to modally aggregate two formulae when one is any A (and thus A can also be a $\square$ formula) and the other is either $t = s$ or $t \neq s$; this does *not* in general entail **C**. Notice that the schema $\mathbf{NI}^* := t = s \to \square(t = s)$ is derivable within this system. Finally, it is worth noting that we will make an essential use of the identity symbol and schemata **NI** and $\mathbf{NI}^*$ to ensure completeness of the system $\mathsf{Q}_=^\circ.\mathsf{MN} \oplus \mathbf{CBF}$, namely, when we add **CBF**. We will show later why this is technically needed.

Given a language $\mathcal{L}$, we shall henceforth refer to the set of its individual constants with the notation $Const(\mathcal{L})$.

Theorem 6.1 (Soundness) *The system* $\mathsf{Q}^{\circ}_{=}.\mathsf{MN}$ *is sound with respect to the class of all multi-relational frames with varying inner domains.*

PROOF. The proof is standard and it is carried out by induction on the length of $\mathcal{D}$, where $\mathcal{D} \coloneqq \mathcal{D}_1, \ldots, \mathcal{D}_n$ is a deduction in n steps in the axiomatic system $Q^{\circ}_{=}.\mathsf{MN}$ with $A = \mathcal{D}_n$, i.e., $A \in Q^{\circ}_{=}.\mathsf{MN}_{\vdash}$.

(a) If $k = 1$, A is an axiom. Let us consider just a few cases. For the propositional schemata, please refer to Theorem 3.4. If A has the form $\forall y(\forall x B(x) \rightarrow B(y/x))$, then assume by reductio that there are a multi-relational frame $\mathcal{F}$, a model $\mathcal{M} = \langle W, \mathcal{R}, D, U, I \rangle$ on $\mathcal{F}$, a world w and a w-assignment σ such that $\mathcal{M} \nvDash^{\sigma}_{w} \forall y(\forall x B(x) \rightarrow B(y/x))$. Then there is a w-assignment τ which is an y-variant of σ such that $\mathcal{M} \nvDash^{\tau}_{w} \forall x B(x) \rightarrow B(y/x)$ and $\tau(y) \in D_w$. Hence $\mathcal{M} \vDash^{\tau}_{w} \forall x B(x)$ and $\mathcal{M} \nvDash^{\tau}_{w} B(y/x)$ and $\tau(y) \in D_w$. It follows that for any w-assignment ϑ, where ϑ is an x-variant of τ such that $\vartheta(x) \in D_w$, $\mathcal{M} \vDash^{\vartheta}_{w} B(x)$ and $\mathcal{M} \nvDash^{\tau}_{w} B(y/x)$ for $\tau(y) \in D_w$. Since one must consider all the x-variants of τ which map x to an element of the inner domain of w and since $\tau(y) \in D_w$, there is an assignment $\vartheta^{\star}$ such that it is an x-variant of τ, $\vartheta^{\star}(x) = \tau(y)$, $\mathcal{M} \vDash^{\vartheta^{\star}}_{w} B(x)$ and $\mathcal{M} \nvDash^{\tau}_{w} B(y/x)$.

If A has the form $B \rightarrow \forall x B$ for x not free in B, then suppose by reductio that there are a multi-relational frame $\mathcal{F}$, a model $\mathcal{M} = \langle W, \mathcal{R}, D, U, I \rangle$ on $\mathcal{F}$, a world w and a w-assignment σ such that $\mathcal{M} \nvDash^{\sigma}_{w} B \rightarrow \forall x B$, thus $\mathcal{M} \vDash^{\sigma}_{w} B$ and $\mathcal{M} \nvDash^{\sigma}_{w} \forall x B$, i.e., $\mathcal{M} \vDash^{\sigma}_{w} \exists x \neg B$. From this it follows that $\mathcal{M} \nvDash^{\tau}_{w} B$ for some x-variant τ of σ such that $\tau(x) \in D_w$. Since x is not free in B, σ and τ coincide on any free variable occurring in B, hence, by Lemma 4.8 it holds that $\mathcal{M} \vDash^{\sigma}_{w} B$ and $\mathcal{M} \nvDash^{\sigma}_{w} B$.

If A has the form $\forall x(B \rightarrow C) \rightarrow (\forall x B \rightarrow \forall x C)$, then suppose by reductio that there are a multi-relational frame $\mathcal{F}$, a model $\mathcal{M} = \langle W, \mathcal{R}, D, U, I \rangle$ on $\mathcal{F}$, a world w and a w-assignment σ such that $\vDash^{\sigma}_{w} \forall x(B \rightarrow C)$, $\vDash^{\sigma}_{w} \forall x B$ and $\vDash^{\sigma}_{w} \exists x \neg C$. Thus (i) $\mathcal{M} \vDash^{\tau}_{w} B \rightarrow C$ for any x-variant τ of σ such that $\tau(x) \in D_w$; (ii) $\mathcal{M} \vDash^{\vartheta}_{w} B$ for any x-variant ϑ of σ such that $\vartheta(x) \in D_w$; (iii) $\mathcal{M} \nvDash^{\upsilon}_{w} C$ for some x-variant υ of σ such that $\upsilon(x) \in D_w$. From (i) it follows that for any x-variant τ of σ such that $\tau(x) \in D_w$, either $\mathcal{M} \nvDash^{\tau}_{w} B$, contradicting (ii), or $\mathcal{M} \vDash^{\tau}_{w} C$, contradicting (iii).

If A has the form $\Box B \wedge s = t \rightarrow \Box(B \wedge s = t)$, then assume that there are a model, a world and an assignment such that $\vDash^{\sigma}_{w} \Box B \wedge s = t$. This entails the existence of a relation R_i such that $R_i(w) \subseteq \|B\|^{\sigma}_{I}$. Take any $v \in R_i(w)$. If t (or s) is an individual constant, then $I^{\sigma}_{w}(t) = I^{\sigma}_{v}(t)$, since constants are rigid designators; on the other hand if t (or s) is a variable, than, since $U_w \subseteq U_v$, it holds that $I^{\sigma}_{w}(t) = \sigma(t) = I^{\sigma}_{v}(t)$. Thus $\vDash^{\sigma}_{w} \Box(B \wedge (t = s))$.

(b) If $k = n + 1$, then A is an axiom (see previous cases) or it has been obtained either via **MP**, or via **RM**, or else via **UG**. For the first two cases, please refer to Theorem 3.4. Let us focus on the latter case. If A has been obtained via the rule **UG**, it has the form $\forall x B$ and it has been derived applying **UG** to a formula B. By IH, $\vDash^{\sigma}_{w} B$ for any valuation I, any world w and any assignment σ. In particular this holds true for any assignment τ which is an x-variant of σ such that $\tau(x) \in D_w$ and hence $\vDash^{\sigma}_{w} \forall x B$. ∎

6.2 Some Auxiliary Results

Following Corsi (2002), let us establish some auxiliary results.

Definition 6.2 *Let* L *be a logic on the language* $\mathcal{L}$, $\Delta \subseteq \mathsf{L}$ *and* $Q \subseteq Const(\mathcal{L})$.
- Δ *is* L-*consistent iff* $\Delta \nvdash_{\mathsf{L}} \bot$.
- Δ *is* L-*deductively closed iff for any sentence* A *of* $\mathcal{L}$, $\Delta \vdash_{\mathsf{L}} A$ *iff* $A \in \Delta$.

