

QUANTIFIER ELIMINATION FOR LEXICOGRAPHIC PRODUCTS OF ORDERED ABELIAN GROUPS

By

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Abstract. Let $L_{\text{ag}} = \{+, -, 0\}$ be the language of the abelian groups, L an expansion of $L_{\text{ag}}(<)$ by relations and constants, and $L_{\text{mod}} = L_{\text{ag}} \cup \{\equiv_n\}_{n \geq 2}$ where each \equiv_n is defined as follows: $x \equiv_n y$ if and only if $n \mid x - y$. Let H be a structure for L such that $H \upharpoonright L_{\text{ag}}(<)$ is a totally ordered abelian group and K a totally ordered abelian group. We consider a product interpretation of $H \times K$ with a new predicate I for $\{0\} \times K$ defined by N. Suzuki [9].

Suppose that H admits quantifier elimination in L .

1. If K is a Presburger arithmetic with smallest positive element 1_K then the product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, 1) \cup L_{\text{mod}}$ with $1^G = (0^H, 1_K)$.
2. If K is dense regular and K/nK is finite for every integer $n \geq 2$ then the product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$ for some set D of constant symbols where $G \models I(d)$ for each $d \in D$.
3. If K admits quantifier elimination in $L_{\text{mod}}(<, D)$ for some set D of constant symbols then the product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$ unless K is dense regular with K/nK being infinite for some n .

Conversely, if the product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$ for some

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set D of constant symbols such that $G \models I(d)$ for each $d \in D$ then H admits quantifier elimination in $L \cup L_{\text{mod}}$, and K admits quantifier elimination in $L_{\text{mod}}(<, D)$.

We also discuss the axiomatization of the theory of the product interpretation of $H \times K$.

Introduction

Throughout the paper, “ordered abelian group” will stand for “totally ordered abelian group”.

Komori [7] and Weispfenning [12] had shown that the direct product $\mathbf{Z} \times \mathbf{Q}$ equipped with the lexicographic ordering admits quantifier elimination in a language expanding the language of the ordered abelian groups $\{+, -, 0, <\}$. Here, \mathbf{Z} is a Presburger arithmetic (the ordered abelian group of the integers), and \mathbf{Q} a divisible ordered abelian group (the ordered abelian group of rational numbers). They also gave a concrete axiomatization (recursive axiomatization) for the theory of $\mathbf{Z} \times \mathbf{Q}$. Weispfenning [12] extensively studied quantifier elimination in the language

$$\{+, -, 0, <\} \cup \{\equiv_n^i\}_{i \leq k, n < \omega} \cup \{I_i\}_{i \leq k}$$

where the I_i for $i \leq k$ represent convex subgroups such that $I_k \supseteq I_{k-1} \supseteq \cdots \supseteq I_0$ and each \equiv_n^i is a binary relation defined by $x \equiv_n^i y \Leftrightarrow \exists z (I_i(z) \wedge n \mid (x - y - z))$. Suzuki [9] has defined a product interpretation of $H \times K$ in the language $L(I)$ equipped with the lexicographic ordering where H is an L -structure for a language L expanding $\{+, -, 0, <\}$ by adding relation symbols and constant symbols such that the reduct of H to $\{+, -, 0, <\}$ is an ordered abelian group, K is also an ordered abelian group, and I is interpreted as the set $\{0\} \times K$. He has shown that if H admits quantifier elimination in L and K is a divisible ordered abelian group then the product interpretation of $H \times K$ admits quantifier elimination in the language $L(I)$. Moreover, the theory of $H \times K$ is determined by the theory of H and it is recursively axiomatizable if the theory of H is. Tanaka and Yokoyama [11] gave another proof. We will show a similar result when K is a Presburger arithmetic or a dense regular abelian group instead of a divisible ordered abelian group. We also show a similar result when K is an ordered abelian group which admits quantifier elimination in $L_{\text{mod}}(<, D)$ for some set D of constant symbols. In the case that H admits quantifier elimination in $L_{\text{mod}}(<, C)$ for some set C of constant symbols, our results follow from Weispfenning’s results [12, 13]. But we believe that our proof is simpler. Choose an ordered

abelian group H_0 , and let H be an expansion of H_0 by relations and constants which admits quantifier elimination. If the form of the language of H is different from $L_{\text{mod}}(<, C)$ for any set of constant symbols C , then we get a new example of product interpretation of $H \times K$ which admits quantifier elimination.

Tanaka and Yokoyama have shown that if $H \equiv H'$ and $K \equiv K'$ in appropriate languages then $H \times K \equiv H' \times K'$. Let us denote the theory of a structure M by $Th(M)$. We present an axiomatization of $Th(H \times K)$ depending on $Th(H)$ and $Th(K)$. Furthermore, if $Th(H)$ and $Th(K)$ are recursively axiomatizable then so is $Th(H \times K)$.

1. Preliminaries

We follow the notation of Hodges' book [5] in general. Throughout the paper, we use the symbols “+”, “−”, “0”, “<” and “ I ”, where “+” is a binary function symbol, “−” a unary function symbol, “0” a constant symbol, “<” a binary relation symbol, and “ I ” a unary relation symbol. Let $L_{\text{ag}} = \{+, -, 0\}$. If L is a language, s_1, s_2, \dots, s_n are new symbols and C is a set of new constant symbols, then $L(s_1, s_2, \dots, s_n, C)$ denotes the language $L \cup \{s_1, s_2, \dots, s_n\} \cup C$, and $L(s_1, s_2, \dots, s_n)$ denotes the language $L \cup \{s_1, s_2, \dots, s_n\}$. We say that L' is an *expansion of L by relations and constants* if L' can be obtained by adding relation symbols and constant symbols to L .

If L is a language and M is an L -structure, $\text{dom}(M)$ denotes the domain or the universe of M , s^M denotes the interpretation of s in M for each symbol s of L . We often omit “dom” from “ $\text{dom}(M)$ ”. Hence, “ $x \in M$ ” will stand for “ $x \in \text{dom}(M)$ ”. For a map f and a subset X of the domain of f , $f|X$ denotes the restriction of f to X . If M is an L -structure and $X \subseteq M$, $M|X$ is a structure with domain X such that $R^{M|X} = R^M \cap X^n$ for each n -ary relation symbol R of L , $f^{M|X} = f^M|X^n$ for each n -ary function symbol f of L , and $c^{M|X} = c^M$ for each constant symbol c of L if $c^M \in X$. Note that $f^{M|X}$ might be a partial map on X in general, and $c^{M|X}$ might be non-existing. $M|X$ is an L -substructure of M if $f^{M|X}$ is a total function from X^n to X for every function symbol f of L , and $c^M \in X$ for every constant symbol c of L (i.e., $M|X$ is an L -structure). Let M be an L -structure and M' an expansion of M to a language L' . M' is called a *definitional expansion of M* if every non-logical symbol of L' is definable in M' by an L -formula.

If f is a function and $\bar{a} = (a_1, \dots, a_n)$ is a tuple of elements a_1, \dots, a_n from the domain of f , $f(\bar{a})$ denotes the tuple $(f(a_1), \dots, f(a_n))$. If $\bar{a} = (a_1, \dots, a_n)$ and b is an element, $\bar{a} \hat{\ } b$ denotes the tuple (a_1, \dots, a_n, b) and $b \hat{\ } \bar{a}$ denotes the tuple (b, a_1, \dots, a_n) .

If L is a language and M is an L -structure, we also call M a structure for L . If two structures are elementarily equivalent as L -structures, we also say that the two structures are elementarily equivalent for L . If $\bar{y} = (y_1, \dots, y_n)$ is a tuple of variables, $\forall \bar{y} \varphi(\bar{y})$ stands for $\forall y_1 \cdots \forall y_n \varphi(y_1, \dots, y_n)$. To dispense with parentheses in formulas, we follow the following hierarchy of precedences for logical operators and quantifiers. \neg has higher precedence than any other logical operators, \wedge has higher precedence than \vee , \vee has higher precedence than \rightarrow and \leftrightarrow , and the quantifiers \forall and \exists have lower precedence than any logical operators. For example, the formula

$$\forall x, y \quad x^2 = y^2 \wedge x \neq y \rightarrow x = -y \wedge x \neq 0$$

stands for

$$(\forall x(\forall y((x^2 = y^2 \wedge x \neq y) \rightarrow (x = -y \wedge x \neq 0))))).$$

When we write $s < t$, sometimes we allow s to be $-\infty$ and t to be ∞ . We consider $-\infty < t$ and $s < \infty$ to be formulas that are always true. For example, $s < x < t$ with $s = -\infty$ stands for $x < t$, $s < x < t$ with $t = \infty$ stands for $s < x$, and $s < x < t$ with $s = -\infty$ and $t = \infty$ stands for a formula that is always true.

DEFINITION 1.1. An L -structure M admits *quantifier elimination* if for any formula $\varphi(\bar{y})$ of L with a tuple of free variables \bar{y} , there is a quantifier-free formula $\psi(\bar{y})$ of L such that

$$M \models \forall \bar{y} \quad \varphi(\bar{y}) \leftrightarrow \psi(\bar{y}).$$

A theory T in L admits *quantifier elimination* if for any formula $\varphi(\bar{y})$ of L with a tuple of free variables \bar{y} , there is a quantifier-free formula $\psi(\bar{y})$ of L such that

$$T \vdash \forall \bar{y} \quad \varphi(\bar{y}) \leftrightarrow \psi(\bar{y}).$$

We often consider a definitional expansion M' of M to some extended language L' . When the defining L -formulas of all the new symbols of L' is given, any L -structure can naturally be expanded to an L' -structure. We say that M admits *quantifier elimination in L'* if the definitional expansion M' of M to L' admits quantifier elimination. In the case that L'' is a sublanguage of L' , we also say that M admits *quantifier elimination in L''* if $M'|L''$ admits quantifier elimination.

For the basic definitions and facts on (ordered) abelian groups, we refer the reader to [3] and [4]. Nevertheless, we will review some definitions and facts.

For a set X , id_X denotes the identity map on X . For a term t of L_{ag} , $0 \cdot t$ denotes 0 , $1 \cdot t$ denotes t , $2 \cdot t$ denotes $t + t$, $3 \cdot t$ denotes $t + t + t$, and so on. In this way, $m \cdot t$ is defined for any non-negative integer m . For any negative integer m , $m \cdot t$ denotes the term $-(|m| \cdot t)$. We sometimes write mt for $m \cdot t$ when there will be no confusion. Let $L_{\text{mod}} = L_{\text{ag}} \cup \{\equiv_n : n \geq 2\}$ where each \equiv_n is a binary relation defined by $x \equiv_n y \Leftrightarrow \exists z (x - y = nz)$. Any abelian group can be considered as an L_{mod} -structure with this definition. For a natural number n , $n|x$ denotes the formula $\exists z (x = nz)$.

DEFINITION 1.2 (Abelian Group). An L_{ag} -structure A is called an *abelian group* if

$$A \models \forall x, y, z \quad (x + y) + z = x + (y + z),$$

$$A \models \forall x, y, z \quad x + 0 = 0 + x = x,$$

$$A \models \forall x, y, z \quad x + (-x) = (-x) + x = 0, \quad \text{and}$$

$$A \models \forall x, y \quad x + y = y + x.$$

If an L_{ag} -structure A is an abelian group, L_{ag} -substructure of A is called a *subgroup of A* . If B is a subgroup of an abelian group and $a \in A$, $a + B = \{a + x : x \in B\}$ is called a *coset of B in A* . A coset of B which is different from B is called a *proper coset of B* . For an abelian group A , let $nA = \{nx : x \in A\}$ for an integer n .

DEFINITION 1.3. Suppose an L_{ag} -structure A is an abelian group. A subgroup B of A is called *pure* if for any positive integer n and for any $b \in B$, $A \models \exists x (nx = b)$ implies $B \models \exists x (nx = b)$. If B is a pure subgroup of A , then B is an L_{mod} -substructure of A .

A subgroup B of an abelian group is called *divisible* if $nB = B$ for every positive integer n . An abelian group A is called *torsion-free* if $A \models \forall x (x \neq 0 \rightarrow nx \neq 0)$ for every integer $n > 0$. Suppose A is an abelian group and B and C are subgroups of A . If $A = \{b + c : b \in B, c \in C\}$ and $B \cap C = \{0\}$ then we call A the *direct sum* (or the internal direct sum) of B and C and write $A = B \oplus C$. In this case, B is called a *direct summand* of A . C is also a direct summand of A . Every direct summand of an abelian group is a pure subgroup.

FACT 1.4. *Let A be an abelian group and B its subgroup. B is a direct summand of A if and only if there is a group homomorphism $\pi : A \rightarrow B$ such that $\pi|_B = \text{id}_B$.*

DEFINITION 1.5 (Direct Product). Suppose L_{ag} -structures B and C are abelian groups. Let A be an L_{ag} -structure with $\text{dom}(A) = \text{dom}(B) \times \text{dom}(C)$ (a product set) such that $0^A = (0^B, 0^C)$, $(x_1, y_1) +^A (x_2, y_2) = (x_1 +^B x_2, y_1 +^C y_2)$, and $-^A(x, y) = (-^B x, -^C y)$. A is called the *direct product* (or *external direct sum*) of B and C . Let $B' = \{(b, 0^C) : b \in \text{dom}(B)\}$ and $C' = \{(0^B, c) : c \in \text{dom}(C)\}$. $A|B'$ and $A|C'$ are subgroups of A and are isomorphic to B and C respectively as groups (L_{ag} -structures). A is the (internal) direct sum of $A|B'$ and $A|C'$.

