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**QUANTITATIVE and QUALITATIVE  
MEASURES for the EVALUATION of the  
SUPERQUADRIC MODELS**

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MS-CIS-89-49  
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**September 1989**

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# Quantitative and Qualitative Measures for the Evaluation of the Superquadric Models

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## **Abstract**

In this paper we discuss the evaluation criteria for superquadric models recovered from the range data. We present arguments to support our belief that both quantitative and qualitative measures are required in order to evaluate a superquadric fit. The concept of superquadric contraction and dilation is introduced and used to derive a novel interpretation of the modified superquadric inside–outside function in terms of contraction/expansion factor. The same concept also gives a close initial guess for the numerical procedure computing the minimum Euclidean distance of a point from a superquadric model. The minimum Euclidean distance map is introduced as a qualitative criterion for interpretation of fit. View-dependent qualitative measures like the contour-difference map and the z-distance map are shown to be essential for the complete evaluation of the models. Analytical solution and techniques for the contour generator on superquadric models are presented. Finally, examples of real objects are given to generate the measures.

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## 1 Introduction

In this paper we discuss the various criteria for the evaluation of superquadric models recovered from range data. By *evaluation* we mean the process of determining the suitability of a recovered model given the original data. In order to apply superquadric models in a general vision system it is imperative to evaluate them at different stages of scene description. The model recovery procedure consists of the optimization of an objective function, studied at length by Boulton and Gross [BG87, BG88], Pentland [Pen86], and Bajcsy and Solina [BS87]. The objective function can be the superquadric inside-outside function [BG87, BG88], or its modified form [BS87, Sol87], or a suitable Euclidean distance measure [Pen88]. The superquadric model recovered by a minimization procedure formulated in terms of average values of distance or the inside-outside function may not be acceptable even if the global fitting error is acceptable. The reason being that the model imposes symmetry and gives an overconstrained estimation of a large set of points in terms of a few parameters. While an object with a model in the superquadric model vocabulary will result in an acceptable global fitting error, the converse is not necessarily true. A different set of measures which analyze the recovered model locally as well as globally is needed for the complete evaluation. We present arguments to support our belief that both quantitative and qualitative measures are required to completely evaluate the fit that is given by superquadric model.

A shape recognition system using superquadrics as a part-model shape primitive needs mechanisms to evaluate the intermediate descriptions in order to extract the part-structure. Thus, in this paper we focus on the evaluation of recovered superquadric models rather than on formulation of the model recovery problem. We discuss two quantitative (global) measures and three qualitative measures. As we shall see, each one of them is individually insufficient to provide an absolute criterion for the evaluation of the fit. However, when they are actually combined and integrated they may be used to provide corroboration and guidance to the segmentation procedure.

We will first present the definition of deformable superquadrics as given by Solina [Sol87, BS87], and then outline the model evaluation criteria that we developed. Finally we will provide some examples and discuss the effect of integrating the different criteria.

## 2 Superquadrics : Deformable Part Models

Volumetric primitives give object-centered descriptions of the object parts. Generalized cylinders, [Kli78] proposed for applications in vision by Binford [Bin71], have been used as volumetric primitives for the rich vocabulary of shapes they provide. However, this vocabulary is very difficult to recover from vision data, limiting the actual vocabulary to simple linear-straight-homogeneous cylinders. Recently, Terzopolous *et.al.* [TWK88] suggested a deformable model based on the concept of generalized cylinders. The model requires segmented data and user intervention for the initial approximation and is computationally expensive. Superquadric primitives can model only a subset of generalized cylinders shapes, but provide a good compromise for representational and computational effectiveness. They are capable of modeling tapering and bending deformations, and are recovered effectively by a stable numerical procedure.

Superquadrics are a family of parametric shapes that have been used as primitives for shape representation in computer vision [Pen86, Sol87, BG87] and computer graphics [Bar81, Bar84].

**Definition :** A superquadric surface is defined as the closed surface spanned by the vector  $S$  having x,y and z components specified as functions of the angles  $\eta$  and  $\omega$  in the given intervals :

$$S(\eta, \omega) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 \cos^{\varepsilon_1}(\eta) \cos^{\varepsilon_2}(\omega) \\ a_2 \cos^{\varepsilon_1}(\eta) \sin^{\varepsilon_2}(\omega) \\ a_3 \sin^{\varepsilon_1}(\eta) \end{bmatrix} \quad \begin{array}{l} -\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2} \\ -\pi \leq \omega < \pi \end{array}$$

We identify components as  $S_x(\eta, \omega)$ ,  $S_y(\eta, \omega)$ , and  $S_z(\eta, \omega)$ <sup>1</sup>.

The implicit superquadric equation can be derived from the above definition by eliminating  $\eta$  and  $\omega$  :

$$\left( \left( \frac{x}{a_1} \right)^{\frac{2}{\varepsilon_2}} + \left( \frac{y}{a_2} \right)^{\frac{2}{\varepsilon_2}} \right)^{\frac{\varepsilon_2}{\varepsilon_1}} + \left( \frac{z}{a_3} \right)^{\frac{2}{\varepsilon_1}} = 1.$$

Thus, alternatively we can define the superquadric in terms of its implicit equation, as the locus of the points  $(x, y, z)$  satisfying the above equation.

The parameters  $a_1$ ,  $a_2$ , and  $a_3$  determine the size of the superquadric in the x,y and z directions (in object-centered coordinate system) respectively; while  $\varepsilon_1$  and  $\varepsilon_2$  represent the squareness parameters in the latitude and in the longitude plane. Based on these parameters, superquadrics can

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<sup>1</sup> Actually the component in the z-direction is independent of  $\omega$  but, we include it, at this point, only for symmetry. However, when deformations are applied  $\omega$  becomes an effective component of the z-direction.

model a large set of standard building blocks, such as spheres, cylinders, parallelepipeds as well as shapes in between.

If both  $\varepsilon_1$  and  $\varepsilon_2$  are equal to 1, the surface defines an ellipsoid. Cylindrical shapes are obtained for  $\varepsilon_1 < 1$  and  $\varepsilon_2 = 1$ . Parallelepipeds are obtained for both  $\varepsilon_1$  and  $\varepsilon_2 < 1$ . In our approach, the model recovery procedure allows  $\varepsilon_1$  and  $\varepsilon_2$  to assume values in the interval  $[0 \dots 1]$ . For values of  $\varepsilon_1$  and  $\varepsilon_2 > 1$  the resulting parametrized shapes define objects which are not in the set of primitives we are interested in portraying. For instance,  $\varepsilon_1, \varepsilon_2 = 2$  yield objects which are diamond-shaped bevels and as their value increases they become pinched.

## 2.1 Applying Deformations to Superquadrics

The representational power of superquadrics is augmented by the application of various *deformations* to the basic model. The deformations which we have included in our vocabulary are **tapering** and **bending**. For notation purposes we define  $\mathbf{S}'$  as the model to which deformations have been applied and identify each of the components in the x,y, and z directions respectively by  $S_X, S_Y, S_Z$  and alternatively as  $(X, Y, Z)$ , accordingly to the definition of the implicit equation.

