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Quantitative central limit theorems for the parabolic Anderson model driven by colored noises*

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Abstract

In this paper, we study the spatial averages of the solution to the parabolic Anderson model driven by a space-time Gaussian homogeneous noise that is colored in both time and space. We establish quantitative central limit theorems (CLT) of this spatial statistics under some mild assumptions, by using the Malliavin-Stein approach. The highlight of this paper is the obtention of rate of convergence in the colored-in-time setting, where one can not use Ito calculus due to the lack of martingale structure. In particular, modulo highly technical computations, we apply a modified version of second-order Gaussian Poincaré inequality to overcome this lack of martingale structure and our work improves the results by Nualart-Zheng (Electron. J. Probab. 2020) and Nualart-Song-Zheng (ALEA, Lat. Am. J. Probab. Math. Stat. 2021).

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1 Introduction

In the recent years, the study of spatial averages of the solution to certain stochastic partial differential equations (SPDEs) has received growing attention. The paper [19], being the first of its kind, investigated the nonlinear stochastic heat equation on $\mathbb{R}_+ \times \mathbb{R}$ driven by a space-time white noise $\dot{W}$ and established central limit theorems for the spatial averages of the solution. Consider the stochastic heat equation on $\mathbb{R}_+ \times \mathbb{R}^d$ driven by a Gaussian noise $\dot{W}$:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + \sigma(u)\dot{W} \\ u(0,\bullet) = 1, \end{cases} \tag{1.1}$$

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where the nonlinearity is encoded into a deterministic Lipschitz continuous function $\sigma : \mathbb{R} \to \mathbb{R}$. In Duhamel formulation (mild formulation), the equation (1.1) is equivalent to

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy), \tag{1.2}$$

where the stochastic integral against $W(ds,dy)$ is an extension of the Ito integral, and $p_t(x) = (2\pi t)^{-1/2}e^{-|x|^2/(2t)}$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ denotes the heat kernel. Suppose also $\sigma(1) \neq 0$, which excludes the trivial case $u(t,x) \equiv 1$. One of the main results in [19] can be roughly stated as follows. Let

$$F_R(t) = \int_{-R}^{R} (u(t,x) - 1)dx$$

and let $d_{\rm TV}(X,Y)$ denote the total variation distance between two real random variables X and Y (see (2.11)). Then, it holds that for any $t > 0$, there is some constant C_t that does not depend on R, such that the following quantitative central limit theorem (CLT) holds:

$$d_{\rm TV}\big(F_R(t)/\sigma_R(t), Z\big) \leq C_t R^{-1/2}, \tag{1.3}$$

where $Z \sim \mathcal{N}(0,1)$ is a standard normal random variable and $\sigma_R(t) = \sqrt{\mathrm{Var}(F_R(t))} > 0$ for each $t, R \in (0,\infty)$. The key ideas for obtaining (1.3) are summarized as follows:

(i) By the mild formulation (1.2) and applying stochastic Fubini's theorem, one can write

$$F_R(t) = \int_{[0,t]\times\mathbb{R}} \Big(\int_{-R}^{R} p_{t-s}(x-y)dx\Big)\sigma(u(s,y))W(ds,dy) =: \delta(V_{t,R}),$$

where δ denotes the Skorohod integral (the adjoint of the Malliavin derivative operator; see Section 2.1) and $V_{t,R}$ is the random kernel given by

$$V_{t,R}(s,y) = \sigma(u(s,y)) \int_{-R}^{R} p_{t-s}(x-y)dx.$$

(ii) Via standard computations, one can obtain $\sigma_R(t) \sim \text{constant} \times R^{1/2}$ as $R \uparrow \infty$.

(iii) The Malliavin-Stein bound (c.f. [19, Proposition 2.2]), being the most crucial ingredient, indicates that

$$d_{\rm TV}\big(\sigma_R(t)^{-1}F_R(t), Z\big) \leq \frac{2}{\sigma_R(t)^2}\sqrt{\mathrm{Var}\big(\langle DF_R(t), V_{t,R}\rangle_{L^2(\mathbb{R}_+\times\mathbb{R})}\big)}, \tag{1.4}$$

where $DF_R(t)$ denotes the Malliavin derivative of $F_R(t)$, which is a random function and belongs to the space $L^2(\mathbb{R}_+ \times \mathbb{R})$ under the setting of [19]. Then the obtention of (1.3) follows from a careful analysis of the inner product $\langle DF_R(t), V_{t,R}\rangle_{L^2(\mathbb{R}_+\times\mathbb{R})}$.

We remark here that for a general nonlinearity σ, the computations mentioned in points (ii) and (iii) are made possible through applications of the *Clark-Ocone formula* and *Burkholder-Davis-Gundy inequality*, which are valid only in the white-in-time setting. The noise $\dot{W}$ that is white in time, naturally gives arise to a martingale structure so that Ito calculus techniques come into the picture and enable the careful analysis of the variance term in (1.4).

The above general strategy has also been exploited in several other papers, see [5, 6, 7, 20, 22, 24] for results on stochastic heat equations and see [4, 9, 33] for results on stochastic wave equations, to name a few. The common feature of these papers is that they consider the case where the driving Gaussian noise is white in time so that the aforementioned strategy of [19] is working very well. To the best of our knowledge, the colored-in-time setting has only been considered in [30, 31] for heat equations and in [1] for wave equations.

In the present paper we are interested in the following parabolic Anderson model (that means $\sigma(u) = u$) on $\mathbb{R}_+ \times \mathbb{R}^d$ driven by a Gaussian noise $\dot{W}$, which is colored in both time and space, with the flat initial condition:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + u \diamond \dot{W} \\ u(0, \bullet) = 1, \end{cases} \tag{1.5}$$

where $\diamond$ denotes the Wick product (c.f. [12, Section 6.6]). Here we allow the noise to be colored in time. This requires us to take $\sigma(u) = u$, because the solution theory for nonlinear stochastic heat equations driven by colored-in-time noise is not available.

Let us now introduce some notation to better facilitate the discussion as well as to state our main results. Fix a positive integer d. Heuristically, $\dot{W} = \{\dot{W}(t,x), (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ will be a centered Gaussian family of random variables with homogeneous covariance structure given by

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \gamma_0(t-s)\gamma_1(x-y),$$

where γ_0, γ_1 are (generalized) functions that satisfy one of the following two conditions:

Hypothesis 1. $(d \geq 1)$ $\gamma_0 : \mathbb{R} \to [0, \infty]$ is a nonnegative-definite locally integrable function and $\gamma_1 \geq 0$ is the Fourier transform of some nonnegative tempered measure μ on $\mathbb{R}^d$ (called the spectral measure), satisfying Dalang's condition (see [8]),

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < +\infty. \tag{1.6}$$

Hypothesis 2. $(d = 1)$ There are $H_0 \in [\frac{1}{2}, 1)$ and $H_1 \in (0, \frac{1}{2})$ with $H_0 + H_1 > \frac{3}{4}$, such that

$$\gamma_0(t) = \begin{cases} \delta(t), & H_0 = \frac{1}{2}, \\ |t|^{2H_0 - 2}, & \frac{1}{2} < H_0 < 1, \end{cases}$$

where δ is the Dirac delta function at 0 and γ_1 is the Fourier transform of $\mu(d\xi) = c_{H_1}|\xi|^{1-2H_1}d\xi$ with $c_{H_1} = \pi^{-1}\int_{\mathbb{R}}(1 - \cos x)|x|^{2H_1 - 2}dx$; see (2.8) for the choice of c_{H_1}.

In order to define rigorously the noise, we need some definitions. Let $C_c^\infty(\mathbb{R}_+)$ and $C_c^\infty(\mathbb{R}^d)$ denote the set of real smooth functions with compact support on $\mathbb{R}_+$ and $\mathbb{R}^d$, respectively. Then, we define Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ to be the completion of $C_c^\infty(\mathbb{R}_+)$ and $C_c^\infty(\mathbb{R}^d)$ with respect to the inner products

$$\langle \phi_0, \psi_0 \rangle_{\mathcal{H}_0} = \int_{\mathbb{R}_+^2} dsdt \gamma_0(t-s)\phi_0(t)\psi_0(s)$$

and

$$\begin{aligned} \langle \phi_1, \psi_1 \rangle_{\mathcal{H}_1} &= \int_{\mathbb{R}^{2d}} dxdy \gamma_1(x-y)\phi_1(x)\psi_1(y) \\ &= \int_{\mathbb{R}^d} \mu(d\xi)\widehat{\phi}_1(\xi)\widehat{\psi}_1(-\xi), \end{aligned}$$

respectively, where $\widehat{\phi}_1(\xi) = \int_{\mathbb{R}^d} dx e^{-i\xi x}\phi_1(x)$ stands for the Fourier transform of ϕ_1.

Set $\mathfrak{H} = \mathcal{H}_0 \otimes \mathcal{H}_1$, equipped with the inner product

$$\langle \phi, \psi \rangle_{\mathfrak{H}} = \int_{\mathbb{R}_+^2} dsdt\, \gamma_0(t-s) \int_{\mathbb{R}^{2d}} dxdy \gamma_1(x-y)\phi(t,x)\psi(s,y), \tag{1.7}$$

which can be also written using the Fourier transform as

$$\langle \phi, \psi \rangle_{\mathfrak{H}} = \int_{\mathbb{R}_+^2} dsdt\, \gamma_0(t-s) \int_{\mathbb{R}^d} \mu(d\xi)\widehat{\phi}(t,\xi)\widehat{\psi}(s,-\xi), \tag{1.8}$$

where $\widehat{\phi}(t,\xi)$ and $\widehat{\psi}(t,\xi)$ denote the Fourier transform in the spatial variable.

We also introduce the following hypotheses that will be used to state our main results.

Hypothesis 3a. $\int_0^a \int_0^a \gamma_0(r-v)drdv > 0$ for all $a > 0$ and $0 < \|\gamma_1\|_{L^1(\mathbb{R}^d)} < \infty$.

Hypothesis 3b. $\int_0^a \int_0^a \gamma_0(r-v)drdv > 0$ for all $a > 0$ and $\gamma_1(z) = |z|^{-\beta}$, $z \in \mathbb{R}^d$ for some $\beta \in (0, 2 \wedge d)$.

We remark here that the restriction for β in Hypothesis 3b ensures that Dalang's condition (1.6) is satisfied. Also, we will call

the case under Hypotheses 1 and 3a, and the case under Hypotheses 1 and 3b

the **regular cases**, since the spatial correlation γ_1 is a real-valued function as opposed to the setting under Hypothesis 2. Meanwhile, we call the case under Hypothesis 2 the **rough case**, because the spatial correlation corresponds to fractional Brownian motion with Hurst index $H_1 \in (0, 1/2)$ (thus rougher than the standard Brownian motion).

With these preliminaries, we consider a centered Gaussian family of random variables $W = \{W(h), h \in \mathfrak{H}\}$ with covariance structure

$$\mathbb{E}[W(\phi)W(\psi)] = \langle \phi, \psi \rangle_{\mathfrak{H}}$$

for all $\phi, \psi \in \mathfrak{H}$. The family W is called an isonormal Gaussian process over $\mathfrak{H}$. Heuristically, the noise

$$\dot{W}(t, x_1, \dots, x_d) = \frac{\partial^{d+1} W(t,x)}{\partial t \partial x_1 \cdots \partial x_d}$$

is the (formal) derivative of W in time and space and the mild formulation of equation (1.5) is given by

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y)\sigma(u(s,y))W(ds,dy), \tag{1.9}$$

where the stochastic integral against $W(ds,dy)$ is a Skorohod integral; see e.g. [29, Section 1.3.2]. It has been proved that under either Hypothesis 1 or 2, the parabolic Anderson model (1.5) admits a unique mild solution; see [13, 15, 18, 25, 34].

Due to the temporal correlation in time of the driving noise, we do not have the playground to apply martingale techniques for obtaining central limit theorem for the spatial statistics

$$F_R(t) = \int_{B_R} \big(u(t,x) - 1\big)dx, \tag{1.10}$$

where $B_R = \{x \in \mathbb{R}^d, |x| \le R\}$. Fortunately, because of the explicit chaos expansion (see (2.5) and (2.6)), one can express $F_R(t)/\sigma_R(t)$, with $\sigma_R^2(t) = \mathrm{Var}\big(F_R(t)\big)$, as a series of multiple stochastic integrals. This series falls into the framework of applying the so-called chaotic CLT. The chaotic CLT roughly means that once we have some control of the tail in the series, it would be enough to show the convergence of each chaos, which can be further proved by using the fourth moment theorems; see [27, Section 6.3] for more details. In fact, in the papers [30, 31], the authors investigated the Gaussian fluctuations of $F_R(t)$ along this idea and proved the following results.

Theorem 1.1. *Assume Hypothesis 1 or 2, and let u be the solution to the parabolic Anderson model* (1.5)*. Recall the definition of $F_R(t)$ from* (1.10) *and let $\sigma_R(t)=\sqrt{\mathrm{Var}(F_R(t))}$. Then, the following results hold.*

(1) *Assume Hypotheses 1 and 3a. Then, for any fixed $t\in(0,\infty)$, as $R\uparrow\infty$,*

$$\sigma_R(t)\sim R^{d/2} \text{ and } \frac{F_R(t)}{\sigma_R(t)} \text{ converges in law to } \mathcal{N}(0,1),$$

where

$$a_R\sim b_R \quad \text{means} \quad 0<\liminf_{R\uparrow\infty} a_R/b_R \le \limsup_{R\uparrow\infty} a_R/b_R < +\infty.$$

See [31, Theorem 1.6].

(2) *Assume Hypotheses 1 and 3b. Then, for any fixed $t\in(0,\infty)$, as $R\uparrow\infty$,*

$$\sigma_R(t)\sim R^{d-\frac{\beta}{2}} \text{ and } \frac{F_R(t)}{\sigma_R(t)} \text{ converges in law to } \mathcal{N}(0,1).$$

See [31, Theorem 1.7].

(3) *Under the hypothesis 2, it holds for any fixed $t\in(0,\infty)$ that as $R\uparrow\infty$,*

$$\sigma_R(t)\sim R^{1/2} \text{ and } \frac{F_R(t)}{\sigma_R(t)} \text{ converges in law to } \mathcal{N}(0,1).$$

See [30, Theorem 1.1 and Proposition 1.2].

Remark 1.2. More precisely, additionally to Theorem 1.1-(3), we know from [30, (1.5)] that

$$\lim_{R\uparrow\infty} R^{-1}\mathrm{Var}(F_R(t)) = 2\int_{\mathbb{R}} dz\,\mathbb{E}\big[\mathfrak{g}(\mathcal{I}_z)\big], \tag{1.11}$$

where $\mathfrak{g}(z)=e^z-z-1$ is strictly positive except for $z=0$. Then the above limit vanishes if and only if almost surely $\mathcal{I}_z=0$ for almost every $z\in\mathbb{R}$. Taking into account the explicit expression of $\mathcal{I}_z=\mathcal{I}_{t,t}^{1,2}(z)$ and equation (1.8) in [30, Proposition 1.3], we can conclude that the limit in (1.11) is strictly positive and thus $\sigma_R(t)\sim R^{1/2}$. Indeed, by (1.8) therein and $L^2(\Omega)$-continuity, $\mathbb{E}[u(t,x)u(t,0)]=\mathbb{E}\big[\exp(\mathcal{I}_x)\big]>1$ for x near 0 so that with positive probability $\mathcal{I}_x\neq 0$ for x near zero.