FACT 1.6. *Let A be a torsion-free abelian group. Any equation $nx = a$ with $n \in \mathbf{Z}$ and $a \in A$ has at most one solution in A . Intersections of pure subgroups of A are again pure in A . For every subset S of A , there exists a minimal pure subgroup containing S . This subgroup is called the pure subgroup generated by S .*

The following fact is Theorem 38.1 together with Exercise 4 and 5 on p. 162 in [4]. Eklof and Fisher called an abelian group ω_1 -*equationally compact* if it satisfies condition (5) of this fact, and pointed out this equivalence [2]. By an *equation over A* , we mean a formula of the form $t = a$ with a term t of L_{ag} (with variables) and $a \in A$. Note that any term of L_{ag} can be considered as a \mathbf{Z} -linear combination of variables in abelian groups.

FACT 1.7. *The following conditions on an abelian group A are equivalent:*

- (1) *If B is a pure subgroup of C , C/B is countable, and $f : B \rightarrow A$ is a group homomorphism, then there is a group homomorphism $g : C \rightarrow A$ such that $g|B = f$.*
- (2) *A is pure-injective: If B is a pure subgroup of C , and $f : B \rightarrow A$ a group homomorphism, then there is a group homomorphism $g : C \rightarrow A$ such that $g|B = f$.*
- (3) *A is algebraically compact: If A is a pure subgroup of C then A is a direct summand of C .*
- (4) *If every finite subsystem of a system of equations over A has a solution in A , then the whole system is solvable in A .*
- (5) *If every finite subsystem of a countable system of equations over A has a solution in A , then the whole system is solvable in A .*

FACT 1.8. *Let A be a torsion-free abelian group. Then for any positive integers m, n ,*

- (1) $A \models \forall x, y \ x \equiv_n y \leftrightarrow mx \equiv_{mn} my$,
- (2) $A \models \forall x, y \ x \equiv_n y \rightarrow mx \equiv_n my$, and
- (3) $A \models \forall x_1, x_2, y_1, y_2 \ x_1 \equiv_n y_1 \wedge x_2 \equiv_n y_2 \rightarrow x_1 + x_2 \equiv_n y_1 + y_2$.

The following lemma seems to be well-known but we could not find it in the literature. It is essentially due to Presburger [8].

LEMMA 1.9. *Suppose G is a torsion-free abelian group. Let $t_1(\bar{y}), \dots, t_n(\bar{y})$ be terms of L_{ag} with tuple \bar{y} of variables, and l_1, \dots, l_n positive integers. Then we can effectively find (by a recursive procedure) a quantifier-free formula $\theta(\bar{y})$ of L_{mod} such that*

$$G \models \forall y \left(\exists x \bigwedge_{i=1, \dots, n} x \equiv_{l_i} t_i(\bar{y}) \right) \leftrightarrow \theta(\bar{y}).$$

PROOF. First, we prove a claim.

CLAIM 1. *Let l and m be any positive integers and let d be the greatest common divisor of l and m . Since l/d and m/d are relatively prime integers, we can choose integers u, v such that $ul/d + vm/d = 1$. Then*

$$G \models \forall x, y, z \quad (x \equiv_l y \wedge x \equiv_m z) \leftrightarrow (x \equiv_{lm/d} (vm/d)y + (ul/d)z \wedge y - z \equiv_d 0).$$

Let $x, y, z \in G$ be arbitrary. Suppose $G \models x \equiv_l y$ and $G \models x \equiv_m z$. Then $G \models (m/d)x \equiv_{m/d} (m/d)y$ and $G \models (l/d)x \equiv_{m/d} (l/d)z$. Hence, $G \models (vm/d)x \equiv_{m/d} (vm/d)y$ and $G \models (ul/d)x \equiv_{m/d} (ul/d)z$. By adding terms on each side, we have $G \models x \equiv_{m/d} (vm/d)y + (ul/d)z$.

Also, since $G \models l \mid x - y$, $G \models m \mid x - z$, and $d \mid l, m$, we have $G \models d \mid x - y$ and $G \models d \mid x - z$, and thus $G \models d \mid y - z$.

Conversely, suppose that $G \models x \equiv_{lm/d} (vm/d)y + (ul/d)z$ and $G \models y - z \equiv_d 0$. Choose $w \in G$ such that $G \models y - z = dw$. Then in G ,

$$\begin{aligned} x &\equiv_{m/d} (vm/d)y + (ul/d)z \\ &= (vm/d + ul/d)y + (ul/d)(z - y) \\ &= 1 \cdot y - ulw \\ &\equiv_l y. \end{aligned}$$

Hence, $G \models x \equiv_l y$. Similarly, $G \models x \equiv_m z$. The claim is proved.

We prove the statement of the lemma by induction on the number n of conjuncts in the scope of “ $\exists x$ ”.

If $n = 1$, then we can always choose such x . Therefore, we can choose $0 = 0$ for $\theta(\bar{y})$.

If $n \geq 2$, by Claim 1, we have

$$G \models \forall \bar{y} \left(\exists x \bigwedge_{i=1, \dots, n} x \equiv_{l_i} t_i(\bar{y}) \right) \leftrightarrow t_1(\bar{y}) - t_2(\bar{y}) \equiv_d 0 \wedge \exists x \\ x \equiv_{l_1 l_2 / d} (vl_2 / d)t_1(\bar{y}) + (ul_1 / d)t_2(\bar{y}) \wedge \bigwedge_{i=3, \dots, n} x \equiv_{l_i} t_i(\bar{y})$$

where d is the greatest common divisor of l_1 and l_2 , and v, u are integers such that $ul_1 + vl_2 = d$. Note that l_1/d and l_2/d are integers.

By induction hypothesis, we can effectively eliminate “ $\exists x$ ” from the subformula

$$\exists x \quad x \equiv_{l_1 l_2 / d} (vl_2 / d)t_1(\bar{y}) + (ul_1 / d)t_2(\bar{y}) \wedge \bigwedge_{i=3, \dots, n} x \equiv_{l_i} t_i(\bar{y}). \quad \square$$

Quantifier elimination is known for abelian groups by Szmielew [10]. A shorter proof can be found in a Ziegler’s paper [14].

FACT 1.10 (Szmielew). *Any abelian group admits quantifier elimination in L_{mod} .*

DEFINITION 1.11 (Ordered Abelian Group). An $L_{\text{ag}}(<)$ -structure A is called an *ordered abelian group* if $A|L_{\text{ag}}$ is an abelian group, $<^A$ is a total order on $\text{dom}(A)$, and

$$A \models \forall x, y, z \quad x < y \rightarrow x + z < y + z.$$

If an $L_{\text{ag}}(<)$ -structure A is an ordered abelian group and B is a subgroup of $A|L_{\text{ag}}$, then the $L_{\text{ag}}(<)$ -substructure of A with domain $\text{dom}(B)$ is also an ordered abelian group.

Suppose an $L_{\text{ag}}(<)$ -structure A is an ordered abelian group. A subset B of A is called *convex* if for any $a, b \in B$ and for any $x \in A$, $A \models a < x < b$ implies $x \in B$. A *convex subgroup* of A is a subgroup of A whose domain is a convex subset of A . A subset B of A is called *dense* if for any $a, b \in A$, there is an element $x \in B$ such that $A \models a < x < b$. A *dense subgroup* of A is a subgroup of A whose domain is a dense subset of A .

If an $L_{\text{ag}}(<)$ -structure A is an ordered abelian group then $A|L_{\text{ag}}$ is a torsion-free abelian group, and any convex subgroup of A is a pure subgroup of A .

The ordered abelian groups which admit quantifier elimination in $L_{\text{mod}}(<)$ together with some set of constant symbols have been classified by Weispfenning [13].

DEFINITION 1.12. An ordered abelian group G is *dense regular* if it satisfies the following equivalent conditions:

(1) For any integer $n \geq 2$,

$$G \models \forall y, z \quad 0 < y \rightarrow \exists x \quad (0 < x < y \wedge x \equiv_n z).$$

(2) For any prime p , pG is dense in G .

(3) G is elementarily equivalent to a dense subgroup of the real numbers \mathbf{R} (a dense Archimedean group).

REMARK 1.13. Suppose n is an integer ≥ 2 . Then for any ordered abelian group G ,

$$G \models \forall y, z \exists x \quad y < x \wedge x \equiv_n z.$$

PROOF. Let $y, z \in G$ be arbitrary. If $y < z$ then the statement holds with $x = z$. If $y = z$, choose a positive element d in G . Then the statement holds with $x = z + nd$. If $z < y$, then $0 < y - z$. Then $y - z < n(y - z)$ since $n \geq 2$. Therefore, $y < z + n(y - z) \equiv_n z$. The statement holds with $x = z + n(y - z)$. \square

LEMMA 1.14. Let n be an integer ≥ 2 . For an ordered abelian group G , the following are equivalent:

(1) $G \models \forall b, c \quad 0 < b \rightarrow \exists x \quad (0 < x < b \wedge x \equiv_n c)$.

(2) $G \models \forall a, b, c \quad 0 \leq a < b \rightarrow \exists x \quad (a < x < b \wedge x \equiv_n c)$.

(3) $G \models \forall a, b, c \quad a < b \rightarrow \exists x \quad (a < x < b \wedge x \equiv_n c)$.

PROOF. We work in G .

(3) \Rightarrow (1) is immediate.

(1) \Rightarrow (2). Let $a, b, c \in G$ be arbitrary with $0 \leq a < b$. By (1), we can choose $x_0 \in G$ such that $0 < x_0 < b - a$ and $x_0 \equiv_n c$. Again by (1), we can choose $x_1 \in G$ such that $0 < x_1 < x_0$ and $x_1 \equiv_n a$. Let $x = a - x_1 + x_0$. Since $a - x_1 \equiv_n 0$, $x \equiv_n x_0 \equiv_n c$. On the other hand, $0 < x_1 < x_0 < b - a$ implies $0 < x_0 - x_1 < b - a$. Hence, $a < a + x_0 - x_1 < b$.

(2) \Rightarrow (3). Let $a, b, c \in G$ be arbitrary with $a < b$. If $0 < b$ then $0 \leq a < b$ or $a < 0 < b$. In either cases, we can choose desired x by (2). If $b \leq 0$, then $0 \leq -b < -a$. By (2), we can choose $x' \in G$ such that $-b < x' < -a$ and $x' \equiv_n -c \pmod{n}$. Hence, $a < -x' < b$ and $-x' \equiv_n c \pmod{n}$. \square

The additive group of rational numbers \mathbf{Q} is dense regular. There are many dense regular groups. Let p be a prime number, and let \mathbf{F}_p be the prime field

of characteristic p . For any abelian group G , G/pG is a \mathbf{F}_p -vector space. Let $\beta_p(G) = \dim_{\mathbf{F}_p} G/pG$. $\beta_p(G)$ is called a *Szmielew invariant*. Note that G/nG is finite for every positive integer n if and only if $\beta_p(G)$ is finite for every prime number p .

FACT 1.15 (Zakon). *For any function f from the set of prime numbers to $\omega \cup \{\omega\}$, there is a dense regular group G such that $\beta_p(G) = f(p)$ for any prime number p . Here ω is the first infinite ordinal number.*

PROOF. We present a construction by Weispfenning [12]. Let $\{r_{p,n} : p \text{ is a prime, } n < \omega\}$ be a set of linearly independent real numbers over \mathbf{Q} . Let $\mathbf{Z}_p = \{a/b \in \mathbf{Q} : b \not\equiv 0 \pmod{p}\}$, and

$$G = \bigoplus_{p:\text{prime}} \bigoplus_{n < f(p)} \mathbf{Z}_p \cdot r_{p,n}.$$

Then G is a dense subgroup of the additive group of the real number field and $\beta_p(G) = f(p)$ for every prime p . \square

FACT 1.16 (Weispfenning). *Let G be an ordered abelian group, and D a pure subgroup of G . Consider each element of D as a constant symbol. Then G admits quantifier elimination in $L_{\text{mod}}(<, D)$ if and only if*

- (1) G is dense regular or
- (2) there exists a finite sequence $\{G_i\}_{0 \leq i \leq m}$ of convex subgroups of G and a sequence $\{(k_i, d_i)\}_{1 \leq i \leq m}$ such that
 - (i) $G_m = G$;
 - (ii) for $1 \leq i \leq m$, k_i is a positive integer, $d_i \in D$, $d_i \in G_i - G_{i-1}$, G_i/G_{i-1} is a \mathbf{Z} -group with smallest positive element $1_i + G_{i-1}$, $k_i \cdot 1_i - d_i \in G_{i-1}$;
 - (iii) G_0 is dense regular, and for every prime p , $\beta_p(G_0)$ is finite and every coset of pG_0 in G_0 has a representative in D .

The following is a corollary to this fact.

FACT 1.17 (Weispfenning). *Let G be an ordered abelian group.*

- (1) G admits quantifier elimination in $L_{\text{mod}}(<)$ if and only if G is dense regular.
- (2) Let d be an element of G . G admits quantifier elimination in $L_{\text{mod}}(<, d)$ if and only if G is dense regular, or there exists a divisible convex subgroup G_0 of G and an integer $k \neq 0$ such that G/G_0 is a \mathbf{Z} -group with smallest positive element $1 + G_0$ and $k \cdot 1 - d \in G_0$.