**Tapering:** Linear tapering along the  $z$ -axis transforms the basic superquadric model from  $\mathbf{S}$  to  $\mathbf{S}'$ , where  $(x, y, z)$  is transformed to  $(X, Y, Z)$ . The transformed model is given by :

$$\begin{aligned} X &= f_x(z)x \quad \text{where } f_x(z) = \frac{K_x}{a_3}z + 1 \\ Y &= f_y(z)y \quad \text{where } f_y(z) = \frac{K_y}{a_3}z + 1 \\ Z &= z \end{aligned}$$

where  $K_x, K_y, -1 \leq K_x, K_y \leq 1$ , represent the tapering with respect to the x and y plane relative to the z direction.

**Bending:** Bending deformation of the superquadric surface vector is defined by the following transformation :

$$\begin{aligned} X &= x + \cos(\alpha)(R - r), \\ Y &= y + \sin(\alpha)(R - r), \\ Z &= \sin(\gamma)\left(\frac{1}{k} - r\right). \end{aligned}$$

Where  $k$  is the curvature and  $r$  is the projection of  $x$  and  $y$  components onto the bending plane  $z - r$  :

$$r = \cos(\alpha - \tan^{-1}(\frac{y}{x}))\sqrt{x^2 + y^2}$$

Bending transforms  $r$  into

$$R = k^{-1} - \cos(\gamma)(k^{-1} - r),$$

Where  $\gamma$  is the bending angle

$$\gamma = zk^{-1}$$

**Combination of Tapering and Bending:** The two independent deformations are applied by computing the corresponding homogeneous transformation matrices. It is possible to apply both transformations to a superquadric model sequentially. However, since matrix multiplication is not commutative, the order in which deformations are applied is important. The model recovery procedure has adopted the following structure to transform an object-centered superquadric model to a deformed superquadric in general position and orientation.

$$\mathbf{S}' = \text{Translation}(\text{Rotation}(\text{Bending}(\text{Tapering}(\mathbf{S}))))$$

Thus, bending and tapering introduce two parameters each in the final superquadric equation, bringing the total parameter count to 15. The minimization procedure is capable of recovering all 15 parameters simultaneously for a given data set. The above equation identifies the volumetric model used to describe parts in our system. Therefore, whenever we talk about a superquadric model, we shall refer to a model as defined by  $\mathbf{S}'$  above.

### 3 Criteria for Model Evaluation

A superquadric model obtained by least-square fitting the inside-outside function is an overconstrained estimation of data, with more constraints than parameters. This is the case since the set of constraints, or data points, is generally two orders of magnitude larger than the number of parameters. As in any parametric approach the goal is to model a large set of data using the least number of parameters. However, such a compact representation does not always recover correct model for given data . It assigns equal importance to each point, independent of its location, with



the central goal of including the point in the global estimation. The model recovered by such a procedure needs to be analyzed, with respect to quantitative and qualitative criteria, in order to determine its suitability in describing the data. With this in mind, we have identified the following measures for model evaluation in the context of the shape recognition problem :

1. The **Goodness-of-fit** measure,  $\mathcal{G}$ , based on the inside–outside function.
2. The **Mean–distance** measure,  $\mathcal{M}$ , based on the true minimum Euclidean distance of individual points from the model surface.
3. The **Min–distance map** produced by mapping the magnitude of the minimum Euclidean distance of individual points from the model surface in image coordinate system.
4. The **Contour–difference map** produced by comparing the apparent contour formed by the model in the viewpoint direction with the occluding contour of the object.
5. The **Z–distance map** produced by measuring the distance of the points in the range image and the superquadric surface in the viewing direction.

Of the aforementioned measures, the first two yield global and *quantitative* results while the last three give local and *qualitative* outcomes. We will now discuss each of these measures in detail.

### 3.1 Goodness-of-fit measure

The modified inside–outside function for an object–centered superquadric model is given by :

$$F(x, y, z) = \left[ \left[ \left( \frac{x}{a_1} \right)^{\frac{2}{\varepsilon_2}} + \left( \frac{y}{a_2} \right)^{\frac{2}{\varepsilon_2}} \right]^{\frac{\varepsilon_2}{\varepsilon_1}} + \left[ \frac{z}{a_3} \right]^{\frac{2}{\varepsilon_1}} \right]^{\varepsilon_1}$$

It determines where a point lies relative to the superquadric surface. If  $F(x, y, z) = 1$ , point  $(x, y, z)$  lies on the surface of the superquadric. If  $F(x, y, z) < 1$ , the point lies inside and if  $F(x, y, z) > 1$ , the point lies outside the superquadric. The minimization procedure optimizes the inside–outside function of deformed superquadrics in general position given by :

$$F(x, y, z) = F(x, y, z; a_1, a_2, a_3, \varepsilon_1, \varepsilon_2, \phi, \theta, \psi, p_x, p_y, p_z, K_x, K_y, k, \alpha)$$

Where  $\phi, \theta, \psi$  define the orientation and  $p_x, p_y, p_z$  define position of superquadric in space.

Goodness-of-fit is defined as :

$$\mathcal{G} = \frac{1}{n} \sum_{i=1}^n |F(x_i, y_i, z_i) - 1|$$

where  $n$  is the number of points. The minimization procedure gives a best fitting superquadric according to the inside-outside function. The  $\mathcal{G}$  value for a superquadric model reflects how well the model fits the data. Ideally it should be as close to 0 as possible. Intuitively, a high value of  $\mathcal{G}$  indicates bad fit. For a given model, as we go away from the surface, the value of  $F$  increases. It is not related to the true Euclidean distance in the sense that two points equidistant from the model have different values of  $F$  in general. What is the meaning of a particular point having a value of  $F$ ? We provide an interpretation for the modified inside-outside function  $F$  in the next section.

### 3.1.1 Interpretation of $F$

The outermost exponent  $\varepsilon_1$  in the inside-outside function  $F$  was added by Solina [Sol87] to cancel out the effect of  $\varepsilon_1$  in the equation. This modification resulted in a better recovery of cylindrical objects. Solina noticed the qualitative effect of the modification, but no mathematical justification was given for it. We provide an explanation which gives an intuitive meaning to the values of the inside-outside function, and makes it possible to use this measure for model evaluation.