Note that the above CLT results are of qualitative nature and there are also functional version of these results, where the limiting objects are centered Gaussian process with explicit covariance structures. Both CLT in cases (1) and (3) are chaotic, meaning that each chaos[1] contributes to the Gaussian limit, while CLT in case (2) is not chaotic. More precisely, in case (2) the first chaotic component, which is Gaussian, dominates the asymptotic behavior as $R\uparrow\infty$; see the above references for more details. Here we point out that the application of the chaotic CLT does not yield the rate of convergence, that is, the error bound like (1.3) is not accessible through this method. Our paper is devoted to deriving quantitative versions of the above CLT results as stated in the following theorem.

Theorem 1.3. *Let the assumptions in Theorem 1.1 hold and recall cases* (1)–(3) *therein. Then, we have for any fixed $t,R\in(0,\infty)$,*

$$d_{\mathrm{TV}}\big(F_R(t)/\sigma_R(t),Z\big)\le C_t\times\begin{cases} R^{-d/2} & \text{in case (1)}\\ R^{-\beta/2} & \text{in case (2)}\\ R^{-1/2} & \text{in case (3)},\end{cases} \tag{1.12}$$

[1] To be more precise, in case (3), the first chaotic component has negligible contribution in the limit while each of the other chaotic components has a Gaussian limit; see [30, Page 910].

where the constant $C_t > 0$ is independent of R and $Z \sim \mathcal{N}(0,1)$ denotes a standard Gaussian random variable.

Remark 1.4. (i) Note that the convergence rate for cases (1) and (3) in Theorem 1.3 can be written as $O(R^{-\frac{d}{2}})$, yet for case (2) the rate is of a different form. As in the classical CLT for partial sums of i.i.d. random variables, the rate of convergence is roughly of the order of the reciprocal of limiting standard deviation. This explains the rate $O(R^{-\frac{d}{2}})$ in case (1) and (3): More precisely,

- (i-a) in case (1), each chaotic component of $F_R(t)/\sigma_R(t)$ contributes to the Gaussian limit with limiting variance of order R^d.
- (i-b) in case (3), that is when the spatial correlation is given by that of a fractional Brownian noise with Hurst index $H_1 < \frac{1}{2}$ (see Hypothesis 2), it has been pointed out in [30, Page 910] that due to the fact that the associated spectral density vanishes at zero, the first chaotic component of $F_R(t)$ is asymptotically negligible compared to other chaoses (each of them has limiting variance of order R), then the rate $O(R^{-1/2})$ is consistent with the CLT heuristic.

In case (2), the spatial correlation kernel $\gamma_1(z) = |z|^{-\beta}$ with $\beta \in (0, 2 \wedge d)$, see Hypothesis 3b. Equivalently, we can consider instead $\gamma_1(z) = |z|^{\beta - d}$ for $d - 2 < \beta < d$, then the limiting variance of $F_R(t)$ will be of order $R^{d+\beta}$, and the resulting rate of convergence is $O(R^{-(d-\beta)/2})$, so that we see the dependency on the dimension d now. Note that in this case (2), the first chaotic component of $F_R(t)$ is dominant with limiting variance of order $R^{2d-\beta}$. Thus it is not surprising at all to see the rate in case (2) to be in a form other than $O(R^{-d/2})$; see [30, Page 910] for the particular situation $(d, \beta) = (1, 2 - 2H_1)$ with $1/2 < H_1 < 1$, and also [31, Sections 3.4 and 3.5] for more details.
(ii) In our settings, the temporal correlation function γ_0 is locally integrable and nontrivial. Then, on a fixed finite time interval, its L^1 norm is of the same order as in the case of white-in-time case where the temporal correlation function is the Dirac delta function at zero. This will essentially lead to the fact that the temporal correlation structure in our settings does not play a role in the order of the limiting variance of $F_R(t)$. See Lemma 2.1 and the embedding inequality (2.9) for the technical reasons of the matching upper bounds of limiting variances in both white, and colored-in time settings. For the matching lower bounds limiting variances in both settings, it is enough to understand it from bounds like e.g. [31, (3.1)]. Of course, the orders of lower and upper bounds are the same. Even if in our study of the spatial averages, the temporal correlations do not explicitly appear in the convergence rate, it is still necessary to "compensate" the roughness in space ($H_0 + H_1 > \frac{3}{4}$), which ensures the existence and uniqueness of the solution to parabolic Anderson model (1.5). A recent work [25] relaxes the restriction to $2H_0 + H_1 > \frac{5}{4}$. It will be interesting to extend our quantitative CLT to the rough case with $2H_0 + H_1 > \frac{5}{4}$.

In a recent paper [1], the authors face the problem of establishing a quantitative CLT for the hyperbolic Anderson model driven by a colored noise. A basic ingredient in this paper is the so-called second-order Gaussian Poincaré inequality (see Proposition 2.3). With this inequality in mind, it is not difficult to see that, in order to obtain the desired rate of convergence, we need to equip ourselves with fine L^p-bounds of the Malliavin derivatives valued at (almost) *every* space-time points. This will be our approach in the **regular case**, and the majority of the effort will be allocated to show these L^p-bounds, with which we will apply Proposition 2.3 to get the quantitative CLT.

However, in the **rough case** the spatial correlation γ_1 is a generalized function and if one understands the inner product $\langle \bullet, \bullet \rangle_{\mathfrak{H}}$ using the Fourier transform (1.8), Proposition 2.3 does not fit. This is a highly nontrivial difficulty the we overcome by taking advantage of an equivalent formulation of the inner product $\langle \bullet, \bullet \rangle_{\mathfrak{H}}$ based on fractional calculus (see Section 2.2). Starting from such an equivalent expression, we derive in the **rough case** another version of the second-order Gaussian Poincaré inequality (see Proposition 2.4) that is better adapted to our purpose. We also refer the readers to Remark 2.5 for a detailed discussion.

Let us complete this section with a few more remarks on (quantitative) CLT in other settings.

(i) The authors of [30] study the situation where the noise W is colored in space-time $\mathbb{R}_+ \times \mathbb{R}$ with the spectral density φ satisfying a modified Dalang's condition and the concavity condition:

$$\int_{\mathbb{R}} \frac{\varphi(x)^2}{1+x^2} dx < +\infty \quad \text{and} \quad \exists \kappa \in (0,\infty) \text{ such that } \varphi(x+y) \le \kappa(\varphi(x)+\varphi(y))$$

for all $x, y \in \mathbb{R}$. Using the chaotic CLT, they are able to establish the CLT results of qualitative nature. It is not clear to us how to derive the moment estimates for Malliavin derivatives in this setting. A more intrinsic problem is that unlike in our **rough case**, we are not aware of any equivalent real-type expression for the inner product of the underlying Hilbert space $\mathfrak{H}$ and thus we do not see how to put the potential moment estimates in use.

(ii) In a recent paper [32], the spatial ergodicity for certain nonlinear stochastic wave equations with spatial dimension not bigger than 3 is established under some mild assumptions. The condition on the driving noise W can be roughly summarized as follows: W is white in time, the spatial correlation satisfies Dalang's condition and the spectral measure has no atom at zero. The authors of [4, 9, 33] established the corresponding (quantitative) CLTs in dimensions 1 or 2. The obtention of CLT in dimension three has been open for a while until Ebina solved it in [11] by first refining the arguments in [32]. The key difficulty in 3D is the lack of a precise estimate of the Malliavin derivative $Du(t,x)$ of the solution, which *shall* be a random measure supported on some sphere; by going through an intermediate step of using Picard iterations, the CLT result in [11] is essentially of qualitative nature. So the quantitative CLT for the stochastic wave equation in dimension 3 is still open.

(iii) Some recent results related to the spatial average of parabolic and hyperbolic Anderson model with time-independent noises can be found in e.g. [2, 3].

The rest of the paper is organized as follows: Section 2 contains some preliminaries that will be used in this paper. We study the **regular cases** in Section 3 and leave the **rough case** to Section 4.

2 Preliminaries and technical lemmas

In this section, we provide some preliminaries and useful lemmas. Let us first introduce some notation that will be used frequently in this paper. Let n be a positive integer, we make use of notation $\mathbf{x}_n = (x_1, \dots, x_n) \in \mathbb{R}^{nd}$ and $\mathbf{t}_n = (t_1, \dots, t_n) \in \mathbb{R}_+^n$. Given $\mathbf{x}_n \in \mathbb{R}^{nd}$, we write $\mathbf{x}_{k:n} \in \mathbb{R}^{n-k+1}$ short for $(x_k, \dots, x_n)$ with $k = 1, \dots, n$, and $\mathbf{t}_{k:n} \in \mathbb{R}_+^{n-k+1}$ is defined in the same way. Let $0 \le s < t < \infty$. We put $\mathbb{T}_n^{s,t} = \{\mathbf{s}_n \in \mathbb{R}_+^n : s < s_1 < \cdots < s_n < t\}$ and $\mathbb{T}_n^t = \mathbb{T}_n^{0,t}$. We use c, c_1, c_2, and c_3 for some positive constants which may vary from line to line. Finally, we write $A_1 \lesssim A_2$ if there exists a constant c such that $A_1 \le cA_2$.

2.1 Wiener chaos and parabolic Anderson model

Let $\mathfrak{H}$ be a Hilbert space of (generalized) functions on $\mathbb{R}_+\times\mathbb{R}$, and let $W=\{W(h),h\in\mathfrak{H}\}$ be an isonormal Gaussian process over $\mathfrak{H}$. Denote by $\mathcal{F}=\sigma(W)$ the smallest σ-algebra generated by W. Then, any $\mathcal{F}$-measurable and square integrable random variable F can be unique expanded into a series of multiple Ito-Wiener integrals (see [29, Theorem 1.1.2]),

$$F=\mathbb{E}(F)+\sum_{n=1}^{\infty} I_n(f_n), \tag{2.1}$$

where for $n=1,2,\dots$, $I_n(f_n)$ is the multiple Ito-Wiener integral of f_n, which is a symmetric function on $(\mathbb{R}_+\times\mathbb{R}^d)^n$, meaning

$$\begin{aligned} f_n\in\mathfrak{H}^{\odot n}=\Big\{f\in\mathfrak{H}^{\otimes n}, f(\mathbf{t}_n,\mathbf{x}_n)=f_n(t_{\sigma(1)},\dots,t_{\sigma(n)},x_{\sigma(1)},\dots,x_{\sigma(n)}),\\ \text{for all permutation } \sigma \text{ on } \{1,\dots,n\}\Big\}. \end{aligned} \tag{2.2}$$

In the space-time white noise case, $I_n(f_n)$ can be understood as the n-folder iterate Ito-Walsh's integral (see [36]),

$$I_n(f_n)=n!\int_{0<t_1<\cdots<t_n<\infty}\int_{\mathbb{R}^n} f_n(\mathbf{t}_n,\mathbf{x}_n)W(dt_1,dx_1)\cdots W(dt_n,dx_n).$$

For any n, we denote by $\mathbf{H}_n$ the n-th Wiener chaos of W, that is the collection of random variables of the form $F=I_n(f_n)$, with $f_n\in\mathfrak{H}^{\odot n}$. In any fixed Wiener chaos $\mathbf{H}_n$, the following inequality of hypercontractivity (see [27, Corollary 2.8.14]) holds

$$\|F\|_p\le (p-1)^{n/2}\|F\|_2, \tag{2.3}$$

for $p\ge 2$ and for all $n=1,2,\dots$ and $F\in\mathbf{H}_n$.

Assume Hypothesis 1. Set $\mathfrak{X}=\mathfrak{H}$ when $\gamma_0=\delta$. Then, $\mathfrak{X}=L^2(\mathbb{R}_+;\mathcal{H}_1)$, and we have the following lemmas that are very helpful in Section 3.2.

Lemma 2.1 ([1, Inequality (2.13)]). *For any nonnegative function $f\in\mathfrak{X}^{\otimes n}$ supported in $([0,t]\times\mathbb{R}^d)^n$, we have*

$$\|f\|^2_{\mathfrak{H}^{\otimes n}}\le\Gamma_t^n\|f\|^2_{\mathfrak{X}^{\otimes n}},$$

where $\Gamma_t:=\int_{[-t,t]}\gamma_0(s)ds$.

Lemma 2.2 ([29, Proposition 1.1.2] and [1, Section 2]). *Let $m,n\ge 1$ be integers and let f and g in $\mathfrak{X}^{\odot n}$ and $\mathfrak{X}^{\odot m}$ respectively. Then,*

$$I_n^{\mathfrak{X}}(f)I_m^{\mathfrak{X}}(g)=\sum_{r=0}^{n\wedge m} r!\binom{m}{r}\binom{n}{r}I^{\mathfrak{X}}_{n+m-2r}(f\otimes_r g),$$

where $I_n^{\mathfrak{X}}$ denotes the the multiple Ito-Wiener integral with respect to an isonormal Gaussian process over $\mathfrak{X}$ and $f\otimes_r g$ denotes the r-th contraction between f and g, namely, an element in $\mathfrak{X}^{n+m-2r}$ defined by

$$\big(f\otimes_r g\big)(\mathbf{t}_{n-r},\mathbf{x}_{n-r},\mathbf{s}_{m-r},\mathbf{y}_{m-r})=\big\langle f\big(\mathbf{t}_{n-r},\mathbf{x}_{n-r},\bullet\big),g\big(\mathbf{s}_{m-r},\mathbf{y}_{m-r},\bullet\big)\big\rangle_{L^2(\mathbb{R}_+^r;\mathcal{H}_1^{\otimes r})}.$$

In particular, if in addition f,g have disjoint temporal support[2]*, then*

$$I_n^{\mathfrak{X}}(f)I_m^{\mathfrak{X}}(g)=I^{\mathfrak{X}}_{n+m}(f\otimes g) \tag{2.4}$$

and $I_n^{\mathfrak{X}}(f)$, $I_m^{\mathfrak{X}}(g)$ are independent.

[2]This means $f=0$ outside $(J\times\mathbb{R}^d)^n$ and $g=0$ outside $(J^c\times\mathbb{R}^d)^m$ for some $J\subset\mathbb{R}_+$. Note that for f,g non-symmetric having disjoint temporal support, the equality (2.4) still holds true.