2. Product Interpretations

DEFINITION 2.1 (Lexicographic Product). Let $L_{\text{ag}}(<)$ -structures B and C be ordered abelian groups. An $L_{\text{ag}}(<)$ -structure A is called *the lexicographic product of B and C* if $A|L_{\text{ag}}$ is the direct product of abelian groups $B|L_{\text{ag}}$ and $C|L_{\text{ag}}$, and for any $x, y \in A$ with $x = (x_B, x_C)$, $y = (y_B, y_C)$, $A \models x < y$ if and only if

$$\begin{aligned} B \models x_B < y_B \quad \text{or} \\ B \models x_B = y_B \quad \text{and} \quad C \models x_C < y_C. \end{aligned}$$

Now, we will introduce the notion of product interpretation for the direct product of two ordered abelian groups. The definition was given in [9] and [11]. The following is a slightly generalized one.

DEFINITION 2.2 (Extended Product Interpretation). Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and D a set of constant symbols such that $D \cap L = \emptyset$. Suppose that H is an L -structure such that $H|L_{\text{ag}}(<)$ is an ordered abelian group, K an $L_{\text{ag}}(<, D)$ -structure such that $K|L_{\text{ag}}(<)$ is an ordered abelian group. Let I be a new unary relation symbol which does not appear in L . A structure G for $L(I, D)$ is called *an extended product interpretation of $H \times K$ with new predicate I* , if

1. $G|L_{\text{ag}}(<)$ is a lexicographic product of $H|L_{\text{ag}}(<)$ and $K|L_{\text{ag}}(<)$,
2. for each constant symbol $c \in L$, there is an element $c_K \in K$ such that $c^G = (c^H, c_K)$, and $c_1^H = c_2^H$ implies $c_1^G = c_2^G$ for any constant symbols $c_1, c_2 \in L$,
3. $((x_1, y_1), \dots, (x_n, y_n)) \in R^G$ if and only if $(x_1, \dots, x_n) \in R^H$ for each relation symbol R of $L - \{<\}$,
4. $I^G = \{(0^H, x) : x \in K\}$, and
5. $d^G = (0^H, d^K)$ for each constant symbol $d \in D$.

Note that $K \cong G|I^G$ as $L_{\text{mod}}(<, D)$ -structures. An extended product interpretation of $H \times K$ is not unique because of condition 2. If $c^G = (c^H, 0^K)$ for each constant symbol $c \in L$, then G is called *the product interpretation of $H \times K$ with new predicate I* [9, 11].

Lemmas 2.3 and 2.8 below are essentially proved by Tanaka and Yokoyama [11].

LEMMA 2.3. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and D a set of constant symbols such that $D \cap L = \emptyset$. Suppose that H is an*

L-structure such that $H \mid L_{\text{ag}}(<)$ is an ordered abelian group, K an $L_{\text{mod}}(<, D)$ -structure such that $K \mid L_{\text{ag}}(<)$ is an ordered abelian group, and G an extended product interpretation of $H \times K$ with a new predicate I . If $\varphi(\bar{x})$ is a quantifier-free formula of L with an n -tuple \bar{x} of variables, there is a quantifier-free formula $\varphi^*(\bar{x})$ of $L(I)$ such that for any tuple $\bar{g} = (g_1, \dots, g_n) \in G^n$ with $g_i = (g_{i,H}, g_{i,K})$ for $i = 1, \dots, n$, $H \models \varphi(\bar{g}_H)$ if and only if $G \models \varphi^*(\bar{g})$, where $\bar{g}_H = (g_{1,H}, \dots, g_{n,H})$.

PROOF. Let $\varphi(\bar{x})$ be a quantifier-free formula of L with a tuple \bar{x} of n variables. Then $\varphi(\bar{x})$ is a Boolean combination of formulas of forms $t(\bar{x}) = 0$, $0 < t(\bar{x})$, and $R(s_1(\bar{x}), \dots, s_l(\bar{x}))$, where $t(\bar{x})$, $s_1(\bar{x}), \dots, s_l(\bar{x})$ are terms of L and R is an l -ary relation symbol of L .

Let $\bar{g} = (g_1, \dots, g_n)$ be an arbitrary tuple from G with $g_i = (g_{i,H}, g_{i,K})$ for $i = 1, \dots, n$, and let $\bar{g}_H = (g_{1,H}, \dots, g_{n,H})$ and $\bar{g}_K = (g_{1,K}, \dots, g_{n,K})$.

We can write $t(\bar{x}) = t_1(\bar{x}) + t_2(\bar{c})$ where $t_1(\bar{x})$ is a term of L_{ag} , $t_2(\bar{c})$ a term of L_{ag} with a p -tuple \bar{z} of variables, and $\bar{c} = (c_1, \dots, c_p)$ a tuple of constant symbols of L . Choose $c_{i,K} \in K$ such that $c_i^G = (c_i^H, c_{i,K})$ for $i = 1, \dots, p$ and let $\bar{c}_K = (c_{1,K}, \dots, c_{p,K})$. Then $t^G(\bar{g}) = (t^H(\bar{g}_H), t_1^K(\bar{g}_K) + t_2^K(\bar{c}_K))$. Hence,

$$H \models t(\bar{g}_H) = 0 \Leftrightarrow G \models I(t(\bar{g})), \quad \text{and}$$

$$H \models 0 < t(\bar{g}_H) \Leftrightarrow G \models 0 < t(\bar{g}) \wedge \neg I(t(\bar{g})).$$

Similarly, we have

$$H \models R(s_1(\bar{g}_H), \dots, s_l(\bar{g}_H)) \Leftrightarrow G \models R(s_1(\bar{g}), \dots, s_l(\bar{g})).$$

Let $\varphi^*(\bar{x})$ be the formula obtained from $\varphi(\bar{x})$ by replacing $t(\bar{x}) = 0$ and $0 < t(\bar{x})$ with $I(t(\bar{x}))$ and $0 < t(\bar{x}) \wedge \neg I(t(\bar{x}))$, respectively. Then $H \models \varphi(\bar{g}_H)$ if and only if $G \models \varphi^*(\bar{g})$. \square

DEFINITION 2.4 (Unnested atomic formula). Let L be a language. By an *unnested atomic formula* $\varphi(\bar{x})$ where \bar{x} is a tuple of variables, we mean an atomic formula of one of the following forms:

$$u = v;$$

$$c = v \quad \text{for some constant symbol } c \text{ of } L;$$

$$f(\bar{z}) = y \quad \text{for some function symbol } f \text{ of } L;$$

$$R(\bar{z}) \quad \text{for some relation symbol } R \text{ of } L.$$

Here, u, v, y are variables from \bar{x} , and \bar{z} a tuple of variables from \bar{x} .

DEFINITION 2.5 (Partial isomorphism). Let A and B be structures for a language L . A partial map f from A to B is called a *partial L -isomorphism* if for any tuple \bar{a} from the domain of f and for any unnested formula $\varphi(\bar{x})$ of L with a tuple \bar{x} of free variables such that the length of \bar{x} is equal to the length of \bar{a} ,

$$A \models \varphi(\bar{a}) \Leftrightarrow B \models \varphi(f(\bar{a})).$$

Note that since $u = v$ is an unnested formula, a partial L -isomorphism is a one-to-one map.

We are going to define $A \approx_k B$, which is defined in [5], p. 102. We define it in a different way, but they are equivalent essentially by [5], Lemma 3.3.1.

DEFINITION 2.6. Let A and B be structures for a language L , \bar{a} a tuple from A , and \bar{b} a tuple from B . Suppose that \bar{a} and \bar{b} have the same length. For any integer $k \geq 0$, we define $(A, \bar{a}) \approx_k (B, \bar{b})$ for L by induction on k as the following:

$(A, \bar{a}) \approx_0 (B, \bar{b})$ for L if there is a partial L -isomorphism f from A to B such that $f(\bar{a}) = \bar{b}$.

Suppose $k > 0$. $(A, \bar{a}) \approx_k (B, \bar{b})$ for L if for every element c of A there is an element d of B such that $(A, \bar{a} \hat{\ } c) \approx_{k-1} (B, \bar{b} \hat{\ } d)$ for L , and for every element d of B there is an element c of A such that $(A, \bar{a} \hat{\ } c) \approx_{k-1} (B, \bar{b} \hat{\ } d)$ for L .

For $k \geq 1$, $A \approx_k B$ for L if $(A, ()) \approx_k (B, ())$ for L where $()$ is the empty tuple.

The following is Corollary 3.3.3 in [5].

FACT 2.7 (Fraïssé-Hintikka). *Let A and B be structures for a finite language L . Then the following are equivalent:*

- (1) $A \equiv B$ for L .
- (2) $A \approx_k B$ for L for every integer $k \geq 1$.

LEMMA 2.8. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants and I a new unary predicate. Suppose that $H \equiv H'$ for L , and $K \equiv K'$ for $L_{\text{ag}}(<, D)$ for some set D of new constant symbols. Then the following hold.*

- (1) *The product interpretations $H \times K$ and $H' \times K'$ with new predicate I are elementarily equivalent.*
- (2) *If G is an extended product interpretation of $H \times K$ with new predicate I , G' is an extended product interpretation of $H' \times K'$ with new predicate I ,*

and for each constant symbol c in L there is a constant symbol $d_c \in D \cup \{0\}$ such that $c^G = (c^H, d_c^K)$ and $c^{G'} = (c^{H'}, d_c^{K'})$, then $G \equiv G'$ for the language $L(I, D)$.

PROOF. It is enough to prove (2). Let G and G' be as above. We only have to show that $G \equiv G'$ for any finite sublanguage L' of $L(I, D)$ such that $L_{\text{ag}}(<, I) \subseteq L'$. We can assume that for any constant symbol $c \in L \cap L'$, there is a constant symbol $d \in (D \cap L') \cup \{0\}$ such that $c^G = (c^H, d^K)$ and $c^{G'} = (c^{H'}, d^{K'})$.

CLAIM 1. Let $a_i \in H$, $a'_i \in H'$, $b_i \in K$ and $b'_i \in K'$ for $i = 1, \dots, m$ with $m \geq 0$. For any integer $k \geq 0$, if $(H, (a_1, a_2, \dots, a_m)) \approx_k (H', (a'_1, a'_2, \dots, a'_m))$ for $L \cap L'$ and $(K, (b_1, b_2, \dots, b_m)) \approx_k (K', (b'_1, b'_2, \dots, b'_m))$ for $L_{\text{ag}}(<, D) \cap L'$ then $(G, (g_1, g_2, \dots, g_m)) \approx_k (G', (g'_1, g'_2, \dots, g'_m))$ for L' where $g_i = (a_i, b_i)$ and $g'_i = (a'_i, b'_i)$ for $i = 1, \dots, m$.

We prove the claim by induction on k .

Suppose $k = 0$. Assume $m > 0$. By the assumption, there is a partial $(L \cap L')$ -isomorphism f_1 from H to H' such that $f_1(a_i) = a'_i$ for $i = 1, \dots, m$, and there is a partial $(L_{\text{ag}}(<, D) \cap L')$ -isomorphism f_2 from K to K' such that $f_2(b_i) = b'_i$ for $i = 1, \dots, m$. Let f be a partial map from G to G' defined by $f(g_i) = f((a_i, b_i)) = (a'_i, b'_i) = (f_1(a_i), f_2(b_i)) = g'_i$ for $i = 1, \dots, m$. It is straightforward to prove that f is well-defined and it is a partial L' -isomorphism. We show that f is a partial $C \cup \{I\}$ -isomorphism where C is the set of constant symbols of $L \cap L'$. The remaining cases can be treated similarly.

If $G \models I(g_i)$ then $a_i = 0^H$ since $g_i = (a_i, b_i)$. We have $f(g_i) = f((0^H, b_i)) = (f_1(0^H), f_2(b_i)) = (0^{H'}, b'_i)$. Hence, $G' \models I(f(g_i))$. By symmetry, $G \models I(g_i)$ if and only if $G' \models I(f(g_i))$. Therefore, f is a partial $\{I\}$ -isomorphism.

Suppose $G \models g_i = c$ for a constant symbol $c \in L \cap L'$. Then $g_i = (c^H, d_c^K)$ for some $d_c \in D \cap L'$. We have $f(g_i) = f((c^H, d_c^K)) = (f_1(c^H), f_2(d_c^K)) = (c^{H'}, d_c^{K'})$. Hence, $G' \models f(g_i) = c$. By symmetry, $G \models g_i = c$ if and only if $G' \models f(g_i) = c$. Therefore, f is a partial C -isomorphism.

Now, we turn to the induction step. Suppose $k > 0$. We are going to show that $(G, (g_1, g_2, \dots, g_m)) \approx_k (G', (g'_1, g'_2, \dots, g'_m))$ for L' . By symmetry, it is enough to show that for any $g_{m+1} \in G$, there is $g'_{m+1} \in G'$ such that $(G, (g_1, g_2, \dots, g_m, g_{m+1})) \approx_{k-1} (G', (g'_1, g'_2, \dots, g'_m, g'_{m+1}))$ for L' .

