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on the generalized α -z-relative Rényi
entropy

by

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Quantifying quantum coherence based on the generalized $\alpha - z$ -relative Rényi entropy

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We present a family of coherence quantifiers based on the generalized $\alpha - z$ -relative Rényi entropy. These quantifiers satisfy all the standard criteria for well-defined measures of coherence, and include some existing coherence measures as special cases.

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I. INTRODUCTION

Coherence, being at the heart of interference phenomena, plays a central role in quantum physics as it enables applications that are impossible within classical mechanics or ray optics. Coherence is also a vital physical resource with various applications in biology [1–3], thermodynamical systems [4, 5], transport theory [6, 7] and nanoscale physics [8]. Recent developments in our understanding of quantum coherence [9–14] and nonclassical correlation have come from the burgeoning field of quantum information science. One important pillar of the field is the study on quantification of coherence.

In Ref. [15] the authors established a rigorous framework (BCP framework) for quantifying coherence. The BCP framework consists of the following postulates that any quantifier of coherence C should fulfill:

(C₁) Faithfulness: $C(\rho) \geq 0$, with equality if and only if ρ is incoherent.

(C₂) Monotonicity: C does not increase under the action of an incoherent operation, i.e.,

$$C[\Phi_{\mathcal{I}}(\rho)] \leq C(\rho),$$

for any incoherent operation $\Phi_{\mathcal{I}}$.

(C₃) Convexity: C is a convex function of the state, i.e.,

$$\sum_n p_n C(\rho_n) \geq C(\sum_n p_n \rho_n),$$

where $p_n \geq 0$, $\sum_n p_n = 1$.

(C₄) Strong monotonicity: C does not increase on average under selective incoherent operations, i.e.,

$$C(\rho) \geq \sum_n p_n C(\varrho_n),$$

with probabilities $p_n = \text{tr}(\mathcal{K}_n \rho \mathcal{K}_n^\dagger)$, post measurement states $\rho_n = \frac{\mathcal{K}_n \rho \mathcal{K}_n^\dagger}{p_n}$, and incoherent operators \mathcal{K}_n .

The authors of Ref. [16] provided a simple and interesting condition to replace (C₃) and (C₄) with the additivity of coherence for block-diagonal states,

$$C(p\rho_1 \oplus (1-p)\rho_2) = pC(\rho_1) + (1-p)C(\rho_2), \quad (1)$$

for any $p \in [0, 1]$, $\rho_i \in \varepsilon(\mathcal{H}_i)$, $i = 1, 2$, and $p\rho_1 \oplus (1-p)\rho_2 \in \varepsilon(\mathcal{H}_1 \oplus \mathcal{H}_2)$, where $\varepsilon(\mathcal{H})$ denotes the set of density matrices on the Hilbert space \mathcal{H} .

For a given d -dimensional Hilbert space \mathcal{H} , let us fix an orthonormal basis $\{|i\rangle\}_{i=1}^d$. We call all density matrices that are diagonal in this basis incoherent and label this set of quantum states by $\mathcal{I} \in \mathcal{H}$. All density operators $\delta \in \mathcal{I}$ are of the form:

$$\delta = \sum_i p_i |i\rangle\langle i|,$$

where $p_i \geq 0$ and $\sum_i p_i = 1$. Otherwise the states are coherent. Let Λ be a completely positive trace preserving (CPTP) map:

$$\Lambda(\rho) = \sum_i \mathcal{K}_i \rho \mathcal{K}_i^\dagger,$$

where $\{\mathcal{K}_n\}$ is a set of Kraus operators satisfying $\sum_n \mathcal{K}_n^\dagger \mathcal{K}_n = I_d$, with I_d the identity operator. If $\mathcal{K}_n^\dagger \mathcal{I} \mathcal{K}_n \in \mathcal{I}$ for all n , we call $\{\mathcal{K}_n\}$ a set of incoherent Kraus operators, and the corresponding operation Λ an incoherent operational one.

II. THE FUNCTION $f_{\alpha,z}(\rho, \sigma)$

Quantifying coherence is a key task in both quantum mechanical theory and practical applications. In Ref. [17, 18] the following function has been presented,

$$f_{\alpha,z}(\rho, \sigma) = \text{Tr}(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}})^z, \quad (2)$$

for arbitrary two density matrices ρ and σ . Here, $\alpha, z \in \mathbb{R}$. To study the limit when $\alpha \rightarrow 1$ and $z \rightarrow 0$, the authors in Ref. [18] parameterized z in terms of α as $z = r(\alpha - 1)$, where r is a non-zero finite real number, and considered the limit when $\alpha \rightarrow 1$: $\lim_{\alpha \rightarrow 1} f_{\alpha,r(\alpha-1)}(\rho, \sigma) = \rho$. For fixed $\alpha \neq 1$, $z \rightarrow 0$ is exactly related to the anti Lie-Trotter problem [19].