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- Δ is L-*complete iff for any sentence A of $\mathcal{L}$, either $A \in \Delta$ or $\neg A \in \Delta$.*
- Δ is L-*maximal iff Δ is L-consistent and L-complete.*
- Δ is Q-*universal iff if $\forall x A(x) \in \Delta$, then $A(c/x) \in \Delta$, for all individual constants $c \in Q$.*
- Δ is Q-*existential iff if $A(c/x) \in \Delta$ for some individual constant $c \in Q$, then $\exists x A(x) \in \Delta$.*
- Δ is Q-*inductive iff if $A(c/x) \in \Delta$ for all individual constants $c \in Q$, then $\forall x A(x) \in \Delta$.*
- Δ is Q-*rich iff if $\exists x A(x) \in \Delta$ for some individual constant $c \in Q$, then $A(c/x) \in \Delta$.*
- Δ is L-*saturated iff if Δ is L-maximal and for some $Q \subseteq Const(\mathcal{L})$, Δ is Q-universal and Q-rich.*

Lemma 6.3 *Let $L \subseteq Q^{\circ}_{\underline{=}}.MN$ be a logic on $\mathcal{L}$. Let C be a denumerable set of individual constants not occurring in $\mathcal{L}$, let $\mathcal{L}^C$ be the language obtained adding C to $Const(\mathcal{L})$ and L^C be the logic on $\mathcal{L}^C$.*

i. If $\vdash_{L^C} A(c_1, \dots, c_n)$, then $\vdash_{L^C} A(x_1/c_1, \dots, x_n/c_n)$ where $x_1, \dots, x_n$ are variables not occurring in $A(c_1, \dots, c_n)$.

ii. If $\vdash_{L^C} A$ and no constants of C occur in A, then $\vdash_L A$.

iii. If Δ is an L-consistent set of sentences and no constant from C occurs in Δ, then Δ is L^C-consistent. (Cf. (Corsi, 2002, Lemma 1.6))

Lemma 6.4 *Given an L-maximal set Δ of sentences in $\mathcal{L}$ and $Q \subseteq Const(\mathcal{L})$, if Δ is Q-universal then Δ is Q-existential.(Cf. (Corsi, 2002, Lemma 1.11))*

PROOF. Assume $A(c/x) \in \Delta$ for some individual constant $c \in Const(\mathcal{L})$ and $\exists x A \notin \Delta$. Since Δ is L-maximal, $\neg \exists x A(x) \in \Delta$ and hence $\forall x \neg A(x) \in \Delta$. Thus, since Δ is Q-universal by definition, $\neg A(c/x) \in \Delta$ and hence $\bot \in \Delta$, contradicting the consistency of Δ. ∎

Lemma 6.5 (Lindenbaum's Lemma) *Given a logic L, if Δ is a L-consistent set of formulae, then there is a L-maximal set Δ^+ such that $\Delta \subseteq \Delta^+$.*

Lemma 6.6 *Let Δ be an L-consistent set of sentences of $\mathcal{L}$. Then for some not-empty denumerable set C of new constants, there is a set Π of sentences of $\mathcal{L}^C$ such that $\Delta \subseteq \Pi$, Π is L^C-maximal, Π is Q-universal and Q-rich for some set $Q \subseteq Const(\mathcal{L}^C)$. (cf. (Corsi, 2002, Lemma 1.16))*

6.3 Canonical Models

In order to define canonical models for a language with identity, we need to introduce a binary equivalence relation on individual constants:

$$a \sim b \text{ if and only if } (a = b) \in w.$$

Any individual constant c may be interpreted on its equivalence class $[c]$, where $[c] := \{a \mid c \sim a\}$.

Definition 6.7 (Non-normal canonical model for $Q^{\circ}_{\underline{=}}.MN$) *Let $L \supseteq Q^{\circ}_{\underline{=}}.MN$ be a logic based on $\mathcal{L}$. Let Q be a set of constants with cardinality $\aleph_0$ such that $Q \supset Const(\mathcal{L})$ and $|Q - Const(\mathcal{L})| = \aleph_0$. A non-normal canonical model for L is a tuple $\mathcal{M} = \langle W, \mathcal{R}, D, I \rangle$ such that:*

- W is the class of all L_w-saturated sets of sentences w, where $\mathcal{L}_w = \mathcal{L}^S$ for some set S of constants such that $Const(\mathcal{L}^S) \neq \varnothing$, $S \subset Q$ and $\|Q - Const(\mathcal{L}^S)\| = \aleph_0$ (where $\mathcal{L}^S$ is defined as in Lemma 6.3).

- For any formula $A \in Fma(\mathcal{L})$ let R_A be a binary relation over W. For all $w, v \in W$, $w R_A v$ iff $\Box A \in w \Rightarrow A \in v$ and for any constant $c \in Const(\mathcal{L}_w)$, $[c]_w = [c]_v$, where $[c]_v = \{b \in Const(\mathcal{L}_v) : (b = c) \in v\}$. The set of relations $\mathcal{R}$ is the collection of all such relations.

- $D_w = \{[c]_w : \exists x(x = c) \in w\}$.

- $U_w = \{[c]_w : c \in Const(\mathcal{L}_w)\}$.

- $I_w(c) = [c]_w$.

- $I_w(P^n) = \{\langle [c_1]_w, \dots, [c_n]_w \rangle : P^n(c_1, \dots, c_n) \in w\}$.

Notice that this definition implies that whenever a formula $\Box A$ does not belong to a state w, the relation associated to A for w would be $R_A(w) = W$, i.e., the whole universe. Otherwise if $\Box A \in w$, $R_A(w)$ would be exactly $|A|_{\mathsf{Q}^\circ_=.\mathsf{MN}}$, where $|A|_{\mathsf{Q}^\circ_=.\mathsf{MN}} := \{v \in W \mid A \in v\}$. Moreover, it is important to notice that the frame of a $\mathsf{Q}^\circ_=.\mathsf{MN}$-canonical model is not always generally serial. Indeed it allows the presence of empty relations for any world, and hence the schema $\mathbf{CON}:=\Diamond\top$ is not valid on the canonical frame.

Lemma 6.8 (Existence lemma) *Given a canonical model $\mathcal{M}$ for $\mathsf{Q}^\circ_=.\mathsf{MN}$, for any $w \in W$, if $\Diamond A \in w$, then for any formula $B \in Fma(\mathcal{L})$ there is a set v such that:*

1. *v belongs to the base set of $\mathcal{M}$, i.e., in the set of possible worlds W*
2. *$A \in v$*
3. *If $\Box B \in w$ then $B \in v$*
4. *$Const(\mathcal{L}_w) \subseteq Const(\mathcal{L}_v)$*
5. *for any individual constant $c \in Const(\mathcal{L}_w)$, $[c]_w = [c]_v$.*

PROOF. Below we shall define a procedure to construct any such v for any formula B.

Let C be a denumerable set of constants which do not belong to $\mathcal{L}_w$ and let $\mathcal{L}_w^C = \mathcal{L}_w \cup C$. Let $H_1, H_2, H_3, \ldots$ be an enumeration of all the existential formulae of $\mathcal{L}_w^C$ where any formula occurs infinitely often.
Let Γ be a chain of sets as defined below:

1. Γ_0:

    (1.a) If $\Box B \in w$, then $\Gamma_0 := \{B\} \cup \{A\}$

    (1.b) otherwise, $\Gamma_0 := \{A\}$.

2. $\Gamma_1 := \Gamma_0 \cup \{(a = b) \mid (a = b) \in w\}$

3. Let Γ_n be already defined for $1 \le n$ and let $H_{n+1} = \exists x F(x)$. The set Γ_{n+1} is defined as:

    (3.a) If $\exists x F(x)$ contains at least one constant c such that $c \notin Const(\Gamma_n)$, then $\Gamma_{n+1} := \Gamma_n$.

    (3.b) otherwise, if any constant occurring within $\exists x F(x)$ is already in Γ_n, then there are a few cases:

    (3.b.1) If $\Gamma_n \cup \{\exists x F(x)\}$ is L_w^C-consistent, then

    (3.b.1.1) $\Gamma_{n+1} := \Gamma_n \cup \{\exists x F(x)\} \cup \{F(b/x)\}$, where $b \in Const(\Gamma_n)$ and $\Gamma_n \cup \{F(b/x)\}$ is L_w^C-consistent;

    (3.b.1.2) $\Gamma_{n+1} := \Gamma_n \cup \{\exists x F(x)\} \cup \{F(c/x)\} \cup \{(c \ne b) \mid b \in Const(\Gamma_n)\}$, if $\Gamma_n \cup \{F(b/x)\}$ is not L_w^C-consistent for any $b \in Const(\Gamma_n)$, and $c \in C$ is a constant not occurring in Γ_n.

    (3.b.2) Otherwise if $\Gamma_n \cup \{\exists x F(x)\}$ is not L_w^C-consistent, then $\Gamma_{n+1} = \Gamma_n$.

4. $\Gamma = \bigcup_{n \in \mathcal{N}}(\Gamma_n)$.

In order to show that Γ is L_w^C-consistent, we have to check that for any n, Γ_n is L_w^C-consistent.

