Quantifying coherence in terms of Fisher information

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In quantum metrology, the parameter estimation accuracy is bounded by quantum Fisher information. In this paper, we present coherence measures in terms of (quantum) Fisher information by directly considering the post-selective non-unitary parametrization process. This coherence measure demonstrates the apparent operational meaning by the exact connection between coherence and parameter estimation accuracy. We also discuss the distinction between our coherence measure and the quantum Fisher information subject to unitary parametrization. The analytic coherence measure is given for qubit states.

I. INTRODUCTION

Quantum coherence, as a fundamental feature in guantum physics, attracts a lot of attention in recent years. Many works have investigated the role of coherence in quantum optics [1-4], quantum thermodynamics [5-7], quantum phase transitions [8], quantum biology [9, 10], and quantum information science [11–18]. These researches not only promote the development of related applications but also the development of the resource theory of coherence [19, 20], where coherence is treated as a physical resource under some limited conditions. Benefiting from the operational view and axiomatic approach, one can quantify coherence in a rigorous manner, study the transformation of coherence, and reveal the connection between coherence with other fundamental quantum features [21–32]. In particular, some coherence measures contain obvious operational meanings, which provide us with a way to understand (interpret) coherence from the viewpoint of quantum information processes (QIP) and find out the potential relation between coherence with some characteristics in QIP [33–39].

It is shown that the coherence of the probing state in many quantum metrology processes is often a key ingredient [11–13]. For instance, in the usual phase estimation for parameter θ with unitary parametrization $\mathcal{U}_{\theta}(\cdot) = e^{-i\theta H}(\cdot)e^{i\theta H}$, coherence with regard to the eigenvectors of Hermitian operator H is necessary. Furthermore, the optimal estimation accuracy of an unknown parameter could be obtained by the state with maximal coherence in the sequential protocol [13]. The estimation accuracy is bounded by quantum Fisher information (QFI), a crucial ingredient in quantum metrology [40-42]. A simple calculation can show that QFI subject to unitary parametrization $\mathcal{U}_{\theta}(\cdot)$ in the qubit case [43] is monotonous with some coherence measures (such as l_1 norm coherence). Many works have investigated the relation between quantum coherence with Fisher information (FI) and QFI [44–53]. Coherence within some *particular* settings could be understood by QFI (or FI) [46, 47, 52].

Significantly, QFI in unitary parametrization is closely connected with unspeakable coherence [47, 54], a special case of resource theory of asymmetry [55–57]. In addition, based on QFI concerning the dephasing parameter, coherence measure has been given in the sense of strictly incoherent operations as free operations [46]. However, up to now the estimation accuracy and FI (or QFI) have not been used to directly quantify quantum coherence in *general* scenarios. An intuitive challenge is that QFI with unitary parametrization $\mathcal{U}_{\theta}(\cdot)$ in the usual sense is not a coherence measure in the *general* resource theory of coherence [20]. For example, 2-dimensional maximally coherent states (MCS) could be obtained under incoherent operations from 3-dimensional MCS [33, 48, 58], but the QFI of the former is strictly larger than the latter, which directly violates the monotonicity of a good measure. Therefore, it is significant to find an appropriate parametrization process for establishing coherence measures and further investigating the role of coherence in quantum metrology.

In this paper, we successfully establish several equivalent coherence measures in the general resource theory of coherence by the FI (and QFI) subject to a type of non-unitary parametrization. Since the optimal estimation accuracy is bounded by FI which is asymptotically attained with maximum likelihood estimators [40, 41], our measure naturally inherits the operational meaning of FI through the optimal estimation accuracy with nonunitary parametrization. We also show that in the qubit case, our coherence measure can be equivalently understood through unitary parametrization and the analytic expression can be obtained. Our coherence measure not only builds a direct relation between coherence and parameter estimation accuracy (or FI) but also sheds new light on the roles of the non-unitary parametrization process. The remainder of this paper is organized as follows. In Sec. II, we first introduce the fundamental concepts of resource theory of coherence and our parametrization process, then present several main theorems to build the coherence measure based on FI. In Sec. III, we give the analytic result of the coherence measure in the qubit case and discuss the equivalence with the unitary parametrization. Finally, we draw our conclusion in Sec. IV.

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II. COHERENCE IN TERMS OF QFI

In this section, we'd like to first introduce the resource theory of coherence established mainly based on the incoherent (free) operations and incoherent (free) states [20]. Considering the preferred basis $\{|n\rangle\}$, the incoherent state is defined by $\rho = \sum_n q_n |n\rangle \langle n|$ with $\mathcal I$ denoting the set of incoherent states, and the incoherent operations (IO) with the Kraus representation $\{K_n : \sum_l K_l^{\dagger} K_l = \mathbb{I}\}$ is a special type of completely positive and trace-preserving (CPTP) map defined by $\frac{K_l \varrho K_l^{\dagger}}{\operatorname{tr}(K_l \varrho K_l^{\dagger})} \in \mathcal{I} \text{ for } \varrho \in \mathcal{I}. \text{ In this sense, a good coherence}$

measure $C(\rho)$ for any state ρ should satisfy

(A1) Non-Negativity: $C(\rho) \ge 0$ is saturated iff $\rho \in \mathcal{I}$; (A2) Monotonicity: $C(\mathcal{E}(\rho)) \leq C(\rho)$ for any incoherent operation $\mathcal{E}(\cdot)$;

(A3) Strong Monotonicity: $\sum_{n} p_n C(K_n \rho K_n^{\dagger}/p_n) \leq$

 $C(\rho)$ for any IO $\{K_n\}$, with $p_n = \operatorname{Tr}[K_n \rho K_n^{\dagger}];$ (A4) Convexity: $C(\rho) \leq \sum_i p_i C(\rho_i)$ for any $\rho =$ $\sum_{i} p_i \rho_i$.

To present a valid coherence measure, we begin with the following parametrization process. Considering a state ρ undergoing quantum channel \mathcal{E}_{θ} depending on parameter θ , the unknown parameter could be estimated from measurements on $\mathcal{E}_{\theta}(\rho)$. Here we are interested in the "free" parametrization processes $\mathcal{E}_{\theta} = \{E_x(\theta)\}$:

$$E_x(\theta) = \sum_n b_n^x(\theta) |g_x(n)\rangle \langle n|, \sum_x E_x(\theta)^{\dagger} E_x(\theta) = \mathbb{I}, \quad (1)$$

where $\{|n\rangle\}$ is the preferred incoherent basis, and $g_x(\cdot)$ is a map from an integer to another.

In order to focus on the role of coherence, we desire that within the parametrization process, the incoherent probe can not take effect on parameter estimation. That is, the measurement outcomes $\mathcal{E}_{\theta}(\varrho)$ and $\{\varrho_x, p_x\}$ obtained from incoherent probe $(\varrho \in \mathcal{I})$ do not depend on parameter θ , where $p_x = \text{tr}[E_x(\theta)\varrho E_x(\theta)^{\dagger}]$ and $\varrho_x = E_x(\theta) \varrho E_x(\theta)^{\dagger} / p_x$. Thus $|b_n^x(\theta)|$ does not depend on parameter θ , and $E_x(\theta)$ can be rewritten as

$$E_x(\theta) = \sum_n c_n^x e^{ih_n^x(\theta)} |g_x(n)\rangle \langle n|, \qquad (2)$$

where c_n^x is parameter-independent, and h_n^x is a real function. In fact, it is very similar to the case of the usual phase estimation $\mathcal{U}_{\theta}(\cdot) = e^{-i\theta H}(\cdot)e^{i\theta H}$ mentioned in Introduction. One can find that the measurement outcomes of an incoherent probe in the phase estimation do not depend on parameter θ , either. In addition, \mathcal{U}_{θ} can be expressed based on $e^{-iH\theta} = \sum_{n} e^{-ih_{n}\theta} |n\rangle \langle n|$ (h_{n} is eigenvalue of H), which is analogous to Eq. (2). In this sense, parametrization process \mathcal{E}_{θ} can be understood as a generalization of unitary phase estimation to non-unitary case.