Consider a superquadric  $S = (X, Y, Z)$  defined in terms of the explicit superquadric equation. Let  $P = (x, y, z)$  be an arbitrary point in space. Now we can scale the three axes of  $S$  by a factor  $\beta$  such that the point  $P$  will lie on the scaled superquadric  $S' = (X', Y', Z')$  :

$$S'(\eta, \omega) = \begin{bmatrix} \beta a_1 \cos^{\varepsilon_1}(\eta) \cos^{\varepsilon_2}(\omega) \\ \beta a_2 \cos^{\varepsilon_1}(\eta) \sin^{\varepsilon_2}(\omega) \\ \beta a_3 \sin^{\varepsilon_1}(\eta) \end{bmatrix} \quad \begin{array}{l} -\frac{\pi}{2} \leq \eta \leq \frac{\pi}{2} \\ -\pi \leq \omega < \pi \end{array}$$

We will now show that  $F$  and  $\beta$  are related.

The implicit form of  $S'(\eta, \omega)$  can be written as :

$$\left[ \left( \frac{x}{\beta a_1} \right)^{\frac{2}{\varepsilon_2}} + \left( \frac{y}{\beta a_2} \right)^{\frac{2}{\varepsilon_2}} \right]^{\frac{\varepsilon_2}{\varepsilon_1}} + \left[ \frac{z}{\beta a_3} \right]^{\frac{2}{\varepsilon_1}} = 1.$$

Now, solving for  $\beta$  :

$$\beta = \left[ \left[ \left( \frac{x}{a_1} \right)^{\frac{2}{\varepsilon_2}} + \left( \frac{y}{a_2} \right)^{\frac{2}{\varepsilon_2}} \right]^{\frac{\varepsilon_2}{\varepsilon_1}} + \left[ \frac{z}{a_3} \right]^{\frac{2}{\varepsilon_1}} \right]^{\frac{\varepsilon_1}{2}}$$

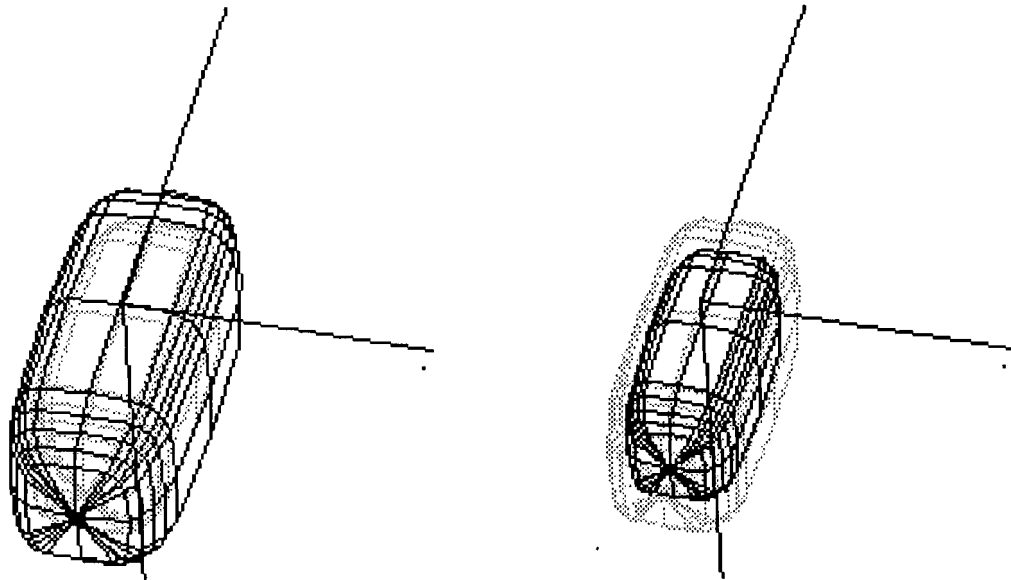


Figure 1:  $\beta$  expansion and contraction of a superquadric model. left:  $\beta = 1.2$ , right:  $\beta = 0.8$ .

It follows from the definition of  $F$  that :

$$F = \beta^2.$$

This result shows that the value of the inside–outside function  $F$  for a point  $(x, y, z)$  is given by the square of the factor by which the superquadric  $S$  is scaled to make it pass through  $(x', y', z')$ . This factor can be seen as the amount a superquadric has to be *expanded* or *contracted* (figure 1) to make it pass through an arbitrary point in 3 space. This result provides an intuitive explanation for the values of  $F$ , with values  $> 1$  indicating dilation and  $< 1$  indicating contraction of the superquadric model.

The obvious question to ask is whether this explanation can be extended to the tapered or bent models? Since tapering is defined in terms of  $a_3$  (the dimension along the major axis), it is not possible to obtain a closed form solution for  $\beta$ . Thus, the above interpretation is only approximately true for tapered models. For the models with bending deformation, however, the interpretation is valid. Since the minimization problem is formulated in terms of the inside–outside function, its values are available with the model parameters, and does not require explicit computation.

Based on the quantitative result provided by  $\mathcal{G}$ , one may determine whether the model provides an underestimation or overestimation of the data. Such a conclusion is, however, biased for this

is simply an average of the values. In fact, if the data is obtained from an object which has a small concavity or convexity, then relying only on this measure might lead to the false conclusion where the superquadric provides a good model despite the fact that such irregularity can not be accounted for. Thus, this measure alone can not provide for a qualitative description of the data. This consideration leads us to the conclusion that other measures are needed to corroborate the hypothesis put forward by the value of  $\mathcal{G}$ .

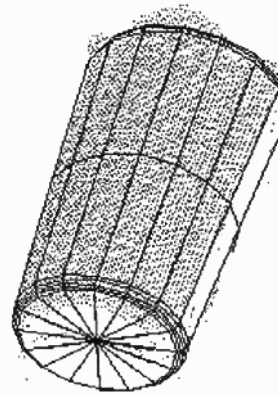
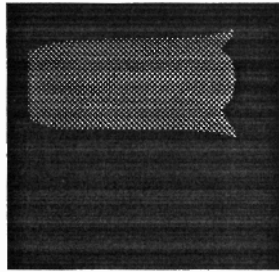
To elaborate on this point, we provide examples of real objects and the corresponding superquadric fits obtained by the model recovery program developed by Solina [Sol87] (shown in figure 2). The Goodness-of-fit values for the models are listed in table 1. From the table we can observe that the model for the vase provides the best fitting value, while the composite object (a cylinder glued to a box) has the worst fitting value as expected. The other two examples, however, indicate that  $\mathcal{G}$  is insufficient to explain the quality of the fit. In fact, in case of the arch, the model is intuitively more acceptable than that of the cup, but has a higher value of  $\mathcal{G}$ . This shows that the Goodness-of-fit measure cannot be the only measure to evaluate the quality of the fit.

In the next section we propose an iterative solution to the true Euclidean distance from a point in 3 space and a superquadric surface. We then use the computed distance to derive a quantitative interpretation of fit, which we call the *Mean-distance* measure.

### 3.2 Euclidean distance measure

The distance of an arbitrary point in 3 space from a given superquadric model is difficult to compute because of multiple solutions of the analytical formulation of the problem as the non-linear root finding problem. Furthermore, it is not possible to obtain a closed-form solution for the problem.