In the rest of this subsection, we provide the definition for the solution to the parabolic Anderson model (1.5). Let $u = \{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ be a $\sigma(W)$-measurable random field such that $\mathbb{E}(u(t,x)^2) < \infty$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Then, due to (2.1), we can write

$$u(t,x) = \mathbb{E}(u(t,x)) + \sum_{n=1}^{\infty} I_n(f_n(\bullet,t,x)) = \mathbb{E}(u(t,x)) + \sum_{n=1}^{\infty} I_n(f_{t,x,n}(\bullet)),$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, where $f_n : (\mathbb{R}_+ \times \mathbb{R}^d)^{n+1} \to \mathbb{R}$ and for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $f_{t,x,n} = f_n(\bullet,t,x) \in \mathcal{H}^{\odot n}$ (see (2.2)). Then, u is said to be Skorohod integrable with respect to W, if the following series is convergent in $L^2(\Omega)$,

$$\delta(u) = \int_0^{\infty} \int_{\mathbb{R}^d} u(t,x) W(dt,dx) = \int_0^{\infty} \int_{\mathbb{R}^d} f_0(t,x) W(dt,dx) + \sum_{n=1}^{\infty} I_{n+1}(\widetilde{f}_n),$$

where $\widetilde{f}_n$ denotes the symmetrization of f_n in $(\mathbb{R}_+ \times \mathbb{R}^d)^{n+1}$. Additionally, u is said to be a (mild) solution to the parabolic Anderson model (1.5), if for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, as a random field with parameters (s,y), $p_{t-s}(x-y)u(s,y)\mathbf{1}_{[0,t]}(s)$ is Skorohod integrable and the following equation holds almost surely,

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u(s,y) W(ds,dy).$$

It has been proved (see [16, Section 4.1]) that u is a solution to (1.5), if and only if it has the following Wiener chaos expansion

$$u(t,x) = 1 + \sum_{n=1}^{\infty} I_n(f_{t,x,n}), \tag{2.5}$$

where the integral kernels $f_{t,x,n}$ are given by

$$f_{t,x,n}(\mathbf{s}_n, \mathbf{x}_n) = \frac{1}{n!} p_{t-s_{\sigma(n)}}(x - x_{\sigma(n)}) \cdots p_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)}), \tag{2.6}$$

with σ the permutation of $\{1,\dots,n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$ and $p_t(x)$ being the heat kernel in $\mathbb{R}^d$. The chaos expansion (2.5) and the expression (2.6) will be used frequently in this paper.

2.2 Fractional Sobolev spaces and an embedding theorem

In this subsection, we give a basic introduction to fractional Sobolev spaces. They are closely related to the Hilbert space $\mathcal{H}_1$ under Hypothesis 2. We also provide an embedding theorem for $\mathcal{H}_0$. These will be used in Section 4. For a more detailed account on applications of this topic to SPDEs, we refer the readers to papers [13, 14, 17] and the references therein.

Following the notation in [10], given parameters $s \in (0,1)$ and $p \geq 1$, the fractional Sobolev space $W^{s,p}(\mathbb{R})$ is the completion of $C_c^{\infty}(\mathbb{R})$ with the norm

$$\|\phi\|_{W^{s,p}} = \left(\|\phi\|_{L^p}^p + [\phi]_{W^{s,p}}^p\right)^{\frac{1}{p}},$$

where $[\bullet]_{W^{s,p}}$ denotes the Gagliardo (semi)norm

$$[\phi]_{W^{s,p}} = \Big(\int_{\mathbb{R}^2} dxdy \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{1+sp}}\Big)^{\frac{1}{p}}. \tag{2.7}$$

In particular, if $p = 2$, $W^{s,2}$ turns out to be a Hilbert space. By using the Fourier transformation, one can write (c.f. [10, Proposition 3.4])

$$[\phi]^2_{W^{s,2}} = C(s)\int_{\mathbb{R}} d\xi |\xi|^{2s} |\widehat{\phi}(\xi)|^2, \quad \text{with} \quad C(s) = \pi^{-1}\int_{\mathbb{R}} \frac{1-\cos(\zeta)}{|\zeta|^{1+2s}} d\zeta.$$

Here the constant $C(s)$ is slightly different from that in [10], because we use another version of the Fourier transformation. Therefore, assuming Hypothesis 2, we see immediately that

$$\|\phi\|^2_{\mathcal{H}_1} = c_{H_1}\int_{\mathbb{R}} d\xi |\xi|^{1-2H_1} |\widehat{\phi}(\xi)|^2 = [\phi]^2_{W^{\frac{1}{2}-H_1,2}}. \tag{2.8}$$

In this paper, we will use both representations of the norm in $\mathcal{H}_1$ via the Fourier transformation and the Gagliardo formulation.

In the next part of this subsection, we introduce an embedding property for the Hilbert space $\mathcal{H}_0$. Assume Hypothesis 2 with $H_0 \in (\frac{1}{2}, 1)$, thanks to the Hardy-Littlewood-Sobolev inequality, there exist a continuous embedding $L^{1/H_0}(\mathbb{R}_+) \hookrightarrow \mathcal{H}_0$ (see [26, Theorem 1.1]), namely, there exists a constant c_{H_0} depending only on H_0 such that

$$|\langle f, g\rangle_{\mathcal{H}_0}| \le c_{H_0}\|f\|_{L^{1/H_0}}\|g\|_{L^{1/H_0}} = c_{H_0}\Big(\int_{\mathbb{R}^2_+} dsdt |f(t)g(s)|^{\frac{1}{H_0}}\Big)^{H_0} \tag{2.9}$$

for all $f, g \in \mathcal{H}_0$. Combining this fact and Cauchy-Schwarz's inequality on the Hilbert space $\mathcal{H}_1$, we can show that for all $H_0 \in [\frac{1}{2}, 1)$,

$$\begin{aligned}
\big|\langle \phi, \psi\rangle_{\mathfrak{H}}\big| &\le \int_{\mathbb{R}^2_+} dsdt\, \gamma_0(t-s)\big|\langle \phi(t,\bullet), \psi(s,\bullet)\rangle_{\mathcal{H}_1}\big| \\
&\le \int_{\mathbb{R}^2_+} dsdt\, \gamma_0(t-s)\big\|\phi(t,\bullet)\big\|_{\mathcal{H}_1} \times \big\|\psi(s,\bullet)\big\|_{\mathcal{H}_1} \\
&\le c_{H_0}\|\phi\|_{L^{1/H_0}(\mathbb{R}_+;\mathcal{H}_1)}\|\psi\|_{L^{1/H_0}(\mathbb{R}_+;\mathcal{H}_1)}.
\end{aligned}$$

By iteration, we can write

$$\big|\langle \phi, \psi\rangle_{\mathfrak{H}^{\otimes n}}\big| \le c^n_{H_0}\|\phi\|_{L^{1/H_0}(\mathbb{R}^n_+;\mathcal{H}_1^{\otimes n})}\|\psi\|_{L^{1/H_0}(\mathbb{R}^n_+;\mathcal{H}_1^{\otimes n})} \tag{2.10}$$

for all $\phi, \psi \in \mathfrak{H}^{\otimes n}$ and $n = 1, 2, 3, \dots$.

2.3 Second-order Gaussian Poincaré inequalities

In this subsection, we provide two versions of the second-order Gaussian Poincaré inequality (see [28, Theorem 1.1] for the first version). As stated in Section 1, they will be used in estimating the total variance distance in **regular** and **rough** cases respectively.

Denote by D the Malliavin differential operator (see [29, Section 1.2]). Let $\mathbb{D}^{2,4}$ stand for the set of twice Malliavin differentiable random variables F with

$$\begin{aligned}
\|F\|^4_{2,4} =& \|F\|^4_4 + \big\|\|DF\|_{\mathfrak{H}}\big\|^4_4 + \big\|\|D^2F\|_{\mathfrak{H}\otimes\mathfrak{H}}\big\|^4_4 \\
=& \mathbb{E}[F^4] + \mathbb{E}\big[\|DF\|^4_{\mathfrak{H}}\big] + \mathbb{E}\big[\|D^2F\|^4_{\mathfrak{H}\otimes\mathfrak{H}}\big] < \infty.
\end{aligned}$$

We denote by $\mathbb{D}^{2,4}_*$ the set of random variables $F \in \mathbb{D}^{2,4}$ such that we can find versions of the derivatives DF and D^2F, which are measurable functions on $\mathbb{R}_+ \times \mathbb{R}^d$ and $(\mathbb{R}_+ \times \mathbb{R}^d)^2$, respectively such that $|DF| \in \mathfrak{H}$ and $|D^2F| \in \mathfrak{H}\otimes\mathfrak{H}$ almost surely.

Let F and G be random variables. The total variance distance between F and G is defined by

$$d_{\mathrm{TV}}(F, G) = \sup\{|\mu(A) - \nu(A)|, A \subset \mathbb{R} \text{ is Borel measurable}\}, \tag{2.11}$$

where μ, ν are the probability laws of F and G respectively.

The next proposition cited from [1, Proposition 1.8] will be used in the **regular case**.

Proposition 2.3. *Assume Hypothesis 1. Let $F \in \mathbb{D}_*^{2,4}$ be a random variable with mean zero and standard deviation $\sigma \in (0,\infty)$. Then*

$$d_{\rm TV}\big(F/\sigma, Z\big) \le \frac{4}{\sigma^2}\sqrt{\mathcal{A}},$$

where $Z \sim \mathcal{N}(0,1)$ and

$$\begin{aligned}\mathcal{A} := \int_{\mathbb{R}_+^6\times\mathbb{R}^{6d}} & drdr'dsds'd\theta d\theta' dzdz'dwdw'dydy'\gamma_0(\theta-\theta')\gamma_0(s-s')\gamma_0(r-r') \\ & \times \gamma_1(z-z')\gamma_1(w-w')\gamma_1(y-y')\big\|D_{r,z}D_{\theta,w}F\big\|_4 \\ & \times \big\|D_{s,y}D_{\theta',w'}F\big\|_4\big\|D_{r',z'}F\big\|_4\big\|D_{s',y'}F\big\|_4.\end{aligned}$$

Inspired by [35, Theorem 2.1], we also have the following proposition, which will be used in the **rough case**.

Proposition 2.4. *Assume Hypothesis 2. Let $F \in \mathbb{D}_*^{2,4}$ be a random variable with mean zero and standard deviation $\sigma \in (0,\infty)$. Then,*

$$d_{\rm TV}\big(F/\sigma, Z\big) \le \frac{2\sqrt{3}}{\sigma^2}\sqrt{\mathcal{A}},$$

where $Z \sim \mathcal{N}(0,1)$ and

$$\begin{aligned}\mathcal{A} := \int_{\mathbb{R}_+^6} & dsds'd\theta d\theta' drdr'\gamma_0(s-s')\gamma_0(\theta-\theta')\gamma_0(r-r') \\ & \times \int_{\mathbb{R}^6} dydy'dzdz'dwdw'|y-y'|^{2H_1-2}|z-z'|^{2H_1-2}|w-w'|^{2H_1-2} \\ & \times \|D_{r',z}F - D_{r',z'}F\|_4\|D_{\theta',w}F - D_{\theta',w'}F\|_4 \\ & \times \big\|D_{r,z}D_{s,y}F - D_{r,z}D_{s,y'}F - D_{r,z'}D_{s,y}F + D_{r,z'}D_{s,y'}F\big\|_4 \\ & \times \big\|D_{\theta,w}D_{s',y}F - D_{\theta,w}D_{s',y'}F - D_{\theta,w'}D_{s',y}F + D_{\theta,w'}D_{s',y'}F\big\|_4.\end{aligned} \tag{2.12}$$

Proof. We begin with the Malliavin-Stein bound[3]

$$d_{\rm TV}\big(F/\sigma, Z\big) \le \frac{2}{\sigma^2}\sqrt{{\rm Var}\big(\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\big)}$$

with L^{-1} denoting the pseudo-inverse of the Ornstein-Uhlenbeck operator. Denote by P_t the Ornstein-Uhlenbeck semigroup. Then, following the arguments verbatim in [1, Appendix 2], we have

$$\begin{aligned}{\rm Var}\big(\langle DF, -DL^{-1}F\rangle_{\mathfrak{H}}\big) \le & 2\mathbb{E}\int_0^\infty dte^{-t}\big\langle D^2F\otimes_1 D^2F, P_t(DF)\otimes P_t(DF)\big\rangle_{\mathfrak{H}^{\otimes 2}} \\ & + 2\mathbb{E}\int_0^\infty dte^{-2t}\big\langle P_t(D^2F)\otimes_1 P_t(D^2F), DF\otimes DF\big\rangle_{\mathfrak{H}^{\otimes 2}},\end{aligned} \tag{2.13}$$

where the two terms on the right-hand side can be dealt with in the same manner. In what follows, we only estimate the second term. Put

$$\begin{aligned}g(r,z,\theta,w) :&= P_t(D_{r,z}D_\bullet F)\otimes_1 P_t(D_{\theta,w}D_\bullet F) \\ &= \int_{\mathbb{R}_+^2} dsds'\gamma_0(s-s')\int_{\mathbb{R}^2} dydy'|y-y'|^{2H_1-2} \\ &\qquad \times P_t\big(D_{r,z}D_{s,y}F - D_{r,z}D_{s,y'}F\big)P_t\big(D_{\theta,w}D_{s',y}F - D_{\theta,w}D_{s',y'}F\big),\end{aligned}$$

[3] See [27, Theorem 5.1.3] and also equation (1.26) and footnote 4 in [1], where the latter reference points out that we do not need to assume the existence of density for F.

where we have used the expression (2.7) for the inner product in $\mathfrak{H}_1$. With this notation we can write

$$
\begin{aligned}
&\big\langle P_t(D^2F)\otimes_1 P_t(D^2F), DF\otimes DF\big\rangle_{\mathfrak{H}^{\otimes 2}} = (g\otimes_1 DF)\otimes_1 (DF)\\
&= \int_{\mathbb{R}_+^4} d\theta d\theta' dr dr' \gamma_0(\theta-\theta')\gamma_0(r-r')\\
&\quad\times \int_{\mathbb{R}^4} dz dz' dw dw' |z-z'|^{2H_1-2}|w-w'|^{2H_1-2}(D_{r',z}F - D_{r',z'}F)\\
&\quad\times (D_{\theta',w}F - D_{\theta',w'}F)\big[g(r,z,\theta,w) - g(r,z',\theta,w) - g(r,z,\theta,w') + g(r,z',\theta,w')\big]\\
&= \int_{\mathbb{R}_+^6} ds ds' d\theta d\theta' dr dr' \gamma_0(s-s')\gamma_0(\theta-\theta')\gamma_0(r-r')\\
&\qquad\times \int_{\mathbb{R}^6} dy dy' dz dz' dw dw' |y-y'|^{2H_1-2}|z-z'|^{2H_1-2}|w-w'|^{2H_1-2}\\
&\qquad\times (D_{r',z}F - D_{r',z'}F)(D_{\theta',w}F - D_{\theta',w'}F)\\
&\qquad\times P_t\big(D_{r,z}D_{s,y}F - D_{r,z}D_{s,y'}F - D_{r,z'}D_{s,y}F + D_{r,z'}D_{s,y'}F\big)\\
&\qquad\times P_t\big(D_{\theta,w}D_{s',y}F - D_{\theta,w}D_{s',y'}F - D_{\theta,w'}D_{s',y}F + D_{\theta,w'}D_{s',y'}F\big).
\end{aligned}
$$

Therefore, by using Hölder inequality and the contraction property of P_t on $L^4(\Omega)$, we get

$$
\begin{aligned}
&2\mathbb{E}\int_0^\infty dt e^{-2t}\big\langle P_t(D^2F)\otimes_1 P_t(D^2F), DF\otimes DF\big\rangle_{\mathfrak{H}^{\otimes 2}}\\
&\le \int_{\mathbb{R}_+^6} ds ds' d\theta d\theta' dr dr' \gamma_0(s-s')\gamma_0(\theta-\theta')\gamma_0(r-r')\\
&\qquad\times \int_{\mathbb{R}^6} dy dy' dz dz' dw dw' |y-y'|^{2H_1-2}|z-z'|^{2H_1-2}|w-w'|^{2H_1-2}\\
&\qquad\times \|D_{r',z}F - D_{r',z'}F\|_4\|D_{\theta',w}F - D_{\theta',w'}F\|_4\\
&\qquad\times \big\|D_{r,z}D_{s,y}F - D_{r,z}D_{s,y'}F - D_{r,z'}D_{s,y}F + D_{r,z'}D_{s,y'}F\big\|_4\\
&\qquad\times \big\|D_{\theta,w}D_{s',y}F - D_{\theta,w}D_{s',y'}F - D_{\theta,w'}D_{s',y}F + D_{\theta,w'}D_{s',y'}F\big\|_4.
\end{aligned}
$$

We have the same bound for the first term (2.13) except for the multiplicative constant 2, due to $2\int_0^\infty dte^{-t} = 2$. Hence, the proof of Proposition 2.4 is complete. ☐

Remark 2.5. (i) Compared to the **regular case**, the expression of $\mathcal{A}$ is much more complicated in the **rough case**, where we need not only to control $\|D_{r,z}u(t,x)\|_p$, but we also have to estimate the more notorious differences $\|D_{r',z}u(t,x) - D_{r',z'}u(t,x)\|_p$ and $\|D_{r,z}D_{s,y}u(t,x) - D_{r,z}D_{s,y'}u(t,x) - D_{r,z'}D_{s,y}u(t,x) + D_{r,z'}D_{s,y'}u(t,x)\|_p$ for the proof of part (3) in Theorem 1.3. This is the current paper's highlight in regard of the technicality.