In addition, we could restrict $\partial_{\theta} h_n^x(\theta) \in [0,1]$, and the conclusion in a more general case could be derived from this case, the detailed discussion is shown in Appendix

A. Based on the Stinespring dilation theorem, the operations could be implemented by a controlled unitary operator and an operation swapping specified states. The details are shown in Appendix B. All the operations of interest (operations in Eq. (2) with $\partial_{\theta} h_n^x(\theta) \in [0,1]$) comprise a set denoted by G. One will find that IO satisfying $Rank \left[E_x(\theta)^{\dagger} E_x(\theta) \right] = 1$ has particular interest in the paper, so we use G_1 to represent the IO set with this particular property.

If the post-selection is allowed, the IO \mathcal{E}_{θ} performed on a quantum state ρ will directly lead to the probability distribution

$$P^{\varepsilon}(x|\theta) = \operatorname{tr}(E_x(\theta)\rho E_x(\theta)^{\dagger}).$$
(3)

If the post-selection isn't allowed, the state after the IO will become $\mathcal{E}_{\theta}(\rho)$. One can operate a positive operator value measure (POVM) $\mathcal{M} = \{M_x\}$ on the state $\mathcal{E}_{\theta}(\rho)$, and obtain the probability distribution family as

$$P^{\varepsilon}_{\mathcal{M}}(x|\theta) = \operatorname{tr}(M_x \mathcal{E}_{\theta}(\rho)), \qquad (4)$$

where the subscript \mathcal{M} denotes the general POVM. The FI of distribution $P(x|\theta)$ is given by

$$F(P,\theta_0) = \sum_{x} P(x|\theta_0) \left[\frac{\partial \ln P(x|\theta)}{\partial \theta} \Big|_{\theta_0} \right]^2, \quad (5)$$

and the QFI of $P^{\varepsilon}_{\mathcal{M}}(x|\theta)$ for any given θ_0 can be written as

$$F_{Q}(\rho, \mathcal{E}, \theta_{0}) = \max_{\mathcal{M}} F(P_{\mathcal{M}}^{\mathcal{E}}, \theta_{0}).$$
(6)

Based on above FI and QFI, we can establish two coherence measures, respectively, which will be given by the following two theorems.

Theorem 1.-The coherence of a state ρ can be quantified by the maximal FI for a given parameter θ_0 as

$$C^{\theta_0}(\rho) = \max_{\mathcal{E} \in G} F(P^{\mathcal{E}}, \theta_0), \tag{7}$$

where $F(P^{\varepsilon}, \theta_0)$ is FI of distribution in Eq. (3).

Proof: We need to prove $C^{\theta_0}(\rho)$ satisfying A1-A4.

(A1) Non-Negativity. If ρ is incoherent, for any \mathcal{E} and x, we have

$$E_{x}(\theta)\rho E_{x}(\theta)^{\dagger}$$

$$=\sum_{n} b_{n}^{x}(\theta)|g_{x}(n)\rangle\langle n|\rho\sum_{m} b_{m}^{x*}(\theta)|m\rangle\langle g_{x}(m)|$$

$$=\sum_{nm} b_{n}^{x}(\theta)b_{m}^{x*}(\theta)\rho_{nm}|g_{x}(n)\rangle\langle g_{x}(m)|$$

$$=\sum_{n} |b_{n}^{x}(\theta)|^{2}\rho_{nn}|g_{x}(n)\rangle\langle g_{x}(n)|, \qquad (8)$$

which doesn't depend on θ due to $b_n^x(\theta) = c_n^x e^{ih_n^x(\theta)}$. Thus $P^{\varepsilon}(x|\theta)$ doesn't depend on θ either, which means

$$F(P^{\varepsilon},\theta_0) = \sum_{x} \left[\frac{\partial P^{\varepsilon}(x|\theta)}{\partial \theta} \Big|_{\theta_0} \right]^2 \frac{1}{P^{\varepsilon}(x|\theta_0)} = 0.$$
(9)

Eq. (9) leads to $C^{\theta_0}(\rho) = 0$.

Conversely, if a *d*-dimensional ρ has non-zero offdiagonal entries, without loss of generality, one can let $\rho_{12} = |\rho_{12}|e^{i\alpha}$. There exists an IO $\{E_i\} \in G$,

$$E_{1}(\theta) = \frac{\sqrt{2}}{2} e^{i(\theta+\gamma)} |1\rangle \langle 1| + \frac{\sqrt{2}}{2} |1\rangle \langle 2|,$$

$$E_{2}(\theta) = -\frac{\sqrt{2}}{2} e^{i(\theta+\gamma)} |2\rangle \langle 1| + \frac{\sqrt{2}}{2} |2\rangle \langle 2|,$$

$$E_{3}(\theta) = \sum_{n=3}^{d} |n\rangle \langle n|,$$
(10)

with $\alpha + \theta_0 + \gamma \in [-\pi/2, 0) \bigcup (0, \pi/2]$, such that

$$P^{\varepsilon}(1|\theta_0) = \operatorname{tr}(E_1(\theta_0)\rho E_1(\theta_0)^{\dagger}) \neq 0,$$

$$\partial_{\theta}\operatorname{tr}(E_1(\theta)\rho E_1(\theta)^{\dagger})|_{\theta_0} \neq 0, \qquad (11)$$

which obviously shows $C^{\theta_0}(\rho) \neq 0$ and $C^{\theta_0}(\rho) > 0$.

(A3) Strong monotonicity. Suppose ρ undergoes an arbitrary IO

$$K_l = \sum_n a_n^l |f_l(n)\rangle \langle n|, \qquad (12)$$

the post-measurement ensemble $\{t_l, \rho_l\}$ reads

$$t_l = \operatorname{tr}(K_l \rho K_l^{\dagger}), \rho_l = K_l \rho K_l^{\dagger} / t_l.$$
(13)

Let $\mathcal{E}^{(l)} = \{E_x^l(\theta)\}_x$ be the optimal IO for ρ_l such that

$$C^{\theta_0}(\rho_l) = F(P_l, \theta_0), \qquad (14)$$

where $E_x^l(\theta) = \sum_n b_n^{lx}(\theta) |g_{lx}(n)\rangle \langle n|$ and

$$P_{l}(x|\theta) = \operatorname{tr}(E_{x}^{l}(\theta)\rho_{l}E_{x}^{l}(\theta)^{\dagger})$$

$$= \operatorname{tr}(E_{x}^{l}(\theta)K_{l}\rho K_{l}^{\dagger}E_{x}^{l}(\theta)^{\dagger})/t_{l}$$

$$= P(x,l|\theta)/t_{l}.$$
 (15)