For a finite dimensional Hilbert space \mathcal{H} , the set of linear operators is denoted by $\mathcal{L}(\mathcal{H})$. The adjoint of $X \in \mathcal{L}(\mathcal{H})$ is denoted by X^\dagger . For $X \in \mathcal{L}(\mathcal{H})$ and real $p \neq 0$, the norm $\|X\|_p$ is defined by [20],

$$\|X\|_p = (\text{tr}|X|^p)^{\frac{1}{p}},$$

where $|X| = \sqrt{X^\dagger X}$. Here, for a self-adjoint operator X , X^{-1} means the inverse restricted to $\text{supp}(X)$, so $X^{-1}X = XX^{-1}$ equals to the orthogonal projection on $\text{supp}(X)$.

The Hölder's inequality belongs to a richer family of inequalities. For every $p_1, \dots, p_k, r > 0$ with $\frac{1}{r} = \frac{1}{p_1} + \dots + \frac{1}{p_k}$ one has [20]:

$$\|X_1 \dots X_k\|_r \leq \|X_1\|_{p_1} \dots \|X_k\|_{p_k}. \quad (3)$$

From this inequality and the fact that $\|X^{-1}\|_{-p} = \|X\|_p^{-1}$, the following reverse Hölder's inequality is derived. Let $r > 0$ and p_1, \dots, p_k be such that $\frac{1}{r} = \frac{1}{p_1} + \dots + \frac{1}{p_k}$ and that exactly one of p_i 's is positive and the rests are negative [20]:

$$\|X_1 \dots X_k\|_r \geq \|X_1\|_{p_1} \dots \|X_k\|_{p_k}. \quad (4)$$

Moreover, equalities holds in (3) and (4) if and only if $|X_i|^{p_i}, i = 1, 2, \dots, k$, are proportional.

Lemma 1 For states ρ and σ ,

(1) If $0 < \alpha < 1$ and $z > 0$, we have

$$f_{\alpha,z}(\rho, \sigma) \leq 1;$$

(2) If $\alpha > 1$ and $z > 0$, we have

$$f_{\alpha,z}(\rho, \sigma) \geq 1.$$

(3) $f_{\alpha,z}(\rho, \sigma) = 1$ if and only if $\rho = \sigma$, for $\alpha \in (0, 1) \cup (1, +\infty)$ and $z > 0$.

[Proof] Let $r = z$, $p_1 = \frac{2z}{1-\alpha}$, $p_2 = \frac{z}{\alpha}$, $X_1 = \sigma^{\frac{1}{p_1}}$, $X_2 = \rho^{\frac{1}{p_2}}$. When $\alpha \in (0, 1)$ and $z > 0$, we have

$$\begin{aligned} f_{\alpha,z}(\rho, \sigma) &= \text{tr}(X_1 X_2 X_1)^z \\ &= \text{tr}(|X_1 X_2 X_1|)^r \\ &= (\|X_1 X_2 X_1\|_r)^r \\ &\leq (\|X_1\|_{p_1} \|X_2\|_{p_2} \|X_1\|_{p_1})^r \\ &= 1, \end{aligned} \quad (5)$$

where the second equality is due to $X_i^\dagger = X_i$ for $i = 1, 2$. From (3), we obtain the first inequality.

When $\alpha > 1$ and $z > 0$, we have

$$\begin{aligned} f_{\alpha,z}(\rho, \sigma) &= (\|X_1 X_2 X_1\|_r)^r \\ &\geq (\|X_1\|_{p_1} \|X_2\|_{p_2} \|X_1\|_{p_1})^r \\ &= 1, \end{aligned} \quad (6)$$

where the first inequality is due to (4).