1. Γ_0 is consistent. Indeed:

    (1.a) If $\Gamma_0 := \{A, B\}$ and $\Box B \in w$, then suppose $\vdash_{\mathsf{MN}} A \wedge B \to \bot$, then $\vdash_{\mathsf{MN}} B \to (A \to \bot)$, $\vdash_{\mathsf{MN}} B \to \neg A$ and $\vdash_{\mathsf{MN}} \Box B \to \Box \neg A$ by the $\mathbf{RM}$ rule. By definition, it follows that $w \vdash_{\mathbf{MN}} \Box \neg A$, $\neg \Diamond A \in w$, leading to a contradiction as w is consistent by assumption.

    (1.b) If $\Gamma_0 := \{A\}$, then assume it is not consistent and hence $\vdash_{\mathsf{MN}} A \to \bot$, i.e., $\vdash_{\mathsf{MN}} \neg A$. Thus by the necessitation rule we have $\vdash_{\mathsf{MN}} \Box \neg A$ and this implies that $\neg \Diamond A \in w$ and hence $\Diamond A \notin w$, which is in contradiction with our hypothesis.

2. Γ_1 is consistent. Indeed assume it is not. There are two cases:

(2.a) $\Gamma_0 := \{A, B\}$ and $\Box B \in w$, then $\vdash_{\mathsf{MN}} B \to (a = b \to (A \to \bot))$ for some $a = b \in w$. Then $\vdash_{\mathsf{MN}} B \to (a = b \to \neg A)$, and $\vdash_{\mathsf{MN}} \Box B \to \Box((a = b) \to \neg A)$ by the **RM** rule. By definition of w and **MP**, it follows that $w \vdash_{\mathsf{MN}} \Box((a = b) \to \neg A)$. Thus by the **RM** rule $\vdash_{\mathsf{MN}} \Box(a = b) \to \Box\neg A$. But $\Box(a = b) \in w$ by the **NI*** schema, hence $\neg \Diamond A \in w$, $\Diamond A \notin w$, which contradicts the hypothesis.

(2.b) If $\Gamma_0 := \{A\}$, then $\vdash_{\mathsf{MN}} (a = b) \to A \to \bot$, i.e., $\vdash_{\mathsf{MN}} (a = b) \to \neg A$ for some $(a = b) \in w$. Thus by the **RM** rule $\vdash_{\mathsf{MN}} \Box(a = b) \to \Box\neg A$. But $\Box(a = b) \in w$ by the **NI** schema, hence $\neg \Diamond A \in w$, $\Diamond A \notin w$, which contradicts the hypothesis.

3. Γ_{n+1} is also consistent, in fact:

(3.a) If $\exists x F(x)$ contains at least one constant c such that $c \notin Const(\Gamma_n)$, then $\Gamma_{n+1} = \Gamma_n$ and thus it is $\mathsf{L}_w^{\mathsf{C}}$-consistent by the Inductive Hypothesis (IH henceforth).

(3.b) If any constant occurring within $\exists x F(x)$ is already in Γ_n, then

(3.b.1) If $\Gamma_n \cup \{\exists x F(x)\}$ is $\mathsf{L}_w^{\mathsf{C}}$-consistent, then

(3.b.1.1) The set $\Gamma_{n+1} = \Gamma_n \cup \{\exists x F(x)\} \cup \{F(b/x)\}$ (where $b \in Const(\Gamma_n)$ and $\Gamma_n \cup \{F(b/x)\}$ is $\mathsf{L}_w^{\mathsf{C}}$-consistent) is consistent by construction and IH;

(3.b.1.2) Assume $\Gamma_{n+1} = \Gamma_n \cup \{\exists x F(x)\} \cup \{F(c/x)\} \cup \{(c \neq b) \mid b \in Const(\Gamma_n)\}$, if $\Gamma_n \cup \{F(b/x)\}$ is not $\mathsf{L}_w^{\mathsf{C}}$-consistent for any $b \in Const(\Gamma_n)$, and $c \in C$ is a constant not occurring in Γ_n. We know that the set $\Gamma_n \cup \{\exists x F(x)\}$ is consistent by hypothesis. First of all, let us show that $\Gamma_n \cup \{F(c/x)\}$ is $\mathsf{L}_w^{\mathsf{C}}$-consistent. Assume by reductio that $\Gamma_n \cup \{F(c/x)\}$ is not $\mathsf{L}_w^{\mathsf{C}}$-consistent. Then there are sentences $\{D_1, ..., D_k\} \in \Gamma_n$ such that $\vdash_{\mathsf{L}_w^{\mathsf{C}}} D_1 \wedge ... \wedge D_k \wedge F(c/x) \to \bot$
$\Gamma_n \vdash_{\mathsf{L}_w^{\mathsf{C}}} \neg(F(c/x)$
$\Gamma_n \vdash_{\mathsf{L}_w^{\mathsf{C}}} \neg F(y/c)$
$\Gamma_n \vdash_{\mathsf{L}_w^{\mathsf{C}}} \forall y \neg F(y/c)$
$\Gamma_n \vdash_{\mathsf{L}_w^{\mathsf{C}}} \neg \exists y F(y)$
contrary to the fact that $\Gamma_n \cup \{\exists x F(x)\}$ is $\mathsf{L}_w^{\mathsf{C}}$-consistent by hypothesis, thus $\Gamma_n \cup \{F(c/x)\}$ is $\mathsf{L}_w^{\mathsf{C}}$-consistent.
Assume by reductio that Γ_{n+1} is not $\mathsf{L}_w^{\mathsf{C}}$-consistent. Hence for some finite set of individual constants $\{b_1, ..., b_h\} \subseteq Const(\Gamma_n)$,
$\Gamma_n \vdash_{\mathsf{L}_w^{\mathsf{C}}} F(c/x) \wedge (c \neq b_1 \wedge ... \wedge c \neq b_h) \to \bot$
$\Gamma_n \vdash_{\mathsf{L}_w^{\mathsf{C}}} F(c/x) \to \neg(c \neq b_1 \wedge ... \wedge c \neq b_h)$
$\Gamma_n \vdash_{\mathsf{L}_w^{\mathsf{C}}} F(c/x) \to (c = b_1 \vee ... \vee c = b_h)$
$\Gamma_n \cup \{F(c/x)\} \vdash_{\mathsf{L}_w^{\mathsf{C}}} c = b_1 \vee ... \vee c = b_h$
hence for some i, $1 \leq i \leq h$, $\Gamma_n \cup \{F(b_i/x)\}$ is $\mathsf{L}_w^{\mathsf{C}}$-consistent, in contradiction with the assumption that there is no constant $b \in Const(\Gamma_n)$ such that $\Gamma_n \cup \{F(b/x)\}$ is $\mathsf{L}_w^{\mathsf{C}}$-consistent. Therefore Γ_{n+1} is $\mathsf{L}_w^{\mathsf{C}}$-consistent.

(3.b.2) If $\Gamma_n \cup \{\exists x F(x)\}$ is not $\mathsf{L}_w^{\mathsf{C}}$-consistent, then $\Gamma_{n+1} = \Gamma_n$ and it is thus consistent by IH.

4. Since Γ is the big union of a chain of consistent sets, then Γ is $\mathsf{L}_w^{\mathsf{C}}$-consistent.

Let $Q = Const(\Gamma)$. We start by showing that
($*$) For any existential formula $\exists x F(x)$ of $\mathcal{L}_w^Q$ there is some Γ_k such that either $\exists x F(x) \in \Gamma_{k+1}$ or $\Gamma_k \cup \{\exists x F(x)\}$ is $\mathcal{L}_w^C$-inconsistent.
Let $c_1, \ldots, c_j$ be all the constants occurring in $\exists x F(x)$. Since $\exists x F(x) \in \mathcal{L}_w^Q$, $\{c_1, \ldots, c_j\} \subseteq Q$, hence for some j, $\{c_1, \ldots, c_j\} \subseteq Const(\Gamma_j)$. Since $\exists x F(x)$ occurs infinitely many times within $H_1, H_2, H_3, ...$, then $\exists x F(x) = H_k$ for some $k > j$. Therefore the (b) step of our construction is applied to $\exists x F(x)$ and hence ($*$) is proved. It follows that Γ is $\mathcal{L}_w^Q$-rich.