Above $P(x, l|\theta)$ represents the probability distribution from $\mathcal{E}' = \{E'_{xl}(\theta)\}_{xl}$ with

$$E'_{xl}(\theta) = E^l_x(\theta)K_l = \sum_n a^l_n b^{lx}_{f_l(n)}(\theta)|g_{lx}[f_l(n)]\rangle\langle n|, \quad (16)$$

which implies $\mathcal{E}' \in G$. Therefore, one can arrive at

$$\sum_{l} t_{l} C^{\theta_{0}}(\rho_{l}) = \sum_{l} t_{l} F(P_{l}, \theta_{0})$$

$$= \sum_{l} t_{l} \sum_{x \in S_{l}} \left[\frac{\partial P_{l}(x|\theta)}{\partial \theta} \Big|_{\theta_{0}} \right]^{2} \frac{1}{P_{l}(x|\theta_{0})}$$

$$= \sum_{l} t_{l} \sum_{x \in S_{l}} \left[\frac{\partial P(l, x|\theta)}{\partial \theta} \Big|_{\theta_{0}} \right]^{2} \frac{1}{P(l, x|\theta_{0}) t_{l}}$$

$$= \sum_{l} \sum_{x \in S_{l}} \left[\frac{\partial P(l, x|\theta)}{\partial \theta} \Big|_{\theta_{0}} \right]^{2} \frac{1}{P(l, x|\theta_{0})}$$

$$= F(P, \theta_{0}) \leq C^{\theta_{0}}(\rho), \qquad (17)$$

where S_l indicates the region of x in P_l , and the last inequality is from that \mathcal{E}' may not be the optimal one for ρ .

(A4) Convexity. For any ensemble $\{t_i, \sigma_i\}$ with the corresponding mixed state $\rho = \sum_i t_i \sigma_i$, let $\mathcal{E} = \{E_x(\theta)\}$ be the optimal IO for ρ in the sense of $C^{\theta_0}(\rho) = F(P, \theta_0)$ with $P(x|\theta) = \operatorname{tr}(E_x(\theta)\rho E_x^{\dagger}(\theta))$. For the state σ_i , denote

$$P_i(x|\theta) = \operatorname{tr}(E_x(\theta)\sigma_i E_x^{\dagger}(\theta)), \qquad (18)$$

then

$$\sum_{i} t_{i} P_{i}(x|\theta) = \sum_{i} t_{i} \operatorname{tr}(E_{x}(\theta)\sigma_{i}E_{x}^{\dagger}(\theta))$$
$$= \operatorname{tr}(E_{x}(\theta)\rho E_{x}^{\dagger}(\theta))$$
$$= P(x|\theta).$$
(19)

However, \mathcal{E} may not be optimal for σ_i , which implies

$$C^{\theta_0}(\sigma_i) \ge F(P_i, \theta_0), \tag{20}$$

so one can immediately get

$$\sum_{i} t_{i} C^{\theta_{0}}(\sigma_{i}) \geq \sum_{i} t_{i} F(P_{i}, \theta_{0})$$
$$\geq F(\sum_{i} t_{i} P_{i}, \theta_{0}) = F(P, \theta_{0})$$
$$= C^{\theta_{0}}(\rho), \qquad (21)$$

where the second inequality is due to the convexity of FI.

Since (A3) and (A4) hold, it is natural that (A2) is satisfied. The proof is completed. $\hfill \Box$

From Theorem 1, coherence could be quantified by FI of probability distribution in Eq. (3), in some sense, this implies the connection between coherence and estimation accuracy for incoherent non-unitary parametrization. In fact, G in the definition Eq. (7) could be replaced by its subset G_1 from the lemma below.

Lemma 1.-For any $\mathcal{E} = \{E_x(\theta)\} \in G$, there always exists another $\mathcal{E}' = \{\tilde{E}_x(\theta)\} \in G_1$, such that

$$F(P^{\varepsilon}, \theta_0) \le F(P^{\varepsilon'}, \theta_0), \tag{22}$$

where $F(P^{\varepsilon}, \theta_0)$ and $F(P^{\varepsilon'}, \theta_0)$ are FI of $P^{\varepsilon}(x|\theta)$ and $P^{\varepsilon'}(x|\theta)$ respectively.

Proof: Let $\mathcal{E} = \{E_x(\theta)\} \in G$, one can rewrite $\{E_x(\theta)\}$ as

$$E_{x}(\theta) = \sum_{n} c_{n}^{x} e^{ih_{n}^{x}(\theta)} |g_{x}(n)\rangle\langle n|$$

$$= \sum_{n} c_{n}^{x} |g_{x}(n)\rangle\langle n| \sum_{m} e^{ih_{m}^{x}(\theta)} |m\rangle\langle m|$$

$$= A_{x} U_{x}(\theta), \qquad (23)$$

where $A_x = \sum_n c_n^x |g_x(n)\rangle \langle n|$, and $U_x(\theta) = \sum_m e^{ih_m^x(\theta)}$

 $|m\rangle\langle m|$. Thus we have

$$E_{x}(\theta)^{\dagger}E_{x}(\theta) = U_{x}(\theta)^{\dagger}A_{x}^{\dagger}A_{x}U_{x}(\theta)$$

$$= U_{x}(\theta)^{\dagger}(\sum_{i}|\psi_{i}^{x}\rangle\langle\psi_{i}^{x}|)U_{x}(\theta)$$

$$= \sum_{i}U_{x}(\theta)^{\dagger}|\psi_{i}^{x}\rangle\langle\psi_{i}^{x}|U_{x}(\theta)$$

$$= \sum_{i}|\phi_{i}^{x}(\theta)\rangle\langle\phi_{i}^{x}(\theta)| = \sum_{i}\tilde{E}_{x,i}(\theta)^{\dagger}\tilde{E}_{x,i}(\theta), \quad (24)$$

where $\sum_{i} |\psi_{i}^{x}\rangle\langle\psi_{i}^{x}|$ denotes the eigen-decomposition of $A_{x}^{\dagger}A_{x}$ (the eigenvalue is absorbed in $|\psi_{i}^{x}\rangle$), $|\phi_{i}^{x}(\theta)\rangle = U_{x}(\theta)^{\dagger}|\psi_{i}^{x}\rangle$ and $\tilde{E}_{x,i}(\theta) = |i\rangle\langle\phi_{i}^{x}(\theta)|$. It is obvious that $\mathcal{E}' = \{\tilde{E}_{x,i}(\theta)\}_{xi} \in G_{1}$. From Cauchy-Schwarz inequality $(|\langle v|w\rangle|^{2} \leq \langle v|v\rangle\langle w|w\rangle)$ [59], one can obtain

$$\partial_{\theta} P(x|\theta)|_{\theta_0}]^2 \le \sum_i \frac{[\partial_{\theta} P_i(x|\theta)|_{\theta_0}]^2}{P_i(x|\theta_0)} \sum_i P_i(x|\theta_0), \quad (25)$$

where $P(x|\theta) = \operatorname{tr}(\rho E_x^{\dagger} E_x), P_i(x|\theta) = \operatorname{tr}(\rho \tilde{E}_{x,i}^{\dagger} \tilde{E}_{x,i})$, thus

$$\frac{[\partial_{\theta} P(x|\theta)|_{\theta_0}]^2}{P(x|\theta_0)} \le \sum_i \frac{[\partial_{\theta} P_i(x|\theta)|_{\theta_0}]^2}{P_i(x|\theta_0)}, \qquad (26)$$

the inequality holds for every x, which implies $F(P^{\varepsilon}, \theta_0) \leq F(P^{\varepsilon'}, \theta_0)$.