In the above proof of inequalities (5) and (6), $\|X_1 X_2 X_1\|_r = \|X_1\|_{p_1} \|X_2\|_{p_2} \|X_1\|_{p_1}$ if and only if $|X_1|^{p_1}$ and $|X_2|^{p_2}$ are proportional, i.e, there is a number k which satisfies $\sigma = k\rho$. Since $\text{tr}(\rho) = \text{tr}(\sigma) = 1$, then we obtain $k = 1$. \square

Let $\mathcal{P}(\mathcal{H})$ be the set of positive semidefinite operators on \mathcal{H} . For non-normalized states $\rho: \forall \rho, \sigma \in \mathcal{P}(\mathcal{H})$ with $\text{supp } \rho \subseteq \text{supp } \sigma$, it has been defined in Ref. [18],

$$D_{\alpha,z}(\rho||\sigma) := \frac{1}{\alpha-1} \log \frac{f_{\alpha,z}(\rho, \sigma)}{\text{tr} \rho}. \quad (7)$$

For any states ρ, σ such that $\text{supp } \rho \subseteq \text{supp } \sigma$, and for any CPTP map $\Lambda: D_{\alpha,z}(\Lambda(\rho)||\Lambda(\sigma)) \leq D_{\alpha,z}(\rho||\sigma)$ holds in each of the following cases [18]:

- $\alpha \in (0, 1]$ and $z \geq \max\{\alpha, 1 - \alpha\}$;
- $\alpha \in [1, 2]$ and $z = 1$;
- $\alpha \in [1, 2]$ and $z = \frac{\alpha}{2}$.
- $\alpha \geq 1$ and $z = \alpha$.

For two states ρ and σ , one has $f_{\alpha,z}(\rho, \sigma) = e^{(\alpha-1)D_{\alpha,z}(\rho||\sigma)}$. Hence $f_{\alpha,z}(\rho, \sigma)$ has the following properties:

Lemma 2 For any quantum states ρ and σ , such that $\text{supp } \rho \subseteq \text{supp } \sigma$, and for any CPTP map Λ , we have

- If $\alpha \in (0, 1]$ and $z \geq \max\{\alpha, 1 - \alpha\}$, then

$$f_{\alpha,z}(\Lambda(\rho), \Lambda(\sigma)) \geq f_{\alpha,z}(\rho, \sigma);$$

- If $\alpha \in [1, 2]$ and $z \in \{1, \frac{\alpha}{2}\}$; or $\alpha \geq 1$ and $z = \alpha$, then

$$f_{\alpha,z}(\Lambda(\rho), \Lambda(\sigma)) \leq f_{\alpha,z}(\rho, \sigma).$$

III. COHERENCE QUANTIFICATION

The coherence $C(\rho)$ in Ref. [21] can be expressed as

$$C(\rho) = 1 - \left[\max_{\sigma \in \mathcal{I}} f_{\frac{1}{2}, 1}(\rho, \sigma) \right]^2. \quad (8)$$

In Ref. [22] a bona fide measure of quantum coherence $C(\rho)$ has been presented by utilizing the Hellinger distance: $D_H(\rho, \sigma) = \text{Tr}(\sqrt{\rho} - \sqrt{\sigma})^2$,

$$\begin{aligned} C(\rho) &= \min_{\sigma \in \mathcal{I}} D_H(\rho, \sigma) \\ &= 2 \left[1 - \max_{\sigma \in \mathcal{I}} f_{\frac{1}{2}, 1}(\rho, \sigma) \right], \end{aligned} \quad (9)$$

which is the coherence $C_{\frac{1}{2}}(\varepsilon|\rho)$ of Theorem 3 in Ref. [23].

In Ref. [23] the coherence has been quantified based on the Tsallis relative α entropy,

$$D'_\alpha(\rho||\sigma) = \frac{1}{\alpha-1} (f_{\alpha,1}(\rho, \sigma) - 1). \quad (10)$$

But it was shown that it violates the strong monotonicity, even though it can unambiguously distinguish the coherent state from the incoherent ones with the monotonicity. In Ref. [24] a family of coherence quantifiers has been presented, which are closely related to the Tsallis relative α entropy:

$$C'_\alpha(\rho) = \min_{\sigma \in \mathcal{I}} \frac{1}{\alpha-1} \left(f_{\alpha,1}^{\frac{1}{\alpha}}(\rho, \sigma) - 1 \right), \quad (11)$$

where $\alpha \in (0, 2]$.

In the following we define a generalized $\alpha - z$ -relative Rényi entropy:

$$D_{\alpha,z}(\rho, \sigma) = \frac{f_{\alpha,z}^{\frac{1}{\alpha}}(\rho, \sigma) - 1}{\alpha - 1}. \quad (12)$$

It is worthwhile noting that several coherence measures like relative entropy [15], geometric coherence [25], the sandwiched Rényi relative entropy [26] and max-relative entropy [9] are related to the generalized $\alpha - z$ -relative Rényi entropy.