(ii) When γ_0 is the Dirac delta function at zero, the expression of $\mathcal{A}$ reduces to

$$
\begin{aligned}
\mathcal{A} &= \int_{\mathbb{R}_+^3} ds d\theta dr \int_{\mathbb{R}^6} dy dy' dz dz' dw dw' |y-y'|^{2H_1-2}|z-z'|^{2H_1-2}|w-w'|^{2H_1-2}\\
&\quad\times \|D_{r,z}F - D_{r,z'}F\|_4\|D_{\theta,w}F - D_{\theta,w'}F\|_4\\
&\quad\times \big\|D_{r,z}D_{s,y}F - D_{r,z}D_{s,y'}F - D_{r,z'}D_{s,y}F + D_{r,z'}D_{s,y'}F\big\|_4\\
&\quad\times \big\|D_{\theta,w}D_{s,y}F - D_{\theta,w}D_{s,y'}F - D_{\theta,w'}D_{s',y}F + D_{\theta,w'}D_{s,y'}F\big\|_4.
\end{aligned}
$$

This case corresponds to the white-in-time setting where the driving noise W behaves like Brownian motion in time, so as an alternative to using Proposition 2.4, one may adapt the general strategy based on the Clark-Ocone formula (see [19]) to establish the quantitative CLT for $F_R(t)$; however, the roughness in space will anyway force one to use the Gagliardo formulation (see (2.7)) of the inner product on $\mathfrak{H}$ when estimating the variance of $\langle DF_R(t), V_{t,R}\rangle_{\mathfrak{H}}$. This will lead to almost the same level of difficulty as in bounding the expression $\mathcal{A}$, while our computations will be done for a broader range of temporal correlation structures that include the Dirac delta function (white-in-time case).

(iii) One may notice that in either Proposition 2.3 or 2.4, the random variable F needs to be in the space $\mathbb{D}_*^{2,4}$. Observe that Theorem 3.1 in the **regular case** and Proposition 4.1 in the **rough case** provide sharp estimates for the (iterated) Malliavin derivatives and their increments of the solution $u(t,x)$ to (1.5). By using these estimates one can easily show that $\mathbb{E}[\|Du(t,x)\|^2_{\mathfrak{H}}] + \mathbb{E}[\|D^2u(t,x)\|^2_{\mathfrak{H}\otimes\mathfrak{H}}] < \infty$ in all the cases (1)–(3). This implies that $F = F_R(t)/\sigma_R(t) \in \mathbb{D}_*^{2,4}$ for all $R \in (0,\infty)$. Therefore, it is legitimate for us to apply Proposition 2.3 and 2.4 throughout this paper.

2.4 Technical lemmas

In this subsection, we provide some useful results related to the heat kernel and gamma functions. They will be used in Section 4. Let us first introduce a few more notation. Set

$$\Delta_t(x,x') = p_t(x+x') - p_t(x), \tag{2.14}$$
$$R_t(x,x',x'') = p_t(x+x'-x'') - p_t(x+x') - p_t(x-x'') + p_t(x), \tag{2.15}$$

and

$$N_t(x) = t^{\frac{1}{8}-\frac{1}{2}H_0}|x|^{H_0-\frac{1}{4}}\mathbf{1}_{\{|x|\le\sqrt{t}\}} + \mathbf{1}_{\{|x|>\sqrt{t}\}} \tag{2.16}$$

for all $t\in\mathbb{R}_+$ and $x,x',x''\in\mathbb{R}$. The next lemma provides further estimates for Δ_t and R_t. This lemma, as well as operator Λ (see (2.27) below), will be used in Proposition 4.1 combined with the simplified formulas in Lemmas 4.2 and 4.3.

Lemma 2.6. *Let Δ_t, R_t and N_t be given as in* (2.14)–(2.16)*. Then, the following results hold:*

$$\int_{\mathbb{R}} \big[N_t(x)\big]^2|x|^{2H_1-2}dx = 4\frac{1-H_1}{1-2H_1}t^{H_1-\frac{1}{2}}, \tag{2.17}$$
$$|\Delta_t(x,x')| \le c_\beta\big(\Phi^\beta_{t,x'}p_{4t}\big)(x), \tag{2.18}$$

and

$$|R_t(x,x',x'')| \le c_\beta(\Phi^\beta_{t,x'}\Phi^\beta_{t,-x''}p_{4t})(x) \tag{2.19}$$

for any $\beta\in[0,1]$, $t\in\mathbb{R}_+$, $x,x',x''\in\mathbb{R}$ with some constant c_β depending only on β, where $\Phi^\beta_{t,x'}$ is the operator acting on $\mathcal{M}(\mathbb{R})$, the real measurable functions on $\mathbb{R}$, given by

$$(\Phi^\beta_{t,x'}g)(x) = \theta_{x'}g(x)\mathbf{1}_{\{|x'|>\sqrt{t}\}} + g(x)\Big[\mathbf{1}_{\{|x'|>\sqrt{t}\}} + t^{-\frac{\beta}{2}}|x'|^\beta\mathbf{1}_{\{|x'|\le\sqrt{t}\}}\Big],$$

with θ denoting the shift function, that is, $(\theta_{x'}g)(x) = g(x+x')$.

Remark 2.7. For any $t>0$ and $z\in\mathbb{R}$, operator $\Phi^\beta_{t,z}$ can be expressed as

$$\Phi^\beta_{t,z}g = \mathbf{1}_{\{|z|>\sqrt{t}\}}\big(\theta_z + \mathbf{I}\big)g + \mathbf{1}_{\{|z|\le\sqrt{t}\}}\big(|z|t^{-1/2}\big)^\beta\mathbf{I}g, \quad g\in\mathcal{M}(\mathbb{R})$$

with $\mathbf{I}g = g$. It is easy to check that the following commutativity property holds $\Phi^\beta_{t,z}\Phi^\beta_{s,y} = \Phi^\beta_{s,y}\Phi^\beta_{t,z}$ on $\mathcal{M}(\mathbb{R})$. Furthermore, it is also clear that $\big(\Phi_{t,z}\mathbf{1}_{\mathbb{R}}\big)(0) \le 2N_t(z)$.

Proof of Lemma 2.6. Equality (2.17) follows from direct computations, which we omit here. In what follows, we first derive the estimate (2.18) for Δ_t. If $|x'| > \sqrt{t}$, it follows immediately that

$$\big|\Delta_t(x,x')\big| \le p_t(x+x') + p_t(x) \le 2\big[p_{4t}(x+x') + p_{4t}(x)\big]. \tag{2.20}$$

On the other hand, suppose now that $|x'| \le \sqrt{t}$. Notice firstly that

$$|\Delta_t(x,x')| = |p_t(x+x') - p_t(x)| < \max\{p_t(x), p_t(x+x')\} \le \frac{1}{\sqrt{2\pi}} t^{-\frac{1}{2}}.$$

By the mean value theorem, there exists a point z between 0 and x' such that,

$$|\Delta_t(x,x')| = |p_t(x+x') - p_t(x)| = |x'| \frac{|x+z|}{(2\pi)^{1/2} t^{3/2}} e^{-\frac{(x+z)^2}{2t}}.$$

In view of the fact that for all $\alpha > 0$,

$$\sup_{x>0} x^\alpha e^{-x} = \alpha^\alpha e^{-\alpha}, \tag{2.21}$$

we know that $\frac{|x+z|}{\sqrt{t}} e^{-\frac{(x+z)^2}{4t}}$ is uniformly bounded, from which it follows that

$$|\Delta_t(x,x')| \le c_1 \frac{|x'|}{t} e^{-\frac{(x+z)^2}{4t}}.$$

Since $\frac{(x+z)^2}{4t} \ge \frac{x^2}{8t} - \frac{z^2}{4t}$ and $|z| \le |x'| \le \sqrt{t}$, we get

$$e^{-\frac{(x+z)^2}{4t}} \le e^{\frac{z^2}{4t}} e^{-\frac{x^2}{8t}} \le e^{\frac{1}{4}} e^{-\frac{x^2}{8t}},$$

and thus

$$|\Delta_t(x,x')| \le c \frac{|x'|}{\sqrt{t}} p_{4t}(x).$$

for some universal constant $c > 0$. As a consequence, if $|x'| \le \sqrt{t}$, for any $\beta \in [0,1]$, we have

$$|\Delta_t(x,x')| \le c_\beta t^{-\frac{\beta}{2}} |x'|^\beta p_{4t}(x), \tag{2.22}$$

where c_β depends only on β. Putting together the estimates (2.20) and (2.22) yields (2.18).

Next, we prove the inequality (2.19) by considering the following four cases.

Case 1. If $|x'| > \sqrt{t}$ and $|x''| > \sqrt{t}$, we use the estimate

$$|R_t(x,x',x'')| \le p_t(x+x'-x'') + p_t(x+x') + p_t(x-x'') + p_t(x). \tag{2.23}$$

Case 2. If $|x'| \le \sqrt{t}$ and $|x''| > \sqrt{t}$, we deduce from (2.22) that

$$\begin{aligned} |R_t(x,x',x'')| \le& |p_t(x+x'-x'') - p_t(x-x'')| + |p_t(x+x') + p_t(x)| \\ =& |\Delta_t(x-x'',x')| + |\Delta_t(x,x')| \\ \le& c_\beta t^{-\frac{\beta}{2}} |x'|^\beta \big(p_{4t}(x) + p_{4t}(x-x'')\big). \end{aligned} \tag{2.24}$$

Case 3. If $|x'| > \sqrt{t}$ and $|x''| \le \sqrt{t}$, then the same argument from Case 2 leads to

$$|R_t(x,x',x'')| \le c_\beta t^{-\frac{\beta}{2}} |x''|^\beta \big(p_{4t}(x) + p_{4t}(x+x')\big). \tag{2.25}$$

Case 4. If $|x'| \le \sqrt{t}$ and $|x''| \le \sqrt{t}$, we write

$$R_t(x,x',x'') = \int_0^{-x''} \big[p_t'(x+x'+y) - p_t'(x+y)\big]dy$$

with the convention $\int_0^{-a} f(t)dt = -\int_0^a f(-t)dt$ for $a>0$. By mean value theorem, we can write

$$\begin{aligned} p_t'(x+x'+y) - p_t'(x+y) &= p_t''(x+y+z')x' \\ &= \frac{x'}{\sqrt{2\pi}} e^{-\frac{(x+y+z')^2}{2t}} \big(t^{-5/2}(x+y+z')^2 - t^{-3/2}\big), \end{aligned}$$

where z' is some number between 0 and y. Using (2.21) and $(x+y+z')^2 \ge \frac{1}{2}x^2 - (y+z')^2$, we have

$$e^{-\frac{(x+y+z')^2}{2t}} t^{-5/2}(x+y+z')^2 \le c\, e^{-\frac{(x+y+z')^2}{4t}} t^{-3/2} \le c\, p_{4t}(x) t^{-1} e^{\frac{(y+z')^2}{4t}}.$$

Since $|y+z'| \le 2|x''| \le 2\sqrt{t}$, we get

$$e^{-\frac{(x+y+z')^2}{2t}} t^{-5/2}(x+y+z')^2 \le c\, p_{4t}(x) t^{-1}$$

and

$$e^{-\frac{(x+y+z')^2}{2t}} t^{-3/2} \le c\, e^{\frac{(y+z')^2}{2t}} \frac{p_{2t}(x)}{t} \le c\, \frac{p_{4t}(x)}{t}.$$

It follows that

$$|R_t(x,x',x'')| \le c_\beta t^{-\beta} |x'|^\beta |x''|^\beta p_{4t}(x), \tag{2.26}$$

provided $|x'| \le \sqrt{t}$ and $|x''| \le \sqrt{t}$. Therefore, inequality (2.19) follows from (2.23)–(2.26). The proof of this lemma is complete. □

We also introduce the operator $\Lambda : \mathcal{M}(\mathbb{R}) \times \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R}^2)$ as follows. This operator will be used in Section 4. Let $0 < r < s$ and let $z', y' \in \mathbb{R}$. Then,

$$\begin{aligned} \Lambda_{r,z',s,y'}(g_1,g_2)(x,y) &= g_1(x)(\Phi^\beta_{s-r,y'} g_2)(y) N_r(z') + g_1(x)(\Phi^\beta_{s-r,y'}\Phi^\beta_{s-r,-z'} g_2)(y) \\ &\quad + (\Phi^\beta_{s-r,-y'} g_1)(x)(\theta_{y'} g_2)(y) N_r(z') + (\Phi^\beta_{s-r,-y'} g_1)(x)(\theta_{y'}\Phi^\beta_{s-r,-z'} g_2)(y) \end{aligned} \tag{2.27}$$

for any $(g_1,g_2,x,y) \in \mathcal{M}(\mathbb{R}) \times \mathcal{M}(\mathbb{R}) \times \mathbb{R}^2$.

Remark 2.8. It is not difficult to see that if $\|g_1\|_{L^1(\mathbb{R})} = 1$, then

$$\int_{\mathbb{R}} \big(\Phi^\beta_{t,x'} g_1\big)(x)dx = \big(\Phi^\beta_{t,x'} \mathbf{1}_{\mathbb{R}}\big)(0)$$

with $\mathbf{1}_{\mathbb{R}}(x) = 1$ for all $x \in \mathbb{R}$. As a result, we have

$$\int_{\mathbb{R}} dx \Lambda_{r,z',s,y'}(g_1,g_2)(x,y) = \Lambda_{r,z',s,y'}(\mathbf{1}_{\mathbb{R}}, g_2)(0,y)$$

provided $\|g_1\|_{L^1(\mathbb{R})} = 1$.

We complete this subsection by the following results about the gamma functions.

Lemma 2.9. *Let*

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy, \quad x > 0$$

be the usual gamma function[4]*, then we have the following bounds.*
(i) (Stirling's formula; c.f. [21, Theorem 1]) *For all $x > 0$, the following inequality holds,*

$$\sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x} \le \Gamma(x) \le \sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x+\frac{1}{12x}}.$$

(ii) (Asymptotic bound of the Mittag-Leffler function; c.f. [23, Formula (1.8.10)])

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \le c_1 \exp\left(c_2 z^{\frac{1}{\alpha}}\right)$$

for all $z \in \mathbb{R}_+$ and $\alpha \in (0, 2)$, where $c_1, c_2 > 0$ depend only on α.