The set Γ can be extended to some v which is L_w^Q-consistent and L_w^Q-maximal. Let $\mathcal{L}_v = \mathcal{L}_w^Q$. The extension v does not compromise richness. Indeed, if an existential formula of $\mathcal{L}_w^Q$ belongs to v, by ($*$) it is also in Γ, and hence some exemplification of it is also in

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Γ and since $\Gamma \subseteq v$, it is also in v. Therefore v is $L_v^=$-saturated and it hence belongs to the canonical base set. Therefore:

1. $v \in W$.
2. Since $A \in \Gamma_0$ and $v \supseteq \bigcup_{n \in \mathcal{N}}(\Gamma_n)$, it follows that $A \in v$.
3. Moreover it holds true that if $\Box B \in w$, then $B \in v$.
4. Since $Const(\mathcal{L}_w) \subseteq Const(\mathcal{L}_w^Q)$ it holds that $Const(\mathcal{L}_w) \subseteq Const(\mathcal{L}_v)$.
5. The set v is L_w^Q-maximal: the only constants occurring in v are those already present in Γ. Since by construction $\{(a = b) \mid (a = b) \in w\} \subseteq \Gamma$, it follows that for any b, $[b]_w \subseteq [b]_v$. On the other hand, suppose that there is some c from $Const(\mathcal{L}_v)$ which does not occur in $Const(\mathcal{L}_w)$. Then c has been added at some point in the construction of the Γ_j sets. The step (3.b.1.2) is the only possible way to add the new constant c to Γ_{j+1} and it guarantees that for any $d \in Const(\mathcal{L}(\Gamma_j))$, $c \neq d \in \Gamma_{j+1}$, so $c \notin [b]_v$ for any $b \in Const(\mathcal{L}_w)$.

∎

Lemma 6.9 (Truth Lemma) *Given a canonical model* $\mathcal{M}^L = \langle W, \mathcal{R}, D, U, I \rangle$ *for a quantified* **N***-Monotonic modal logic* L *extending* $Q_=^\circ$.MN*, for any formula* $A \in Fma(\mathcal{L})$*, for any world* $w \in W$*, the following holds:* $\vDash_w^\sigma A(x_i) \Leftrightarrow A(\sigma(x_i)/x_i) \in w$.

PROOF. By induction on the length of a formula A. We omit details of the induction base.

Suppose the length of A is $n+1$ and A has the form $\exists x B(x, y_1, \ldots, y_m)$.

(i) $\vDash_w^\sigma \exists x B(x, y_1, \ldots, y_m)$ iff for some x-variant τ of σ such that $\tau(x) \in D_w$, $\vDash_w^\tau B(x, y_1, \ldots, y_m)$. Suppose $\tau(x) = c$. Since by assumption all constants are rigid designators, by Lemma 4.7 it holds that $\vDash_w^\sigma B(c/x, y_1, \ldots, y_m)$. Hence $B(c/x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$ by IH. Since w is D_w-universal, w is also D_w-existential by Lemma 6.4 and hence $\exists x B(x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$. (ii) Assume $\exists x B(x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$. Since w is D_w-rich, $B(c/x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$ for some $c \in D_w$. Thus, by IH, $\vDash_w^\sigma B(c/x, y_1, \ldots, y_m)$ and, by Lemma 4.7 it holds that $\vDash_w^\tau B(x, y_1, \ldots, y_m)$ for some w-assignment τ which is an x-variant of σ such that $\tau(x) = I_w(c) = c$. Therefore $\vDash_w^\sigma \exists x B(x, y_1, \ldots, y_m)$.

Assume that $\Diamond A \in w$. Since the schema $\Diamond A \rightarrow \Diamond \top$ is a theorem[6] of $Q_=^\circ$.MN, it follows that $\Diamond \top \in w$ and by Lemma 6.13 in (Calardo and Rotolo, 2014), w is locally serial, i.e., for any formula $B \in Fma(\mathcal{L})$ there is a state z such that wR_Bz. It remains to show that for any formula $B \in Fma(\mathcal{L})$, $R_B(w) \cap |A|_{Q_=^\circ.MN} \neq \varnothing$, i.e., for any B there is some z such that $z \in R_B(w)$ and $A \in z$. (i) If A is a theorem, then $|A|_{Q_=^\circ.MN} = W$ (A belongs to any maximal consistent set in W). (ii) A cannot be a contradiction, otherwise $\vdash_{Q_=^\circ.MN} \neg A$ by **EFQ**, $\vdash_{Q_=^\circ.MN} \Box \neg A$, $\Box \neg A \in w$, $\neg \Box \neg A \notin w$, i.e., $\Diamond A \notin w$ which leads to a contradiction. (iii) If A is neither a theorem nor a contradiction, both $\{A\}$ and $\{\neg A\}$ are $Q_=^\circ$.MN-consistent and both $|A|_{Q_=^\circ.MN}$ and $|\neg A|_{Q_=^\circ.MN}$ are not empty. Assume by reductio that for some $C \in Fma(\mathcal{L})$ $R_Const(w) \subseteq |\neg A|_{Q_=^\circ.MN}$. This implies that $\Box C \in w$ (otherwise we would have $R_C(w) = W$ which is inconsistent with our assumption). Hence we have that in all $Q_=^\circ$.MN-maximal sets z, $C \rightarrow \neg A \in z$ and therefore it is a theorem of $Q_=^\circ$.MN. Thus $\vdash_{Q_=^\circ.MN} C \rightarrow \neg A$, $\vdash_{Q_=^\circ.MN} \Box C \rightarrow \Box \neg A$, $\Box \neg A \in w$ and hence $\Diamond A \notin w$ which is a contradiction.

∎

Lemma 6.10 *Let* $\mathcal{M}^L := \langle W, \mathcal{R}, D, U, I \rangle$ *be a canonical model for a logic* $L \supseteq Q_=^\circ$.MN*. If* Δ *is an* L*-consistent set of formulae, then for some* $w \in W$ *and some* w*-assignment* σ*,* $\mathcal{M}^L \vDash_w^\sigma D$ *for any* $D \in \Delta$*. (cf. Corsi, 2002, Lemma 1.19)*

Let L be any logic $L \supseteq Q_=^\circ$.MN. Consider any formula A such that $\nvdash_L A$. Then $\{\neg A\}$ is L-consistent. By Lemma 6.10 there is a world w of a canonical model $\mathcal{M}^L$ for L and a w-assignment σ such that $\mathcal{M}^L \nvDash_w^\sigma A$ and hence $\mathcal{M}^L \nvDash A$.

Corollary 6.11 (Completeness of $Q_=^\circ$.MN) *The logic* $Q_=^\circ$.MN *is strongly complete with respect to the class of all multi-relational frames.*

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[6]Indeed $\vdash_{Q_=^\circ.MN} \bot \rightarrow \neg A$ *ex falso quodlibet*, $\vdash_{Q_=^\circ.MN} \Box \bot \rightarrow \Box \neg A$ by **RM**, $\vdash_{Q_=^\circ.MN} \Diamond A \rightarrow \Diamond \top$ by contraposition.

6.4 Completeness and Increasing Domains

The class of frames with restricted increasing inner domains is quite interesting on many respects. As shown in Section 5 it is closely connected to the **CBF** schema. In Kripke semantics the class of frames with increasing inner domains is captured by the logic $\mathsf{Q}^\circ_=.\mathsf{K} \oplus$ **CBF**, thus showing that **CBF** is sufficient to achieve the result. The case of multi-relational frames with restricted inner domains is, however, more complex. In this section we focus on finding an axiomatisation to capture the semantic concept of restricted increasing inner domains.

It turns out that merely adding **CBF** to our system is not enough to axiomatise the logic semantically generated by the class of multi-relational frames with restricted increasing inner domains: Something more is needed. In particular, as it will become clear in the following theorem, a weak version of schema **C** is needed, namely

$$\Box \exists := \Box A \wedge \exists x_1 \ldots \exists x_n (x_1 = t_1 \wedge \ldots \wedge x_n = t_n) \to \Box (A \wedge \exists x_1 \ldots \exists x_n (x_1 = t_1 \wedge \ldots \wedge x_n = t_n))$$

Let us define $\mathsf{Q}^\circ_=.\mathsf{MN} \oplus$ **CBF** $\oplus\, \Box\exists$ as the system obtained by adding **CBF** and $\Box\exists$ to $\mathsf{Q}^\circ_=.\mathsf{MN}$. Notice that although we added a weak version of **C** (or, in other words, a stronger version of **NI**), this system is still non-normal. Moreover the schema $\Box\exists$ does not hold in general, although it is sound in the class of multi-relational frames with finite restricted increasing inner domains.