Based on the relation $f_{\alpha,z}(\rho, \sigma)$ and $D_{\alpha,z}(\rho, \sigma)$, and Lemma 2, we have

Corollary 1 For any quantum states ρ and σ for which $\text{supp } \rho \subseteq \text{supp } \sigma$, and for any CPTP map Λ : $D_{\alpha,z}(\Lambda(\rho), \Lambda(\sigma)) \leq D_{\alpha,z}(\rho, \sigma)$ holds in each of the following case:

- $\alpha \in (0, 1]$ and $z \geq \max\{\alpha, 1 - \alpha\}$;
- $\alpha \in [1, 2]$ and $z = 1$;
- $\alpha \in [1, 2]$ and $z = \frac{\alpha}{2}$;
- $\alpha \geq 1$ and $z = \alpha$.

With the above properties, based on the generalized $\alpha - z$ -relative Rényi entropy we define the quantity: $C_{\alpha,z}(\rho) = \min_{\sigma \in \mathcal{I}} D_{\alpha,z}(\rho, \sigma)$. The following statement takes place.

Theorem 1 The quantum coherence $C_{\alpha,z}(\rho)$ of a state ρ given by

$$C_{\alpha,z}(\rho) = \min_{\sigma \in \mathcal{I}} D_{\alpha,z}(\rho, \sigma) \quad (13)$$

is a well-defined measure of coherence for the following case:

- $\alpha \in (0, 1)$ and $z \geq \max\{\alpha, 1 - \alpha\}$;
- $\alpha \in (1, 2]$ and $z = 1$;
- $\alpha \in (1, 2]$ and $z = \frac{\alpha}{2}$;
- $\alpha > 1$ and $z = \alpha$.

[Proof] Because of (2), (12) and (13), we have

$$C_{\alpha,z}(\rho) = \begin{cases} \frac{1 - \max_{\sigma \in \mathcal{I}} f_{\alpha,z}^{\frac{1}{z}}(\rho, \sigma)}{1 - \alpha}, & 0 < \alpha < 1, \\ \frac{\min_{\sigma \in \mathcal{I}} f_{\alpha,z}^{\frac{1}{z}}(\rho, \sigma) - 1}{\alpha - 1}, & \alpha > 1. \end{cases}$$

From Lemma 1, we have $C_{\alpha,z}(\rho) \geq 0$, and $C_{\alpha,z}(\rho) = 0$ if and only if $\rho = \sigma$. Let σ be the optimal incoherent state such that $C_{\alpha,z}(\rho) = D_{\alpha,z}(\rho, \sigma)$. Taking into account Corollary 1, we have that $C_{\alpha,z}(\rho)$ does not increase under any incoherent operations.

Next we prove that $C_{\alpha,z}(\rho)$ satisfies Eq. (1). Suppose ρ is block-diagonal in the reference basis $\{|j\rangle\}_{j=1}^d$, $\rho = p_1 \rho_1 \oplus p_2 \rho_2$ with $p_1 \geq 0, p_2 \geq 0, p_1 + p_2 = 1$, ρ_1 and ρ_2 are density operators. Let $\sigma = q_1 \sigma_1 \oplus q_2 \sigma_2$ with $q_1 \geq 0, q_2 \geq 0, q_1 + q_2 = 1$, and σ_1, σ_2 are diagonal states similar to ρ_1, ρ_2 , respectively.

Denote Δ either max or min. Set $t_i = \Delta_{\sigma_i} \text{tr}(\sigma_i^{\frac{1-\alpha}{2z}} \rho_i^{\frac{\alpha}{z}} \sigma_i^{\frac{1-\alpha}{2z}})^z$, $i = 1, 2$. We have

$$\begin{aligned} & \Delta_{\sigma \in \mathcal{I}} \text{tr}(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}})^z \\ &= \Delta_{q_1, q_2} (q_1^{1-\alpha} p_1^\alpha t_1 + q_2^{1-\alpha} p_2^\alpha t_2). \end{aligned} \quad (14)$$