Let $g_{m+1} = (a_{m+1}, b_{m+1}) \in G$ be arbitrary. Since

$$(H, (a_1, a_2, \dots, a_m)) \approx_k (H', (a'_1, a'_2, \dots, a'_m))$$

for $L \cap L'$ and $a_{m+1} \in H$, we can choose $a'_{m+1} \in H'$ such that

$$(H, (a_1, a_2, \dots, a_m, a_{m+1})) \approx_{k-1} (H', (a'_1, a'_2, \dots, a'_m, a'_{m+1}))$$

for $L \cap L'$. Also, since

$$(K, (b_1, b_2, \dots, b_m)) \approx_k (K', (b'_1, b'_2, \dots, b'_m))$$

for $L_{\text{ag}}(<, D) \cap L'$, we can choose $b'_{m+1} \in K'$ such that

$$(K, (b_1, b_2, \dots, b_m, b_{m+1})) \approx_{k-1} (H', (b'_1, b'_2, \dots, b'_m, b'_{m+1}))$$

for $L_{\text{ag}}(<, D) \cap L'$. Let $g'_{m+1} = (a'_{m+1}, b'_{m+1})$. Then by the induction hypothesis,

$$(G, (g_1, g_2, \dots, g_m, g_{m+1})) \approx_{k-1} (G', (g'_1, g'_2, \dots, g'_m, g'_{m+1}))$$

for L' . We have proved the claim.

Now we turn to the proof of the lemma. Let $k \geq 1$ be any integer. Since $H \equiv H'$ for $L \cap L'$ and $K \equiv K'$ for $L_{\text{ag}}(<, D) \cap L'$, we have $H \approx_k H'$ for $L \cap L'$ and $K \approx_k K'$ for $L_{\text{ag}}(<, D) \cap L'$ by Fact 2.7. Hence, $G \approx_k G'$ for L' by Claim 1. Since $G \approx_k G'$ for L' for any integer $k \geq 1$, $G \equiv G'$ for L' by Fact 2.7. \square

LEMMA 2.9. *If G is an ordered abelian group, A a convex subgroup of G , B a subgroup of G , and $G = B \oplus A$ as an abelian group, then G is isomorphic to the lexicographic product of B and A .*

PROOF. Assume $b + a \leq b' + a'$ with $b, b' \in B$ and $a, a' \in A$.

Suppose $b < b'$ is not the case. Then $b \geq b'$ and we have $0 \leq b - b' \leq a' - a \in A$. Hence, $b - b' \in A$ by convexity of A and thus $b - b' \in A \cap B = \{0\}$. Hence, $b = b'$ and $a \leq a'$. \square

PROPOSITION 2.10 (Theory of an Extended Product Interpretation). *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and H a structure for L such that $H \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group, and K an $L_{\text{ag}}(<, D)$ -structure for some set D of constant symbols such that $K \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group. Let G be an extended product interpretation of $H \times K$ with a new predicate I . Suppose that for each constant symbol $c \in L$, there is a constant symbol $d_c \in D$ such that $c^G = (c^H, d_c^K)$. Then $M \equiv G$ for $L(I, D)$ if and only if M satisfies the following axioms:*

1. $M \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group;
2. I^M is a convex subgroup;
3. $I^M \equiv K$ for $L_{\text{ag}}(<, D)$;

4. for each relation symbol R of $L - \{<\}$, truth value of R is fixed modulo I , i.e., if R has the arity m ,

$$M \models \forall x_1, \dots, x_m \forall y_1, \dots, y_m$$

$$I(y_1) \wedge \dots \wedge I(y_m) \rightarrow (R(x_1, \dots, x_m) \leftrightarrow R(x_1 - y_1, \dots, x_m - y_m));$$

5. $M/I^M \equiv H$ for L ;

6. for each term $t(y_1, \dots, y_n)$ of L_{ag} and a tuple (c_1, \dots, c_n) of constant symbols of L ,

$$M \models t(c_1 - d_{c_1}, \dots, c_n - d_{c_n}) \neq 0 \rightarrow \neg I(t(c_1 - d_{c_1}, \dots, c_n - d_{c_n}))$$

and for each positive integer n ,

$$\begin{aligned} M \models \forall x \quad I(x) \wedge n \mid x + t(c_1 - d_{c_1}, \dots, c_n - d_{c_n}) \\ \rightarrow n \mid x \wedge n \mid t(c_1 - d_{c_1}, \dots, c_n - d_{c_n}). \end{aligned}$$

Note that assuming condition 4, M/I^M can naturally be considered as an L -structure.

In particular, if the theory of H in L and the theory of K in $L_{\text{ag}}(<, D)$ are recursively axiomatizable and the function mapping each constant symbol c of L to a constant symbol d_c of D is a recursive function, then the theory of G in $L(I, D)$ is recursively axiomatizable.

PROOF. It is straightforward to check that G satisfies the axioms 1–6.

Let M be any model of the axioms 1–6. To show that $M \equiv G$ for $L(I, D)$, we can replace M by an elementary extension of M . So, we can assume that M is ω_1 -saturated. Let us denote the $L_{\text{ag}}(<, D)$ -substructure of M with domain I^M by I^M also. Let C be the set of constant symbols of L and P the pure subgroup of M generated by $\{(c - d_c)^M : c \in C\}$. Then $P \cap I^M = \{0\}$ and $P \oplus I^M$ is a pure subgroup of M by Axiom 6. Therefore, there is a group homomorphism g from $P \oplus I^M$ to I^M such that $g|_{I^M} = \text{id}$ and $g(x) = 0^M$ for every $x \in P$. Since M is ω_1 -saturated, I^M satisfies condition (5) of Fact 1.7 (ω_1 -equationally compact). Hence, I^M is pure-injective by Fact 1.7. Therefore, we can extend g to a homomorphism $g' : M \rightarrow I^M$. Since $g'(x) = g(x) = x$ for every $x \in I^M$, $M = \text{Ker } g' \oplus I^M$. Since $P \subseteq \text{Ker}(g) \subseteq \text{Ker}(g')$, M is isomorphic to an extended product interpretation of $\text{Ker}(g')$ and I^M by Lemma 2.9, and $\text{Ker}(g') \equiv H$ as L -structures by Axiom 4. Therefore, $M \cong \text{Ker}(g') \times I^M \equiv G$ in the language $L(I, D)$ by Lemma 2.8. \square

3. Lemmas for Quantifier Elimination

In this section, we present some lemmas used in common later.

REMARK 3.1. Suppose that $L = L'(C)$ for some set C of constant symbols. Then to show that a theory T admits quantifier elimination in L , it is enough to show that every existential formula of L' is equivalent to a quantifier-free formula of $L = L'(C)$ modulo T .

LEMMA 3.2. Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and D a set of constant symbols such that $D \cap L = \emptyset$. Suppose H is an L -structure such that $H \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group, K an $L_{\text{ag}}(<, D)$ -structure such that $K \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group, and G an extended product interpretation of $H \times K$ with a new predicate I . Let L_{R} be the set of relation symbols of L other than $<$. Then the following are equivalent:

- (1) G admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$.
- (2) Let x be a variable and \bar{y} an n -tuple of variables. Suppose that p, q are natural numbers such that $p \leq q$, m is a non-zero integer, $\varphi(x, \bar{y})$ a conjunction of literals of $L_{\text{R}}(+, -, 0, I)$, $t_i(\bar{y})$ a term of L_{ag} for $i = 1, \dots, q$, $s_1(\bar{y})$ a term of L_{ag} or $-\infty$, $s_2(\bar{y})$ a term of L_{ag} or ∞ , $\Psi_1(x, \bar{y})$ the formula

$$s_1(\bar{y}) < mx < s_2(\bar{y}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}),$$

and $\Psi_2(x, \bar{y})$ the formula

$$mx = s_1(\bar{y}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}).$$

We assume that $s_1(\bar{y})$ is a term of L_{ag} in $\Psi_2(x, \bar{y})$.

Then for any n -tuple \bar{a} from G , each of the statements $G \models \exists x \varphi(x, \bar{a})$, $G \models \exists x \Psi_1(x, \bar{a})$ and $G \models \exists x \Psi_2(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ for some quantifier-free formula $\theta(\bar{y})$ of $L(I, D) \cup L_{\text{mod}}$.

PROOF. Let C be the set of constant symbols of L . Let L' be the language $L_{\text{R}}(I) \cup L_{\text{mod}}$. Then $L(I, D) \cup L_{\text{mod}} = L'(C \cup D)$. By Remark 3.1, it is enough to show that any existential formula of $L_{\text{R}}(I) \cup L_{\text{mod}}$ is equivalent to a quantifier-free formula of $L(I, D) \cup L_{\text{mod}}$ modulo the theory of G .

Since G is totally ordered by $<^G$, any quantifier-free formula of $L_{\mathcal{R}}(I) \cup L_{\text{mod}}$ with free variables $x \hat{\ } \bar{y}$ is equivalent to a disjunction of formulas of forms $\Psi_1(x, \bar{y})$ and $\Psi_2(x, \bar{y})$ allowing m to be 0. In the case with $m = 0$, it is enough to eliminate the quantifier from $\exists x \varphi(x, \bar{y})$. Now, the lemma is clear. \square

The statements $G \models \exists x \varphi(x, \bar{a})$ and $G \models \exists x \Psi_2(x, \bar{a})$ of Lemma 3.2 (2) are reduced by the following lemma.

LEMMA 3.3. *Assume the assumption of Lemma 3.2, and the assumption of Lemma 3.2 (2). Let $\varphi^1(x, \bar{y})$ be the formula obtained from $\varphi(x, \bar{y})$ by replacing each subformula “ $I(t)$ ” with “ $t = 0$ ”. Then the following hold:*

- (1) *Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n -tuple from G , and \bar{a}_H the n -tuple (b_1, \dots, b_n) . Then $G \models \exists x \varphi(x, \bar{a})$ if and only if $H \models \exists x_1 \varphi^1(x_1, \bar{a}_H)$.*
- (2) *Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n -tuple from G , and \bar{a}_H the n -tuple (b_1, \dots, b_n) . Then $G \models \exists x \Psi_2(x, \bar{a})$ if and only if the conjunction of the following statements holds:*

$$H \models \exists x_1 \quad mx_1 = s(\bar{a}_H) \wedge \varphi^1(x_1, \bar{a}_H),$$

$$G \models s(\bar{a}) \equiv_m 0 \wedge \bigwedge_{1 \leq i \leq p} s(\bar{a}) \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} s(\bar{a}) \equiv_{l_j} t_j(\bar{a}).$$

- (3) *If H admits quantifier elimination in L then for any n -tuple \bar{a} from G , each of the statements $G \models \exists x \varphi(x, \bar{a})$ and $G \models \exists x \Psi_2(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I) \cup L_{\text{mod}}$.*

PROOF. (1) and (2) are immediate. We have (3) by (1), (2) and Lemma 2.3. \square

Statement $G \models \exists x \Psi_1(x, \bar{a})$ of Lemma 3.2 (2) will be reduced with several lemmas.

LEMMA 3.4. *Assume the assumption of Lemma 3.2, and the assumption of Lemma 3.2 (2). Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n -tuple from G , and \bar{a}_H the n -tuple (b_1, \dots, b_n) . Then $G \models \exists x \Psi_1(x, \bar{a})$ is equivalent to the disjunction of the following statements (a) and (b):*

- (a) $H \models s_1(\bar{a}_H) < s_2(\bar{a}_H)$ and $G \models \exists x \Psi_1(x, \bar{a})$.
- (b) $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H)$ and $G \models \exists x \Psi_1(x, \bar{a})$.

Statement (a) of Lemma 3.4 is reduced by the following lemma.

LEMMA 3.5. *Assume the assumption of Lemma 3.2, and the assumption of Lemma 3.2 (2). Let $\varphi^1(x, \bar{y})$ be the formula obtained from $\varphi(x, \bar{y})$ by replacing each subformula “ $I(t)$ ” with “ $t = 0$ ”.*

Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n -tuple from G , \bar{a}_H the n -tuple (b_1, \dots, b_n) , and \bar{a}_K the n -tuple (c_1, \dots, c_n) . Then the following statements (1) and (2) are equivalent:

- (1) $H \models s_1(\bar{a}_H) < s_2(\bar{a}_H)$ and $G \models \Psi_1(x, \bar{a})$.
(2) For some $W \subseteq \{1, \dots, p\}$,

$$\begin{aligned} H \models s_1(\bar{a}_H) < s_2(\bar{a}_H) \wedge \exists x_1 \\ s_1(\bar{a}_H) \leq mx_1 \leq s_2(\bar{a}_H) \wedge \varphi^1(x_1, \bar{a}_H) \wedge \bigwedge_{k \in W^c} mx_1 \not\equiv_{l_k} t_k(\bar{a}_H) \\ \wedge \bigwedge_{i \in W} mx_1 \equiv_{l_i} t_i(\bar{a}_H) \wedge \bigwedge_{p+1 \leq j \leq q} mx_1 \equiv_{l_j} t_j(\bar{a}_H) \end{aligned}$$

and

$$K \models \exists x_2 \bigwedge_{i \in W} mx_2 \not\equiv_{l_i} t_i(\bar{a}_K) \wedge \bigwedge_{p+1 \leq j \leq q} mx_2 \equiv_{l_j} t_j(\bar{a}_K).$$

PROOF. (1) \Rightarrow (2). Assume (1). Then there is $x = (x_H, x_K) \in G$ such that

$$G \models s_1(\bar{a}) < mx < s_2(\bar{a}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{a}) \wedge \varphi(x, \bar{a}).$$

First, we have

$$H \models s_1(\bar{a}_H) \leq mx_H \leq s_2(\bar{a}_H) \wedge \varphi^1(x_H, \bar{a}).$$

Let $W = \{1 \leq i \leq p : H \models mx_H \equiv_{l_i} t_i(\bar{a}_H)\}$. For $i \in W$, if $K \models mx_K \equiv_{l_i} t_i(\bar{a}_K)$ then $G \models m(x_H, x_K) \equiv_{l_i} (t_i(\bar{a}_H), t_i(\bar{a}_K))$. Therefore, $K \models mx_K \not\equiv_{l_i} t_i(\bar{a}_K)$ for $i \in W$. (2) holds with $x_1 = x_H \in H$ and $x_2 = x_K \in K$.