From the lemma, maximizing the FI over the set G can be realized by the optimization over the set G_1 , which effectively reduces the range of the optimized IO.

Theorem 1 mainly focuses on the FI with the related probability distribution generated via the post-selective IO on a state. Next, we would build another coherence measure defined by QFI with respect to parametrization in G,

$$C_Q^{\theta_0}(\rho) = \max_{\mathcal{E} \in G} F_Q(\rho, \mathcal{E}, \theta_0).$$
(27)

To do this, we would first give a lemma.

Lemma 2.- The maximal QFI subject to parametrization in G is upper bounded by the FI directly induced by the optimal post-selective IO parametrization process, namely,

$$\max_{\mathcal{E}\in G} F_Q(\rho, \mathcal{E}, \theta_0) \le \max_{\mathcal{E}\in G} F(P^{\varepsilon}, \theta_0),$$
(28)

where P^{ε} is the distribution in Eq. (3).

Proof: Suppose $\tilde{\mathcal{E}}$ and \mathcal{M} are the optimal parametrization and measurement for the optimal F_Q respectively, from Eq. (4), we have $P_{\mathcal{M}}^{\mathcal{E}}(x|\theta) = \operatorname{tr}(\sum_i |\psi_i^x\rangle \langle \psi_i^x| \tilde{\mathcal{E}}_{\theta}(\rho)) = \sum_i P_i(x|\theta)$ where $\sum_i |\psi_i^x\rangle \langle \psi_i^x|$ represents the eigen-decomposition of M_x . In particu-

lar, $P_i(x|\theta) = \langle \psi_i^x | \tilde{\mathcal{E}}_{\theta}(\rho) | \psi_i^x \rangle$, which can be rewritten as

$$P_{i}(x|\theta) = \operatorname{tr}(|i\rangle \langle \psi_{i}^{x}| \tilde{\mathcal{E}}_{\theta}(\rho) |\psi_{i}^{x}\rangle \langle i|)$$

$$= \sum_{ynn'} \operatorname{tr}(|i\rangle \langle \psi_{i}^{x}| b_{n}^{y}(\theta) |g_{y}(n)\rangle \langle n|\rho|n'\rangle \langle g_{y}(n')|b_{n'}^{y*}(\theta) |\psi_{i}^{x}\rangle \langle i|)$$

$$= \sum_{ynn'} \operatorname{tr}(b_{n}^{ixy}(\theta) |i\rangle \langle n|\rho|n'\rangle \langle i| b_{n'}^{ixy*}(\theta))$$

$$= \sum_{y} \operatorname{tr}(E_{ixy}(\theta)\rho E_{ixy}(\theta)^{\dagger}) = \sum_{y} P(ixy|\theta), \quad (29)$$

where $b_n^{ixy}(\theta) = \langle \psi_i^x | b_n^y(\theta) | g_y(n) \rangle$ and $E_{ixy}(\theta) = \sum_n b_n^{ixy}(\theta) | i \rangle \langle n |$. It is obvious that $\mathcal{E}'_{\theta} = \{ E_{ixy}(\theta) \} \in G$, then

$$\max_{\mathcal{E}} F_{Q}(\rho, \mathcal{E}, \theta_{0}) = F(P_{\mathcal{M}}^{\varepsilon}, \theta_{0})$$

$$= \sum_{x} \frac{[\partial_{\theta} P_{\mathcal{M}}^{\varepsilon}(x|\theta)|_{\theta_{0}}]^{2}}{P_{\mathcal{M}}^{\varepsilon}(x|\theta_{0})} = \sum_{x} \frac{[\sum_{i} \partial_{\theta} P_{i}(x|\theta)|_{\theta_{0}}]^{2}}{\sum_{i} P_{i}(x|\theta_{0})}$$

$$\leq \sum_{ix} \frac{[\partial_{\theta} P_{i}(x|\theta)|_{\theta_{0}}]^{2}}{P_{i}(x|\theta_{0})} = \sum_{ix} \frac{[\sum_{y} \partial_{\theta} P(ixy|\theta)|_{\theta_{0}}]^{2}}{\sum_{y} P(ixy|\theta_{0})}$$

$$\leq \sum_{ixy} \frac{[\partial_{\theta} P(ixy|\theta)|_{\theta_{0}}]^{2}}{P(ixy|\theta_{0})} = F(P,\theta_{0}) \leq \max_{\mathcal{E}} F(P^{\varepsilon},\theta_{0}),$$
(30)

where P is distribution from \mathcal{E}'_{θ} , the first two inequality could be derived based on Cauchy-Schwarz inequality, and the derivation process is similar as Eq. (25) and (26), namely, from

$$\left[\sum_{i} \partial_{\theta} P_{i}(x|\theta)|_{\theta_{0}}\right]^{2} \leq \sum_{i} \frac{\left[\partial_{\theta} P_{i}(x|\theta)|_{\theta_{0}}\right]^{2}}{P_{i}(x|\theta_{0})} \sum_{i} P_{i}(x|\theta_{0})$$

we could obtain the first inequality, and from

$$\left[\sum_{y} \partial_{\theta} P(ixy|\theta)|_{\theta_{0}}\right]^{2} \leq \sum_{y} \frac{\left[\partial_{\theta} P(ixy|\theta)|_{\theta_{0}}\right]^{2}}{P(ixy|\theta_{0})} \sum_{y} P(ixy|\theta_{0})$$

we could reach the second inequality. Thus one can complete the proof. $\hfill \Box$

Next, we show that $C_Q^{\theta_0}(\rho)$ in Eq. (27) is equivalent to $C^{\theta_0}(\rho)$, and can also quantify the quantum coherence of ρ .

Theorem 2.-For a given density matrix ρ ,

$$C_{Q}^{\theta_{0}}(\rho) = C^{\theta_{0}}(\rho). \tag{31}$$

Proof: From Lemma 1, $C^{\theta_0}(\rho)$ could be written as

$$C^{\theta_0}(\rho) = \max_{\mathcal{E} \in G_1} F(P^{\mathcal{E}}, \theta_0).$$
(32)