Due to the Hölder inequality with $0 < \alpha < 1$, we have

$$q_1^{1-\alpha} p_1^\alpha t_1 + q_2^{1-\alpha} p_2^\alpha t_2 \leq \left(\sum_{i=1,2} p_i t_i^{\frac{1}{\alpha}} \right)^\alpha,$$

where the equality holds if and only if $q_1 = l p_1 t_1^{\frac{1}{\alpha}}$ and $q_2 = l p_2 t_2^{\frac{1}{\alpha}}$ with $l = \left[p_1 t_1^{\frac{1}{\alpha}} + p_2 t_2^{\frac{1}{\alpha}} \right]^{-1}$, i.e.,

$$\max_{q_1, q_2} (q_1^{1-\alpha} p_1^\alpha t_1 + q_2^{1-\alpha} p_2^\alpha t_2) = \left(\sum_{i=1,2} p_i t_i^{\frac{1}{\alpha}} \right)^\alpha. \quad (15)$$

Similarly, for the inequality with $\alpha > 1$, we have

$$q_1^{1-\alpha} p_1^\alpha t_1 + q_2^{1-\alpha} p_2^\alpha t_2 \geq \left(\sum_{i=1,2} p_i t_i^{\frac{1}{\alpha}} \right)^\alpha.$$

When $q_1 = l p_1 t_1^{\frac{1}{\alpha}}$ and $q_2 = l p_2 t_2^{\frac{1}{\alpha}}$, we obtain

$$\min_{q_1, q_2} (q_1^{1-\alpha} p_1^\alpha t_1 + q_2^{1-\alpha} p_2^\alpha t_2) = \left(\sum_{i=1,2} p_i t_i^{\frac{1}{\alpha}} \right)^\alpha. \quad (16)$$

Combining (14), (15) and (16), we have

$$\begin{aligned} \Delta_{\sigma \in \mathcal{I}} f_{\alpha, z}^{\frac{1}{\alpha}}(\rho, \sigma) &= p_1 \Delta_{\sigma_1 \in \mathcal{I}} f_{\alpha, z}^{\frac{1}{\alpha}}(\rho_1, \sigma_1) \\ &\quad + p_2 \Delta_{\sigma_2 \in \mathcal{I}} f_{\alpha, z}^{\frac{1}{\alpha}}(\rho_2, \sigma_2). \end{aligned}$$

Thus, $C_{\alpha, z}$ satisfies additivity of coherence for block-diagonal states: $C_{\alpha, z}(p_1 \rho_1 \oplus p_2 \rho_2) = p_1 C_{\alpha, z}(\rho_1) + p_2 C_{\alpha, z}(\rho_2)$. \square

$C_{\alpha, z}(\rho)$ actually defines a family of coherence measures which includes several typical coherence measures.

- The coherence $C_{\alpha, z}(\rho)$ with $\alpha = \frac{1}{2}, z = 1$, i.e., $C_{\frac{1}{2}, 1}(\rho)$ is the coherence $C(\rho)$ of (8) in Ref. [21].
- $\alpha \in (0, 1)$ and $z = 1$ the coherence $C_{\alpha, 1}(\rho)$ is the coherence $C_a^\alpha(\rho)$ in Ref. [17], where the difference of a constant factor $\frac{1}{1-\alpha}$ in defining the coherence has already been taken into account.
- $\alpha \in (0, 1) \cup (1, 2]$ and $z = 1$, the coherence $C_{\alpha, 1}(\rho)$ is the coherence $C(\rho)$ in Ref. [24].
- $\alpha \in [\frac{1}{2}, 1)$ and $z = \alpha$; $\alpha > 1$ and $z = \alpha$, the coherence $C_{\alpha, z}(\rho)$ is the coherence $C_{s, \alpha}(\rho)$ in Ref. [26].

In particular, from the relation between the α affinity of coherence [17] and $C_{\alpha, z}$, we have that $\frac{1}{2} C_{\frac{1}{2}, 1}(\rho)$ is just the error probability to discriminate $\{|\varphi\rangle_i, \eta_i\}_{i=1}^d$ with von Neumann measurement, where $|\varphi\rangle_i = \eta_i^{-\frac{1}{2}} \sqrt{\rho} |i\rangle$, $\eta_i = \rho_{ii}$ and $d = \sqrt{\rho}$. Furthermore, if ρ is an incoherent state, the coherence $C_{\frac{1}{2}, 1}(\rho) = 0$, which means that a set of linearly independent pure states can be perfectly discriminated by the least square measurement.