(2) \Rightarrow (1). Assume (2). Choose $W \subseteq \{1, \dots, p\}$, $x_1 \in H$ and $x_2 \in K$ such that

$$\begin{aligned} H \models s_1(\bar{a}_H) \leq mx_1 \leq s_2(\bar{a}_H) \wedge \varphi^1(x_1, \bar{a}_H) \\ \wedge \bigwedge_{k \in W^c} mx_1 \not\equiv_{l_k} t_k(\bar{a}_H) \wedge \bigwedge_{i \in W} mx_1 \equiv_{l_i} t_i(\bar{a}_H) \wedge \bigwedge_{p+1 \leq j \leq q} mx_1 \equiv_{l_j} t_j(\bar{a}_H) \end{aligned}$$

and

$$K \models \bigwedge_{i \in W} mx_2 \not\equiv_{l_i} t_i(\bar{a}_K) \wedge \bigwedge_{p+1 \leq j \leq q} mx_2 \equiv_{l_j} t_j(\bar{a}_K).$$

Since $H \models s_1(\bar{a}_H) < s_2(\bar{a}_H)$ and $H \models s_1(\bar{a}_H) \leq mx_1 \leq s_2(\bar{a}_H)$, we have $H \models mx_1 < s_2(\bar{a}_H)$ or $H \models mx_1 = s_2(\bar{a}_H)$.

Case $H \models mx_1 < s_2(\bar{a}_H)$. Let l be a common multiple of l_1, \dots, l_q and m . By Remark 1.13, we can choose an element $d \in K$ satisfying $K \models d \equiv_l 0$ and $K \models s_1(\bar{a}_K) - mx_2 < d$. Since $K \models d \equiv_l 0$, $K \models d \equiv_{|m|} 0$. Pick $d' \in K$ such that $K \models d = md'$. Put $x_K = x_2 + d' \in K$ and $x = (x_1, x_K)$. Then $K \models s_1(\bar{a}_K) < mx_2 + d = m(x_2 + d') = mx_K$. Since $H \models s_1(\bar{a}_H) \leq mx_1$, $K \models s_1(\bar{a}_K) < mx_K$, and $s_1^G(\bar{a}) = (s_1^H(\bar{a}_H), s_1^K(\bar{a}_K))$, we have $G \models s_1(\bar{a}) < mx$. Since $H \models mx_1 < s_2(\bar{a}_H)$, we have $G \models mx < s_2(\bar{a})$.

Since $K \models d \equiv_l 0$ for each l_i , we have $K \models mx_K = mx_2 + d \equiv_{l_i} mx_2$ for each i . Hence,

$$K \models \bigwedge_{i \in W} mx_K \not\equiv_{l_i} t_i(\bar{a}_K) \wedge \bigwedge_{p+1 \leq j \leq q} mx_K \equiv_{l_j} t_j(\bar{a}_K).$$

Therefore, we have (1):

$$H \models s_1(\bar{a}_H) < s_2(\bar{a}_H) \quad \text{and}$$

$$G \models s_1(\bar{a}) < mx < s_2(\bar{a}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{a}) \wedge \varphi(x, \bar{a}).$$

Case $H \models mx_1 = s_2(\bar{a}_H)$. Let l be a common multiple of l_1, \dots, l_q and m . By Remark 1.13, we can choose an element $d \in K$ satisfying $K \models d \equiv_l 0$ and $K \models -s_2(\bar{a}_K) + mx_2 < d$. Since $K \models d \equiv_l 0$, $K \models d \equiv_{|m|} 0$. Pick $d' \in K$ such that $K \models d = md'$. Put $x_K = x_2 - d' \in K$ and $x = (x_1, x_K)$. Then $K \models mx_K = m(x_2 - d') = mx_2 - d < s_2(\bar{a}_K)$. Since $H \models mx_1 = s_2(\bar{a}_H)$, $K \models mx_K < s_2(\bar{a}_K)$, and $s_2^G(\bar{a}) = (s_2^H(\bar{a}_H), s_2^K(\bar{a}_K))$, we have $G \models mx < s_2(\bar{a})$. Since $H \models s_1(\bar{a}_H) < s_2(\bar{a}_H) = mx_1$, we have $G \models s_1(\bar{a}) < mx$.

Now, with an argument similar to the case $H \models mx_1 < s_2(\bar{a}_H)$, we can deduce (1). \square

LEMMA 3.6. *Assume the assumption of Lemma 3.2, and the assumption of Lemma 3.2 (2). Suppose H admits quantifier elimination in L and for any positive integer l , K/lK is finite and there is a set D_l of variable-free terms of $L_{\text{ag}}(D)$ such that $D_l^K = \{d^K : d \in D_l\}$ forms a set of representatives of the proper cosets of lK in K . Let $\psi(x, \bar{y})$ be a formula of L , and W a subset of $\{1, \dots, p\}$. Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n -tuple from G , and put $\bar{a}_H = (b_1, \dots, b_n)$ and $\bar{a}_K = (c_1, \dots, c_n)$. Then the conjunction of the statements*

$$(e) \quad H \models \exists x_1 \quad \psi(x_1, \bar{a}_H) \wedge \bigwedge_{i \in W} mx_1 \equiv_{l_i} t_i(\bar{a}_H) \wedge \bigwedge_{p+1 \leq j \leq q} mx_1 \equiv_{l_j} t_j(\bar{a}_H)$$

and

$$(f) \quad K \models \exists x_2 \bigwedge_{i \in W} mx_2 \not\equiv_{l_i} t_i(\bar{a}_K) \wedge \bigwedge_{p+1 \leq j \leq q} mx_2 \equiv_{l_j} t_j(\bar{a}_K)$$

is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I, D) \cup L_{\text{mod}}$.

PROOF. Let l be an arbitrary integer such that $l \geq 2$, and let D_l be a set of variable-free terms of $L_{\text{ag}}(D)$ such that $D_l^K = \{d^K : d \in D_l\}$ forms a set of representatives of the proper cosets of lK in K . Then (f) is equivalent to (f1):

$$(f1) \quad K \models \exists x_2 \bigwedge_{i \in W} \left(\bigvee_{d \in D_{l_i}} mx_2 \equiv_{l_i} t_i(\bar{a}_K) + d \right) \wedge \bigwedge_{p+1 \leq j \leq q} mx_2 \equiv_{l_j} t_j(\bar{a}_K).$$

Assuming (e), (f1) is equivalent to

$$(f2) \quad G \models \exists x \bigwedge_{i \in W} \left(\bigvee_{d \in D_{l_i}} mx \equiv_{l_i} t_i(\bar{a}) + d \right) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{a}).$$

Hence, the conjunction of (e) and (f) is equivalent to the conjunction of (e) and (f2).

By the assumption that H admits quantifier elimination in L and Lemma 2.3, (e) is equivalent to a statement of the form $G \models \theta(\bar{a})$ with $\theta(\bar{y})$ a quantifier-free formula of $L(I)$.

It is enough to show that (f2) is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with $\theta(\bar{y})$ a quantifier-free formula of $L(I, 1) \cup L_{\text{mod}}$. (f2) is equivalent to a finite disjunction of statements of the form

$$(f3) \quad G \models \exists x \bigwedge_{1 \leq i \leq n'} mx \equiv_{l'_i} t'_i(\bar{a})$$

with terms $t'_i(\bar{y})$ of $L_{\text{ag}}(D)$.

By Lemma 1.9,

$$G \models \forall z_1, \dots, z_{n'} \left(\exists x \bigwedge_{i=1, \dots, n'} x \equiv_{l'_i} z_i \right) \leftrightarrow \theta_2(z_1, \dots, z_{n'})$$

for some quantifier-free formula $\theta_2(z_1, \dots, z_{n'})$ in L_{mod} . Therefore, (f3) is equivalent to

$$G \models \theta_2(t_1(\bar{a}), \dots, t_{n'}(\bar{a}))$$

with a quantifier-free formula $\theta_2(t'_1(\bar{y}), \dots, t'_{n'}(\bar{y}))$ of $L_{\text{mod}}(D)$. The lemma is proved. \square

4. Products with a Presburger Arithmetic

DEFINITION 4.1. An ordered abelian group G is called a *Presburger arithmetic* or a \mathbf{Z} -group if it is elementarily equivalent to the structure \mathbf{Z} of integers for $L_{\text{ag}}(<)$.

THEOREM 4.2. Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and H an L -structure such that $H \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group and H admits quantifier elimination in L , and K a Presburger arithmetic (\mathbf{Z} -group) with smallest positive element 1_K . Then any extended product interpretation G of $H \times K$ with new predicate I admits quantifier elimination in $L(I, d) \cup L_{\text{mod}}$ with a new constant symbol d when d^G is any non-zero multiple of $(0^H, 1_K)$.

Moreover, if there is a recursive procedure for quantifier elimination of H in L and there is a recursive map f from the set C of constant symbols of L to K such that $c^G = (c^H, f(c))$ for each $c \in C$, then there is a recursive procedure for quantifier elimination of G in $L(I, d) \cup L_{\text{mod}}$.

PROOF. First, we introduce a constant symbol 1 such that $1^G = (0^H, 1_K)$. In G , d can be represented as $m_0 \cdot 1$ for some non-zero integer m_0 . At some stage, we use 1 for quantifier elimination and then eliminate the constant 1 using d .

We show the statement of Lemma 3.2 (2). Let x be a variable and \bar{y} an n -tuple of variables. Suppose that p, q , and m are natural numbers with $p \leq q$, $\varphi(x, \bar{y})$ is a conjunction of literals of $L_{\mathbf{R}}(+, -, 0, I)$, $t_i(\bar{y})$ a term of L_{ag} for $i = 1, \dots, q$, $s_1(\bar{y})$ a term of L_{ag} or $-\infty$, $s_2(\bar{y})$ a term of L_{ag} or ∞ , $\Psi_1(x, \bar{y})$ the formula

$$s_1(\bar{y}) < mx < s_2(\bar{y}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}),$$

and $\Psi_2(x, \bar{y})$ the formula

$$mx = s_1(\bar{y}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}).$$

We assume that $s_1(\bar{y})$ is a term of L_{ag} in $\Psi_2(x, \bar{y})$.

By Lemma 3.3, we have the following:

CLAIM 1. For any n -tuple \bar{a} from G , each of the statements $G \models \exists x \varphi(x, \bar{a})$ and $G \models \exists x \Psi_2(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I)$.

Now, we turn to the reduction of $G \models \exists x \Psi_1(x, \bar{a})$ for any n -tuple \bar{a} from G .

CLAIM 2. *Let l be a common multiple of all the l_i 's and m . Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n -tuple from G , and put $\bar{a}_H = (b_1, \dots, b_n)$. Then the following statements (b) and (b1) are equivalent:*

- (b) $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H)$ and $G \models \Psi_1(x, \bar{a})$.
 (b1) $H \models \exists x_1 \ s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1 \wedge \varphi^1(x_1, \bar{a}_H)$, and for some natural number k such that $1 \leq k \leq l$,

$$G \models s_1(\bar{a}) + k \cdot 1 < s_2(\bar{a}) \wedge s_1(\bar{a}) + k \cdot 1 \equiv_m 0 \\
\wedge \bigwedge_{1 \leq i \leq p} s_1(\bar{a}) + k \cdot 1 \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} s_1(\bar{a}) + k \cdot 1 \equiv_{l_j} t_j(\bar{a}).$$

PROOF OF CLAIM 2. Suppose (b) holds. Choose $x = (x_H, x_K) \in G$ such that

$$G \models s_1(\bar{a}) < mx < s_2(\bar{a}) \wedge \varphi(x, \bar{a}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{a}).$$

Since $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H)$, we have $H \models s_1(\bar{a}_H) = mx_H = s_2(\bar{a}_H)$. Hence, $H \models \exists x_1 \ s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1 \wedge \varphi^1(x_1, \bar{a}_H)$.

Since $G \models I(s_2(\bar{a}) - s_1(\bar{a}))$, we have $G \models mx = s_1(\bar{a}) + z$ for some $z \in I^G$ with $G \models 0 < z$. Let $z = (0^H, z_K)$. Since K is a \mathbf{Z} -group, there is an integer k such that $1 \leq k \leq l$ and $K \models k \cdot 1 \equiv_l z_K$. Also, $K \models k \cdot 1 \leq z_K$ because 1^K is the least positive element of K . Therefore, $G \models s_1(\bar{a}) + k \cdot 1 \leq s_1(\bar{a}) + z = mx < s_2(\bar{a})$. Also, $G \models s_1(\bar{a}) + k \cdot 1 \equiv_l mx$. By the choice of l , we have $G \models s_1(\bar{a}) + k \cdot 1 \equiv_m mx \equiv_m 0$ and $G \models s_1(\bar{a}) + k \cdot 1 \equiv_i mx$ for each i . Therefore, we have (b1).