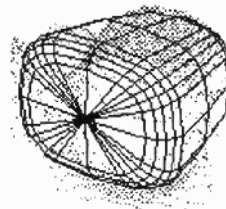
Boult and Gross [BG88] present a minimization of the error of fit function, based on the Euclidean distance between each data point and the corresponding point on the superquadric along the line connecting the center of the superquadric and the data point. They note that the measure thus derived overestimates the distance and is in particular less accurate for squarish models. Attempting to use such an approach in evaluation of deformed superquadrics yields a less reliable source of information. For this reason we have sought to determine the true Euclidean measure for each individual point.



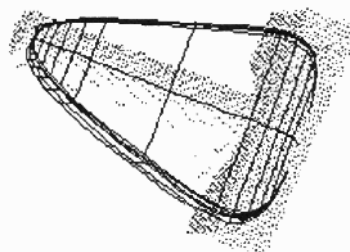
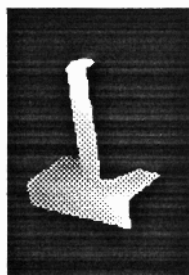
(a) A Vase.



(b) An Arch.



(c) A Cup.



(d) A Composite object.

Figure 2: Range Images of real objects and their corresponding superquadric models.

The formulation of the superquadric recovery procedure does not require the computation of the Euclidean measure at any stage. Furthermore, the inside–outside function and the distance measure are not related in the sense that two points which are located at the same distance from the superquadric surface do *not* have the same value of  $F$  in general. Thus, we have posed the problem in terms of an iterative procedure to minimize the distance  $d$  for a given point and a given (deformed) superquadric (figure 3).

Superquadric surfaces are parametrized by  $\eta$  and  $\omega$ , and are differentiable everywhere. The distance function  $d$  is convex for the points lying outside the superquadric model, but not for the points inside the model. So global minima is not guaranteed for the inside points in case of squarish models. Since our method for computing the initial guess traces the locus of  $(\eta, \omega)$ , that could lead the approximation to choose the side of the squarish models with local minima instead of global minima. Therefore, for the inside points in such models, we also need to investigate the directions orthogonal to the direction of the local minima.

The problem is formulated as :

**Problem definition** : Given a point  $P = (x_1, y_1, z_1)$  in space, minimize the following function of two variables :

$$d(\eta, \omega) = \sqrt{(x(\eta, \omega) - x_1)^2 + (y(\eta, \omega) - y_1)^2 + (z(\eta, \omega) - z_1)^2}$$

where  $x(\eta, \omega), y(\eta, \omega), z(\eta, \omega)$  are the position vectors of the (deformed) superquadric  $S'$ .

To ensure a fast convergence to the right solution a good initial approximation was required, as often is the case with iterative methods. Guided by the intuitive consideration that in a sphere the minimum distance of a point to the sphere’s surface lies in the radial direction, we realized that the same approach for determining minimum distance could be applied superquadrics. However, while in the former case there is a direct correspondence to the minimum distance, in the latter the resulting distance could only be taken as an approximation of the minimum distance. As we later discovered, [BG88] had proposed a similar approach in constructing a minimizing function to be used in determining the best fit model for the data, as discussed above. Our method differs in that we are actually computing the true Euclidean distance and employ it both to corroborate the quality of the fit as well as to guide the fitting of the minimization process.

The approximation is obtained by extending the expansion/contraction approach introduced in the previous section (figure 3).

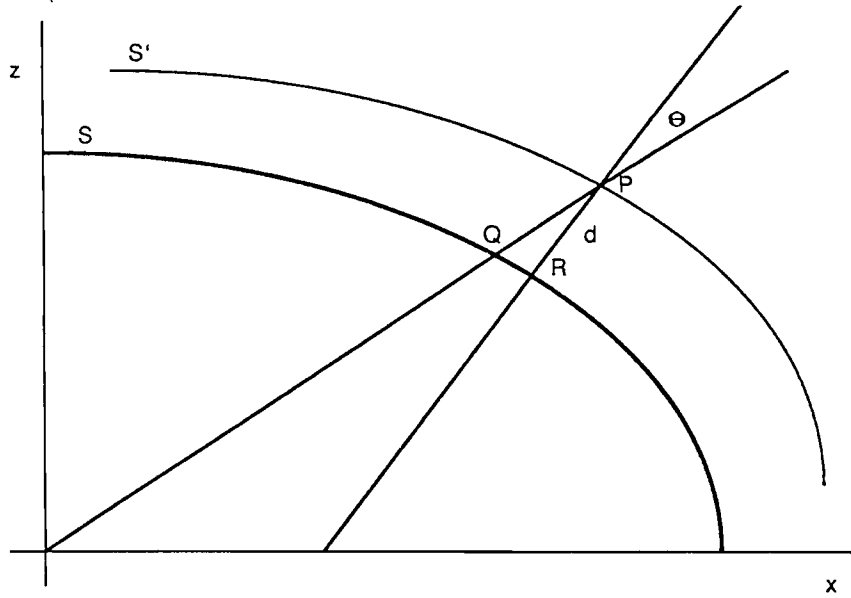


Figure 3: Euclidean distance and initial approximation for the iterative procedure.

Corresponding to the point  $P = (x_p, y_p, z_p)$  in 3 space, there is a point  $Q_0 = (x_0, y_0, z_0)^2$  on the original superquadric  $S$  :

$$\begin{aligned} x_0 &= x_p/\beta, \\ y_0 &= y_p/\beta, \\ z_0 &= z_p/\beta, \end{aligned}$$

The point  $Q$  in Cartesian coordinate system can be written as  $Q_0 = (\eta_0, \omega_0)$  in the parametrized form. Thus, an initial approximation of  $\eta$  and  $\omega$  is easily obtained. If the superquadric in consideration is deformed, then deformations are also applied to the point  $P$ . Such deformations allow us to determine  $\beta$  in function of the deformed superquadric for we would like  $\beta$  to closely relate the two superquadrics in question and yield the best initial approximation possible. We observed, in the previous section, that while there is a closed-form solution for  $\beta$  when the deformation in question is simply bending, no such solution is possible when dealing with tapering. However, we have noticed that by applying the tapering deformation to  $P$  we can recover an approximation for  $\beta$  which is effectively more accurate than if  $\beta$  is recovered without the tapering deformation altogether.

The points  $P$  and  $Q_0$  correspond to the same  $\eta$  and  $\omega$  values, and  $Q_0$  is likely to be very close to the point  $Q^* = (\eta^*, \omega^*)$ , denoted in figure 3 by  $R$ , such that the distance between  $Q^*$  and  $P$  is

<sup>2</sup>The subscript is meant to indicate that it represents the first approximation

	Vase	Arch	Cup	Composite object
Goodness-of-fit ( $\mathcal{G}$ )	0.0012	0.0056	0.0037	0.008
Mean-distance ( $\mathcal{M}$ )	3.18 mm.	3.15 mm.	7.52 mm.	16.29 mm.