Suppose $\mathcal{E} = \{E_z(\theta)\}$ is the optimal operation in G_1 , such that

$$C^{\theta_0}(\rho) = F(P^{\varepsilon}, \theta_0), \qquad (33)$$

here

$$P^{\varepsilon}(z|\theta) = \operatorname{tr}(E_z(\theta)\rho E_z(\theta)^{\dagger}).$$
(34)

and $Rank[E_z(\theta)^{\dagger}E_z(\theta)] = 1$, without lose of generality, $E_z(\theta)$ could be written as

$$E_z(\theta) = |z\rangle \langle \phi_z(\theta)|. \tag{35}$$

Denote

$$P_{\mathcal{P}}^{\mathcal{E}}(z|\theta) = \operatorname{tr}(|z\rangle\langle z|\mathcal{E}_{\theta}(\rho)|z\rangle\langle z|), \qquad (36)$$

where \mathcal{P} indicates the projective measurements on the parameterized state. Note that

$$P^{\varepsilon}(z|\theta) = \operatorname{tr}(E_{z}\rho E_{z}^{\dagger})$$

= tr(|z\rangle\langle z|($\sum_{z'} E_{z'}\rho E_{z'}^{\dagger}$)|z\rangle\langle z|)
= tr(|z\rangle\langle z|\mathcal{E}_{\theta}(\rho)|z\rangle\langle z|) = P_{\mathcal{P}}^{\varepsilon}(z|\theta), (37)

thus

$$C^{\theta_0}(\rho) = F(P^{\varepsilon}, \theta_0) = F(P^{\varepsilon}_{\mathcal{P}}, \theta_0) \leq \max_{\mathcal{M}} F(P^{\varepsilon}_{\mathcal{M}}, \theta_0)$$
$$= F_Q(\rho, \mathcal{E}, \theta_0) \leq \max_{\mathcal{E} \in G} F_Q(\rho, \mathcal{E}, \theta_0) = C^{\theta_0}_Q(\rho).$$
(38)

Conversely, from Lemma 2, we can immediately reach that

$$C^{\theta_0}(\rho) \ge C^{\theta_0}_Q(\rho),\tag{39}$$

thus one can get the $C_Q^{\theta_0}(\rho) = C^{\theta_0}(\rho)$, which finishes the proof.

We have shown that the coherence measures based on QFI and FI subject to the post-selective parametrization are equivalent to each other. The most distinct advantage of this type of coherence measure is that it can be straightforwardly connected with the parameter estimation process in terms of the Cramér-Rao bound [41, 42, 60].

Let's consider an incoherent non-unitary parametrization $\mathcal{E} = \{E_x(\theta)\} \in G$ on ρ as introduced previously, then one will obtain a probability distribution $P^{\varepsilon}_{\mathcal{M}}(x|\theta)$ through a POVM on ρ_{θ} or obtain $P^{\varepsilon}(x|\theta)$ directly through post-selection of \mathcal{E} . With maximum likelihood estimators $\hat{\theta}_{\mathcal{M}}$ with respect to $P_{\mathcal{M}}^{\varepsilon}$ or $\hat{\theta}$ with respect to P^{ε} , the Cramér-Rao bound can be asymptotically attained. That is, the mean square error $(\delta \hat{\theta}_{\mathcal{M}})^2 = E[(\hat{\theta}_{\mathcal{M}} - \theta)^2]$ and $(\delta\hat{\theta})^2 = E[(\hat{\theta} - \theta)^2]$ approach $\frac{1}{nF}$ in the asymptotic sense, where E indicates the expectation value, θ is the true value and n denotes the runs of detection. Thus in the asymptotic limit, the estimation accuracy $\frac{1}{n(\delta\hat{\theta}_{\mathcal{M}})^2}$ approaches $F(P^{\varepsilon}_{\mathcal{M}}, \theta)$, which is naturally bounded by $C^{\theta}_{\mathcal{Q}}(\rho)$ based on Eq. (26). In particular, the bound $C^{\theta}_{\varphi}(\rho)$ can be asymptotically achieved with the optimal parametrization process and optimal POVM. Similarly, $\frac{1}{n(\delta \hat{\theta})^2}$ approaches $F(P^{\varepsilon}, \theta)$ in the asymptotic scenario, and simultaneously reach $C^{\theta}(\rho)$ in an asymptotic sense with an

optimal parametrization process. Note that the two measures are equivalent, therefore our coherence measure can be understood as the optimal accuracy through two different estimation processes as well as the corresponding incoherent non-unitary parametrization.

In fact, the optimized \mathcal{M} in $C_Q^{\theta_0}$ (Eq. (27)) can be replaced by \mathcal{P} , the projective measurement on the preferred basis. In this sense, the above two coherence measures have an equivalent expression as $C_{\mathcal{P}}^{\theta_0}(\rho) = \max_{\mathcal{E}\in G} F(P_{\mathcal{P}}^{\varepsilon}, \theta_0)$. This can be understood as follows. We first have $C^{\theta_0}(\rho) \leq C_{\mathcal{P}}^{\theta_0}(\rho)$ from the second equality in Eq. (38). Note that \mathcal{M} in $C_Q^{\theta_0}(\rho)$ contains projective measurement, which implies $C_Q^{\theta_0}(\rho) \geq C_{\mathcal{P}}^{\theta_0}(\rho)$. Combine the above two inequalities with Theorem 2, one can reach $C_{\mathcal{P}}^{\theta_0}(\rho) = C^{\theta_0}(\rho) = C_Q^{\theta_0}(\rho)$. Although they are identical in value, they imply different details of operational meanings and give us different ways to understand coherence.

III. CONNECTION WITH QFI BASED ON UNITARY PARAMETRIZATION

Although the coherence measure has obvious operation meaning based on quantum metrology, an analytically computable expression seems not to be easy. Next, we will show that for a 2-dimensional quantum state, the analytic result could be obtained, and the coherence measure can be realized by FI with unitary parametrization. However, our measure is not equivalent to that based on unitary parametrization in high-dimensional cases, which is proved later.

Theorem 3.-For a 2-dimensional state ρ , the coherence based on Theorem 1 can be given as

$$C^{\theta_0}(\rho) = F_Q(\rho, U_\theta, \theta_0), \tag{40}$$

where F_Q is QFI of ρ subject to unitary parametrization $U_{\theta} = e^{i\theta} |1\rangle \langle 1| + |2\rangle \langle 2|.$

Proof: For qubit states ρ , let the IO $\{E_x\} \in G$ read

$$E_x(\theta) = a_1'^x e^{ih_1'^x(\theta)} |f_x(1)\rangle \langle 1| + a_2'^x e^{ih_2'^x(\theta)} |f_x(2)\rangle \langle 2|,$$
(41)

where $a_1^{\prime x}$ or $a_2^{\prime x}$ may be zero. The Kraus operator could be written as

$$E_x(\theta) = a_1^x e^{ih_1^x(\theta)} |f_x(1)\rangle \langle 1| + a_2^x e^{ih_2^x(\theta)} |f_x(2)\rangle \langle 2|, \quad (42)$$

where $a_j^x = a_j'^x e^{ih_j'^x(\theta_0)}$ and $h_j^x(\theta) = h_j'^x(\theta) - h_j'^x(\theta_0)$ for j = 1, 2. According to Lemma 1 and its proof, the optimal IO can be rank-1 with the form $\{|i\rangle \langle \psi_i^x(\theta)|\}$, which means $f_x(1) = f_x(2)$ for any x. Then we have

$$P(x|\theta) = \operatorname{tr}(|a_1^x|^2\rho_{11}|f_x(1)\rangle\langle f_x(1)| + |a_2^x|^2\rho_{22}|f_x(2)\rangle\langle f_x(2)| + \rho_{12}a_1^xa_2^{x*}e^{i[h_1^x(\theta) - h_2^x(\theta)]}|f_x(1)\rangle\langle f_x(2)| + \rho_{21}a_1^{x*}a_2^xe^{-i[h_1^x(\theta) - h_2^x(\theta)]}|f_x(2)\rangle\langle f_x(1)|) = |a_1^x|^2\rho_{11} + |a_2^x|^2\rho_{22} + \rho_{12}a_1^xa_2^{x*}e^{i[h_1^x(\theta) - h_2^x(\theta)]} + \rho_{21}a_1^{x*}a_2^xe^{-i[h_1^x(\theta) - h_2^x(\theta)]},$$
(43)