3 Regular cases under Hypothesis 1

In this section, we prove the first two error bounds in (1.12). As already mentioned in the introduction, the majority of this section will devoted to proving the following L^p estimates of Malliavin derivatives.

Theorem 3.1. Assume that Hypothesis 1 holds true. Given $(t, x) \in (0, \infty) \times \mathbb{R}^d$ and $(p, m) \in [2, \infty) \times \mathbb{N}^*$, then for almost every $(\mathbf{s}_m, \mathbf{y}_m) \in ([0, t] \times \mathbb{R}^d)^m$, we have

$$\left\| D^m_{\mathbf{s}_m, \mathbf{y}_m} u(t, x) \right\|_p \le C(t) f_{t,x,m}\big(\mathbf{s}_m, \mathbf{y}_m\big), \tag{3.1}$$

where $f_{t,x,m}$ is the chaos coefficient defined as in (2.6) and the constant $C(t)$ depends on $(t, p, m, \gamma_0, \gamma_1)$ and is increasing in t.

The proof of Theorem 3.1 is deferred to Section 3.2. In Section 3.1, we prove the first two error bounds in (1.12) by using Theorem 3.1.

3.1 Proof of quantitative CLTs in regular cases

Assume Hypothesis 1. With

$$F_R(t) = \int_{B_R} [u(t, x) - 1] dx \quad \text{and} \quad \sigma_R(t) = \sqrt{\text{Var}\big(F_R(t)\big)},$$

we have the following facts from [31].

(i) Under Hypothesis 3a, $\sigma_R(t) \sim R^{d/2}$.

(ii) Under Hypothesis 3b, one has $\sigma_R(t) \sim R^{d-\frac{\beta}{2}}$.

3.1.1 Proof of (1.12) under Hypotheses 1 and 3a

Using Minkowski's inequality, we have

$$\|D_{r,z} D_{s,y} F_R(t)\|_4 = \left\| \int_{B_R} D_{r,z} D_{s,y} u(t, x) dx \right\|_4 \le \int_{B_R} \left\| D_{r,z} D_{s,y} u(t, x) \right\|_4 dx.$$

Then it follows from (3.1) that

$$\|D_{r,z} D_{s,y} F_R(t)\|_4 \lesssim \int_{B_R} f_{t,x,2}(r, z, s, y) dx, \tag{3.2}$$

with

$$f_{t,x,2}(r, z, s, y) = \frac{1}{2} \left[p_{t-r}(x - z) p_{r-s}(z - w) \mathbf{1}_{\{r>s\}} + p_{t-s}(x - y) p_{s-r}(z - y) \mathbf{1}_{\{r<s\}} \right].$$

[4]This shall not be confusing with Γ_t defined as in Lemma 2.1.

In the same way, we have

$$\|D_{s,y}F_R(t)\|_4 \lesssim \int_{B_R} p_{t-s}(x-y)dx, \tag{3.3}$$

where the implicit constants in (3.2) and (3.3) do not depend on (R, r, z, s, y) and are increasing in t.

Apply Proposition 2.3 and plugging (3.2) and (3.3) into the expression of $\mathcal{A}$, we get

$$\begin{aligned}\mathcal{A} \lesssim \int_{[0,t]^6\times\mathbb{R}^{6d}} & drdr'dsds'd\theta d\theta' dzdz'dydy'dwdw'\gamma_0(r-r')\gamma_0(s-s')\gamma_0(\theta-\theta')\gamma_1(z-z') \\ & \times \int_{B_R^4} d\mathbf{x}_4 \gamma_1(w-w')\gamma_1(y-y')f_{t,x_1,2}(r,z,\theta,w)f_{t,x_2,2}(s,y,\theta',w')p_{t-r'}(x_3-z') \\ & \times p_{t-s'}(x_4-y').\end{aligned}$$

Taking the expression of $f_{t,x,2}$ into account, we need to consider four terms depending on $r>\theta$ or not, and depending on $s>\theta'$ or not. Since the computations are similar, it suffices to provide the estimate for case $r>\theta$ and $s>\theta'$. In other words, we need to show that

$$\begin{aligned}\mathcal{A}^* := \int_{[0,t]^6\times\mathbb{R}^{6d}} & drdr'dsds'd\theta d\theta' dzdz'dydy'dwdw'\gamma_0(r-r')\gamma_0(s-s')\gamma_0(\theta-\theta') \\ & \times \int_{B_R^4} d\mathbf{x}_4 p_{t-r}(x_1-z)p_{r-\theta}(z-w)p_{t-s}(x_2-y)p_{s-\theta'}(y-w')p_{t-r'}(x_3-z') \\ & \times p_{t-s'}(x_4-y')\gamma_1(w-w')\gamma_1(y-y')\gamma_1(z-z') \\ \lesssim & R^d.\end{aligned}$$

In fact, the above estimate follows from integrating with respect to dx_1, dx_2, dx_4, dy', dy, dw', dw, dz, dz', dx_3 one by one and using the local integrability of γ_0. The desired bound follows immediately. □

3.1.2 Proof of (1.12) under Hypotheses 1 and 3b

Similarly as in Section 3.1.1, we need to show $\mathcal{A}^* \lesssim R^{4d-3\beta}$. Making the change of variables

$$(\mathbf{x}_4, z, z', y, y', w, w') \to R(\mathbf{x}_4, z, z', y, y', w, w')$$

and using the scaling properties of the Riesz and heat kernels[5] yields

$$\mathcal{A}^* = R^{4d-3\beta}\int_{[0,t]^6} drdr'dsds'd\theta d\theta'\gamma_0(r-r')\gamma_0(s-s')\gamma_0(\theta-\theta')\mathbf{S}_R,$$

with

$$\begin{aligned}\mathbf{S}_R := \int_{B_1^4\times\mathbb{R}^{6d}} & d\mathbf{x}_4 dzdz'dydy'dwdw'|w-w'|^{-\beta}|y-y'|^{-\beta}|z-z'|^{-\beta} \\ & \times p_{\frac{t-r}{R^2}}(x_1-z)p_{\frac{r-\theta}{R^2}}(z-w)p_{\frac{t-s}{R^2}}(x_2-y)p_{\frac{s-\theta'}{R^2}}(y-w')p_{\frac{t-r'}{R^2}}(x_3-z')p_{\frac{t-s'}{R^2}}(x_4-y').\end{aligned}$$

Making the following change of variables

$$\boldsymbol{\eta}_6 = (z-x_1, z-w, y-x_2, y-w', z'-x_3, y'-x_4)$$

[5] $p_t(Rz) = R^{-d}p_{tR^{-2}}(z)$ for $z\in\mathbb{R}^d$.

(so $w = \eta_1 - \eta_2 + x_1, w' = \eta_3 - \eta_4 + x_2$) yields

$$\begin{aligned}\mathbf{S}_R = &\int_{B_1^4\times\mathbb{R}^{6d}} d\mathbf{x}_4 d\boldsymbol{\eta}_6 p_{\frac{t-r}{R^2}}(\eta_1)p_{\frac{r-\theta}{R^2}}(\eta_2)p_{\frac{t-s}{R^2}}(\eta_3)p_{\frac{s-\theta'}{R^2}}(\eta_4)p_{\frac{t-r'}{R^2}}(\eta_5)p_{\frac{t-s'}{R^2}}(\eta_6)\\ &\times|\eta_1-\eta_2-\eta_3+\eta_4+x_1-x_2|^{-\beta}|\eta_3-\eta_6+x_2-x_4|^{-\beta}|\eta_1-\eta_5+x_1-x_3|^{-\beta}\\ = &\int_{B_1^4} d\mathbf{x}_4\mathbb{E}\Big[\Big|\frac{\sqrt{t-r}}{R}Z_1-\frac{\sqrt{r-\theta}}{R}Z_2-\frac{\sqrt{t-s}}{R}Z_3+\frac{\sqrt{s-\theta'}}{R}Z_4+x_1-x_2\Big|^{-\beta}\\ &\times\Big|\frac{\sqrt{t-s}}{R}Z_3-\frac{\sqrt{t-s'}}{R}Z_6+x_2-x_4\Big|^{-\beta}\Big|\frac{\sqrt{t-r}}{R}Z_1-\frac{\sqrt{t-r'}}{R}Z_5+x_1-x_3\Big|^{-\beta}\Big],\end{aligned}$$

where $Z_1,\dots,Z_6$ are i.i.d. standard Gaussian vectors on $\mathbb{R}^d$. Notice that

$$\begin{aligned}\mathcal{K} := \sup_{z\in\mathbb{R}^d}\int_{B_1}|y+z|^{-\beta}dy &\le \mathbf{1}_{\{|z|\le 2\}}\int_{B_3}|y|^{-\beta}dy + \mathbf{1}_{\{|z|>2\}}\int_{B_1}\big(|z|-|y|\big)^{-\beta}dy\\ &\le \int_{B_3}|y|^{-\beta}dy + \int_{B_1}1dy < \infty.\end{aligned}$$

Therefore, we deduce that

$$\mathbf{S}_R \le \mathcal{K}^3\int_{B_1} 1\, dx_1 = \mathcal{K}^3\mathrm{Vol}(B_1).$$

So $\mathcal{A}^* \le \mathcal{K}^3\mathrm{Vol}(B_1)(t\Gamma_t)^3R^{4d-3\beta}$, with $\Gamma_t = \int_{-t}^t\gamma_0(s)ds$. Hence applying Proposition 2.3 yields the desired conclusion. ☐

3.2 Proof of Theorem 3.1

Recall the Wiener chaos expansion (2.5) and (2.6) for $u(t,x)$. Then, for any positive integer m, the m-th Malliavin derivative valued at $(s_1,y_1,\dots,s_m,y_m)$ is given by

$$\begin{aligned}D^m_{\mathbf{s}_m,\mathbf{y}_m}u(t,x) &= D_{s_1,y_1}D_{s_2,y_2}\cdots D_{s_m,y_m}u(t,x)\\ &= \sum_{n=m}^{\infty}\frac{n!}{(n-m)!}I_{n-m}\big(f_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m,\bullet)\big),\end{aligned}$$

whenever the series converges in L^2. By definition, it is easy to check that $f_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m,\bullet)\in\mathfrak{H}^{\odot(n-m)}$ and by symmetry again, we can assume $t > s_m > s_{m-1} > \cdots > s_1 > 0$.

For any $n\ge m$, we make use of the notation,

$$[n]_< = \{\mathbf{i}_m = (i_1,\dots,i_m), 1\le i_1<\cdots<i_m\le n\}.$$

We also define the function $f^{(\boldsymbol{i}_m)}_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m;\bullet) : \mathbb{R}_+^{n-m}\times(\mathbb{R}^d)^{n-m}$ by

$$\begin{aligned}f^{(\boldsymbol{i}_m)}_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m;\bullet) =& f^{(i_1)}_{t,x,i_1}(s_m,y_m;\bullet)\otimes f^{(i_2-i_1)}_{s_m,y_m,i_2-i_1}(s_{m-1},y_{m-1};\bullet)\\ &\otimes\cdots\otimes f^{(i_m-i_{m-1})}_{s_2,y_2,i_m-i_{m-1}}(s_1,y_1;\bullet)\otimes f_{s_1,y_1,n-i_m},\end{aligned}\tag{3.4}$$

where $f^{(1)}_{t,x,1}(r,z;\bullet) = p_{t-r}(x-z)$, and for all $k\ge 2$,

$$\begin{aligned}f^{(k)}_{t,x,k}(r,z;\mathbf{s}_{k-1},\mathbf{y}_{k-1}) :=& \frac{1}{k!}p_{t-s_{\sigma(k-1)}}(x-y_{\sigma(k-1)})p_{s_{\sigma(k-1)}-s_{\sigma(k-2)}}(y_{\sigma(k-1)}-y_{\sigma(k-2)})\\ &\times\cdots\times p_{s_{\sigma((1)}-r}(y_{\sigma(1)}-z),\end{aligned}$$

and σ denotes the permutation of $\{1,\dots,k-1\}$ such that $r < s_{\sigma(1)} < \cdots < s_{\sigma(k-1)} < t$. Let $h^{(\boldsymbol{i}_m)}_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m;\bullet)$ be the symmetrization of $f^{(\boldsymbol{i}_m)}_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m;\bullet)$. Then, for any $p\in[2,\infty)$, we deduce from Minkowski's inequality and (2.3) that

$$\left\|D^m_{\mathbf{s}_m,\mathbf{y}_m}u(t,x)\right\|_p \le \sum_{n=m}^{\infty}(p-1)^{\frac{n-m}{2}}\left\|I_{n-m}\Big(\frac{n!}{(n-m)!}f_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m,\bullet)\Big)\right\|_2$$

and

$$\frac{n!}{(n-m)!}f_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m,\bullet) = \sum_{\boldsymbol{i}_m\in[n]_<} h^{(\boldsymbol{i}_m)}_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m;\bullet).$$

It follows that

$$\left\|I_{n-m}\Big(\frac{n!}{(n-m)!}f_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m,\bullet)\Big)\right\|_2^2 \le \binom{n}{m}\sum_{\boldsymbol{i}_m\in[n]_<}\left\|I_{n-m}\big(f^{(\boldsymbol{i}_m)}_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m;\bullet)\big)\right\|_2^2. \tag{3.5}$$

Additionally, due to Lemma 2.1, we deduce that

$$\|I_{n-m}(f^{(\boldsymbol{i}_m)}_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m;\bullet))\|_2^2 \le \Gamma_t^{n-m}\|I^{\mathfrak{X}}_{n-m}(f^{(\boldsymbol{i}_m)}_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m;\bullet))\|_2^2. \tag{3.6}$$

The inequalities (3.5) and (3.6) together with the product formula given in Lemma (2.2) and the decomposition (3.4), imply

$$\left\|D^m_{\mathbf{s}_m,\mathbf{y}_m}u(t,x)\right\|_p \le \sum_{n=m}^{\infty}\big[(p-1)\Gamma_t\big]^{\frac{n-m}{2}}\sqrt{\mathcal{Q}^{\mathfrak{X}}_{n,m}}, \tag{3.7}$$

where

$$\mathcal{Q}^{\mathfrak{X}}_{n,m} = \binom{n}{m}\sum_{\boldsymbol{i}_m\in[n]_<}\left\|I_{n-m}\big(f^{(\boldsymbol{i}_m)}_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m;\bullet)\big)\right\|_2^2.$$

Using the independence among the random variables inside the expectation, see Lemma 2.2, and the notation $(i_0, s_{m+1}, y_{m+1}) = (0, t, x)$, we can write

$$\begin{aligned}&\left\|I^{\mathfrak{X}}_{n-m}\big(f^{(\boldsymbol{i}_m)}_{t,x,n}(\mathbf{s}_m,\mathbf{y}_m;\bullet)\big)\right\|_2^2\\ &= \left\|I^{\mathfrak{X}}\big(f_{s_1,y_1,n-i_m}\big)\right\|_2^2 \times \prod_{j=1}^{m}\left\|I^{\mathfrak{X}}_{i_j-i_{j-1}-1}\Big(f^{(i_j-i_{j-1})}_{s_{m-j+2},y_{m-j+2},i_j-i_{j-1}}(s_{m-j+1},y_{m-j+1};\bullet)\Big)\right\|_2^2.\end{aligned}$$