Table 1: Table of the quantitative measures of the four objects.

minimal. Effectively we are updating the position of  $Q_i$  such that the angle  $\Theta$  between the normal to the surface and the vector through  $Q_i$  and  $P$  decreases. We have noticed that when  $P$  is located in proximity to one of the axes the initial approximation was often extremely good, within one or two decimal digits of the computed minimum distance. Therefore, the objective is to find  $R$ .

The function  $d(\eta, \omega)$  is minimized given the initial approximation  $\eta_0$  and  $\omega_0$  using a quasi-Newton method<sup>3</sup>. A line search is used to locate a new point, as described in [DS83].

The method requires only function values; a finite-difference method is used to estimate the gradient internally. Though  $d$  is differentiable at all points (even with deformations), we have found that supplying an external gradient does not speed up the iterative process in general. The method was found to be accurate up to the sixth decimal place for experimental data. However, we can settle for lower accuracy for faster convergence. The method has been successfully tested on deformed superquadrics. We now use the computed Euclidean distance to derive a quantitative interpretation of the fit.

### 3.2.1 Mean-distance measure

As was pointed out by Boulton [BG88], obtaining the true Euclidean measure is helpful in interpreting the recovered model since such a measure is not affected by the scaling factor of the superquadric, i.e. it can be considered as an absolute fit measure. We define the measure of the *Mean-distance* for a given superquadric,  $\mathcal{M}(S)$ , as the average of the minimum distance over all the points :

$$\mathcal{M} = \frac{1}{n} \sum_{i=1}^n d_i(\eta, \omega)$$

---

<sup>3</sup>Minimization routine *dbconf* from the IMSL version 10.0 library was used with double precision mathematics.



This measure provides us with a value which is in effect the distance from the model to the data point. Thus, a low distance value accounts, in general, for a good overall fit.

Since  $\mathcal{M}$  is a quantitative measure, and in particular an average, it may erroneously suggest a good fit when effectively that is not the case. If, in fact, we have an object which presents some surface irregularities, we may be induced into believing that we not only have a good fit but that the model accounts for the object thoroughly. The low value for  $\mathcal{M}$  is derived by averaging over all the points. Hence, we notice that the quantitative interpretation falls short of giving enough information to build our confidence on the quality of the fit. In view of this limitation we now discuss the qualitative interpretation of the Euclidean distance as a map of the magnitude of the Euclidean distance associated with individual points and represented like a range image.

### 3.3 Min-Distance Map

The minimum distance map, interpreted as a range image, provides an interpretation for the quality of the fit. As one would expect, large values identify points which are not well accounted for by the proposed model. One would like to be able to use this information to identify areas of *poor* fit. These areas could suggest potential *local deformation* or/and *object segmentation*.

The distance map provides values which are in general not immediately fitted by a continuous function of second or third order. That, in particular, holds for a surface which is in itself not regular. Thus, we propose to cluster points which have equal distance values and then identify regions. However, we expect, in general, that the distance value varies from point to point. Hence, limiting a given region to points which have exactly equal distance may yield a map which, in general, would be no more informative than the original one.

#### 3.3.1 Clustering Criteria

In order to successfully cluster points into the same region we require that the given points meet the following criteria : *Equal Discretized Distance* and *Spatial Adjacency*.

The first criterion actually defines a filter for the range image while the second one describes a criterion to spatially relate points which have equal filtered value.

The discretization of the distance for any given point is obtained by incrementing or decrementing it to the multiple of the thresholding interval  $t$  which is closer to it. The thresholding value  $t$

is not arbitrary but is related to the value of  $\mathcal{M}$ . This process emphasizes regions which have large discrepancy. By focusing on these particular areas, the Min-Distance Map may be employed as a guide to successively refine the model.

To simplify the process of determining Spatial Adjacency, we take advantage of the data structure representation of the image provided in image-coordinate system. Hence, we apply two inverse transformations. The first one takes the point from object-coordinate system to world-coordinate system; while the second transforms the point from world-coordinate system to image-coordinate system. We note that due to rounding-off errors the mapping fails to provide a bijection between object-coordinate system and image-coordinate system. However, this lack of correspondence impairs our ability to fully recover the regions, but only apparently. In fact, the rounding-off errors due to the inverse transformations will be uniformly distributed over the image and hence not effectively influence the region identification process.

### 3.3.2 Region Properties

Regions are recovered by a simple region growing technique which allows us to group points together. Once the regions have been identified in the map, we would like to study their properties and relations.

We apply a preliminary screening to the study of the regions. We would like at first to eliminate those regions which present little information and are effectively a handicap to obtain a global understanding of the fit. Therefore, we look at the small regions (few pixels in area) and observe whether their distribution may actually be symptomatic of a highly patterned or irregular surface. If this is not the case they can be “absorbed” by the enveloping region.

We then identify intrinsic as well as relative properties of the regions. The former category is comprised of :

- *Size* thresholded value (effectively the thresholded value), area, volume.
- *Shape* defined in terms of its eccentricity, convexity, etc.
- *In-Out Value* determined by the value of the inside-outside function on the region.

While the first two give quantitative information on the fit by relating the “Sizes” of the regions, the last one yields qualitative information on the type of estimate given by the fit. Namely there

might be regions for which the superquadric provides an underestimation or an overestimation of fit. These can be plainly identified by picking a point in a given region and applying the inside–outside function.

The latter category focuses on relation amongst the properties such as :

- *Size* distribution and hierarchy.
- *On-Top-of* relation and hierarchy–graph.

These relations and others give information on the areas of the superquadric which least account for the data.

In the remaining portion of this section we present Min–distance map for the objects in figure 2, and then discuss the limitations of this measure.

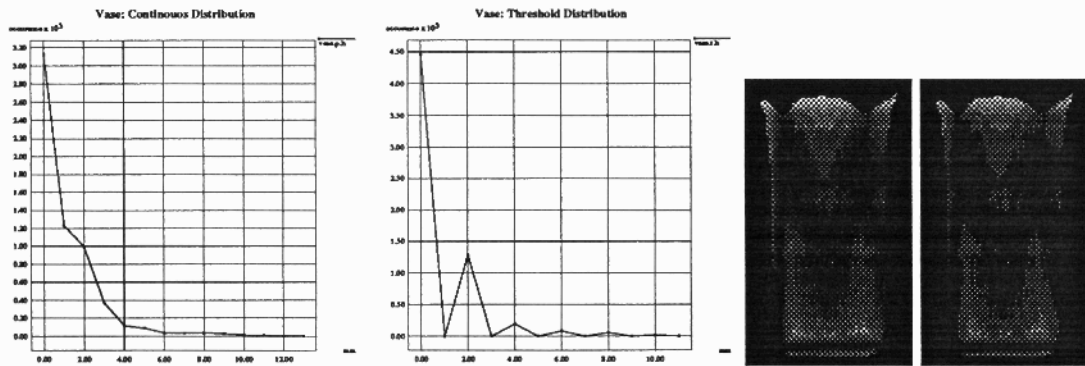
### 3.3.3 Min–Distance Map: Examples

The example we discuss are representative both of the value that the Min–Distance Map provides as well as of the limitations that are encountered. The figures 4.a–4.d have all been scaled so that the respective regions are visible. Regions with high value (white) identify points which are further away from the model. The adjacent histograms are meant to give a preliminary interpretation on the type of the fit. They are presented in relation to both the table 1 and the respective figures.