Lemma 6.12 *The schema $\Box\exists$ is valid in the class of multi-relational frames with restricted increasing inner domains (as described in Lemma 5.11).*

PROOF. Suppose by reductio ad absurdum that there are a model $\mathcal{M} := \langle W, \mathcal{R}, D, U, I \rangle$ based on a frame with restricted increasing inner domains and a world w such that (a) $\vDash^\sigma_w \Box A$, (b) $\vDash^\sigma_w \exists x_1 \ldots \exists x_n (x_1 = a_1 \wedge \ldots \wedge x_n = a_n)$, where $\{a_1, \ldots, a_n\} \subseteq D_w$, and (c) $\vDash^\sigma_w \Diamond(\neg A \vee \forall x_1 \ldots \forall x_n (x_1 \neq a_1 \vee \ldots \vee x_n \neq a_n))$. Clearly by (c) w is locally serial and by (a) there is a relation R_i such that $R_i(w) \subseteq \|A\|^\sigma_I$. By definition there is a relation R_k such that $R_k(w) \subseteq R_i(w)$ and the inner domains are increasing. Since $R_k(w) \subseteq \|A\|^\sigma_I$, by (c) there must be a world $v \in R_k(w)$ such that $\vDash^\sigma_v \forall x_1 \ldots \forall x_n (x_1 \neq a_1 \vee \ldots \vee x_n \neq a_n)$. Since any a_i belongs to D_w, the inner domains are not increasing, thus reaching a contradiction. ∎

Theorem 6.13 (Completeness of $\mathsf{Q}^\circ_=.\mathsf{MN} \oplus$ CBF $\oplus\,\Box\exists$) *The logic $\mathsf{Q}^\circ_=.\mathsf{MN} \oplus$ **CBF** $\oplus\,\Box\exists$ is complete with respect to the class of multi-relational frames with finite restricted increasing inner domains, i.e. frames with the following property: $\forall w \in W$, if w is locally serial, then for all R_j, there is some relation R_i such that $R_i(w) \subseteq R_j(w)$ and for all worlds v in $R_i(w)$, $D_w \subseteq D_v$, as in Lemma 5.11.*

PROOF. Let $\mathcal{M}$ be a canonical model for $\mathsf{Q}^\circ_=.\mathsf{MN} \oplus$ **CBF**. Take any world w and assume it is locally serial. Then the schema $\Diamond\top$ belongs to w. By Lemma 6.8, it follows that for any formula $B_i \in Fma(\mathcal{L})$ there is a world v_i such that $wR_{B_i}v_i$ and $Const(\mathcal{L}_w) \subseteq Const(\mathcal{L}_{v_i})$ and for any individual constant $c \in Const(\mathcal{L}_w)$, $[c]_w = [c]_{v_i}$. This implies that $U_w \subseteq U_{v_i}$. Take any $[c] \in D_w$; by definition it holds that $\exists x(x = c) \in w$. From these facts it follows that $\vDash^\sigma_w x = c$ for some σ such that $\sigma(x) \in D_w$ and, moreover, for each i, $\vDash^\sigma_{v_i} x = c$. Hence $\vDash^\sigma_w \Diamond(x = c)$ and since $\sigma(x) \in D_w$, $\exists x \Diamond(x = c) \in w$. By **CBF**, $\Diamond\exists x(x = c) \in w$. This means that for every *existing* individual b from D_w, for any relation R_B, there is a world $t \in R_B(w)$ such that $b \in D_t$.

Now we want to show that the property expressed by Lemma 5.11 holds. Take any relation R_A. By definition of canonical model, the set $R_A(w)$ is such that either (1) $R_A(w) = \|A\|_I$, or (2) $R_A(w) = \|\top\|_I$. If (1), then $\Box A \in w$, hence the formula $\Box A \wedge \bigwedge_{a \in D_w} \exists x(x = a) \in w$. By axiom $\Box\exists := \Box A \wedge \exists x_1(x_1 = t_1) \wedge \ldots \wedge \exists x_n(x_n = t_n) \to \Box(A \wedge \exists x_1(x_1 = t_1) \wedge \ldots \wedge \exists x_n(x_n = t_n))$, we get $\Box(A \wedge \bigwedge_{a \in D_w} \exists x(x = a)) \in w$. Let us denote this formula by the letter C. Clearly, $R_C(w) \subseteq R_A(w)$ and for any world $v \in R_C(w)$, it holds that $D_w \subseteq D_v$. Likewise, if (2) holds, $\Box\top \in w$ and by a similar argument we can show that the domains are increasing. ∎

7 The Case of Barcan Formula

7.1 BF and CGF: Semantic Considerations

If compared to **CBF**, additional technical difficulties make **BF** harder to handle. The remainder of this section provides some preliminary but interesting results.

Similar to the case with **CBF**, the invalidity of the **BF** schema follows immediately from its analogous result in Kripke semantics.

Lemma 7.1 BF *is not valid in the class of multi-relational frames.*

PROOF. Consider the following model: $\mathcal{M} := \langle W, R_1, D, U, I \rangle$ where $W := \{w, z\}$, $R_1 := \{\langle w, z \rangle\}$, $D_w = \{a\}$, $D_z = \{b\}$, $U_w = U_z = \{a, b\}$, $P(a) \in I_z(P)$ and $P(b) \notin I_z(P)$. Then it holds that for any $x - variant$ of a σ-assignment τ such that $\tau(x) \in D_w$, $\vDash_w^\tau \Box P(x)$ and hence $\vDash_w^\sigma \forall x \Box P(x)$. Moreover there is an x-variant θ of σ such that $\theta(x) \in D_z$ and $P(x) \in I_z^\theta(P)$, i.e. $\theta(x) = b$ and since $w R_1 z$ it holds that $\vDash_w^\sigma \Diamond \exists x \neg P(x)$. (See Figure 3) ■
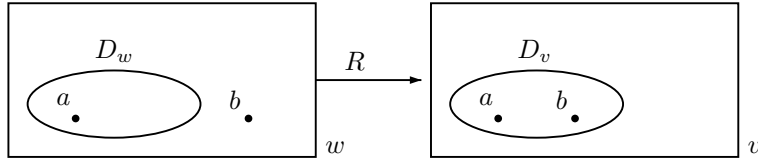

Figure 3: A Kripke frame to build a countermodel for BF.

Lemma 7.2 (BF Characterisation Result) *For any multi-relational frame $\mathcal{F} := \langle W, \mathcal{R}, D, U \rangle$, $\mathcal{F} \vDash \forall x \Box A \to \Box \forall x A$ iff $\forall w \in W$, either there is an $R_i \in \mathcal{R}$ such that $R_i(w) = \varnothing$ or for any (possibly infinite) sequence of worlds $z_1, z_2, \ldots$ of length n (if the sequence is infinite n is $\aleph_0$) such that n is also the cardinality of $\mathcal{R}$ and $w R_k z_k$ for any $R_k \in \mathcal{R}$, for any (possibly infinite) sequence $R_{j_1}, R_{j_2}, \ldots$ of length m where $\|D_w\| = m$, i.e., m is the number of individuals belonging to the inner domain of w, there is a world $t \in \{z_1, z_2, \ldots\} \cap \bigcap_{i \leq m} R_{j_i}(w)$ such that $D_w \supseteq D_t$.*

PROOF. ($\Rightarrow$) Assume that there is a $w \in W$ such that for all $R_i \in \mathcal{R}$, $R_i(w) \neq \varnothing$ and for some (possibly infinite) sequence of worlds $z_1, z_2, \ldots$ such that $w R_k z_k$ for any $R_k \in \mathcal{R}$, for some (possibly infinite) sequence of relations $R_{j_1}, R_{j_2}, \ldots$ the following holds: for all $t \in W$, if $t \in \{z_1, z_2, \ldots\} \cap R_{j_1}(w) \cap \ldots \cap R_{j_m}(w)$ then $D_w \subset D_t$. We shall now define an interpretation I in order to build a countermodel for **BF** in this frame. Let $D_w := \{d_1, d_2, \ldots\}$ of cardinality m (not necessarily finite), and consider the w-assignments $\sigma_1, \sigma_2, \ldots$ such that for any σ_i, $\sigma_i(x) = d_i$. For some unary predicate P, for any i, let $\|P(x)\|_I^{\sigma_i} := R_{j_i}(w)$. Then clearly $\vDash_w^{\sigma_i} \Box P(x)$ for each i and hence, since $\sigma_1, \sigma_2, \ldots$ are all the x-variant of a w-assignment σ such that $\sigma_i \in D_w$ for each i, $\vDash_w^\sigma \forall x \Box P(x)$. Consider now any z_i from the sequence $z_1, z_2, \ldots$ and define a z_i-assignment ϑ_i such that $\vartheta_i(x) \notin I_{z_i}^{\vartheta_i}(P(x))$ and $\vartheta_i(x) \in D_{z_i}$. Notice that it is always possible to define such an assignment. In fact even if $z_i \in R_{j_1}(w) \cap \ldots \cap R_{j_m}(w)$, by assumption we have that $D_{z_i} \supset D_w$. Hence $\vDash_{z_i}^{\vartheta_i} \neg P(x)$ and hence $\vDash_{z_i}^\sigma \neg \exists x P(x)$ which implies that $\nvDash_w^\sigma \Box \forall x P(x)$.