Thanks to the isometry property between the space $\mathfrak{H}^{\odot n}$ (see (2.2)), equipped with the modified norm $\sqrt{n!}\|\bullet\|_{\mathfrak{H}^{\otimes n}}$, and the n-th Wiener chaos $\mathbf{H}_n$, we can write

$$\begin{aligned}\mathcal{Q}^{\mathfrak{X}}_{n,m} =& \binom{n}{m}\sum_{\boldsymbol{i}_m\in[n]_<}(n-i_m)!\left\|f_{s_1,y_1,n-i_m}\right\|^2_{\mathfrak{X}^{\otimes(n-i_m)}}\\ &\times\prod_{j=1}^{m}(i_j-i_{j-1}-1)!\left\|f^{(i_j-i_{j-1})}_{s_{m-j+2},y_{m-j+2},i_j-i_{j-1}}(s_{m-j+1},y_{m-j+1};\bullet)\right\|^2_{\mathfrak{X}^{\otimes(i_j-i_{j-1})}}.\end{aligned} \tag{3.8}$$

We first estimate $\left\|f_{s,y,k}\right\|_{\mathfrak{X}^{\otimes k}}$ and begin with

$$\begin{aligned}\left\|f_{s,y,k}\right\|^2_{\mathfrak{X}^{\otimes k}} &= k!\int_{\mathbb{T}^s_k}d\mathbf{r}_k\left\|f_{s,y,k}(\mathbf{r}_k,\bullet)\right\|^2_{\mathcal{H}_1^{\otimes k}}\\ &= \frac{1}{k!}\int_{\mathbb{T}^s_k}d\mathbf{r}_k\int_{\mathbb{R}^{dk}}\mu^{\otimes k}(d\boldsymbol{\xi}_k)\prod_{j=1}^{k}\left|\widehat{p}_{r_{j+1}-r_j}(\xi_j+\cdots+\xi_k)\right|^2,\end{aligned} \tag{3.9}$$

with $r_{k+1} = s$ and $\widehat{p}_t(\xi) = e^{-t|\xi|^2/2}$. In the current **regular case**, we can deduce from the maximal principle (c.f. [31, Lemma 4.1]) that

$$\sup_{z\in\mathbb{R}^d}\int_{\mathbb{R}^d}\mu(dy)e^{-s|y+z|^2} = \int_{\mathbb{R}^d}\mu(dy)e^{-s|y|^2} < \infty.$$

Then, preforming change of variables $w_j = r_{j+1} - r_j$, and using Lemma 3.3 in [15], we have

$$\begin{aligned}\big\|f_{s,y,k}\big\|^2_{\mathfrak{X}^{\otimes k}} \le& \frac{1}{k!}\int_{\mathcal{T}_k(s)} d\mathbf{w}_k \int_{\mathbb{R}^{dk}}\mu(d\boldsymbol{\xi}_k)\prod_{j=1}^k e^{-w_j|\xi_j|^2} \\ \le& \frac{1}{k!}\sum_{\ell=0}^k \binom{k}{\ell}\frac{s^\ell}{\ell!}D_N^\ell(2C_N)^{k-\ell} \\ \le& \frac{2^k}{k!}\sum_{\ell=0}^k \frac{s^\ell}{\ell!}D_N^\ell(2C_N)^{k-\ell},\end{aligned} \tag{3.10}$$

where $\mathcal{T}_k(s) := \{\mathbf{w}_k \in \mathbb{R}_+^k : w_1 + \cdots + w_k \le s\}$ and

$$C_N := \int_{\mathbb{R}^d}\frac{\mu(d\xi)}{|\xi|^2}\mathbf{1}_{\{|\xi|\ge N\}} \quad \text{and} \quad D_N := \mu\big(\{\xi\in\mathbb{R}^d : |\xi|\le N\}\big), \tag{3.11}$$

are finite quantities under Dalang's condition (1.6).

Finally, in what follows, we estimate $\big\|f^{(k)}_{s,y,k}(r,z;\bullet)\big\|_{\mathfrak{X}^{\otimes(k-1)}}$. It is trivial that for $k=1$,

$$\big\|f^{(1)}_{s,y,1}(r,z;\bullet)\big\|^2_{\mathfrak{X}^{\otimes 0}} = p_{s-r}(y-z)^2. \tag{3.12}$$

For $k=2$, we can write

$$\begin{aligned}\big\|f^{(2)}_{s,y,2}(r,z;\bullet)\big\|^2_{\mathfrak{X}} &= \frac{1}{4}\int_r^s dv\big\|p_{s-v}(y-\bullet)p_{v-r}(\bullet - z)\big\|^2_{\mathcal{H}_1} \\ &= \frac{1}{4}p_{s-r}(y-z)^2\int_r^s dv\Big\|\frac{p_{s-v}(y-\bullet)p_{v-r}(\bullet-z)}{p_{s-r}(y-z)}\Big\|^2_{\mathcal{H}_1}.\end{aligned}$$

Using the fact (c.f. [7, Formula (1.4)]) that

$$\frac{p_t(a)p_s(b)}{p_{t+s}(a+b)} = p_{st/(t+s)}\Big(b - \frac{s}{s+t}(a+b)\Big), \tag{3.13}$$

we get

$$f(x) := \frac{p_{s-v}(y-x)p_{v-r}(x-z)}{p_{s-r}(y-z)} = p_{(v-r)(s-v)/(s-r)}\Big(x - z - \frac{v-r}{s-r}(y-z)\Big).$$

The Fourier transform of f is given by

$$\widehat{f}(\xi) = \exp\Big(-i\Big(z + \frac{v-r}{s-r}(y-z)\Big)\xi\Big)\exp\Big(-\frac{(v-r)(s-v)}{2(s-r)}|\xi|^2\Big).$$

This implies that

$$\Big\|\frac{p_{s-v}(y-\bullet)p_{v-r}(\bullet-z)}{p_{s-r}(y-z)}\Big\|^2_{\mathcal{H}_1} = c_{H_1}\int_{\mathbb{R}^d}\mu(d\xi)\exp\Big(-\frac{(v-r)(s-v)}{s-r}|\xi|^2\Big),$$

and thus

$$\begin{aligned}\big\|f^{(2)}_{s,y,2}(r,z;\bullet)\big\|^2_{\mathfrak{X}} &= c_{H_1}\frac{1}{4}p^2_{s-r}(y-z)\int_r^s dv\int_{\mathbb{R}^d}\mu(d\xi)\exp\Big(-\frac{(v-r)(s-v)}{s-r}|\xi|^2\Big) \\ &\le \frac{1}{4}c_{H_1}p_{s-r}(y-z)^2\big[(s-r)D_N + 4C_N\big],\end{aligned} \tag{3.14}$$

for any $N>0$. Indeed,

$$\int_r^s dv \int_{\mathbb{R}^d} \mu(d\xi) \exp\left(-\frac{(v-r)(s-v)}{s-r}|\xi|^2\right)$$
$$\leq (s-r)D_N + \int_{|\xi|\geq N} \mu(d\xi) \int_r^s dv \exp\left(-\frac{(v-r)(s-v)}{s-r}|\xi|^2\right)$$

and

$$\int_r^s dv \exp\left(-\frac{(v-r)(s-v)}{s-r}|\xi|^2\right) = \int_0^{s-r} dv \exp\left(-\frac{v(s-r-v)}{s-r}|\xi|^2\right)$$
$$= (s-r)\int_0^1 dv e^{-v(1-v)(s-r)|\xi|^2} = 2(s-r)\int_0^{1/2} dv e^{-v(1-v)(s-r)|\xi|^2}$$
$$\leq 2(s-r)\int_0^{1/2} dv e^{-v\frac{(s-r)|\xi|^2}{2}} = \frac{4}{|\xi|^2}\left(1-e^{-(s-r)|\xi|^2/4}\right) \leq \frac{4}{|\xi|^2},$$

so that

$$\int_r^s dv \int_{\mathbb{R}^d} \mu(d\xi) \exp\left(-\frac{(v-r)(s-v)}{s-r}|\xi|^2\right)$$
$$= (s-r)\int_0^1 dv \int_{\mathbb{R}^d} \mu(d\xi) \exp\left(-(s-r)v(1-v)|\xi|^2\right) \leq (s-r)D_N + 4C_N,$$

for all $N>0$.

For $k\geq 3$, we obtain

$$\left\|f_{s,y,k}^{(k)}(r,z;\bullet)\right\|_{\mathfrak{X}^{\otimes(k-1)}}^2 = k!\int_{\mathbb{T}_{k-1}^{r,s}} d\mathbf{r}_{k-1}\left\|f_{s,y,k}^{(k)}(r,z;\mathbf{r}_{k-1},\bullet)\right\|_{\mathcal{H}_1^{\otimes k-1}}^2$$

In order to estimate $\|f_{s,y,k}^{(k)}(r,z;\mathbf{r}_{k-1},\bullet)\|\|_{\mathcal{H}_1^{\otimes k-1}}$, we apply formula (3.13) several times and get another expression for $f_{s,y,k}^{(k)}(r,z;\mathbf{r}_{k-1},\mathbf{z}_{k-1})$ as follows,

$$f_{s,y,k}^{(k)}(r,z;\mathbf{r}_{k-1},\boldsymbol{z}_{k-1}) = \frac{1}{k!}p_{s-r}(y-z)\frac{p_{s-r_{k-1}}(y-z_{k-1})p_{r_{k-1}-r_{k-2}}(z_{k-1}-z_{k-2})}{p_{s-r_{k-2}}(y-z_{k-2})}$$
$$\times \frac{p_{s-r_{k-2}}(y-z_{k-2})p_{r_{k-2}-r_{k-3}}(z_{k-2}-z_{k-3})}{p_{s-r_{k-3}}(y-z_{k-3})} \times\cdots\times \frac{p_{s-r_1}(y-z_1)p_{r_1-r}(z_1-z)}{p_{s-r}(y-z)}$$
$$= \frac{1}{k!}p_{s-r}(y-z)\prod_{i=1}^{k-1} p_{\frac{(s-r_i)(r_i-r_{i-1})}{s-r_{i-1}}}\left(z_i - \frac{s-r_i}{s-r_{i-1}}z_{i-1} - \frac{r_i-r_{i-1}}{s-r_{i-1}}y\right), \tag{3.15}$$

where by convention, $r_0=r$ and $z_0=z$. In the next step, we compute the Fourier transform of $f_{s,y,k}^{(k)}(r,z;\mathbf{r}_{k-1},\bullet)$. Using the representation (3.15), we integrate the following expression $f_{s,y,k}^{(k)}(r,z;\mathbf{r}_{k-1},\mathbf{z}_{k-1})e^{i\langle\boldsymbol{\xi}_{k-1},\mathbf{z}_{k-1}\rangle}$ subsequently in $z_{k-1},z_{k-2},\dots,z_1$ using some elementary Fourier computation formulas (c.f. [14, Section 3.2] for similar computations) and get

$$\widehat{f}_{s,y,k}^{(k)}(r,z;\mathbf{r}_{k-1},\boldsymbol{\xi}_{k-1}) = \frac{1}{k!}p_{s-r}(y-z)\exp\Big(-iz\frac{s-r_1}{s-r}\sum_{k_1=1}^{k-1}\Big(\xi_{k_1}\prod_{k_2=2}^{k_1}\frac{s-r_{k_2}}{s-r_{k_2-1}}\Big)\Big)$$
$$\times\exp\Big[-iy\sum_{j=1}^{k-1}\frac{r_j-r_{j-1}}{s-r_{j-1}}\sum_{k_1=j}^{k-1}\Big(\xi_{k_1}\prod_{k_2=j+1}^{k_1}\frac{s-r_{k_2}}{s-r_{k_2-1}}\Big)\Big]$$
$$\times\exp\Big[-\sum_{j=1}^{k-1}\frac{(s-r_j)(r_j-r_{j-1})}{2(s-r_{j-1})}\Big[\sum_{k_1=j}^{k-1}\Big(\xi_{k_1}\prod_{k_2=j+1}^{k_1}\frac{s-r_{k_2}}{s-r_{k_2-1}}\Big)\Big]^2\Big], \tag{3.16}$$

where by convention $\prod_{k_2=j+1}^{j}\frac{s-r_{k_2}}{s-r_{k_2-1}}=1$. Notice that

$$\sum_{k_1=j}^{k-1}\Big(\xi_{k_1}\prod_{k_2=j+1}^{k_1}\frac{s-r_{k_2}}{s-r_{k_2-1}}\Big)=\xi_j+\sum_{k_1=j+1}^{k-1}\Big(\xi_{k_1}\prod_{k_2=j+1}^{k_1}\frac{s-r_{k_2}}{s-r_{k_2-1}}\Big).$$

Thus, by the the maximal principle ([31, Lemma 4.1]) again, we have

$$\begin{aligned}\big\|f^{(k)}_{s,y,k}(r,z;\bullet)\big\|^2_{\mathfrak{X}^{\otimes(k-1)}}=&k!\int_{\mathbb{T}^{r,s}_{k-1}}d\mathbf{r}_{k-1}\int_{\mathbb{R}^{k-1}}\mu^{\otimes(k-1)}(d\boldsymbol{\xi}_{k-1})\big|\widehat{f}^{(k)}_{s,y,k}(r,z;\mathbf{r}_{k-1},\boldsymbol{\xi}_{k-1})\big|^2\\ \le&\frac{1}{k!}p^2_{s-r}(y-z)\mathcal{I},\end{aligned}$$

where, by the change of variables $v_i=\frac{s-r_{k-i}}{s-r}$ for all $i=1,\dots,k$,

$$\begin{aligned}\mathcal{I}=&\int_{\mathbb{T}^{r,s}_{k-1}}d\mathbf{r}_{k-1}\prod_{j=1}^{k-1}\int_{\mathbb{R}^d}\mu(d\xi_j)\exp\Big(-\frac{(s-r_j)(r_j-r_{j-1})}{(s-r_{j-1})}|\xi_j|^2\Big)\\ =&(s-r)^{k-1}\int_{\mathbb{T}^1_{k-1}}d\mathbf{v}_{k-1}\int_{\mathbb{R}^{(k-1)d}}\mu^{\otimes(k-1)}(d\boldsymbol{\xi}_{k-1})\prod_{j=1}^{k-1}\exp\Big(-\frac{(s-r)(v_{j+1}-v_j)}{v_{j+1}}|\xi_j|^2\Big).\end{aligned}$$

We estimate the term $\mathcal{I}$ as follows. First we make the decomposition

$$\begin{aligned}\mathcal{I}=&(s-r)^{k-1}\sum_{J\subset\{1,\dots,k-1\}}\int_{\mathbb{T}^1_{k-1}}d\mathbf{v}_{k-1}\int_{\mathbb{R}^{(k-1)d}}\mu^{\otimes(k-1)}(d\boldsymbol{\xi}_{k-1})\\ &\times\Big(\prod_{j\in J}\mathbf{1}_{\{|\xi_j|\le N\}}\Big)\Big(\prod_{\ell\in J^c}\mathbf{1}_{\{|\xi_\ell|>N\}}\Big)\prod_{\iota=1}^{k-1}\exp\Big(-(s-r)\frac{(v_{\iota+1}-v_\iota)v_\iota}{v_{\iota+1}}|\xi_\iota|^2\Big)\\ \le&(s-r)^{k-1}\sum_{J\subset\{1,\dots,k-1\}}D_N^{|J|}\int_{\mathbb{T}^1_{k-1}}d\mathbf{v}_{k-1}\int_{\mathbb{R}^{|J^c|d}}\Big(\prod_{\ell\in J^c}\mu(d\xi_\ell)\Big)\\ &\times\prod_{\ell\in J^c}\mathbf{1}_{\{|\xi_\ell|>N\}}\exp\Big(-(s-r)\frac{(v_{\ell+1}-v_\ell)v_\ell}{v_{\ell+1}}|\xi_\ell|^2\Big),\end{aligned}$$

where D_N and C_N appearing below are introduced as in (3.11) and $J^c=\{1,\dots,k-1\}\setminus J$.