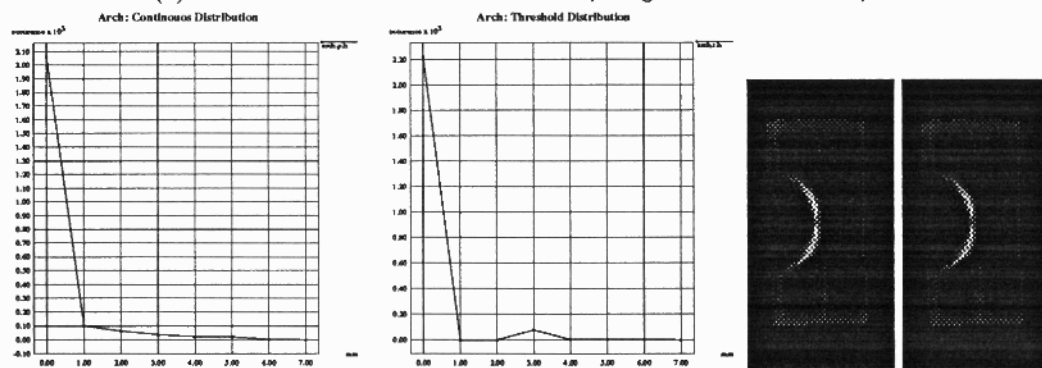
**Vase:** As we can see in table 1, the vase has a small value for  $\mathcal{M}$ . That, as was previously pointed out, is a sign of good fit. As we notice in figure 4.a and the respective histogram, that is indeed the case. The histogram gives us some summary information on the cumulative quality for the regions. Most of them are identifiable in the lower range quite below the average distance.

**Arch:** In this example the low value in the table 1 would seem to be indicative of a good fit. While for most of the points the fit is rather satisfactory (see the histogram in figure 4.b), there is a rather well defined area which is clearly overestimated by the superquadric.

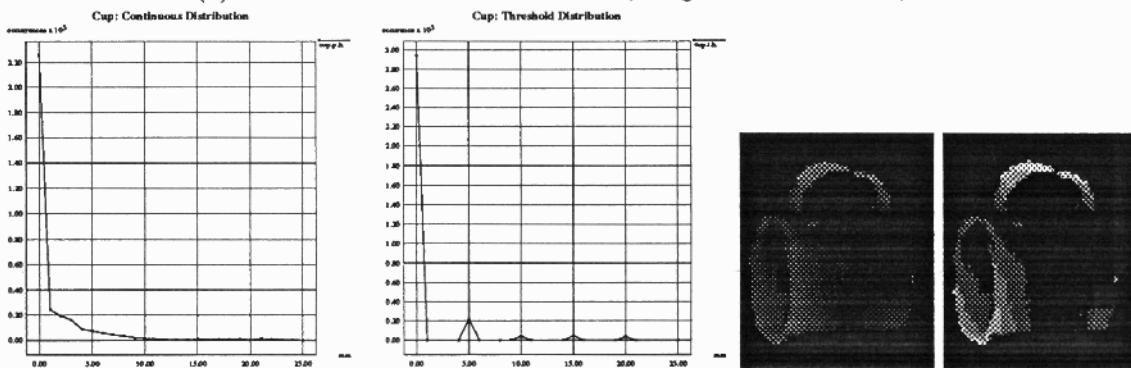
**Cup:** In figure 4.c we notice that both the measure as well as the figure present several areas where the fit is insufficient to describe the data. In particular it is interesting that the region of the handle as well as the concavity of the object are highlighted. Those are regions in which further



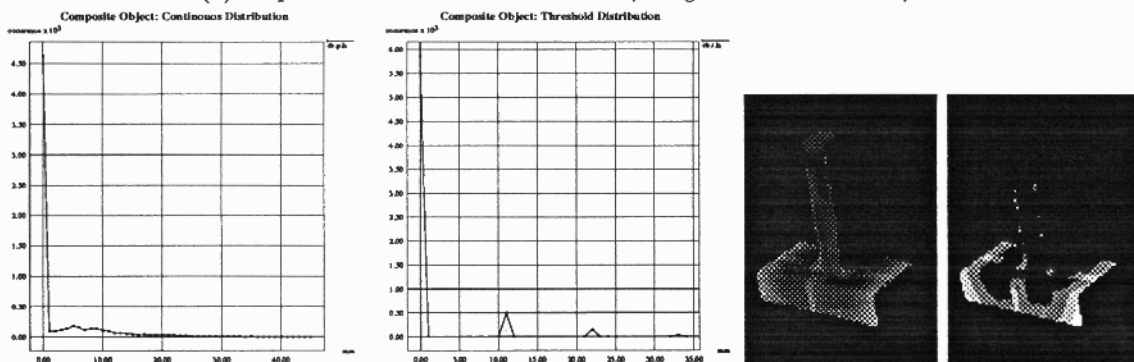
(a) Vase. Left: Continuous difference map. Right: Thresholded map.



(b) Arch. Left: Continuous difference map. Right: Thresholded map.



(c) Cup. Left: Continuous difference map. Right: Thresholded map.



(d) Composite object. Left: Continuous difference map. Right: Thresholded map.

Figure 4: Continuous and Thresholded difference maps: Histograms and respective images.

action must be taken toward a possible segmentation as well as the incorporation of the presence of the cavity in the object.

**Composite Object:** This, perhaps, is the most interesting of the object so far observed. The distance map in figure 4.d highlights regions which are clearly not well interpreted by the fit, see figure 7. The region information will be now relevant toward determining a better model when seen in conjunction to the maps defined later on.

### 3.3.4 Limitations of Min-Distance Map

This measure provides valuable information, as we have seen; however, there are situations in which it is insufficient to use it as the only measure to determine the quality of the fit. Observe, for instance, that if the object in figure 6 (the arch) had been scanned from a viewpoint where no points in the cavity were visible, the Min-Distance Map would concur with the quantitative measures. That would lead us to believe that the model  $S$ , which was derived by the fitting procedure, is actually optimal. Yet, one can verify by inspection, that there is actually more to the data representation which is not accounted for by the model provided.

This consideration simply tells us that the minimum distance, as well as the ones previously discussed, are insufficient criteria for deciding on the feasibility of a proposed model.