Conversely, suppose (b1) holds. Choose $x_1 \in H$ and a positive integer k as in (b1). Since $G \models s_1(\bar{a}) + k \cdot 1 \equiv_m 0$, there is $x \in G$ such that $G \models mx = s_1(\bar{a}) + k \cdot 1$. Let $x = (x'_1, x_2)$. Then clearly, $H \models mx_1 = s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx'_1$, and thus $x'_1 = x_1$. Hence $G \models \varphi(x, \bar{a})$. Note also that $G \models s_1(\bar{a}) < s_1(\bar{a}) + k \cdot 1$ by $k \geq 1$. Replacing $s_1(\bar{a}) + k \cdot 1$ with mx , we get (b). The claim is proved.

CLAIM 3. *For any n -tuple \bar{a} from G , $G \models \exists x \Psi_1(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I, 1) \cup L_{\text{mod}}$.*

PROOF OF CLAIM 3. Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n -tuple from G , and put $\bar{a}_H = (b_1, \dots, b_n)$. By Lemma 3.4, $G \models \exists x \Psi_1(x, \bar{a})$ is equivalent to the disjunction of the statements (a), (b) of Lemma 3.4. Statement (a) of

Lemma 3.4 is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I, 1) \cup L_{\text{mod}}$ by Lemma 3.6 with $D = \{1\}$ and Lemma 2.3. Statement (b) of Lemma 3.4 is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I, 1) \cup L_{\text{mod}}$ by Claim 2 and Lemma 2.3. The claim is proved.

CLAIM 4. *G admits quantifier elimination in $L(I, d) \cup L_{\text{mod}}$ if $d^G = m_0 \cdot 1^G$ with an integer $m_0 \neq 0$.*

PROOF OF CLAIM 4. 1 occurs only in subformulas of one of the forms $s(\bar{y}) = t(\bar{y})$, $s(\bar{y}) \equiv_l t(\bar{y})$ and $s(\bar{y}) < t(\bar{y})$ with terms $s(\bar{y})$, $t(\bar{y})$ of $L_{\text{ag}}(1)$. For any n -tuple \bar{a} from G , $G \models s(\bar{a}) = t(\bar{a}) \leftrightarrow |m_0|s(\bar{a}) = |m_0|t(\bar{a})$, $G \models s(\bar{a}) \equiv_l t(\bar{a}) \leftrightarrow |m_0|s(\bar{a}) \equiv_{l/|m_0|} |m_0|t(\bar{a})$, and $G \models s(\bar{a}) < t(\bar{a}) \leftrightarrow |m_0|s(\bar{a}) < |m_0|t(\bar{a})$. Since $|m_0|s(\bar{y})$ and $|m_0|t(\bar{y})$ can be considered as terms of $L_{\text{ag}}(d)$, G admits quantifier elimination in $L(I, d) \cup L_{\text{mod}}$. \square

5. Products with a Dense Regular Group

THEOREM 5.1. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and D a set of constant symbols such that $D \cap L = \emptyset$. Suppose H is an L -structure such that $H \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group, K an $L_{\text{mod}}(<, D)$ -structure such that $K \upharpoonright L_{\text{ag}}(<)$ is a dense regular ordered abelian group, and K/nK is finite and every proper coset of nK intersects with $D^K = \{d^K : d \in D\}$ for any integer $n \geq 2$. If H admits quantifier elimination in L then any extended product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$.*

Moreover, if there is a recursive procedure for quantifier elimination of H in L and for quantifier elimination of K in $L_{\text{mod}}(<, D)$, and there is a recursive map f from the set C of constant symbols of L to K such that $c^G = (c^H, f(c))$ for each $c \in C$, then there is a recursive procedure for quantifier elimination of G in $L(I, D) \cup L_{\text{mod}}$.

PROOF. We show the statement of Lemma 3.2 (2). Let x be a variable and \bar{y} an n -tuple of variables. Suppose that p, q are natural numbers such that $p \leq q$, m is a non-zero integer, $\varphi(x, \bar{y})$ is a conjunction of literals of $L_{\text{R}}(+, -, 0, I)$, $t_i(\bar{y})$ a term of L_{ag} for $i = 1, \dots, q$, $s_1(\bar{y})$ a term of L_{ag} or $-\infty$, $s_2(\bar{y})$ a term of L_{ag} or ∞ , $\Psi_1(x, \bar{y})$ the formula

$$s_1(\bar{y}) < mx < s_2(\bar{y}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}),$$

and $\Psi_2(x, \bar{y})$ the formula

$$mx = s_1(\bar{y}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}).$$

We assume that $s_1(\bar{y})$ is a term of L_{ag} in $\Psi_2(x, \bar{y})$.

By Lemma 3.3, we have the following:

CLAIM 1. *For any n -tuple \bar{a} from G , each of the statements $G \models \exists x \varphi(x, \bar{a})$ and $G \models \exists x \Psi_2(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I)$.*

Now, we turn to the reduction of $G \models \exists x \Psi_1(x, \bar{a})$ for any n -tuple \bar{a} from G .

CLAIM 2. *Let l be a common multiple of all the l_i 's and m , and D_l a subset of D such that $D_l^K = \{d^K : d \in D\}$ forms a set of representatives of all the proper cosets of lK in K . Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n -tuple from G , and put $\bar{a}_H = (b_1, \dots, b_n)$. Then the following statements (b) and (b1) are equivalent:*

(b) $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H)$ and $G \models \Psi_1(x, \bar{a})$.

(b1) $H \models \exists x_1 \ s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1 \wedge \varphi^1(x_1, \bar{a}_H)$, and for some $d \in D_l \cup \{0\}$,

$$G \models s_1(\bar{a}) < s_2(\bar{a}) \wedge s_1(\bar{a}) + d \equiv_m 0 \\ \wedge \bigwedge_{1 \leq i \leq p} s_1(\bar{a}) + d \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} s_1(\bar{a}) + d \equiv_{l_j} t_j(\bar{a}).$$

PROOF OF CLAIM 2. Suppose (b) holds. Choose $x = (x_H, x_K) \in G$ such that

$$G \models s_1(\bar{a}) < mx < s_2(\bar{a}) \wedge \varphi(x, \bar{a}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{a}).$$

Since $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H)$, we have $H \models s_1(\bar{a}_H) = mx_H = s_2(\bar{a}_H)$. Hence, $H \models \exists x_1 \ s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1 \wedge \varphi^1(x_1, \bar{a}_H)$.

Since $G \models I(s_2(\bar{a}) - s_1(\bar{a}))$, we have $G \models mx = s_1(\bar{a}) + z$ for some $z \in I^G$. Since $D_l \cup \{0\}$ is a set of representatives of all the cosets of lK in K and $K \cong I^G$, there is $d \in D_l$ such that $I^G \models z \equiv_l d$, and thus $G \models z \equiv_l d$. Therefore, $G \models s_1(\bar{a}) + d \equiv_l mx$. By the choice of l , we have $G \models s_1(\bar{a}) + d \equiv_m mx \equiv_m 0$ and $G \models s_1(\bar{a}) + d \equiv_{l_i} mx$ for each i . Therefore, we have (b1).

Conversely, suppose (b1) holds. Choose $x_1 \in H$ and $d \in D_l \cup \{0\}$ as in (b1). We have $G \models 0 < s_2(\bar{a}) - s_1(\bar{a})$ and $G \models I(s_2(\bar{a}) - s_1(\bar{a}))$. Since K is dense regular and $K \cong G|I^G$ as $L_{\text{mod}}(D)$ -structures, we can pick $x_2 \in I^G$ such that $G \models 0 < x_2 < s_2(\bar{a}) - s_1(\bar{a}) \wedge x_2 \equiv_l d$.

Then we have

$$G \models s_1(\bar{a}) < s_1(\bar{a}) + x_2 < s_2(\bar{a}) \wedge s_1(\bar{a}) + x_2 \equiv_m 0 \\ \wedge \bigwedge_{1 \leq i \leq p} s_1(\bar{a}) + x_2 \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} s_1(\bar{a}) + x_2 \equiv_{l_j} t_j(\bar{a}).$$

Let $x = (x_H, x_K) \in G$ be such that $G \models mx = s_1(\bar{a}) + x_2$. Since $x_2 \in I^G$, $x_2 = (0, z)$ for some $z \in K$. Hence, $s_1^G(\bar{a}) + x_2 = (s_1^H(\bar{a}_H), s_1^K(\bar{a}_K) + d)$. Therefore, $H \models mx_H = s_1(\bar{a}_H)$. Since $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1 \wedge \varphi^1(x_1, \bar{a}_H)$, we have $H \models mx_H = s_1(\bar{a}_H) = s_2(\bar{a}_H) = mx_1$. Hence, $H \models x_H = x_1$. Therefore, $G \models \varphi(x, \bar{a})$ since $H \models \varphi^1(x_H, \bar{a}_H)$. Now, we have (b). The claim is proved.

CLAIM 3. *For any n -tuple \bar{a} from G , $G \models \exists x \Psi_1(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I, D) \cup L_{\text{mod}}$.*

PROOF OF CLAIM 3. Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n -tuple from G , and put $\bar{a}_H = (b_1, \dots, b_n)$. By Lemma 3.4, $G \models \exists x \Psi_1(x, \bar{a})$ is equivalent to the disjunction of the statements (a), (b) of Lemma 3.4. Statement (a) of Lemma 3.4 is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I, D) \cup L_{\text{mod}}$ by Lemma 3.6 and Lemma 2.3. Statement (b) of Lemma 3.4 is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I, D) \cup L_{\text{mod}}$ by Claim 2 and Lemma 2.3. The claim is proved. \square

For the case that K is a dense regular ordered abelian group such that K/nK is infinite for some n , we have the following.

THEOREM 5.2. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants. Suppose H is an L -structure such that $H \upharpoonright L_{\text{ag}}(<)$ is a divisible ordered abelian group, and K an $L_{\text{ag}}(<)$ -structure which is a dense regular ordered abelian group. If H admits quantifier elimination in L then any extended product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I) \cup L_{\text{mod}}$.*

PROOF. We show the statement of Lemma 3.2 (2). Let x be a variable and \bar{y} an n -tuple of variables. Suppose that p, q are natural numbers such that $p \leq q$, m is a non-zero integer, $\varphi(x, \bar{y})$ a conjunction of literals of $L_{\text{R}}(+, -, 0, I)$, $t_i(\bar{y})$ a

term of L_{ag} for $i = 1, \dots, q$, $s_1(\bar{y})$ a term of L_{ag} or $-\infty$, $s_2(\bar{y})$ a term of L_{ag} or ∞ , $\Psi_1(x, \bar{y})$ the formula

$$s_1(\bar{y}) < mx < s_2(\bar{y}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}),$$

and $\Psi_2(x, \bar{y})$ the formula

$$mx = s_1(\bar{y}) \wedge \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{y}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{y}) \wedge \varphi(x, \bar{y}).$$

We assume that $s_1(\bar{y})$ is a term of L_{ag} in $\Psi_2(x, \bar{y})$.

By Lemma 3.3, we have the following:

CLAIM 1. *For any n -tuple \bar{a} from G , each of the statements $G \models \exists x \varphi(x, \bar{a})$ and $G \models \exists x \Psi_2(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I)$.*

Now, we turn to the reduction of $G \models \exists x \Psi_1(x, \bar{a})$ for any n -tuple \bar{a} from G .

CLAIM 2. *Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n -tuple from G , and put $\bar{a}_H = (b_1, \dots, b_n)$. Then the following statements (a) and (a1) are equivalent:*

- (a) $H \models s_1(\bar{a}_H) < s_2(\bar{a}_H)$ and $G \models \exists x \Psi_1(x, \bar{a})$.
- (a1) $H \models s_1(\bar{a}_H) < s_2(\bar{a}_H) \wedge \exists x_1 s_1(\bar{a}_H) \leq mx_1 \leq s_2(\bar{a}_H) \wedge \varphi^1(x, \bar{a})$ and

$$G \models \exists x \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{a}).$$

PROOF OF CLAIM 2. (a) \Rightarrow (a1) is immediate.

(a1) \Rightarrow (a). Suppose (a1) holds. Let $\bar{a}_K = (c_1, \dots, c_n)$. Choose $x = (x_H, x_K) \in G$ such that

$$G \models \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{a}).$$

Since H is divisible, $mx_H \equiv_{l_i} t_i(\bar{a}_H)$ for $i = 1, \dots, p, \dots, q$. Therefore,

$$K \models \bigwedge_{1 \leq i \leq p} mx_K \not\equiv_{l_i} t_i(\bar{a}_K) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{a}_K).$$

Now, we can show (a) by an argument similar to the proof of (2) \Rightarrow (1) for Lemma 3.5. Claim 2 is proved.

CLAIM 3. Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n -tuple from G , and put $\bar{a}_H = (b_1, \dots, b_n)$. Then the following statements (b) and (b1) are equivalent:

(b) $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H)$ and $G \models \exists x \Psi_1(x, \bar{a})$.

(b1) $H \models s_1(\bar{a}_H) = s_2(\bar{a}_H) \wedge \exists x_1 \ mx_1 \leq s_1(\bar{a}_H) \wedge \phi^1(x, \bar{a})$ and

$$G \models s_1(\bar{a}) < s_2(\bar{a}) \wedge \exists x \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{a}).$$

PROOF OF CLAIM 3. (b) \Rightarrow (b1) is immediate.