thus

$$F(P,\theta_0) = \sum_x \frac{[2Im(\rho_{12}a_1^x a_2^{x*})]^2 [\partial_\theta h_1^x(\theta)|_{\theta_0} - \partial_\theta h_2^x(\theta)|_{\theta_0}}{|a_1^x|^2 \rho_{11} + |a_2^x|^2 \rho_{22} + 2Re(\rho_{12}a_1^x a_2^{x*})} \le \sum_x \frac{[2Im(\rho_{12}a_1^x a_2^{x*})]^2}{|a_1^x|^2 \rho_{11} + |a_2^x|^2 \rho_{22} + 2Re(\rho_{12}a_1^x a_2^{x*})},$$
(44)

where the inequality could be saturated by the function taken as $h_1^x(\theta) = \theta$, $h_2^x(\theta) = 0$, and the corresponding IO reads $E_x(\theta) = K_x U_\theta$ with

$$K_x = a_1^x |f_x(1)\rangle \langle 1| + a_2^x |f_x(2)\rangle \langle 2|,$$

$$U_\theta = e^{i\theta} |1\rangle \langle 1| + |2\rangle \langle 2|,$$
(45)

where $f_x(1) = f_x(2)$ and $\{K_x\} \in G_1$. In this sense, the probability distribution can be rewritten as

$$P^{\varepsilon}(x|\theta) = \operatorname{tr}(E_x(\theta)\rho E_x(\theta)^{\dagger}) = \operatorname{tr}(K_x U_{\theta}\rho U_{\theta}^{\dagger} K_x^{\dagger})$$
$$= \operatorname{tr}(U_{\theta}\rho U_{\theta}^{\dagger} K_x^{\dagger} K_x) = P_{\mathcal{M}}(x|\theta).$$
(46)

above $P_{\mathcal{M}}$ can be understood as distribution generated by a unitary parametrization U_{θ} followed by a rank-1 POVM $\mathcal{M} = \{K_x^{\dagger}K_x\}$. Considering the above optimal IO, one can arrive at

$$C^{\theta_0}(\rho) = \max_{\mathcal{E} \in G_1} F(P^{\mathcal{E}}, \theta_0)$$

=
$$\max_{\mathcal{M}} F(P_{\mathcal{M}}, \theta_0) = F_{\mathcal{Q}}(\rho, U_{\theta}, \theta_0), \qquad (47)$$

which finishes the proof.

In fact, in general high-dimensional case, C^{θ_0} is distinct from FI with unitary parametrization. To demonstrate the difference, we will give a concrete example. Consider a state with maximal coherence,

$$|\phi\rangle = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})^T,$$
 (48)

and the parametrization $\mathcal{E} = \{E_x(\theta)\}$ expressed as

$$E_x(\theta) = a_1^x e^{ih_1^x \theta} |f_x(1)\rangle \langle 1| + a_2^x e^{ih_2^x \theta} |f_x(1)\rangle \langle 2| + a_3^x e^{ih_3^x \theta} |f_x(1)\rangle \langle 3|, \qquad (49)$$

with a_n^x and h_n^x $(x = 1, \dots, 9)$ to be given at the end. Denote $\rho = |\phi\rangle\langle\phi|$. The probability distribution is

$$P(x|0) = \operatorname{tr}(E_x(\theta)|\phi\rangle\langle\phi|E_x(\theta)^{\dagger})$$

=\rho_{11}|a_1^x|^2 + \rho_{22}|a_2^x|^2 + \rho_{33}|a_3^x|^2
+2\operatorname{Re}[\rho_{12}a_1^xa_2^{x*} + \rho_{12}a_2^xa_3^{x*} + \rho_{31}a_3^xa_1^{x*}], (50)

and

$$\begin{aligned} \partial_{\theta} P(x|\theta)|_{0} &= 2 \operatorname{Im}[\rho_{12} a_{1}^{x} a_{2}^{x*} (h_{1}^{x} - h_{2}^{x}) + \rho_{23} a_{2}^{x} a_{3}^{x*} (h_{2}^{x} - h_{3}^{x}) \\ &+ \rho_{31} a_{3}^{x} a_{1}^{x*} (h_{3}^{x} - h_{1}^{x})]. \end{aligned}$$
(51)

Therefore, the corresponding FI reads

$$F(P^{\mathcal{E}}, 0) = \sum_{x} \frac{[\partial_{\theta} P(x|\theta)|_{0}]^{2}}{P(x|0)} = 0.9410.$$
(52)

From the definition, we have $C^0(\rho) \ge F(P^{\mathcal{E}}, 0)$.

To compare our measure with QFI subject to the optimal unitary parametrization in G, we calculate $\max_{U_{\theta} \in G} F_{Q}(|\phi\rangle, U_{\theta}, 0)$, where U_{θ} is the unitary operator expressed as

$$U_{\theta} = \sum_{n} e^{ih_{n}(\theta)} |n\rangle \langle n| \tag{53}$$

with $\partial_{\theta}h_n(\theta) \in [0,1]$ (based on Appendix A, other cases with different range of $\partial_{\theta}h_n(\theta)$ lead the same conclusion). When eigenvalues of the parameterized state $U_{\theta}\rho U_{\theta}^{\dagger}$ are parameter-independent, QFI could be calculated from the following equation [43, 61],

$$F_{Q}(\rho, U_{\theta}, \theta_{0}) = \sum_{ij} \frac{2(P_{i} - P_{j})^{2}}{P_{i} + P_{j}} |\langle \varphi_{i} | \partial_{\theta} \varphi_{j} \rangle|^{2}, \quad (54)$$

where $\{P_i\}$ and $\{|\varphi_i\rangle\}$ denote the eigenvalues and eigenvectors of $U_{\theta}\rho U_{\theta}^{\dagger}$ respectively, and we use $|\partial_{\theta}\varphi_j\rangle$ to briefly express the partial derivative $\frac{\partial|\varphi_j\rangle}{\partial\theta}|_{\theta_0}$. Besides, the terms with $P_i = P_j = 0$ are not included in the summation. In addition, for a pure state $\rho = |\psi\rangle\langle\psi|$, let $\{|\psi_i\rangle\}$ be the basis vectors satisfying $|\psi\rangle = |\psi_1\rangle$, then the corresponding $P_1 = 1$ and residual eigenvalues P_i $(i \neq 1)$ are zero. Then the eigenvectors of $U_{\theta}\rho U_{\theta}^{\dagger}$ are $\{U_{\theta}|\psi_i\rangle\}$. Denote

$$H_{\theta} = \sum_{n} \partial_{\theta} h_{n}(\theta) |n\rangle \langle n|, \qquad (55)$$

we have

$$F_{Q}(|\psi\rangle, U_{\theta}, \theta_{0})$$

$$= \sum_{i} \frac{2(1-P_{i})^{2}}{1+P_{i}} \langle \psi | U_{\theta_{0}}^{\dagger} U_{\theta_{0}} H_{\theta_{0}} | \psi_{i} \rangle \langle \psi_{i} | H_{\theta_{0}} U_{\theta_{0}}^{\dagger} U_{\theta_{0}} | \psi \rangle$$

$$+ \sum_{i} \frac{2(P_{i}-1)^{2}}{P_{i}+1} \langle \psi_{i} | U_{\theta_{0}}^{\dagger} U_{\theta_{0}} H_{\theta_{0}} | \psi \rangle \langle \psi | H_{\theta_{0}} U_{\theta_{0}}^{\dagger} U_{\theta_{0}} | \psi_{i} \rangle$$

$$= 4 \langle \psi | H_{\theta_{0}} \sum_{i} | \psi_{i} \rangle \langle \psi_{i} | H_{\theta_{0}} | \psi \rangle - 4 \langle \psi | H_{\theta_{0}} | \psi \rangle \langle \psi | H_{\theta_{0}} | \psi \rangle$$

$$= 4 \langle \psi | H_{\theta_{0}}^{2} | \psi \rangle - 4 \langle \psi | H_{\theta_{0}} | \psi \rangle^{2}. \tag{56}$$