## 3.4 Viewpoint-dependent Qualitative Measures

Having discussed the global measures and the qualitative interpretation of the true Euclidean distance measure, we now discuss the view-dependent measures which analyze the superquadric model in the viewing direction. The two qualitative measures are the *contour-difference map* and the *z-distance map*. These measures are easy to compute and analyze, and provide local analysis of regions of underestimation and overestimation. The contour-difference map is obtained by comparing the occluding contour of the object with the projected apparent contour of the superquadric model. The z-depth difference map encodes the distance of each point from the model in z direction. It differs from the min-distance map in the sense that it is not the minimum distance and is available only if the image point has a corresponding point on the model in the given direction. In this section we will outline methods for generating the apparent contour of the

superquadrics and the z-distance map.

### 3.4.1 Apparent Contours of Superquadrics

**Definition:** The **Contour-generator** (or occluding contour) is defined as the locus of the points (a closed curve) on the superquadric surface where the surface-normal vector is perpendicular to the viewpoint vector.

Let  $\mathbf{V} = (V_x, V_y, V_z)$  be the viewpoint vector, and  $\mathbf{N} = (n_x, n_y, n_z)$  be any surface-normal vector. The Occluding contour is then given by :

$$\mathbf{V} \cdot \mathbf{N} = 0$$

We now derive a closed-form solution for the contour generator on a non-deformed superquadric surface :

$$V_x n_x + V_y n_y + V_z n_z = 0$$

Substituting for  $\mathbf{N}$  gives :

$$\frac{V_x}{a_1} \cos^{2-\varepsilon_1}(\eta) \cos^{2-\varepsilon_2}(\omega) + \frac{V_y}{a_2} \cos^{2-\varepsilon_1}(\eta) \sin^{2-\varepsilon_2}(\omega) + \frac{V_z}{a_3} \sin^{2-\varepsilon_1}(\eta) = 0$$

Solving for  $\eta$  gives the closed-form solution for generating the apparent contour :

$$\eta = \tan^{-1} \left( \left( -\frac{a_3}{V_z} \left( \frac{V_x}{a_1} \cos^{2-\varepsilon_2}(\omega) + \frac{V_y}{a_2} \sin^{2-\varepsilon_2}(\omega) \right) \right)^{\frac{1}{2-\varepsilon_1}} \right).$$

Figure 5 (a and b) shows the apparent contours of superquadrics generated by the above equation. Unfortunately, there is no closed-form solution for a general deformed superquadric, as the surface-normal vector  $\mathbf{N}$  has to undergo deformation by the following rule (derived by Barr [Bar84]) :

$$\mathbf{N}' = \det \mathbf{J} \mathbf{J}^{-1T} \mathbf{N}$$

where  $\mathbf{J}$  is the Jacobian of the deformed superquadric. To trace the apparent contour of a deformed superquadric, the angles  $\eta$  and  $\omega$  are varied systematically on the superquadric Surface. Points on the contour are accumulated in such a way that a closed contour is formed (see figure 5(c)). This contour is then orthographically projected on the image-coordinate system to make comparisons with the image contour.

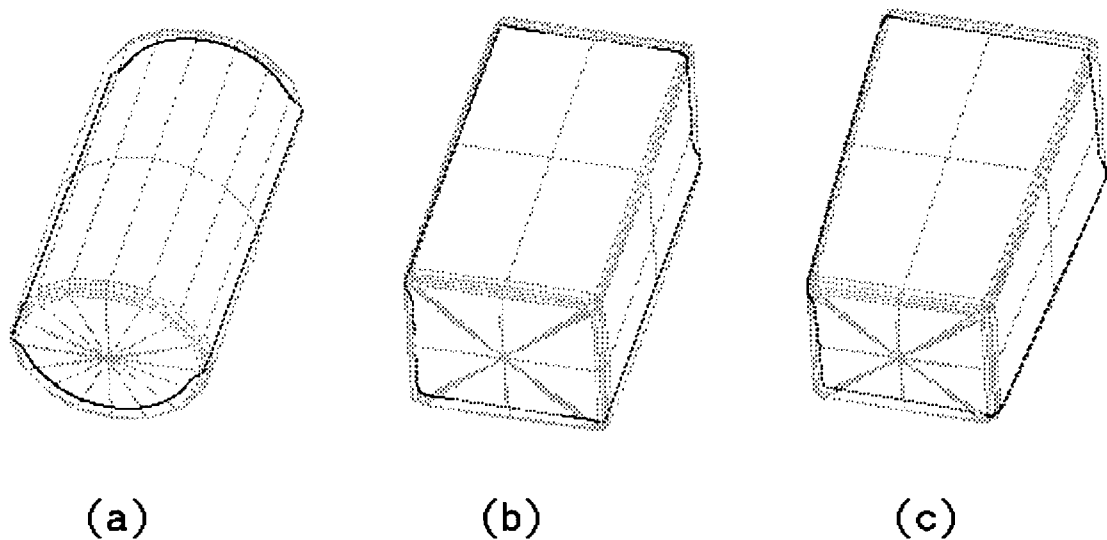


Figure 5: **Apparent contours of Superquadrics** : for non-deformed box and cylinder, and for a tapered box.

### 3.4.2 Z-distance map

For the purpose of comparing the superquadric model with given surface points to generate a difference map, we have to compute the distance of every given point from the superquadric surface *along a given direction*<sup>4</sup>. There are two ways of doing this :

1. Compute the distance in world-coordinate system.
2. Reconstruct the superquadric surface in the image-coordinate system by sampling the superquadric surface and then perform point by point comparison in  $z$  direction to compute the difference map.

The first method needs the occluding contour of the superquadric to determine if a point has distance from the superquadric surface along the given direction. The second method samples the superquadric surface and transforms the superquadric into image-coordinate system, where the  $z$ -distance map is obtained by simply subtracting the depth values at each image point. Given the projected superquadric, the occluding contour of the superquadric can also be traced by the same

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<sup>4</sup>Since the range images are stored in Z-depth format, We are effectively “Looking from above” along Z.

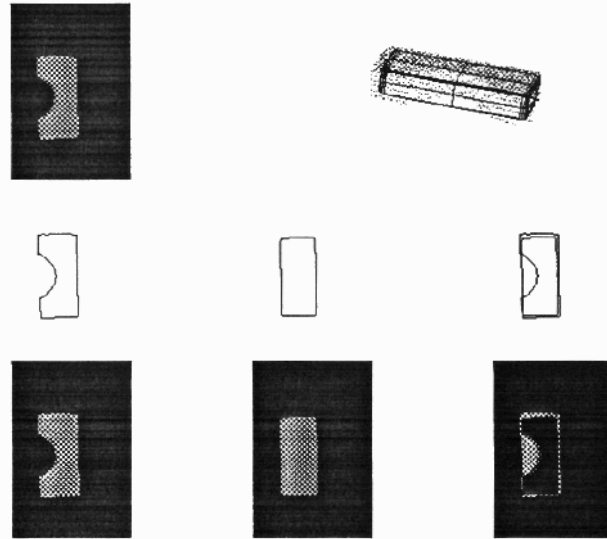


Figure 6: **Box with a circular cutout (an arch)** : Though the volumetric model gives acceptable fit in terms of quantitative measures, it does not account for the cutout.

method as image–contour tracing. We have used the second method to generate  $z$ –difference and contour–difference maps.