($\Leftarrow$) Assume that for some frame $\mathcal{F}$, $\mathcal{F} \nvDash \forall x \Box A \to \Box \forall x A$ for some formula A. Then there are a world w, an interpretation I and an assignment σ such that (a) $\vDash_w^\sigma \forall x \Box A(x)$ and (b) $\vDash_w^\sigma \Diamond \exists x \neg A(x)$. Given that $D_w := \{d_1, d_2, \ldots\}$ and $\|D_w\| = m$, from (a) it follows that there are m w-assignments $\sigma_1, \sigma_2, \ldots$ which are all the x-variants of σ such that $\sigma_i(x) \in D_w$ and $\vDash_w^{\sigma_i} \Box A(x)$. Thus there is a set of relations $\{R_1, R_2, \ldots\}$ of cardinality m such that for any R_i, $R_i(w) \subseteq \|A(x)\|_I^{\sigma_i}$.

From (b) it follows that there is a sequence $z_1, z_2, \ldots$ of worlds such that for each R_i, $w R_i z_i$ (hence for each R_i, $R_i(w) \neq \varnothing$) and for each i, $\vDash_{z_i}^\sigma \exists x \neg A(x)$. Thus for each z_i there is some z_i-assignment ϑ_i such that $\vartheta_i(x) \in D_{z_i}$ and $\vDash_{z_i}^{\vartheta_i} \exists x \neg A(x)$.

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Clearly all the worlds t belonging to the intersection $\bigcap_{i \leq m} R_{j_i}(w)$ are such that $\vDash_t^{\sigma_i} A(x)$ for all the x-variants of σ such that $\sigma_i(x) \in D_w$ and hence if a world z_i belongs to such intersection, the z_i-assignment ϑ_i must be such that $\vartheta_i(x) \in D_{z_i}$ but $\vartheta_i(x) \notin D_w$ and therefore for any such world $D_w \subset D_{z_i}$. ∎

Again, mirroring Kripke semantics, the schemata **BF** and **CGF** := $\Box \exists x A \to \exists x \Box A$ are equivalent.

Lemma 7.3 *The schemata* **BF** *and* **CGF** *characterise the same class of multi-relational frames.*

PROOF. ($\Rightarrow$) Assume that $\exists w \in W$ such that for all $R_i \in \mathcal{R}$, $R_i(w) \neq \varnothing$ and for some (possibly infinite) sequence $z_1, z_2, \ldots$ such that $wR_k z_k$ for any $R_k \in \mathcal{R}$, for some (possibly infinite) sequence of relations $R_{j_1}, R_{j_2}, \ldots$ of length m (if the sequence is infinite m is $\aleph_0$), where m is the number of individuals belonging to D_w, the following holds: for all $t \in W$, if $t \in \{z_1, z_2, \ldots\} \bigcap_{i \leq m} R_{j_i}(w)$ then $D_w \subset D_t$.

We shall now define an interpretation I in order to build a countermodel for **CGF** in this frame. Let $D_w := \{d_1, d_2, \ldots\}$ of cardinality m and consider the sequence of w-assignments $\sigma_1, \sigma_2, \ldots$ of length m such that for any σ_i, $\sigma_i(x) = d_i$. For some unary predicate P, for any i, let $\|P(x)\|_I^{\sigma_i} := R_{j_i}(w)$. Then clearly $\vDash_w^{\sigma_i} \Box P(x)$ for each i and hence, since $\sigma_1, \sigma_2, \ldots$ are all the x-variants of a w-assignment σ such that $\sigma_i \in D_w$ for each i, $\vDash_w^{\sigma} \forall x \Box P(x)$.

Consider now any z_i from the sequence $z_1, z_2, \ldots$ and define a z_i-assignment ϑ_i such that $\vartheta_i(x) \notin I_{z_i}^{\vartheta_i}(P(x))$ and $\vartheta_i(x) \in D_{z_i}$. Notice that it is always possible to define such an assignment. In fact even if $z_i \in \bigcap_{i \leq m} R_{j_i}(w)$, by assumption we have that $D_{z_i} \supset D_w$. Hence $\vDash_{z_i}^{\vartheta_i} \neg P(x)$ and hence $\vDash_{z_i}^{\sigma} \neg \exists x P(x)$ which implies that $\nvDash_w^{\sigma} \Box \forall x P(x)$.

($\Leftarrow$) Assume that for some frame $\mathcal{F}$, $\mathcal{F} \nvDash \forall x \Box A \to \Box \forall x A$ for some formula A. Then there are a world w, an interpretation I and an assignment σ such that (a) $\vDash_w^{\sigma} \forall x \Box A(x)$ and (b) $\vDash_w^{\sigma} \Diamond \exists x \neg A(x)$. Given that $D_w := \{d_1, \ldots, d_m\}$, from (a) it follows that there are $\sigma_1, \sigma_2, \ldots$ w-assignments which are all the x-variants of σ such that $\sigma_i(x) \in D_w$ and $\vDash_w^{\sigma_i} \Box A(x)$. Thus there is a set of relations $R_1, R_2, \ldots$ such that for any R_i, $R_i(w) \subseteq \|A(x)\|_I^{\sigma_i}$.

From (b) it follows that there is a sequence $z_1, z_2, \ldots$ of length n of worlds such that for each R_i, $wR_i z_i$ (hence for each R_i, $R_i(w) \neq \varnothing$) and for each i, $\vDash_{z_i}^{\sigma} \exists x \neg A(x)$. Thus for each z_i there is some z_i-assignment ϑ_i such that $\vartheta_i(x) \in D_{z_i}$ and $\vDash_{z_i}^{\vartheta_i} \exists x \neg A(x)$.

Clearly all the worlds t belonging to the intersection $\bigcap_{i \leq m} R_{j_i}(w)$ are such that $\vDash_t^{\sigma_i} A(x)$ for all the x-variants of σ such that $\sigma_i(x) \in D_w$ and hence if a world z_i belongs to such intersection, the z_i-assignment ϑ_i must be such that $\vartheta_i(x) \in D_{z_i}$ but $\vartheta_i(x) \notin D_w$ and therefore for any such world $D_w \subset D_{z_i}$. ∎

By translating in terms of multi-relational semantics also Stolpe (2003)'s CUPI condition (see Theorem 2.8) it is quite straightforward to prove a weaker correspondence result for **BF** in multi-relational models. However this implies conditions on models instead of frames.