This result does not depend on the choice of $|\psi_i\rangle$ as long as $|\psi\rangle = |\psi_1\rangle$, and the optimal QFI $\max_{U_{\theta}\in G} F_Q(|\phi\rangle, U_{\theta}, 0)$ can be calculated as

$$\max_{U_{\theta} \in G} F_{Q}(|\phi\rangle, U_{\theta}, 0) = \max_{H \in S} 4\langle \phi | H^{2} | \phi \rangle - 4\langle \phi | H | \phi \rangle^{2}$$
$$= 8/9, \tag{57}$$

where $|\phi\rangle$ is the 3-dimensional MCS in Eq. (48), S is the set of operator $H = h_1 |1\rangle \langle 1| + h_2 |2\rangle \langle 2| + h_3 |3\rangle \langle 3|$ $(h_i \in [0,1])$. Thus $C^0(\rho) > \max_{U_{\theta} \in G} F(|\phi\rangle, U_{\theta}, 0)$, which indicates that C^{θ_0} is different from FI with unitary parametrization.

Finally, we'd like to present all the coefficients of E_x in above calculation by defining $A^x = [a_1^x, a_2^x, a_3^x]$, where

$$A^1 = [0, \sqrt{0.4}, \sqrt{0.6}] / \sqrt{3},$$

$$A^{2} = [0, \sqrt{0.4}e^{-i2\pi/3}, \sqrt{0.6}e^{i2\pi/3}]/\sqrt{3},$$

$$A^{3} = [0, \sqrt{0.4}e^{-i4\pi/3}, \sqrt{0.6}e^{i4\pi/3}]/\sqrt{3},$$

$$A^{4} = [\sqrt{0.4}, \sqrt{0.6}, 0]/\sqrt{3},$$

$$A^{5} = [\sqrt{0.4}, \sqrt{0.6}e^{i2\pi/3}, 0]/\sqrt{3},$$

$$A^{6} = [\sqrt{0.4}, \sqrt{0.6}e^{i4\pi/3}, 0]/\sqrt{3},$$

$$A^{7} = [\sqrt{0.6}, 0, \sqrt{0.4}]/\sqrt{3},$$

$$A^{8} = [\sqrt{0.6}, 0, \sqrt{0.4}e^{i2\pi/3}]/\sqrt{3},$$

$$A^{9} = [\sqrt{0.6}, 0, \sqrt{0.4}e^{i4\pi/3}]/\sqrt{3}.$$

In addition,

$$h_1^x = 0, \ h_2^x = 1, \ h_3^x = 0, \ x = 1, 2, 3$$

 $h_1^x = 1, \ h_2^x = 0, \ h_3^x = 0, \ x = 4, \cdots, 9.$ (59)

IV. CONCLUSIONS

In this paper, we have established coherence measures based on FI subject to the incoherent non-unitary parametrization process. The coherence measure could be defined by two forms based on FI or QFI, which both imply the direct operational meaning by the connection with the parameter estimation accuracy. In addition, we compare our measure with QFI in unitary parametrization and find that in the qubit case, our coherence measure can be equivalently understood through unitary parametrization, and can be analytically calculated. Our coherence also sheds new light on the roles of the non-unitary parametrization process.

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Appendix A: Region of $\partial_{\theta} h(\theta)$

In the previous sections, $C_Q^{\theta_0}$ and C^{θ_0} are defined under a certain condition $\frac{\partial h_n^x(\theta)}{\partial \theta} \in [0, 1]$. In fact, measures defined under other conditions can be transformed to the original C^{θ_0} .

We first consider the case that $\frac{\partial h_n^x(\theta)}{\partial \theta}$ is finite, and suppose $\max_{n,x} \left| \frac{\partial h_n^x(\theta)}{\partial \theta} \right| \leq k$ (k is finite). Denote $\tilde{C}_k^{\theta_0}$ as the function defined in a similar way as C^{θ_0} (in Theorem 1) but with the different condition $\max_{n,x} \left| \frac{\partial h_n^x(\theta)}{\partial \theta} \right| \leq k$, and $G^{(k)}$ as the set of the corresponding channels, namely,

$$\tilde{C}_{k}^{\theta_{0}}(\rho) = \max_{\mathcal{E}\in G^{(k)}} F(\tilde{P}^{\mathcal{E}}, \theta_{0}),$$
(A1)

where

(58)

$$\tilde{P}^{\mathcal{E}}(x|\theta) = \operatorname{tr}(\tilde{E}_{x}(\theta)\rho\tilde{E}_{x}(\theta)^{\dagger}),$$
$$\tilde{E}_{x}(\theta) = \sum_{n} a_{n}^{x}e^{ih_{n}^{x}(\theta)}|f_{x}(n)\rangle\langle n|, \qquad (A2)$$

and $\max_{n,x} \left| \frac{\partial h_n^x(\theta)}{\partial \theta} \right| \le k.$

We find that $\tilde{C}_k^{\theta_0}$ has a connection with the previous coherence measure.

Lemma 3.-The function $\tilde{C}_k^{\theta_0}$ satisfies that

$$C^{\gamma_0}(\rho) = \frac{1}{4k^2} \tilde{C}_k^{\theta_0}(\rho), \qquad (A3)$$

where $\gamma_0 = 2k\theta_0$.

In this sense, investigation under condition $\frac{\partial h_n^x(\theta)}{\partial \theta} \in [0, 1]$ could cover all other situations where k is finite. Next, we give a brief proof.