(b1) \Rightarrow (b). Suppose (b1) holds. Let $\bar{a}_K = (c_1, \dots, c_n)$. As in Claim 2, we can choose $x_K \in K$ such that

$$K \models \bigwedge_{1 \leq i \leq p} mx_K \not\equiv_{l_i} t_i(\bar{a}_K) \wedge \bigwedge_{p+1 \leq j \leq q} mx_K \equiv_{l_j} t_j(\bar{a}_K).$$

Let l be a common multiple of all the l_i 's. Choose $d \in K$ such that $K \models 0 < d < s_2(\bar{a}_K) - s_1(\bar{a}_K) \wedge d \equiv_{l} x_K$. Let $x = (s_1^H(\bar{a}_H), s_1^K(\bar{a}_K) + d)$. Then we have (b). The claim is proved.

CLAIM 4. For any n -tuple \bar{a} from G , $G \models \exists x \Psi_1(x, \bar{a})$ is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I) \cup L_{\text{mod}}$.

PROOF OF CLAIM 4. Let $\bar{a} = ((b_1, c_1), \dots, (b_n, c_n))$ be an arbitrary n -tuple from G , and put $\bar{a}_H = (b_1, \dots, b_n)$.

By Lemma 3.4, $G \models \exists x \Psi_1(x, \bar{a})$ is equivalent to the disjunction of the statements (a) and (b) of Lemma 3.4. By Fact 1.10, the statement

$$G \models \exists x \bigwedge_{1 \leq i \leq p} mx \not\equiv_{l_i} t_i(\bar{a}) \wedge \bigwedge_{p+1 \leq j \leq q} mx \equiv_{l_j} t_j(\bar{a}).$$

is equivalent to a statement of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of L_{mod} . Hence, the statement (a) of Lemma 3.4 is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I) \cup L_{\text{mod}}$ by Claim 2 and Lemma 2.3, and the statement (b) of Lemma 3.4 is equivalent to a Boolean combination of statements of the form $G \models \theta(\bar{a})$ with a quantifier-free formula $\theta(\bar{y})$ of $L(I) \cup L_{\text{mod}}$ by Claim 3 and Lemma 2.3. The claim is proved. \square

QUESTION 5.3. Is there any ordered abelian group H other than divisible ordered abelian group such that an extended product interpretation of $H \times K$ admits quantifier elimination?

EXAMPLE 5.4. Let R be a dense regular ordered abelian group such that R/nR is infinite for some $n > 0$. Let H_0 be the lexicographic product $\mathbf{Z} \times R$. H_0 does not admit quantifier elimination in $L_{\text{mod}}(<)$. Let H be a definitional expansion of H_0 such that H admits quantifier elimination in the expanded language L . Note that L is different from $L_{\text{mod}}(<, D)$ for any set D of constant symbols. Let K be a \mathbf{Z} -group or a dense regular group such that K/nK is finite for any integer $n > 0$. Then any extended product interpretation of $H \times K$ admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$ for some set $D \subseteq K$ of constants.

6. Products with a Quantifier Elimidable Group

The following two lemmas appear in [13] in some different forms.

LEMMA 6.1. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and D a set of constant symbols such that $D \cap L = \emptyset$. Suppose H is an L -structure such that $H \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group, K an $L_{\text{mod}}(<, D)$ -structure such that $K \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group, and G an extended product interpretation of $H \times K$ with a new predicate I . Suppose H has the smallest positive element 1_H and there is a constant symbol c of L such that $c^H = k \cdot 1_H$ for some integer $k \neq 0$. Then I is equivalent to a quantifier-free formula of $L_{\text{ag}}(<, c)$ in G .*

PROOF. Suppose c is a constant symbol of L such that $c^H = k \cdot 1_H$ with an integer $k \neq 0$. Without loss of generality, we can assume that $k > 0$. Since $c^G = (k \cdot 1_H, c_K)$ for some $c_K \in K$, we have $G \models \forall x (I(x) \leftrightarrow -c < kx < c)$. \square

LEMMA 6.2. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and D a set of constant symbols such that $D \cap L = \emptyset$. Suppose H is an L -structure such that $H \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group, K an $L_{\text{mod}}(<, D)$ -structure such that $K \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group, and G an extended product interpretation of $H \times K$ with a new predicate I . Suppose further that n is an integer and there is a binary relation \equiv'_n of L such that $H \models \forall x, y (x \equiv'_n y \leftrightarrow n \mid (x - y))$. Then the following hold:*

- (1) $G \models \forall x, y (x \equiv'_n y \leftrightarrow \exists z (I(z) \wedge x - y - z \equiv_n 0))$.
- (2) If K/nK is finite and every coset of nK in K has a representative of the form t^K for some term t of $L_{\text{ag}}(D)$, then the relation \equiv'_n is definable by a quantifier-free formula of $L_{\text{mod}}(D)$ in G .
- (3) Suppose K is a \mathbf{Z} -group and let 1_K be the smallest positive element of K . If $K \models d = k \cdot 1_K$ for some $d \in D$ with an integer $k \neq 0$, then the relation \equiv'_n is definable by a quantifier-free formula of $L_{\text{mod}}(d)$ in G .

PROOF. (1) Let x, y be arbitrary elements of G . Then we can write $x = (x_H, x_K)$ and $y = (y_H, y_K)$ for some $x_H, y_H \in H$ and $x_K, y_K \in K$. Suppose $G \models x \equiv'_n y$. Then by the definition of an extended product interpretation, $H \models x_H \equiv'_n y_H$, and thus $H \models n \mid (x_H - y_H)$. Let $z = (0^H, x_K -^K y_K)$. Then $z \in G$ and $G \models I(z) \wedge x - y - z \equiv_n 0$.

Conversely, suppose $G \models I(z) \wedge x - y - z \equiv_n 0$ for some $z \in G$. Since $G \models I(z)$, $z = (0^H, z_K)$ for some $z_K \in K$. Hence, $(x - y - z)^G = (x_H -^H y_H, u)$ for some $u \in K$. Since $G \models n \mid (x - y - z)$, $H \models n \mid (x_H - y_H)$. Therefore, $G \models x \equiv'_n y$.

(2) Let S be a finite set of terms of $L_{\text{ag}}(D)$ such that the set $S^K = \{t^K : t \in S\}$ forms a set of representatives of all the cosets of nK in K . Then by (1),

$$G \models \forall x, y \quad x \equiv'_n y \leftrightarrow \bigvee_{t \in S} x - y \equiv_n t.$$

(3) Introduce a constant symbol 1 such that 1^K is the smallest positive element of K . Let $S = \{0, 1, 2 \cdot 1, \dots, (n-1) \cdot 1\}$. Then S^K forms a set of representatives of all the cosets of nK in K .

Let $d \in D$ be such that $K \models d = k \cdot 1$ with an integer $k \neq 0$. Then for each $i < n$ and for any $x, y \in G$, $G \models x - y \equiv_n i \cdot 1$ if and only if $G \models k(x - y) \equiv_{kn} i \cdot d$. By this and (2), the relation \equiv'_n is definable by a quantifier-free formula of $L_{\text{mod}}(d)$ in G . \square

THEOREM 6.3. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and H an L -structure such that $H \mid L_{\text{ag}}(<)$ is an ordered abelian group. Suppose K is an ordered abelian group and $D \subseteq K$ a pure subgroup of K such that K admits quantifier elimination in $L_{\text{mod}}(<, D)$ but K is not dense regular.*

If H admits quantifier elimination in L then any extended product interpretation of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$.

PROOF. Since K admits quantifier elimination in the language $L_{\text{mod}}(<, D)$, by Fact 1.16, there is a finite sequence $\{G_i\}_{0 \leq i \leq m}$ of convex subgroups of K and a sequence $\{(k_i, d_i)\}_{1 \leq i \leq m}$ such that (i) $G_m = K$; (ii) for $1 \leq i \leq m$, k_i is a positive integer, $d_i \in D$, $d_i \in G_i - G_{i-1}$, G_i/G_{i-1} is a \mathbf{Z} -group with smallest positive element $1_i + G_{i-1}$, $k_i \cdot 1_i - d_i \in G_{i-1}$; and (iii) G_0 is dense regular, and for every prime p , $\beta_p(G_0)$ is finite and every coset of pG_0 in G_0 has a representative in D .

Introduce a new predicate I_i representing G_i for each $i \leq m$. Let K' be an ω_1 -saturated elementary extension of K in the expanded language $L_{\text{ag}}(<, D) \cup \{I_i\}_{i \leq m}$. Let $G'_i = I_i(K')$ for each $i = 1, \dots, m$. By Fact 1.7, for each $i = 1, \dots, m$,

there is a subgroup A_i of G'_i such that $G'_i = A_i \oplus G'_{i-1}$. $A_i \cong G'_i/G'_{i-1}$ is a \mathbf{Z} -group. Let 1_{A_i} be the smallest positive element of A_i for each $i = 1, \dots, m$. Then $k_i \cdot 1_{A_i} - d_i^{K'} \in G'_{i-1}$ for each $i = 1, \dots, m$.

Now, let G be an arbitrary extended product interpretation of $H \times K$ with a new predicate I . For each constant symbols c of L , we have $c^G = (c^H, c_K)$ for some $c_K \in K$ by the definition of an extended product interpretation. Let G' be an extended product interpretation of $H \times K' \mid L_{\text{mod}}(<, D)$ with new predicate I such that $c^{G'} = (c^H, c_K)$ for each constant symbols c of L . Then $G' \equiv G$ for $L(I, D)$ by Lemma 2.8. To show that G admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$, it is enough to show that G' admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$. Let $D_{G_0} = \{d \in D : d^K \in G_0\}$. Then by Remark 3.1, it is enough to show that the reduct G'' of G' to $L(I, d_m, d_{m-1}, \dots, d_1, D_{G_0})$ admits quantifier elimination in $L(I, d_m, d_{m-1}, \dots, d_1, D_{G_0}) \cup L_{\text{mod}}$.

Consider A_m as a structure for $L_{\text{ag}}(<, d_m)$ by $d_m^{A_m} = k_m \cdot 1_{A_m}$. Let B_m be a structure for $L(I, d_m)$ which is an extended product interpretation of $H \times A_m$ with new predicate I such that $c^{B_m} = (c^H, c_{A_m})$ for each constant symbol of L where $c^G = (c^H, c_K)$ with $c_K = c_{A_m} + c_{G'_{m-1}}$, $c_{A_m} \in A_m$ and $c_{G'_{m-1}} \in G'_{m-1}$. By Theorem 4.2, B_m admits quantifier elimination in the language $L(I, d_m) \cup L_{\text{mod}}$. Let $L_{\text{mod}}^m = \{\equiv_n^m : n \geq 2\}$ and consider B_m as a structure for $L(I, d_m) \cup L_{\text{mod}}^m$ with $B_m \models \forall x, y (x \equiv_n^m y \leftrightarrow x \equiv_n y)$ for each integer $n \geq 2$. B_m admits quantifier elimination in the language $L(I, d_m) \cup L_{\text{mod}}^m$. Since K' is isomorphic to the lexicographic product of A_m and G'_{m-1} by Lemma 2.9, G'' is isomorphic to a reduct of an extended product interpretation of $B_m \times G'_{m-1}$ with new predicate I_{m-1} . Here, G'_{m-1} is considered as a structure for $L_{\text{ag}}(<, d_{m-1}, \dots, d_1, D_{G_0})$.

Now, consider A_{m-1} as a structure for $L_{\text{ag}}(<, d_{m-1})$ by $d_{m-1}^{A_{m-1}} = k_{m-1} \cdot 1_{A_{m-1}}$. Let B_{m-1} be a structure for $L(I, I_{m-1}, d_m, d_{m-1})$ which is an extended product interpretation of $B_m \times A_m$ with new predicate I_{m-1} such that $c^{B_{m-1}} = (c^H, c_{A_m}, c_{A_{m-1}})$ for each constant symbol of L where $c_K = c_{A_m} + c_{A_{m-1}} + c_{G'_{m-2}}$, $c_{A_m} \in A_m$, $c_{A_{m-1}} \in A_{m-1}$, and $c_{G'_{m-2}} \in G'_{m-2}$. By Theorem 4.2, B_{m-1} admits quantifier elimination in the language $L(I, I_{m-1}, d_m, d_{m-1}) \cup L_{\text{mod}}^m \cup L_{\text{mod}}$. I_{m-1} is definable by a quantifier-free formula of $L_{\text{ag}}(d_m)$ in B_{m-1} by Lemma 6.1, and each relation of L_{mod}^m is definable by a quantifier-free formula of $L_{\text{ag}}(d_{m_1})$ in B_{m-1} by Lemma 6.2. Therefore, B_{m-1} admits quantifier elimination in the language $L(I, d_m, d_{m-1}) \cup L_{\text{mod}}$. Let $L_{\text{mod}}^{m-1} = \{\equiv_n^{m-1} : n \geq 2\}$ and consider B_{m-1} as a structure for $L(I, d_m, d_{m-1}) \cup L_{\text{mod}}^{m-1}$ with $B_{m-1} \models \forall x, y (x \equiv_n^{m-1} y \leftrightarrow x \equiv_n y)$ for each integer $n \geq 2$. B_{m-1} admits quantifier elimination in the language $L(I, d_m, d_{m-1}) \cup L_{\text{mod}}^{m-1}$. Since G'_{m-1} is isomorphic to the lexicographic product of A_{m-1} and G'_{m-2} by Lemma 2.9, G'' is isomorphic to a reduct of an extended product interpretation of $B_{m-1} \times G'_{m-2}$

with new predicate I_{m-2} . Here, G'_{m-2} is considered as a structure for $L_{\text{ag}}(<, d_{m-2}, \dots, d_1, D_{G_0})$.