Suppose $J^c=\{\ell_1,\dots,\ell_j\}$ for some $j=0,\dots,k-1$ with $\ell_1<\cdots<\ell_j$. Then, performing the integral with respect to v_{ℓ_1}, yields

$$\begin{aligned}&\int_0^{v_{\ell_1+1}}dv_{\ell_1}\exp\Big(-(s-r)\frac{(v_{\ell_1+1}-v_{\ell_1})v_{\ell_1}}{v_{\ell_1+1}}|\xi_{\ell_1}|^2\Big)\\ =&v_{\ell_1+1}\int_0^1du\exp\Big(-(s-r)[u(1-u)]v_{\ell_1+1}|\xi_{\ell_1}|^2\Big)\\ \le&2v_{\ell_1+1}\int_0^{1/2}du\exp\Big(-(s-r)[u(1-u)]v_{\ell_1+1}|\xi_{\ell_1}|^2\Big)\\ \le&2v_{\ell_1+1}\int_0^{1/2}du\exp\Big(-\frac{1}{2}u(s-r)v_{\ell_1+1}|\xi_{\ell_1}|^2\Big)\le\frac{4}{(s-r)|\xi_{\ell_1}|^2}.\end{aligned}$$

Applying this procedure for the integrals with respect to the variables $v_{\ell_2},\dots,v_{\ell_j}$ successively, we obtain

$$\begin{aligned}&\int_{\mathbb{T}^1_{k-1}}d\mathbf{v}_{k-1}\int_{\mathbb{R}^{jd}}\prod_{\ell\in J^c}\mu(d\xi_\ell)\mathbf{1}_{\{|\xi_\ell|>N\}}\exp\Big(-(s-r)\frac{(v_{\ell+1}-v_\ell)v_\ell}{v_{\ell+1}}|\xi_\ell|^2\Big)\\ &\le\Big(\frac{4C_N}{s-r}\Big)^j\frac{1}{|k-j-1|!}.\end{aligned}$$

It follows that

$$\begin{aligned}\mathcal{I} \leq & (s-r)^{k-1} \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{[4(s-r)^{-1}C_N]^{k-j-1} D_N^j}{j!} \\ \leq & 2^{k-1}(s-r)^{k-1} \sum_{j=0}^{k-1} \frac{[4(s-r)^{-1}C_N]^{k-j-1} D_N^j}{j!},\end{aligned}$$

which implies

$$\big\| f_{s,y,k}^{(k)}(r,z;\bullet) \big\|_{\mathfrak{X}^{\otimes(k-1)}}^2 \leq \frac{2^{k-1}}{k!} p_{s-r}(y-z)^2 \sum_{\ell=0}^{k-1} \frac{[4C_N]^{k-\ell-1}}{\ell!} [(s-r)D_N]^\ell. \tag{3.17}$$

Combining (3.8), (3.10), (3.12), (3.14) and (3.17), we can write

$$\begin{aligned}\mathcal{Q}_{n,m}^{\mathfrak{X}} \leq & c_1 c_2^n \sum_{\boldsymbol{i}_m \in [n]_<} \sum_{\ell=0}^{n-i_m} \frac{s_1^\ell}{\ell!} D_N^\ell C_N^{n-i_m-\ell} \\ & \times \prod_{j=1}^m \sum_{\ell=0}^{i_j-i_{j-1}-1} \frac{C_N^{i_j-i_{j-1}-\ell-1}}{\ell!} [(s_{m-j+2}-s_{m-j+1})D_N]^\ell f_{s,y,m}(\mathbf{s}_m,\mathbf{y}_m)^2. \end{aligned} \tag{3.18}$$

If $C_N = 0$ for some $N > 0$, then, we have

$$\begin{aligned}\mathcal{Q}_{n,m}^{\mathfrak{X}} \leq & c_1 c_2^n \sum_{\boldsymbol{i}_m \in [n]_<} \frac{s_1^{n-i_m}}{(n-i_m)!} D_N^{n-i_m} \\ & \times \prod_{j=1}^m \frac{1}{(i_j-i_{j-1}-1)!} [(s_{m-j+2}-s_{m-j+1})D_N]^{i_j-i_{j-1}-1} f_{s,y,m}(\mathbf{s}_m,\mathbf{y}_m)^2.\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{n=m}^\infty \big[(p-1)\Gamma_t\big]^{\frac{n-m}{2}} \sqrt{\mathcal{Q}_{n,m}^{\mathfrak{X}}} \\ &\leq \sum_{1\leq i_1<\cdots<i_m\leq n<\infty} c_1 c_2^n \big[(p-1)\Gamma_t\big]^{\frac{n-m}{2}} \frac{s_1^{\frac12(n-i_m)}}{[(n-i_m)!]^{\frac12}} D_N^{\frac12(n-i_m)} \\ &\quad \times \prod_{j=1}^m \frac{1}{[(i_j-i_{j-1}-1)!]^{\frac12}} [(s_{m-j+2}-s_{m-j+1})D_N]^{\frac12(i_j-i_{j-1}-1)} f_{s,y,m}(\mathbf{s}_m,\mathbf{y}_m) \\ &= \sum_{i_1=1}^\infty \sum_{i_2=i_1+1}^\infty \cdots \sum_{n=i_m+1}^\infty c_1 c_2^n \big[(p-1)\Gamma_t\big]^{\frac{n-m}{2}} \frac{s_1^{\frac12(n-i_m)}}{[(n-i_m)!]^{\frac12}} D_N^{\frac12(n-i_m)} \\ &\quad \times \prod_{j=1}^m \frac{1}{[(i_j-i_{j-1}-1)!]^{\frac12}} [(s_{m-j+2}-s_{m-j+1})D_N]^{\frac12(i_j-i_{j-1}-1)} f_{s,y,m}(\mathbf{s}_m,\mathbf{y}_m).\end{aligned}$$

Notice that, by using Lemma 2.9, we have

$$\begin{aligned}&\sum_{n=i_m+1}^\infty c_1 c_2^n \big[(p-1)\Gamma_t\big]^{\frac{n-m}{2}} \frac{s_1^{\frac12(n-i_m)}}{[(n-i_m)!]^{\frac12}} D_N^{\frac12(n-i_m)} \\ &\quad = c_1 c_2^{i_m} \big[(p-1)\Gamma_t\big]^{\frac{i_m-m}{2}} \sum_{k=1}^\infty c_2^k \big[(p-1)\Gamma_t\big]^{\frac{k}{2}} \frac{s_1^{\frac12(k)}}{[\Gamma(k+1)]^{\frac12}} D_N^{\frac12 k} \\ &\quad \leq c_1 c_2^{i_m} \big[(p-1)\Gamma_t\big]^{\frac{i_m-m}{2}} e^{c_3 \Gamma_t t}.\end{aligned}$$

Thus, by iteration, we conclude that

$$\sum_{n=m}^{\infty}\left[(p-1)\Gamma_t\right]^{\frac{n-m}{2}}\sqrt{\mathcal{Q}^{\mathfrak{X}}_{n,m}}\le c_1e^{c_2\Gamma_t t}f_{s,y,m}(\mathbf{s}_m,\mathbf{y}_m). \tag{3.19}$$

Inequality (3.1) is a consequence of inequalities (3.7) and (3.19).

On the other hand, suppose that $C_N>0$ for all $N>0$. From inequality (3.18), we deduce that

$$\begin{aligned}\mathcal{Q}^{\mathfrak{X}}_{n,m}\le& c_1c_2^nC_N^{n-m}\sum_{\boldsymbol{i}_m\in[n]_<}\sum_{\ell=0}^{n-i_m}\frac{s_1^\ell}{\ell!}D_N^\ell C_N^{-\ell}\\ &\times\prod_{j=1}^m\sum_{\ell=0}^{i_j-i_{j-1}-1}\frac{C_N^{-\ell}}{\ell!}[(s_{m-j+2}-s_{m-j+1})D_N]^\ell f_{s,y,m}(\mathbf{s}_m,\mathbf{y}_m)^2\\ \le& c_1c_2^n\sum_{\boldsymbol{i}_m\in[n]_<}C_N^{n-m}e^{\frac{D_N}{C_N}t}f_{s,y,m}(\mathbf{s}_m,\mathbf{y}_m)^2\le c_1c_2^nC_N^{n-m}e^{\frac{D_N}{C_N}t}f_{s,y,m}(\mathbf{s}_m,\mathbf{y}_m)^2.\end{aligned}$$

Notice that by definition $\lim_{N\uparrow\infty}C_N=0$. Therefore, we can find N large enough such that $c_2((p-1)\Gamma_t)^{\frac12}C_N<1$, and thus,

$$\begin{aligned}\sum_{n=m}^{\infty}\left[(p-1)\Gamma_t\right]^{\frac{n-m}{2}}\sqrt{\mathcal{Q}^{\mathfrak{X}}_{n,m}}\le& c_1c_2^me^{\frac{D_N}{C_N}t}f_{s,y,m}(\mathbf{s}_m,\mathbf{y}_m)\sum_{n=0}^{\infty}\left[(p-1)\Gamma_t\right]^{\frac{n}{2}}c_2^nC_N^n\\ =&\frac{c_1c_2^me^{\frac{D_N}{C_N}t}}{1-c_2((p-1)\Gamma_t)^{\frac12}C_N}f_{s,y,m}(\mathbf{s}_m,\mathbf{y}_m).\end{aligned}$$

This completes the proof of Theorem 3.1. □

4 Rough case under Hypothesis 2

In this section, we will deal with the **rough case**. That is, we consider the parabolic Anderson model (1.5) under Hypothesis 2 and, as already mentioned in the introduction, extra effort will be poured into for the spatial roughness. Taking advantage of the Gagliardo representation (2.7) of the inner product on $\mathcal{H}_1$, we apply a modified version of the second-order Gaussian Poincaré inequality (see Proposition 2.4). In order to estimate the quantity $\mathcal{A}$ in Proposition 2.4, we need the next proposition about the upper bounds of the Malliavin derivatives and their increments.

Proposition 4.1. *Assume Hypothesis 2 and let u be the solution to (1.5). Given $t\in(0,\infty)$, for almost any $0<r<s<t$, $x,y,y',z,z'\in\mathbb{R}$ and for every $p\ge2$, the following inequalities hold:*

$$\|D_{s,y+y'}u(t,x)-D_{s,y}u(t,x)\|_p\le C_1(t)\big(\Phi_{t-s,-y'}p_{4(t-s)}\big)(x-y) \tag{4.1}$$

and

$$\begin{aligned}&\|D^2_{r,z+z',s,y+y'}u(t,x)-D^2_{r,z+z',s,y}u(t,x)-D^2_{r,z,s,y+y'}u(t,x)+D^2_{r,z,s,y}u(t,x)\|_p\\ &\le C_1(t)\Lambda_{r,z',s,y'}(p_{4(t-s)},p_{4(s-r)})(x-y,y-z),\end{aligned} \tag{4.2}$$

where we fix $\Phi=\Phi^{H_0-\frac14}$, defined as in Lemma 2.6, Λ is defined as in (2.27), and $C_1(t)=c_1\exp(c_2t^{\frac{2H_0+H_1-1}{H_1}})$ for all $t>0$ with some constants c_1 and c_2 depending on H_0 and H_1.

The proof of Proposition 4.1 is based on the following lemmas. We firstly show how these lemmas imply Proposition 4.1. The proofs of Lemmas 4.2 and 4.3, which heavily rely on the Wiener chaos expansion, are postponed to Section 4.2.

Lemma 4.2. *Assume Hypothesis 2 and let u be the solution to* (1.5)*. Then, for almost every $(s,x,y,y')\in(0,t)\times\mathbb{R}^3$ and for any $p\geq 2$, the following inequalities hold:*

$$\|D_{s,y}u(t,x)\|_p \leq C_1(t)p_{t-s}(x-y) \tag{4.3}$$

and

$$\|D_{s,y+y'}u(t,x)-D_{s,y}u(t,x)\|_p \leq C_1(t)\Big(|\Delta_{t-s}(y-x,y')|+p_{t-s}(x-y)N_{t-s}(y')\Big), \tag{4.4}$$

where Δ_t and N_t are defined as in (2.14) *and* (2.16)*, respectively, and $C_1(t)$ is the same as in Proposition 4.1.*

Lemma 4.3. *Assume Hypothesis 2. Let u be the solution to* (1.5)*. Then, for almost any $0<r<s<t$, $x,y,y',z,z'\in\mathbb{R}$ and for every $p\geq 2$, the following inequality holds:*

$$\begin{aligned}
&\|D^2_{r,z+z',s,y+y'}u(t,x)-D^2_{r,z+z',s,y}u(t,x)-D^2_{r,z,s,y+y'}u(t,x)+D^2_{r,z,s,y}u(t,x)\|_p\\
\leq& C_1(t)\bigg\{p_{t-s}(x-y)N_r(z')\Big[|\Delta_{s-r}(y-z,y')|+p_{s-r}(y-z)N_{s-r}(y')\Big]\\
&+p_{t-s}(x-y)\Big[|R_{s-r}(y-z,y',z')|+|\Delta_{s-r}(y-z,y')|N_{s-r}(z')\\
&\qquad+|\Delta_{s-r}(z-y,z')|N_{s-r}(y')+p_{s-r}(y-z)N_{s-r}(y')N_{s-r}(z')\Big]\\
&+p_{s-r}(y+y'-z)N_r(z')\Big[|\Delta_{t-s}(y-x,y')|+p_{t-s}(x-y)N_{t-s}(y')\Big]\\
&+\Big[|\Delta_{s-r}(z-y-y',z')|+p_{s-r}(y+y'-z)N_{s-r}(z')\Big]\\
&\quad\times\Big[|\Delta_{t-s}(y-x,y')|+p_{t-s}(x-y)N_{t-s}(y')\Big]\bigg\},
\end{aligned} \tag{4.5}$$

where Δ_t, R_t and N_t are defined as in (2.14)–(2.16)*, respectively, and C_1 is the same as in Proposition 4.1.*

Proof of Proposition 4.1. Let us first recall from (2.16) that

$$N_t(x)=t^{\frac18-\frac12 H_0}|x|^{H_0-\frac14}\mathbf{1}_{\{|x|\leq\sqrt{t}\}}+\mathbf{1}_{\{|x|>\sqrt{t}\}}.$$

Then applying Lemma 2.6 with $\beta=H_0-\frac14$ yields immediately that

$$\Phi_{t,x'}g(x)\equiv\Phi^{H_0-\frac14}_{t,x'}g(x)=\theta_{x'}g(x)\mathbf{1}_{\{|x'|>\sqrt{t}\}}+N_t(x')g(x)\geq N_t(x')g(x) \tag{4.6}$$

for any nonnegative function $g\in\mathcal{M}(\mathbb{R})$. Now combining inequalities (4.4) and (2.18), we get (4.1) immediately.