Proof: Suppose $\{E_x\}$ are Kraus operators of channel in $G^{(\frac{1}{2})}$, namely,

$$E_x(\gamma) = \sum_n a_n^x e^{iu_n^x(\gamma)} |f_x(n)\rangle \langle n|, \qquad (A4)$$

where $|\partial_{\gamma} u_n^x| \leq \frac{1}{2}$. Denote

$$\hat{E}_x(\gamma) = e^{i\gamma/2} E_x(\gamma) = \sum_n a_n^x e^{iv_n^x(\gamma)} |f_x(n)\rangle \langle n|, \quad (A5)$$

where $v_n^x(\gamma) = u_n^x(\gamma) + \gamma/2$, thus $\partial_{\gamma} v_n^x \in [0, 1]$. The two channels lead to identical effects, that is

$$\hat{E}_x(\gamma)\rho\hat{E}_x(\gamma)^{\dagger} = e^{i\gamma/2}E_x(\gamma)\rho e^{-i\gamma/2}E_x(\gamma)^{\dagger}$$
$$= E_x(\gamma)\rho E_x(\gamma)^{\dagger}, \qquad (A6)$$

from this, we have

$$C^{\gamma_0}(\rho) = \tilde{C}^{\gamma_0}_{1/2}(\rho).$$
 (A7)

Consider \tilde{E}_x in Eq. (A2), denote S_l^x as the set satisfying $f_x(n) = l$ when $n \in S_l^x$, then

$$\tilde{P}(x|\theta) = \sum_{l} \sum_{n,n' \in S_l^x} \rho_{nn'} a_n^x a_{n'}^{x*} e^{i[h_n^x(\theta) - h_{n'}^x(\theta)]}, \quad (A8)$$

and

$$\partial_{\theta} \tilde{P}(x|\theta)|_{\theta_{0}} = \sum_{l} \sum_{n,n' \in S_{l}^{x}} \{\rho_{nn'} a_{n}^{x} a_{n'}^{x*} e^{i[h_{n}^{x}(\theta_{0}) - h_{n'}^{x}(\theta_{0})]} \\ \times i[\partial_{\theta} h_{n}^{x}(\theta)|_{\theta_{0}} - \partial_{\theta} h_{n'}^{x}(\theta)|_{\theta_{0}}] \}.$$
(A9)

Denote $\gamma = 2k\theta$, then

$$\tilde{P}(x|\theta) = \sum_{l} \sum_{n,n' \in S_l^x} \rho_{nn'} a_n^x a_{n'}^x e^{i[h_n^x(\frac{\gamma}{2k}) - h_{n'}^x(\frac{\gamma}{2k})]}, \quad (A10)$$

thus $\tilde{P}(x|\theta)$ could be rewritten as $P(x|\gamma)$. Define $g_n^x(\gamma) = h_n^x(\frac{\gamma}{2k})$, then $\partial_{\gamma}g_n^x(\gamma) = \frac{\partial_{\theta}h_n^x(\theta)}{2k}$, thus $|\partial_{\gamma}g_n^x(\gamma)| \leq \frac{1}{2}$. In addition,

$$\partial_{\gamma} P(x|\gamma)|_{\gamma_{0}}$$

$$= \sum_{l} \sum_{n,n' \in S_{l}^{x}} \{\rho_{nn'} a_{n}^{x} a_{n'}^{x*} e^{i[h_{n}^{x}(\frac{\gamma_{0}}{2k}) - h_{n'}^{x}(\frac{\gamma_{0}}{2k})]}$$

$$\times i[\partial_{\gamma} h_{n}^{x}(\frac{\gamma}{2k})|_{\gamma_{0}} - \partial_{\gamma} h_{n'}^{x}(\frac{\gamma}{2k})|_{\gamma_{0}}]\}$$

$$= \frac{1}{2k} \sum_{l} \sum_{n,n' \in S_l^x} \{\rho_{nn'} a_n^x a_{n'}^{x*} e^{i[h_n^x(\theta_0) - h_{n'}^x(\theta_0)]} \\ \times i[\partial_\theta h_n^x(\theta)|_{\theta_0} - \partial_\theta h_{n'}^x(\theta)|_{\theta_0}] \}$$
$$= \frac{1}{2k} \partial_\theta \tilde{P}(x|\theta)|_{\theta_0}, \qquad (A11)$$

where $\gamma_0 = 2k\theta_0$. Thus

$$F(P,\gamma_0) = \sum_x \frac{[\partial_\gamma P(x|\gamma)|_{\gamma_0}]^2}{P(x|\gamma_0)}$$
$$= \sum_x \frac{[\partial_\theta \tilde{P}(x|\theta)|_{\theta_0}]^2}{4k^2 \tilde{P}(x|\theta_0)} = \frac{F(\tilde{P},\theta_0)}{4k^2}, \qquad (A12)$$

combine it with Eq. (A7), we have

$$C^{\gamma_0}(\rho) = \frac{1}{4k^2} \tilde{C}^{\theta_0}(\rho).$$
 (A13)

Above proof shows that if $\partial_{\theta}h_n^x(\theta)$ is finite, the investigation under condition $\partial_{\theta}h_n^x(\theta) \in [0,1]$ could cover all other general cases. However, if $\partial_{\theta}h_n^x(\theta)$ is infinite, FI and $C^{\theta_0}(\rho)$ will be infinite. Besides, physical models generally lead to a finite $\partial_{\theta}h_n^x(\theta)$, for example, the parametrization Ramsey interferometer could be written as $U_{\theta} = \exp(-i\theta J_z)$, the corresponding $\partial_{\theta}h_n^x(\theta)$ is finite $(J_z \text{ is the } z-\text{component of the total angular momentum})$. Thus we could focus on finite case.

Appendix B: Dilation of the Optimal Channel in G

For the estimation process in Eq. (3), the optimal channel in G_1 could be written as

$$E_x(\theta) = \sum_n b_n^x(\theta) |1\rangle \langle n|, \qquad (B1)$$

denote \mathcal{H}_A (*d*-dimension) as the space for it. Assume $|x_B\rangle$ are basis vectors in another space \mathcal{H}_B (*L*-dimension), and we construct the following states in \mathcal{H}_B ,

$$|\psi_B^n\rangle = \sum_{x=1}^L b_n^x(\theta) |x_B\rangle, \quad n = 1, 2, \dots, d.$$
 (B2)

From $\sum_{x} E_{x}^{\dagger} E_{x} = \mathcal{I}$, we have $\langle \psi_{B}^{m} | \psi_{B}^{n} \rangle = \delta_{nm}$. With states in Eq. (B2), one can always find the other $|\psi_{B}^{x}\rangle$ $(x = d + 1, \dots, L)$ to form a set of basis together in \mathcal{H}_{B} . Denote

$$U^B = \sum_{x} |\psi^x_B\rangle \langle x_B|, \qquad (B3)$$

clearly, $U^{B\dagger}U^B = U^B U^{B\dagger} = I^B$. Then we can construct a controlled unitary in $\mathcal{H}_A \otimes \mathcal{H}_B$,

$$U^{AB} = |1_A\rangle \langle 1_A| \otimes U^B + \mathbb{I}_1^A \otimes \mathbb{I}^B, \qquad (B4)$$

where \mathbb{I}_1^A is the identity operator in the residual subspace of H^A . Obviously, U^{AB} is an unitary operator in $H_A \otimes H_B$. Denote

$$V = \sum_{n} (|1_{A}\rangle \langle n_{A}| \otimes |n_{B}\rangle \langle 1_{B}| + |n_{A}\rangle \langle 1_{A}| \otimes |1_{B}\rangle \langle n_{B}|) -|1_{A}\rangle \langle 1_{A}| \otimes |1_{B}\rangle \langle 1_{B}|,$$
(B5)

and

$$W = V + \mathbb{I}_2, \tag{B6}$$

where \mathbb{I}_2 is the identity operator from the residual subspace which eliminates VV^{\dagger} in $H^A \otimes H^B$, and W is an unitary operator swapping specified states. It is easy to see that

$$E_x(\theta) = \langle x_B | U^{AB} W | 1_B \rangle, \tag{B7}$$

based on Stinespring dilation theorem, $\{E_x(\theta)\}$ could be implement by an unitary $U^{AB}W$ on $\rho \otimes |1_B\rangle\langle 1_B|$ and projective measurement $\{|x_B\rangle\langle x_B|\}$.

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