IV. THE PROPERTIES OF $C_{\alpha, z}(\rho)$

From Theorem 1, $C_{\alpha, 1}(\rho)$ is a well-defined measure of coherence for $\alpha \in (0, 1) \cup (1, 2]$,

$$C_{\alpha, 1} = \min_{\sigma \in \mathcal{I}} \left[\frac{f_{\alpha}^{\frac{1}{\alpha}}(\rho, \sigma) - 1}{\alpha - 1} \right],$$

where $f(\rho, \sigma) = \text{tr}(\rho^\alpha \sigma^{1-\alpha})$, since for any pair of square matrices A and B , the eigenvalues of AB and BA are the same. For any incoherent state $\sigma = \sum_{k=1}^d \delta_{kk} |k\rangle\langle k|$, we have

$$\begin{aligned} \text{tr}(\sigma^{1-\alpha} \rho^\alpha) &= \sum_{k=1}^d \delta_{kk}^{1-\alpha} \langle k | \rho^\alpha | k \rangle \\ &= Q \sum_{k=1}^d \frac{\langle k | \rho^\alpha | k \rangle}{Q} \delta_{kk}^{1-\alpha}, \end{aligned}$$

where $Q = \left(\sum_{k=1}^d \langle k | \rho^\alpha | k \rangle^{\frac{1}{\alpha}} \right)^\alpha$. Denote

$$\varepsilon(\alpha) = \begin{cases} -1, & 0 < \alpha < 1, \\ 1, & 1 < \alpha. \end{cases}$$

According to the Hölder inequality and the converse Hölder inequality, we have

$$\begin{aligned} \varepsilon(\alpha) \sum_{k=1}^d \frac{\langle k | \rho^\alpha | k \rangle}{Q} \delta_{kk}^{1-\alpha} & \tag{17} \\ \geq \varepsilon(\alpha) \left(\sum_{k=1}^d \delta_{kk} \right) \left[\sum_{k=1}^d \left(\frac{\langle k | \rho^\alpha | k \rangle}{Q} \right)^{\frac{1}{\alpha}} \right]^\alpha & \\ = \varepsilon(\alpha), & \end{aligned}$$

where the equality is attained when $\delta_{kk}^{1-\alpha} = \frac{\langle k | \rho^\alpha | k \rangle}{Q}$. Then one finds the following conclusion.

Corollary 2 For $\alpha \in (0, 1) \cup (1, 2]$,

$$C_{\alpha, 1}(\rho) = \frac{\sum_{k=1}^d \langle k | \rho^\alpha | k \rangle^{\frac{1}{\alpha}} - 1}{\alpha - 1}.$$

And the maximal coherence can be achieved by the maximally coherent states.

That the maximal coherence can be achieved by the maximally coherent states for $C_{\alpha,1}(\rho)$, with $\alpha \in (0, 1) \cup (1, 2]$, can be seen in the following. Based on the eigen-decomposition of a d -dimensional state $\rho = \sum_{j=1}^d \lambda_j |\varphi_j\rangle\langle\varphi_j|$, with λ_j and $|\varphi_j\rangle$ representing the eigenvalue and eigenvectors, we have:

$$\begin{aligned} \varepsilon(\alpha) \sum_{k=1}^d \langle k | \rho^\alpha | k \rangle^{\frac{1}{\alpha}} &= \varepsilon(\alpha) \sum_{k=1}^d \left(\sum_{j=1}^d \lambda_j^\alpha |\langle \varphi_j | k \rangle|^2 \right)^{\frac{1}{\alpha}} \\ &\leq \varepsilon(\alpha) d^{\frac{\alpha-1}{\alpha}} \left[\sum_{k,j=1}^d \lambda_j^\alpha |\langle \varphi_j | k \rangle|^2 \right]^{\frac{1}{\alpha}} \\ &= \varepsilon(\alpha) d^{\frac{\alpha-1}{\alpha}} \left[\sum_{j=1}^d \lambda_j^\alpha \right]^{\frac{1}{\alpha}}, \end{aligned}$$

where the first inequality is due to

$$\sum_{k=1}^n \lambda_k x_k^p \begin{cases} \leq (\sum_{k=1}^n \lambda_k)^{1-p} (\sum_{k=1}^n \lambda_k x_k)^p, & 0 < p \leq 1, \\ \geq (\sum_{k=1}^n \lambda_k)^{1-p} (\sum_{k=1}^n \lambda_k x_k)^p, & p > 1, \end{cases}$$

with $x_k = \sum_{j=1}^d \lambda_j^\alpha |\langle \varphi_j | k \rangle|^2 \geq 0$, $\lambda_k = 1$ ($k = 1, 2, \dots, n$) and $p = \frac{1}{\alpha}$. Then one can easily find that the upper bound of the coherence can be attained by the maximally coherent states $\rho_d = |\varphi\rangle\langle\varphi|$ with $|\varphi\rangle = \frac{1}{\sqrt{d}} \sum_j e^{i\phi_j} |j\rangle$, $C_{\alpha,1}(\rho_d) = \frac{d^{\frac{\alpha-1}{\alpha}} - 1}{\alpha - 1}$. \square