Lemma 7.4 (CUPI models) *For any multi-relational model* $\mathcal{M} := \langle W, \mathcal{R}, D, U, I \rangle$, $\mathcal{M} \vDash \forall x \Box A \to \Box \forall x A$ *if and only if for any world w the following holds: given a w-assignment σ, if for any x-variant σ_i such that $\sigma_i(x) \in D_w$ there exists a relation R_i such that $R_i(w) \subseteq \|A(x)\|_I^{\sigma_i}$, then there is a relation R_k such that $R_k(w) \subseteq \|\forall x A\|_I^{\sigma}$.*

PROOF. The left to right arrow is trivial. For the other direction consider the contrapositive proposition. Assume that for some world $w \in W$ it holds true that, for all w-assignments σ, (a) for any x-variant σ_i of σ, $\sigma_i(x) \in D_w$, there exists a relation R_i such that $R_i(w) \subseteq \|A(x)\|_I^{\sigma_i}$ and (b) for any relation R_k, $R_k(w) \nsubseteq \|\forall x A\|_I^{\sigma}$. From (a) it follows that for any σ_i, $\vDash_w^{\sigma_i} \Box A(x)$ and hence $\vDash_w^{\sigma} \forall x \Box A(x)$ whereas from (b) it follows that $\nvDash_w^{\sigma} \Box \forall x A$, thus $\nvDash_w^{\sigma} \forall x \Box A \to \Box \forall x A$. ∎

7.2 The Independence of the Schemata C and BF

Recall that the schema **C** imposes on frames the following property: $\mathcal{F} \vDash \Box A \wedge \Box B \to \Box(A \wedge B)$ iff for any world w, for any relation R_i, R_k there exists a relation R_j such that $R_j(w) \subseteq R_k(w) \cap R_i(w)$. Hence for any **C**-frame, the characterisation result for **BF** is the following:

Lemma 7.5 (BF on C-frames) *Let $\mathcal{F}$ be any **C**-frame. Then for any world $w \in W$, given that $\|D_w\| = k$, if w is locally serial, then for all sets of relations with cardinality up to k $\{R_1^i, \ldots, R_k^i\}$, for all worlds $v \in R_1^i \cap \ldots \cap R_k^i$ it holds that $D_w \supseteq D_v$.*

PROOF. Indeed if we assume that the **BF**-condition does not hold on a **C**-frame, we would get the following situation. For some world w such that w is locally serial and $\|D_w\| = k$, there is some set $R_1, \ldots, R_k$ of relations such that for some world $v \in R_1(w) \cap \ldots \cap R_k(w)$ it holds that $D_w \not\supseteq D_v$. It is easy to build a counter-model in such situation. In Kripke semantics this is equivalent to denying decreasing domains. ∎

It is easy to find a frame validating **C** but not **BF**: It is enough to consider the frame described in Lemma 7.1, which is Kripkean and hence validates **C**.

It is not hard to show that the schema **BF** does not imply **C**. Below we describe a countermodel for **C** based on a frame for **BF**. Consider the model $\mathcal{M} := \langle W, \mathcal{R}, D, U, I \rangle$ where $W = \{w, z_1, z_2\}$, $\mathcal{R} := \{R_1, R_2\}$, $R_1 := \{\langle w, z_1 \rangle\}$, $R_2 := \{\langle w, z_2 \rangle\}$, $D_w = D_{z_1} = D_{z_2} = \{d\}$, σ is a w-assignment such that $\sigma(x) = d$. Let P and Q be two unary predicates such that $\|P(x)\|_I^\sigma = \{z_1\}$ and $\|Q(x)\|_I^\sigma = \{z_2\}$. Then $\vDash_w^\sigma \Box P(x) \wedge \Box Q(x)$ but $\nvDash_w^\sigma \Box(P(x) \wedge Q(x))$. Moreover $\mathcal{M}$ is built on a frame which fulfills the conditions imposed by **BF**. Indeed w is locally serial but there is only one tuple $\langle z_1, z_2 \rangle$ such that wR_1z_1 and wR_2z_2. The number of individuals from D_w is 1, thus: $z_1 \in \{z_1, z_2\} \cap R_1(w)$ and $D_w \subseteq D_{z_1}$ and $z_2 \in \{z_1, z_2\} \cap R_2(w)$ and $D_w \subseteq D_{z_2}$. Notice that any countermodel for **C** based on a **BF**-frame must contain a locally serial world w whose inner domain contains at most one individual and the frame of such model should have more than one relation.

Corollary 7.6 *The schemata **BF** and **C** are semantically mutually independent.*

The importance of this result is limited at the moment, as it lays only on semantic grounds. In fact, without a completeness result, one cannot infer that adding **BF** to a system does not generate normal systems, i.e., it is not yet clear whether **BF** does or does not syntactically imply **C**.

8 The Role of Identity

The choice of a language with the identity relation can be driven by different reasons. First of all, the expressive power increases greatly. Nevertheless, there is also a technical reason behind it. Working with the identity symbol allowed us to build canonical models for **CBF**-frames, a very difficult goal to be achieved without it. It is actually much easier to prove completeness for systems without identity, at least for those systems which do not include any form of the Barcan schemata.

8.1 Completeness without Identity

Below we present a completeness theorem for an analogous system built on a language without identity. The axiomatic system $\mathsf{Q}^\circ.\mathsf{MN}$ is obtained by deleting all the schemata concerning identity:

The system $\mathsf{Q}^\circ.\mathsf{MN}$ contains the following axioms and inference rules:

- Propositional tautologies;
- $\mathbf{UI}^\circ := \forall y(\forall x A(x) \to A(y/x))$
- $\forall x \forall y A \leftrightarrow \forall y \forall x A$
- $A \to \forall x A$, x not free in A
- $\forall x(A \to B) \to (\forall x A \to \forall x B)$

- $\mathbf{M} := \Box(A \wedge B) \to (\Box A \wedge \Box B)$
- $\mathbf{N} := \Box \top$ to $\mathsf{Q}_=^\circ.\mathsf{E}$
- $\mathbf{MP} := A \to B, A/B$
- $\mathbf{RE} := A \leftrightarrow B/\Box A \leftrightarrow \Box B$
- $\mathbf{UG} := A/\forall x A$

Theorem 8.1 (Soundness) *The system $Q^\circ.MN$ is sound with respect to the class of all multi-relational frames with varying inner domains.*

PROOF. It follows from Theorem 6.1. ∎

Lemma 8.2 *Let Δ be an L-consistent set of sentences of $\mathcal{L}$. Then for some not-empty denumerable set C of new constants, there is a set Π of sentences of $\mathcal{L}^C$ such that $\Delta \subseteq \Pi$, Π is L^C-maximal, Π is Q-universal and Q-rich for some set $Q \subseteq Const(\mathcal{L}^C)$. (cf. (Corsi, 2002, Lemma 1.16))*

Definition 8.3 (Non-normal Canonical Models) *Let L be a non-normal quantified modal logic on the language $\mathcal{L}$ such that $L \supseteq Q^\circ.MN$. Let Q be a set of constants of cardinality $\aleph_0$ such that $Q \supset Const(\mathcal{L})$ and $\|Q - Const(\mathcal{L})\| = \aleph_0$. A non-normal canonical model $\mathcal{M}^L = \langle W, \mathcal{R}, D, U, I \rangle$ for L is defined as follows:*

- W is the class of all L_w-saturated sets of sentences w, where $\mathcal{L}_w = \mathcal{L}^S$ for some set S of constants such that $Const(\mathcal{L}^S) \neq \varnothing$, $S \subset Q$ and $\|Q - Const(\mathcal{L}^S)\| = \aleph_0$.

- For any formula $A \in Fma(\mathcal{L})$ let R_A be a binary relation over W. For all $w, v \in W$, $wR_A v$ iff $\Box A \in w \Rightarrow A \in v$. The set of relations $\mathcal{R}$ is the collections of all such relations.

- $D_w = \{c \in Const(\mathcal{L}_w) \mid \forall x A \to A(c/x) \in w$, for all sentences $\forall x A$ of $\mathcal{L}_w\}$

- $U_w = Const(\mathcal{L}_w)$

- $I_w(c) = c$

- $I_w(P^n) = \{\langle c_1, \ldots, c_n \rangle \mid P^n(c_1, \ldots, c_n) \in w\}$.

Lemma 8.4 (Existence lemma) *Given a canonical model $\mathcal{M}$ for $Q^\circ.MN$, for any $w \in W$, if $\Diamond A \in w$, then for any formula $B \in Fma(\mathcal{L})$ there is a state z such that $wR_B z$ and $A \in z$.*

The proof follows directly from Lemma 6.14 by Calardo and Rotolo (2014).

Lemma 8.5 (Truth Lemma) *Given a canonical model $\mathcal{M}^L = \langle W, \mathcal{R}, D, U, I \rangle$ for a quantified $\mathbf{N}$-Monotonic modal logic L extending $Q^\circ.MN$, for any formula $A \in Fma(\mathcal{L})$, for any world $w \in W$, the following holds: $\vDash_w^\sigma A(x_i) \iff A(\sigma(x_i)/x_i) \in w$.*

PROOF. By induction on the length of a formula A. We omit details of the induction base. Suppose the length of A is $n + 1$ and A has the form $\exists x B(x, y_1, \ldots, y_m)$.

(i) $\vDash_w^\sigma \exists x B(x, y_1, \ldots, y_m)$ iff for some x-variant τ of σ such that $\tau(x) \in D_w$, $\vDash_w^\tau B(x, y_1, \ldots, y_m)$. Suppose $\tau(x) = c$. Since by assumption all constants are rigid designators, by Lemma 4.7 it holds that $\vDash_w^\sigma B(c/x, y_1, \ldots, y_m)$. Hence $B(c/x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$ by IH. Since w is D_w-universal, w is also D_w-existential by Lemma 6.4 and hence $\exists x B(x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$.