Repeating this argument, we get a structure B_1 for $L(I, d_m, d_{m-1}, \dots, d_1) \cup L_{\text{mod}}^1$ with $L_{\text{mod}}^1 = \{\equiv_n^1 : n \geq 2\}$ such that $B_1 \models \forall x, y (x \equiv_n^1 y \leftrightarrow x \equiv_n y)$ for each integer $n \geq 2$, B_1 admits quantifier elimination in its language, and G'' is isomorphic to a reduct of an extended product interpretation B_0 of $B_1 \times G'_0$ with new predicate I_0 . Here, G'_0 is considered as a structure for $L_{\text{ag}}(<, D_{G_0})$. B_0 admits quantifier elimination in the language $L(I, I_0, d_m, d_{m-1}, \dots, d_1) \cup L_{\text{mod}}^1 \cup L_{\text{mod}}$ by Theorem 5.1. I_0 is definable by a quantifier-free formula of $L_{\text{ag}}(d_1)$ in B_0 by Lemma 6.1, and each relation of L_{mod}^1 is definable by a quantifier-free formula of $L_{\text{ag}}(D_{G_0})$ in B_0 by Lemma 6.2. Therefore, B_0 admits quantifier elimination in the language $L(I, d_m, d_{m-1}, \dots, d_1, D_{G_0}) \cup L_{\text{mod}}$ by Theorem 5.1. Since G'' is isomorphic to the reduct of B_0 to the language $L(I, d_m, d_{m-1}, \dots, d_1, D_{G_0})$, G'' admits quantifier elimination in $L(I, d_m, d_{m-1}, \dots, d_1, D_{G_0}) \cup L_{\text{mod}}$. \square

Finally, we show partial converses.

THEOREM 6.4. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and H an L -structure such that $H \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group. Suppose K is an ordered abelian group and $D \subseteq K$ a pure subgroup of K .*

If an extended product interpretation of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$ then H admits quantifier elimination in $L \cup L_{\text{mod}}$.

PROOF. Suppose an extended product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$. We show that H admits quantifier elimination in $L \cup L_{\text{mod}}$. Let x be a variable and \bar{y} a tuple of variables. Let $\exists x \varphi(x, \bar{y})$ be a formula of $L \cup L_{\text{mod}}$, where $\varphi(x, \bar{y})$ is quantifier-free. Since $\varphi(x, \bar{y})$ is a quantifier-free formula of $L \cup L_{\text{mod}}$, the formula $\varphi(x, \bar{y})$ is a Boolean combination of formulas of the forms $mx = t(\bar{y})$, $mx < t(\bar{y})$, $mx + t(\bar{y}) \equiv_n 0$ and $R(s_1(x, \bar{y}), \dots, s_l(x, \bar{y}))$, where R is a relation symbol of $L - \{<\}$, l, m, n are integers such that l is the arity of R and $n \geq 2$, and $t(\bar{y})$, $s_1(x, \bar{y}), \dots, s_l(x, \bar{y})$ are terms of L .

Let $\varphi^*(x, \bar{y})$ be a formula obtained from $\varphi(x, \bar{y})$ by replacing $mx = t(\bar{y})$, $mx < t(\bar{y})$ and $mx + t(\bar{y}) \equiv_n 0$ with $I(t(\bar{y}) - mx)$, $mx < t(\bar{y}) \wedge \neg I(t(\bar{y}) - mx)$, and $\exists z(I(mx + t(\bar{y}) - nz))$, respectively. Let $\bar{h} = (h_1, \dots, h_n)$ be a tuple of elements from the ordered abelian group H . Then, we have

$$H \models \exists x \varphi(x, \bar{h}) \Leftrightarrow G \models \exists x \varphi^*(x, \bar{h}_G),$$

where $\bar{h}_G = ((h_1, 0^K), \dots, (h_n, 0^K))$. Since the ordered abelian group G admits quantifier elimination in the language $L(I, D) \cup L_{\text{mod}}$, there exists some quantifier-free formula $\psi(\bar{y})$ in $L(I, D) \cup L_{\text{mod}}$ such that

$$G \models \exists x \varphi^*(x, \bar{h}_G) \Leftrightarrow G \models \psi(\bar{h}_G).$$

Because $\psi(\bar{y})$ is a quantifier-free formula of $L(I, D) \cup L_{\text{mod}}$, the formula $\psi(\bar{y})$ is a Boolean combination of formulas of the forms $t(\bar{y}) = 0$, $t(\bar{y}) < 0$, $t(\bar{y}) \equiv_n 0$, $R(s_1(\bar{y}), \dots, s_l(\bar{y}))$ and $I(t(\bar{y}))$, where l, n are positive integers, t, s_1, \dots, s_l are terms of $L(D)$ and R is an l -ary relation symbol of L other than “ $<$ ”. Let $t(\bar{y}) = t_1(\bar{y}) + t_2(\bar{c}) + d$, where $t_1(\bar{y})$ is a term of L_{ag} , $t_2(\bar{c})$ a term of L_{ag} with a p -tuple \bar{z} of variables, $\bar{c} = (c_1, \dots, c_p)$ is a tuple of constant symbols from L , and $d \in D$. Choose $c_{i,K} \in K$ such that $c_i^G = (c_i^H, c_{i,K})$ for each $i = 1, \dots, p$ and let $\bar{c}_K = (c_{1,K}, \dots, c_{p,K})$. Note that $t_2(\bar{c})^G = (t_2(\bar{c})^H, t_2^K(\bar{c}_K))$. Then,

$$\begin{aligned} G \models t_1(\bar{h}_G) + t_2(\bar{c}) + d = 0 &\Leftrightarrow \begin{cases} H \models t_1(\bar{h}) + t_2(\bar{c}) = 0 & \text{if } K \models t_2(\bar{c}_K) + d = 0 \\ H \models \neg(0 = 0) & \text{if } K \models t_2(\bar{c}_K) + d \neq 0, \end{cases} \\ G \models t_1(\bar{h}_G) + t_2(\bar{c}) + d < 0 &\Leftrightarrow \begin{cases} H \models t_1(\bar{h}) + t_2(\bar{c}) < 0 & \text{if } K \models t_2(\bar{c}_K) + d \geq 0 \\ H \models t_1(\bar{h}) + t_2(\bar{c}) \leq 0 & \text{if } K \models t_2(\bar{c}_K) + d < 0, \end{cases} \\ G \models t_1(\bar{h}_G) + t_2(\bar{c}) + d \equiv_n 0 &\Leftrightarrow \begin{cases} H \models t_1(\bar{h}) + t_2(\bar{c}) \equiv_n 0 & \text{if } K \models t_2(\bar{c}_K) + d \equiv_n 0 \\ H \models \neg(0 = 0) & \text{if } K \models t_2(\bar{c}_K) + d \not\equiv_n 0, \end{cases} \\ G \models R(s_1(\bar{h}_G), \dots, s_l(\bar{h}_G)) &\Leftrightarrow H \models R(s_1^*(\bar{h}), \dots, s_l^*(\bar{h})), \\ G \models I(t_1(\bar{h}_G) + t_2(\bar{c}) + d) &\Leftrightarrow H \models t_1(\bar{h}) + t_2(\bar{c}) = 0, \end{aligned}$$

where $s_i^*(\bar{y})$ is the term obtained from $s_i(\bar{y})$ by replacing each element of D with 0.

Therefore, there exists some quantifier-free formula $\psi'(\bar{y})$ in $L \cup L_{\text{mod}}$ such that $G \models \psi(\bar{h}_G) \Leftrightarrow H \models \psi'(\bar{h})$. It follows that H admits quantifier elimination in $L \cup L_{\text{mod}}$. \square

THEOREM 6.5. *Let L be an expansion of $L_{\text{ag}}(<)$ by predicates and constants, and H an L -structure such that $H \upharpoonright L_{\text{ag}}(<)$ is an ordered abelian group. Suppose K is an ordered abelian group and $D \subseteq K$ a pure subgroup of K .*

If an extended product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$ and there is a constant symbol $d_c \in D$ such that $c^G = (c^H, d_c^K)$ for each constant symbol c of L , then K admits quantifier elimination in $L_{\text{mod}}(<, D)$.

PROOF. Suppose an extended product interpretation G of $H \times K$ with a new predicate I admits quantifier elimination in $L(I, D) \cup L_{\text{mod}}$. We show that K admits quantifier elimination in $L_{\text{mod}}(<, D)$. Let $\exists x \theta(x, \bar{y})$ be a formula of $L_{\text{mod}}(<, D)$, where $\theta(x, \bar{y})$ is a quantifier-free formula of $L_{\text{mod}}(<, D)$. Then the formula $\theta(x, \bar{y})$ is a Boolean combination of formulas of the forms $mx = t(\bar{y})$, $mx < t(\bar{y})$, and $mx + t(\bar{y}) \equiv_n 0$, where m and n are integers such that $n \geq 2$, and t is a term of $L_{\text{ag}}(D)$. Let $\bar{k} = (k_1, \dots, k_n)$ be a tuple of elements from the ordered abelian group K . Let $\bar{k}_G = ((0, k_1), \dots, (0, k_n))$. Then, we have

$$K \models \exists x \varphi(x, \bar{k}) \Leftrightarrow G \models \exists x I(x) \wedge \varphi(x, \bar{k}_G).$$

Since the ordered abelian group G admits quantifier elimination in the language $L(I, D) \cup L_{\text{mod}}$, there exists some quantifier-free formula $\tau(\bar{y})$ of $L(I, D) \cup L_{\text{mod}}$ such that

$$G \models \exists x I(x) \wedge \varphi(x, \bar{k}_G) \Leftrightarrow G \models \tau(\bar{k}_G).$$

Because $\tau(\bar{y})$ is a quantifier-free formula of $L(I, D) \cup L_{\text{mod}}$, the formula $\tau(\bar{y})$ is a Boolean combination of the forms $t(\bar{y}) = 0$, $t(\bar{y}) < 0$, $t(\bar{y}) \equiv_n 0$, $R(s_1(\bar{y}), \dots, s_l(\bar{y}))$ and $I(t(\bar{y}))$, where l, n are positive integers, t, s_1, \dots, s_l are terms of $L(D)$ and R is an l -ary relation symbol of L . Let $t(\bar{y}) = t_1(\bar{y}) + t_2(\bar{c}) + d$, where $t_1(\bar{y})$ is a term of L_{ag} , $t_2(\bar{c})$ a term of L_{ag} with a p -tuple \bar{c} of variables, $\bar{c} = (c_1, \dots, c_p)$ is a tuple of constant symbols from L , and $d \in D$. Put $\bar{0} = (0, \dots, 0)$. Choose $d_{c_i} \in D$ such that $c_i^G = (c_i^H, d_{c_i}^K)$ for each $i = 1, \dots, p$ and let $\bar{d}_{\bar{c}} = (d_{c_1}, \dots, d_{c_p})$. Note that $t_2(\bar{c})^G = (t_2(\bar{c})^H, t_2(\bar{d}_{\bar{c}})^K)$. Then,

$$\begin{aligned} G \models t_1(\bar{k}_G) + t_2(\bar{c}) + d = 0 &\Leftrightarrow \begin{cases} K \models t_1(\bar{k}) + t_2(\bar{d}_{\bar{c}}) + d = 0 & \text{if } H \models t_2(\bar{c}) = 0 \\ K \models \neg(0 = 0) & \text{if } H \models t_2(\bar{c}) \neq 0, \end{cases} \\ G \models t_1(\bar{k}_G) + t_2(\bar{c}) + d < 0 &\Leftrightarrow \begin{cases} K \models \neg(0 = 0) & \text{if } H \models t_2(\bar{c}) > 0 \\ K \models t_1(\bar{k}) + t_2(\bar{d}_{\bar{c}}) + d < 0 & \text{if } H \models t_2(\bar{c}) = 0 \\ K \models 0 = 0 & \text{if } H \models t_2(\bar{c}) < 0, \end{cases} \\ G \models t_1(\bar{k}_G) + t_2(\bar{c}) + d \equiv_n 0 &\Leftrightarrow \begin{cases} K \models t_1(\bar{k}) + t_2(\bar{d}_{\bar{c}}) + d \equiv_n 0 & \text{if } H \models t_2(\bar{c}) \equiv_n 0 \\ K \models \neg(0 = 0) & \text{if } H \models t_2(\bar{c}) \not\equiv_n 0, \end{cases} \\ G \models R(s_1(\bar{k}_G), \dots, s_l(\bar{k}_G)) &\Leftrightarrow \begin{cases} K \models 0 = 0 & \text{if } H \models R(s_1^*(\bar{0}), \dots, s_l^*(\bar{0})) \\ K \models \neg(0 = 0) & \text{if } H \models \neg R(s_1^*(\bar{0}), \dots, s_l^*(\bar{0})), \end{cases} \\ G \models I(t_1(\bar{k}_G) + t_2(\bar{c}) + d) &\Leftrightarrow \begin{cases} K \models 0 = 0 & \text{if } H \models t_2(\bar{c}) = 0 \\ K \models \neg(0 = 0) & \text{if } H \models t_2(\bar{c}) \neq 0, \end{cases} \end{aligned}$$

where $s_i^*(\bar{y})$ is the term obtained from $s_i(\bar{y})$ by replacing d with 0 .

Therefore, there exists some quantifier-free formula $\tau'(\bar{y})$ in $L_{\text{mod}}(<, D)$ such that $G \models \tau(\bar{k}_G) \Leftrightarrow K \models \tau'(\bar{k})$. It follows that K admits quantifier elimination in $L_{\text{mod}}(<, D)$. \square

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