Theorem 2 For $\alpha \in (0, 1)$, $\beta \in (1, 2]$, $\gamma > 1$, $\max\{\alpha, 1 - \alpha\} \leq z_1 \leq 1$, $z_2 \geq 1$, we have

$$C_{\alpha,z_1}(\rho) \leq C_{\alpha,1}(\rho) \leq C_{\alpha,z_2}(\rho); \quad (18)$$

$$C_{\beta,\beta}(\rho) \leq C_{\beta,1}(\rho) \leq C_{\beta,\frac{\beta}{2}}(\rho); \quad (19)$$

And

$$C_{\gamma,\gamma}(\rho) \leq \frac{\sum_{k=1}^d \langle k | \rho^\gamma | k \rangle^{\frac{1}{\gamma}} - 1}{\gamma - 1}. \quad (20)$$

[Proof] Set

$$\varepsilon(z_i) = \begin{cases} -1, & 0 \leq z_i \leq 1, \\ 1, & z_i > 1, \end{cases}$$

where $i = 1, 2$. According to the Araki-Lieb-Thirring inequality, for matrixes $A, B \geq 0$, $q \geq 0$ and for $0 \leq r \leq 1$, the following inequality holds [28],

$$\text{tr}(A^r B^r A^r)^q \leq \text{tr}(ABA)^{rq}. \quad (21)$$

While for $r \geq 1$, the inequality is reversed [28],

$$\text{tr}(A^r B^r A^r)^q \geq \text{tr}(ABA)^{rq}. \quad (22)$$

From (21) and (22), we have

$$\begin{aligned} \varepsilon(z_i) f_{\alpha,z_i}(\rho, \sigma) &= \varepsilon(z_i) \text{tr}(\sigma^{\frac{1-\alpha}{2z_i}} \rho^{\frac{\alpha}{z_i}} \sigma^{\frac{1-\alpha}{2z_i}})^{z_i} \\ &\leq \varepsilon(z_i) \text{tr}(\sigma^{\frac{1-\alpha}{2}} \rho^\alpha \sigma^{\frac{1-\alpha}{2}}) \\ &= \varepsilon(z_i) \text{tr}(\rho^\alpha \sigma^{1-\alpha}) \\ &= \varepsilon(z_i) f_{\alpha,1}(\rho, \sigma). \end{aligned}$$

Combining (13) and $\alpha \in (0, 1)$, we have $C_{\alpha, z_1}(\rho) \leq C_{\alpha, 1}(\rho) \leq C_{\alpha, z_2}(\rho)$. (19) can be obtained in a similar way.

Since $\gamma > 1$, we have $f_{\gamma, \gamma}(\rho, \sigma) \leq \text{tr}(\rho^\gamma \sigma^{1-\gamma})$. Similar to the proof of (17), $\min_{\sigma \in \mathcal{I}} \text{tr}^{\frac{1}{\gamma}}(\rho^\gamma \sigma^{1-\gamma}) = \sum_{k=1}^d \langle k | \rho^\gamma | k \rangle^{\frac{1}{\gamma}}$, we obtain (20). \square

Example 1: Let us consider a single-qubit pure state,

$$\rho = \frac{1}{2}(I_2 + \sum_i c_i \sigma_i),$$

where $\sum_i c_i^2 = 1$, I_2 is the 2×2 identity matrix and σ_i ($i = 1, 2, 3$) are Pauli matrices. By Ref. [17], one has

$$\max_{\sigma \in \mathcal{I}} \text{tr}^2(\sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}}) = \frac{1}{2}(1 + |c_3|),$$

and

$$\max_{\sigma \in \mathcal{I}} \text{tr}^2(\sqrt{\rho} \sqrt{\sigma}) = \frac{1}{2}(1 + c_3^2).$$

For the single-qubit pure state ρ , one has

$$\rho^{\frac{1}{4}} = \rho = \begin{pmatrix} \frac{1+c_3}{2} & \frac{c_1-ic_2}{2} \\ \frac{c_1+ic_2}{2} & \frac{1-c_3}{2} \end{pmatrix}. \quad (23)$$