(ii) Assume $\exists x B(x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$. Since w is D_w-rich, $B(c/x, \sigma(y_1), \ldots, \sigma(y_m)) \in w$ for some $c \in D_w$. Thus, by IH, $\vDash_w^\sigma B(c/x, y_1, \ldots, y_m)$ and, by Lemma 4.7 it holds that $\vDash_w^\tau B(x, y_1, \ldots, y_m)$ for some w-assignment τ which is an x-variant of σ such that $\tau(x) = I_w(c) = c$. Therefore $\vDash_w^\sigma \exists x B(x, y_1, \ldots, y_m)$. If $lg(A) = n + 1$ and A has the form $\Box B$, please refer to Lemma 48 (Truth Lemma) in (Calardo and Rotolo, 2014). ∎

Lemma 8.6 *Let $\mathcal{M}^L := \langle W, \mathcal{R}, D, U, I \rangle$ be a canonical model for a logic $L \supseteq Q^\circ.MN$. If Δ is an L-consistent set of formulae, then for some $w \in W$ and some w-assignment σ, $\mathcal{M}^L \vDash_w^\sigma D$ for any $D \in \Delta$. (cf. Corsi, 2002, Lemma 1.19)).*

Let L be any logic $L \supseteq Q^\circ.MN$. Consider any formula A such that $\nvdash_L A$. Then $\{\neg A\}$ is L-consistent. By Lemma 6.10 there is a world w of a canonical model $\mathcal{M}^L$ for L and a w-assignment σ such that $\mathcal{M}^L \nvDash_w^\sigma A$ and hence $\mathcal{M}^L \nvDash A$.

Theorem 8.7 (Completeness of $Q^\circ.MN$) *The logic $Q^\circ.MN$ is strongly complete with respect to the class of all multi-relational frames.*

8.2 Some Remarks on Identity

The completeness proof for a system without identity follows from the propositional results (see Calardo and Rotolo, 2014) and from those stated by Corsi (2002). However, things change radically when the identity symbol is included. As Corsi (2002) observes:

> In a language with identity, the fact that constants are rigid designators can be expressed by the schema:
>
> $$(x = a) \rightarrow \Box(x = a)$$
>
> Therefore (...) all the systems of Q.M.L. with identity we are going to discuss are bound to be systems with rigid terms. (Corsi, 2002, 1499)

However this ceases to be true in the broader framework of multi-relational semantics. In fact if there is only one relation within a frame (the Kripkean case), the problem of rigidity can be easily solved by stating that if two names denote the same individual in a world, then this is bound to be the case in *all* accessible worlds. On the other hand, if interpreted in the multi-relational case, this would only state that *under some accessibility relation* this couple of names denote the same individual. This is obviously not enough. Our version of the necessity of identity schema

$$\mathbf{NI} := \Box A \wedge (a = b) \rightarrow \Box(A \wedge (a = b))$$

says something more. It states that *for any formula*, i.e., semantically, for any relation, this must hold. It is a restricted form of axiom **C**, which holds only for specific formulae, namely identities. The same holds for the necessity of diversity, **ND**. The resulting system is still a proper subset of K even though it includes a restricted version of the aggregation schema **C**.

This may explain what happened in Lemma 6.8, in the construction of the set Γ_1. In the proof of Lemma 6.8, the set Γ_0 is built to ensure the truth of modal formulae:

$$\Gamma_0 :=$$

1. If $\Box B \in w$, then $\Gamma_0 := \{B\} \cup \{A\}$
2. otherwise, $\Gamma_0 := \{A\}$.

This is not sufficient to preserve rigidity of denotation. However, it can be amended by adding a further step in the construction:

$$\Gamma_1 := \Gamma_0 \cup \{(a = b) \mid (a = b) \in w\}$$

This further step is necessary to keep rigidity of designation as well as the validity of axiom **NI**. Moreover, the consistency of Γ_1 is guaranteed by the presence of **NI** in the system.

This is even more obvious within the proof of Theorem 6.13, where we make an essential use of identity formulae. In fact, when it comes to prove completeness results for systems including **CBF**, if identity is not present, it is very difficult to denote the same individual across different worlds. One may think about adding an existence predicate, but this would be just a definite description in disguise. The standard way to introduce an existence predicate is to add to the language a unary predicate symbol E whose extension, for any world, is equal to the inner domain:

$$\vDash^{\sigma}_w E(x) \text{ if and only if } \sigma(x) \in D_w$$

This is clearly equivalent to state that $E(a)$ is satisfied in a world if and only if the formula $\exists x(x = a)$ holds. Hence, adding an existence predicate would make very little difference, if not at all, on the technical level and would not take us any closer to proof completeness for systems including **CBF**.

9 Summary and Further Work

This paper provided a semantic study in multi-relational semantics of quantified **N**-Monotonic modal logics with varying domains with and without the identity symbol ($\mathsf{Q}^{\circ}_{=}.\mathsf{MN}$ and $\mathsf{Q}^{\circ}.\mathsf{MN}$). We also identified conditions on frames to characterize Barcan and Ghilardi schemata and presented some completeness results. The characterisation of **BF** and **CBF** in multi-relational frames with varying domains has shown that **BF** and **C** are mutually independent and that **M** does not imply the validity of **CBF** (**M** is an axiom of MN). This fact was suggested for classical modal systems by Stolpe (2003) but unfortunately that work used only models and not frames.

In detail, in this paper

- We provided a completeness result for $\mathsf{Q}^{\circ}_{=}.\mathsf{MN}$;

- We provided a completeness results for $\mathsf{Q}^{\circ}.\mathsf{MN}$;

- We proved the semantic equivalence for **CBF** and **GF**, and for **BF** and **CGF**;

- We provided frame characterisation results for both **CBF**, and **BF**;

- We provided a completeness result for $\mathsf{Q}^{\circ}_{=}.\mathsf{MN} \oplus \mathbf{CBF} \oplus \Box\exists$;

- We showed that Schema **M** does not entail **CBF**;

- We proved that schema **C** and **BF** are semantically independent.

Problems related to the completeness of non-normal systems, with and without identity, that include **BF**, are still open and require a deeper analysis. In systems without identity, this problem concerns both **CBF** and **BF**. Moreover, we see other important directions for future research.

A preliminary question—which is out of scope for this paper—regards the general relation between multi-relational and neighbourhood semantics. For propositional non-normal modal logics the two semantics are equivalent (Governatori and Rotolo, 2005). However, it is not obvious if they are still equivalent (in regard, e.g., to completeness and incompleteness results) above K: Normal systems do not necessarily make Governatori and Rotolo (2005)'s semantics collapse on Kripke's. Regarding quantified modal logics, a full comparison of the two semantics is anyway needed.

An immediate extension of the present work is to consider constant domains ($\mathsf{FOL} \oplus \mathsf{MN}$). It is definitely less trivial to adopt Governatori and Rotolo (2005)'s semantics and study the open problem that Arló-Costa (2011) mentioned for $\mathsf{FOL} \oplus \mathsf{E} \oplus \mathbf{CBF}$, as well as to investigate other quantified (classical, monotonic, and regular) systems with constant and varying domains.

Finally, a question concerning (Kracht and Wolter, 1999; Gasquet and Herzig, 1996)'s proof that non-normal modal logics can be simulated by a normal modal logic with three modalities. The key idea is to translate the neighbourhood models into Kripke models with three suitable relations as well as the $\Box$-formulae into three-modality formulae. This result, which holds for MN, is of great interest in the context of multi-relational semantics, whose structures can in fact recall Kripke frames for multi-modal logics: indeed, $\Box$-formulae are evaluated in multi-relational semantics by considering one existing relation among a set of accessibility relations, a plurality of relations that occurs, too, in Kripke structures for multi-modal logics. However, when predicate calculi are added to the propositional modal base we can obtain unexpected interactions between Barcan schemata and modal axioms: while for propositional multi-relational frame, one can easily adapt (Kracht and Wolter, 1999; Gasquet and Herzig, 1996)'s translation, it is not clear that the translation holds for first-order extensions. Hence, extending Kracht and Wolter (1999)'s case to quantified non-normal modal logics is an open question.

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