#### 4 Quantitative v/s Qualitative measures

We now present examples of real objects shown in figure 2, of varying complexity, which highlight the need for different measures for the complete evaluation of the fit of the superquadric models. For all the objects (figures 6–9) the superquadric model and occluding contour, projected superquadric model and its occluding contour, the contour–difference map, and the  $z$ –depth–difference map are computed.

While the volumetric model gives a holistic explanation of the whole object it can miss details that are beyond the scope of the model. An overall measure of goodness of fit, like the residual from Goodness-of-fit fit, or the distance measure does not always give an accurate evaluation of the appropriateness of the volumetric model. Although models can have acceptable overall goodness–of–fit, like the volumetric model for the box with cut-out (figure 6 and table 1), they need not be the acceptable representations of the objects. On the other hand, for value of the goodness–of–fit



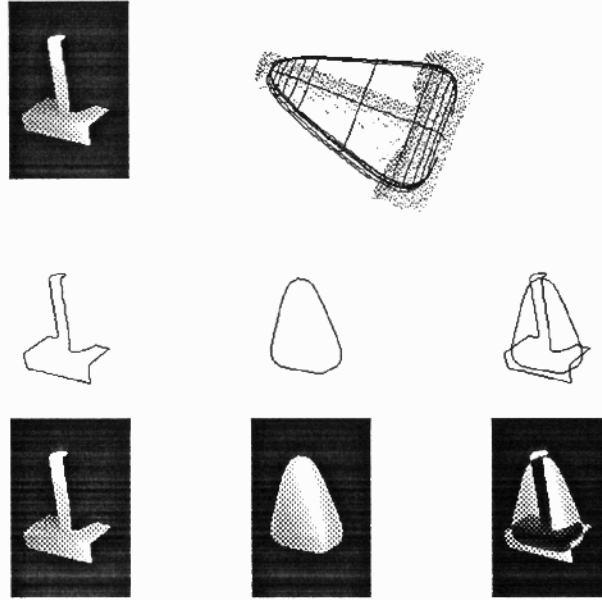


Figure 7: **A composite object** : The two boundaries coincide in only part of the image alerting to the fact that the object has parts.

in same range, volumetric model for the vase (figure 8) is an acceptable volumetric representation of the actual object. In this particular example, as we have seen, it is possible to further account for the irregularity of the surface.

The qualitative measure obtained by comparing the local boundary of the object in the range image with the boundary of the recovered volumetric model can point out the limitations of the volumetric model and suggest improvements in segmentation or refinement in shape representation. When boundaries do not coincide, preference should be given to the actual boundary in the range image, but the possibility of missing data (due to self occlusion) must also be considered. For the arch example (figure 6), contour and z-depth maps provide ample evidence for an unacceptable volumetric model. The z-distance map provides enough information to model the missing part of the box as negative volume by fitting a superquadric model for it.

The vase in figure 8 is formed by three second-order surface patches, collectively organized in a cylindrical shape. At the volumetric level, a cylindrical model is sufficient to describe the overall shape.

Contour analysis signals the presence of details on the object and accepts the superquadric

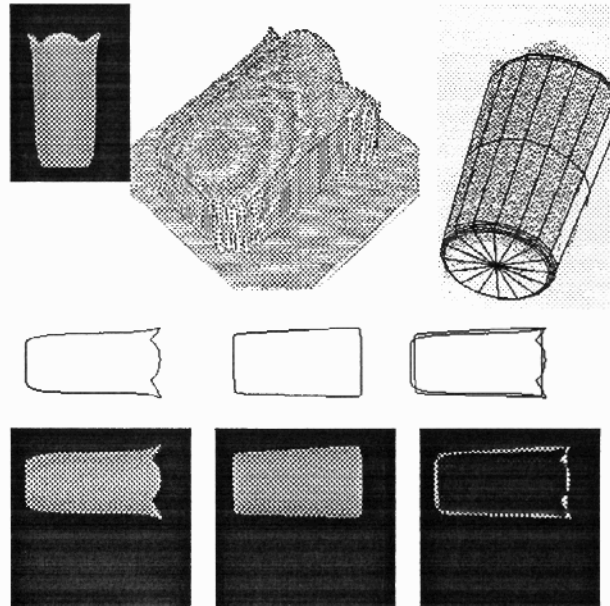


Figure 8: Object with surface detail (A vase) : The difference between the two outlines is negligible compared to the overall size of the object.

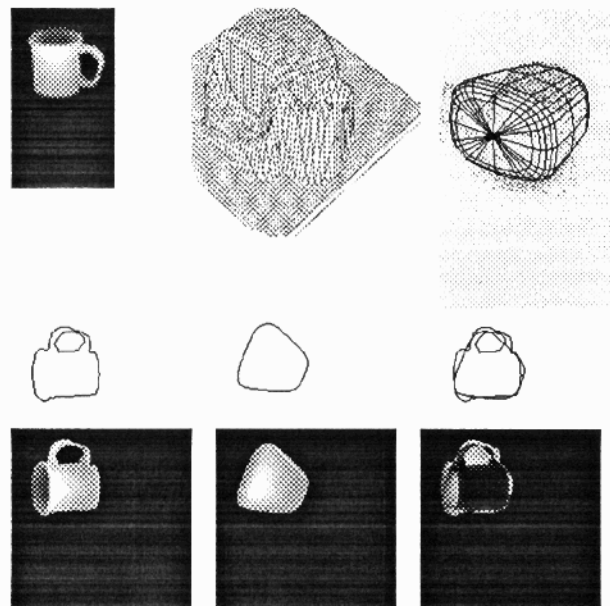


Figure 9: Object with hole and cavity : Both quantitative and qualitative measures indicate unacceptable model. Z-depth-difference and contour-difference help in segmentation of the object.

model. However, the superquadric model is accepted only after the surface comparison yields acceptable error. Thus, both qualitative measures are essential for model evaluation. The min-distance map may be incorporated at this point to account for these irregularities in the surface and guide both the surface-fitting modules as well as give directions for both further deformations or object segmentation.

For objects with parts (figure 7 and 9), both quantitative and qualitative measures will indicate unacceptable superquadric model. Based on difference measures and other considerations [Gup89], part segmentation needs to be performed in order to describe individual parts in terms of superquadrics. Details of how segmentation will be carried out based on these measures are beyond the scope of this paper. In conclusion, we have shown that by considering the various qualitative and quantitative measures it is possible to evaluate the superquadric models recovered from the range data.

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