Since $\text{tr}(\sigma^{\frac{1}{8}} \rho^{\frac{1}{4}} \sigma^{\frac{1}{8}})^2 = \text{tr}(\sigma^{\frac{1}{4}} \rho^{\frac{1}{4}})^2$, we now compute $\max_{\sigma \in \mathcal{I}} [\text{tr}(\sigma^{\frac{1}{4}} \rho^{\frac{1}{4}})^2]^2$. Suppose that $\sigma = \sum_i p_i |i\rangle\langle i|$ with $p_1 + p_2 = 1$ and $0 \leq p_1, p_2 \leq 1$. We have

$$\begin{aligned} \sqrt{\text{tr}(\sigma^{\frac{1}{4}} \rho^{\frac{1}{4}})^2} &= \frac{1+c_3}{2} p_1^{\frac{1}{4}} + \frac{1-c_3}{2} p_2^{\frac{1}{4}} \\ &\leq \left[\left(\frac{1+c_3}{2} \right)^{\frac{4}{3}} + \left(\frac{1-c_3}{2} \right)^{\frac{4}{3}} \right]^{\frac{3}{4}}, \end{aligned}$$

by using the Hölder inequality and that the equality holds if and only if $p_1 = c(\frac{1+c_3}{2})^{\frac{4}{3}}$ and $p_2 = c(\frac{1-c_3}{2})^{\frac{4}{3}}$ with $c = \left[\left(\frac{1-c_3}{2} \right)^{\frac{4}{3}} + \left(\frac{1+c_3}{2} \right)^{\frac{4}{3}} \right]^{-1}$. Therefore we have

$$\max_{\sigma \in \mathcal{I}} [\text{tr}(\sigma^{\frac{1}{4}} \rho^{\frac{1}{4}})^2]^2 = \left[\left(\frac{1+c_3}{2} \right)^{\frac{4}{3}} + \left(\frac{1-c_3}{2} \right)^{\frac{4}{3}} \right]^3.$$

Due to (13), we obtain

$$C_{\frac{1}{2}, z}(\rho) = 2 \left[1 - \max_{\sigma \in \mathcal{I}} \text{tr}^2(\sigma^{\frac{1}{4z}} \rho^{\frac{1}{2z}} \sigma^{\frac{1}{4z}})^z \right],$$

then we have

$$C_{\frac{1}{2}, \frac{1}{2}}(\rho) = 1 - |c_3|,$$

$$C_{\frac{1}{2}, 1}(\rho) = 1 - c_3^2$$

and

$$C_{\frac{1}{2}, 2}(\rho) = 2 - 2 \left[\left(\frac{1+c_3}{2} \right)^{\frac{4}{3}} + \left(\frac{1-c_3}{2} \right)^{\frac{4}{3}} \right]^3.$$

It is obvious that $C_{\frac{1}{2}, \frac{1}{2}}(\rho) \leq C_{\frac{1}{2}, 1}(\rho) \leq C_{\frac{1}{2}, 2}(\rho)$, see Fig. 1.

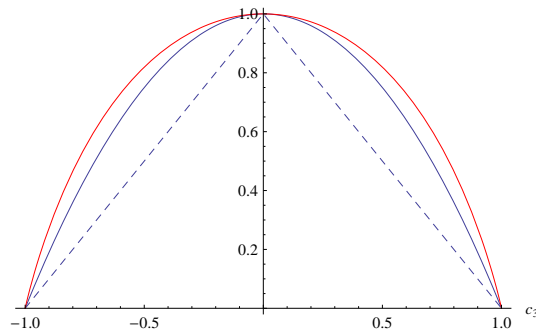


FIG. 1: The red dotted line is the vale of $C_{\frac{1}{2},2}(\rho)$; The blue solid line is the vale of $C_{\frac{1}{2},1}(\rho)$; The dashed line is the vale of $C_{\frac{1}{2},\frac{1}{2}}(\rho)$.

V. CONCLUSION

In summary, we have proposed four classes of coherence $C_{\alpha,z}(\rho)$ measures based on the generalized $\alpha - z$ -relative Rényi entropy. It has been proven that these coherence measures satisfy all the required criteria for a satisfactory coherence measure. Moreover, we have obtained the analytical formulas for special quantifiers with $z = 1$ and also studied relations among the four classes of coherence $C_{\alpha,z}(\rho)$.

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