In the next step, we prove inequality (4.2) that contains more terms. This is because the bound in (4.5) is a sum of 4 terms, which we denote by R_1,R_2,R_3 and R_4.

Firstly, we estimate R_1 and R_2 by using inequality (4.6) and Lemma 2.6 as follows,

$$\begin{aligned}
R_1:=&p_{t-s}(x-y)N_r(z')\big(|\Delta_{s-r}(y-z,y')|+p_{s-r}(y-z)N_{s-r}(y')\big)\\
\leq& c\,p_{4(t-s)}(x-y)N_r(z')\big(\Phi_{s-r,y'}p_{4s-4r}(y-z)+p_{4s-4r}(y-z)N_{s-r}(y')\big)\\
\leq& c\,p_{4(t-s)}(x-y)N_r(z')\big(\Phi_{s-r,y'}p_{4s-4r}\big)(y-z),
\end{aligned}$$

and taking into account Remark 2.7 and the inequality $p_t(x) \le 2p_{4t}(x)$, we can write

$$\begin{aligned}
R_2 :=& \, p_{t-s}(x-y)\big[|R_{s-r}(y-z,y',z')| + |\Delta_{s-r}(y-z,y')|N_{s-r}(z') \\
&+ |\Delta_{s-r}(z-y,z')|N_{s-r}(y') + p_{s-r}(y-z)N_{s-r}(y')N_{s-r}(z')\big] \\
\le& c\, p_{t-s}(x-y)\big[(\Phi_{s-r,y'}\Phi_{s-r,-z'}p_{4(s-r)})(y-z) + (\Phi_{s-r,y'}p_{4(s-r)})(y-z)N_{s-r}(z') \\
&+ (\Phi_{s-r,z'}p_{4(s-r)})(z-y)N_{s-r}(y') + p_{4s-4r}(y-z)N_{s-r}(y')N_{s-r}(z')\big] \\
\le& c\, p_{t-s}(x-y)\big[(\Phi_{s-r,y'}\Phi_{s-r,-z'}p_{4(s-r)})(y-z) + (\Phi_{s-r,-z'}\Phi_{s-r,y'}p_{4(s-r)})(y-z) \\
&+ (\Phi_{s-r,-y'}\Phi_{s-r,z'}p_{4(s-r)})(z-y) + (\Phi_{s-r,y'}\Phi_{s-r,-z'}p_{4s-4r})(y-z)\big] \\
\le& c\, p_{4(t-s)}(x-y)\big(\Phi_{s-r,y'}\Phi_{s-r,-z'}p_{4(s-r)}\big)(y-z).
\end{aligned}$$

Following a similar argument, we also get

$$\begin{aligned}
R_3 :=& p_{s-r}(y+y'-z)N_r(z')\big[|\Delta_{t-s}(y-x,y')| + p_{t-s}(x-y)N_{t-s}(y')\big] \\
\le& c\, p_{4(s-r)}(y+y'-z)N_r(z')\big(\Phi_{t-s,-y'}p_{4(t-s)}\big)(x-y) \\
=& c\,(\theta_{y'}p_{4(s-r)})(y-z)N_r(z')\big(\Phi_{t-s,-y'}p_{4(t-s)}\big)(x-y)
\end{aligned}$$

and

$$\begin{aligned}
R_4 :=& \big[|\Delta_{s-r}(z-y-y',z')| + p_{s-r}(y+y'-z)N_{s-r}(z')\big] \\
&\times \big[|\Delta_{t-s}(y-x,y')| + p_{t-s}(x-y)N_{t-s}(y')\big] \\
\le& c\,(\Phi_{s-r,-z'}p_{4s-4r})(y+y'-z)(\Phi_{t-s,-y'}p_{4(t-s)})(x-y) \\
=& c\,(\theta_{y'}\Phi_{s-r,-z'}p_{4s-4r})(y-z)\big(\Phi_{t-s,-y'}p_{4(t-s)}\big)(x-y).
\end{aligned}$$

Using the above estimates, inequality (4.2) follows immediately. $\square$

In what follows, we first give the remaining proof of (1.12) in Section 4.1. Later, in Section 4.2 provides proofs of several auxiliary results.

4.1 Proof of (1.12) in the rough case

According to Proposition 2.4, we need to estimate the quantity $\mathcal{A}$ defined as in (2.12) with $F = F_R$ given as in (1.10). Our goal is to show $\mathcal{A} \lesssim R$ for large R. Indeed, we already know from Theorem 1.1 that $\sigma_R^2(t) \sim R$, then the desired bound (1.12) (in the **rough case**) follows.

In what follows, we only provide a detailed proof assuming $\gamma_0(s) = |s|^{2H_0-2}$ for $H_0 \in (1/2,1)$, while the other case ($\gamma_0 = \delta_0$) can be dealt with in the same way. We first write

$$\mathcal{A} \le \int_{[0,t]^6} ds ds' dr dr' d\theta d\theta' |s-s'|^{2H_0-2}|r-r'|^{2H_0-2}|\theta-\theta'|^{2H_0-2}\mathcal{A}_0,$$

where, using a changing of variables in space,

$$\begin{aligned}
\mathcal{A}_0 :=& \int_{\mathbb{R}^6} dy dy' dz dz' dw dw' \int_{[-R,R]^4} dx_1 dx_2 dx_3 dx_4 \\
&\times |y'|^{2H_1-2}|z'|^{2H_1-2}|w'|^{2H_1-2} \\
&\times \big\|D_{r',z+z'}u(t,x_1) - D_{r',z}u(t,x_1)\big\|_4 \big\|D_{\theta',w+w'}u(t,x_2) - D_{\theta',w}u(t,x_2)\big\|_4 \\
&\times \Big\|D_{s,y+y'}D_{r,z+z'}u(t,x_3) - D_{s,y+y'}D_{r,z}u(t,x_3) \\
&\qquad - D_{s,y}D_{r,z+z'}u(t,x_3) + D_{s,y}D_{r,z}u(t,x_3)\Big\|_4 \\
&\times \big\|D_{s',y+y'}D_{\theta,w+w'}u(t,x_4) - D_{s',y+y'}D_{\theta,w}u(t,x_4) - D_{s',y}D_{\theta,w+w'}u(t,x_4) \\
&\quad + D_{s',y}D_{\theta,w}u(t,x_4)\big\|_4.
\end{aligned}$$

Next, we will estimate $\mathcal{A}_0$. Suppose $0 < r < s < t$, and $0 < \theta < s' < t$. We deduce from Proposition 4.1 that

$$\begin{aligned}\mathcal{A}_0 \lesssim \int_{[-R,R]^4} dx_1dx_2dx_3dx_4 \int_{\mathbb{R}^6} dydy'dzdz'dwdw' \\ \times |y'|^{2H_1-2}|z'|^{2H_1-2}|w'|^{2H_1-2} \\ \times \big(\Phi_{t-r',z'}p_{4(t-r')}\big)(x_1-z)\big(\Phi_{t-\theta',w'}p_{4(t-\theta')}\big)(x_2-w) \\ \times \Lambda_{r,z',s,y'}(p_{4(t-s)},p_{4(s-r)})(x_3-y,y-z) \\ \times \Lambda_{\theta,w',s',y'}(p_{4(t-s')},p_{4(s'-\theta)})(x_4-y,y-w).\end{aligned}$$

Now we first integrate out x_3, x_4 and x_2 one by one (see Remark 2.8):

$$\begin{aligned}\mathcal{A}_0 \lesssim \int_{-R}^{R} dx_1 \int_{\mathbb{R}^6} dydy'dzdz'dwdw' \\ \times |y'|^{2H_1-2}|z'|^{2H_1-2}|w'|^{2H_1-2} \\ \times \big(\Phi_{t-r',z'}p_{4(t-r')}\big)(x_1-z)\big(\Phi_{t-\theta',w'}\mathbf{1}_{\mathbb{R}}\big)(0) \\ \times \Lambda_{r,z',s,y'}(\mathbf{1}_{\mathbb{R}},p_{4(s-r)})(0,y-z) \\ \times \Lambda_{\theta,w',s',y'}(\mathbf{1}_{\mathbb{R}},p_{4(s'-\theta)})(0,y-w),\end{aligned}$$

and then integrate out w, y and z to get

$$\begin{aligned}\mathcal{A}_0 \lesssim \int_{-R}^{R} dx_1 \int_{\mathbb{R}^3} dy'dz'dw' \\ \times |y'|^{2H_1-2}|z'|^{2H_1-2}|w'|^{2H_1-2} \\ \times \big(\Phi_{t-r',z'}\mathbf{1}_{\mathbb{R}}\big)(0)\big(\Phi_{t-\theta',w'}\mathbf{1}_{\mathbb{R}}\big)(0)\Lambda_{r,z',s,y'}(\mathbf{1}_{\mathbb{R}},\mathbf{1}_{\mathbb{R}})(0,0) \\ \times \Lambda_{\theta,w',s',y'}(\mathbf{1}_{\mathbb{R}},\mathbf{1}_{\mathbb{R}})(0,0).\end{aligned}$$

Applying Cauchy-Schwarz inequality, we can further deduce that

$$\begin{aligned}\mathcal{A}_0 &\lesssim R\int_{\mathbb{R}} dy'|y'|^{2H_1-2}\bigg(\int_{\mathbb{R}^2} dz'dw'|z'|^{2H_1-2}|w'|^{2H_1-2} \\ &\qquad\qquad \times \big|\big(\Phi_{t-r',z'}\mathbf{1}_{\mathbb{R}}\big)(0)\big(\Phi_{t-\theta',w'}\mathbf{1}_{\mathbb{R}}\big)(0)\big|^2\bigg)^{1/2} \\ &\quad \times \bigg(\int_{\mathbb{R}^2} dz'dw'|z'|^{2H_1-2}|w'|^{2H_1-2} \\ &\qquad \times \big|\Lambda_{r,z',s,y'}(\mathbf{1}_{\mathbb{R}},\mathbf{1}_{\mathbb{R}})(0,0)\Lambda_{\theta,w',s',y'}(\mathbf{1}_{\mathbb{R}},\mathbf{1}_{\mathbb{R}})(0,0)\big|^2\bigg)^{1/2} \\ &\lesssim R\bigg(\int_{\mathbb{R}^2} dz'dw'|z'|^{2H_1-2}|w'|^{2H_1-2}\big|\big(\Phi_{t-r',z'}\mathbf{1}_{\mathbb{R}}\big)(0)\big(\Phi_{t-\theta',w'}\mathbf{1}_{\mathbb{R}}\big)(0)\big|^2\bigg)^{1/2} \\ &\quad \times \bigg(\int_{\mathbb{R}^2} dy'dz'|y'|^{2H_1-2}|z'|^{2H_1-2}\big|\Lambda_{r,z',s,y'}(\mathbf{1}_{\mathbb{R}},\mathbf{1}_{\mathbb{R}})(0,0)\big|^2\bigg)^{1/2} \\ &\quad \times \bigg(\int_{\mathbb{R}^2} dy'dw'|y'|^{2H_1-2}w'|^{2H_1-2}\big|\Lambda_{\theta,w',s',y'}(\mathbf{1}_{\mathbb{R}},\mathbf{1}_{\mathbb{R}})(0,0)\big|^2\bigg)^{1/2}.\end{aligned}$$

Due to the fact that $2H_1 + 2H_0 - \frac{5}{2} > -1$ and $2H_1 < 1$, we have

$$\begin{aligned}&\int_{\mathbb{R}} dz'|z'|^{2H_1-2}\big|(\Phi_{t-r',z'}\mathbf{1}_{\mathbb{R}})(0)\big|^2 \\ &= 8\int_{\sqrt{t-r'}}^{\infty} dz'|z'|^{2H_1-2} + 2(t-r')^{\frac{1}{4}-H_0}\int_0^{\sqrt{t-r'}} |z'|^{2H_1+2H_0-\frac{5}{2}}dz' \lesssim (t-r')^{H_1-\frac{1}{2}}. \quad (4.7)\end{aligned}$$

We can also deduce the next inequality by definition of Λ and Remark 2.7,

$$\begin{aligned}\left|\Lambda_{r,z',s,y'}(\mathbf{1}_{\mathbb{R}},\mathbf{1}_{\mathbb{R}})(0,0)\right|^2 &\lesssim \big(N_{s-r}(y')N_r(z')+N_{s-r}(y')N_{s-r}(z')\\ &\quad + N_{t-s}(y')N_r(z')+N_{t-s}(y')N_{s-r}(z')\big)^2\\ &\lesssim N_{s-r}(y')^2N_r(z')^2+N_{s-r}(y')^2N_{s-r}(z')^2\\ &\quad + N_{t-s}(y')^2N_r(z')^2+N_{t-s}(y')^2N_{s-r}(z')^2,\end{aligned}$$

which, together with (2.17), implies

$$\begin{aligned}&\int_{\mathbb{R}^2} dy'dz'|y'|^{2H_1-2}|z'|^{2H_1-2}|\Lambda_{r,z',s,y'}(\mathbf{1}_{\mathbb{R}},\mathbf{1}_{\mathbb{R}})(0,0)|^2\\ &\lesssim (s-r)^{H_1-\frac12}r^{H_1-\frac12}+(s-r)^{2H_1-1}\\ &\quad +(t-s)^{H_1-\frac12}r^{H_1-\frac12}+(t-s)^{H_1-\frac12}(s-r)^{H_1-\frac12}.\end{aligned} \tag{4.8}$$

As a consequence of (4.7) and (4.8), we get $\mathcal{A}_0 \lesssim \mathcal{B}_0 R$ with

$$\begin{aligned}\mathcal{B}_0 &:= (t-r')^{\frac{H_1}{2}-\frac14}(t-\theta')^{\frac{H_1}{2}-\frac14}\Big[(s-r)^{\frac{H_1}{2}-\frac14}r^{\frac{H_1}{2}-\frac14}+(s-r)^{H_1-\frac12}\\ &\qquad + (t-s)^{\frac{H_1}{2}-\frac14}r^{\frac{H_1}{2}-\frac14}+(t-s)^{\frac{H_1}{2}-\frac14}(s-r)^{\frac{H_1}{2}-\frac14}\Big]\\ &\quad\times\Big[(s'-\theta)^{\frac{H_1}{2}-\frac14}\theta^{\frac{H_1}{2}-\frac14}+(s'-\theta)^{H_1-\frac12}+(t-s')^{\frac{H_1}{2}-\frac14}\theta^{\frac{H_1}{2}-\frac14}\\ &\qquad + (t-s')^{\frac{H_1}{2}-\frac14}(s'-\theta)^{\frac{H_1}{2}-\frac14}\Big].\end{aligned}$$

Notice that with $\gamma_0(s)=|s|^{2H_0-2}$ for $H_0\in(1/2,1)$. This allows us to apply the embedding inequality (2.9) and get

$$\begin{aligned}&\int_{\substack{0<r<s<t\\0<\theta<s'<t}} drdsd\theta ds'\int_{[0,t]^2}dr'd\theta'\gamma_0(s-s')\gamma_0(r-r')\gamma_0(\theta-\theta')\mathcal{B}_0\\ &\lesssim \Big\{\int_{0<r<s<t}drds\int_0^t d\theta'(t-\theta')^{\frac{H_1}{2H_0}-\frac{1}{4H_0}}\big((s-r)^{\frac{H_1}{2}-\frac14}r^{\frac{H_1}{2}-\frac14}+(s-r)^{H_1-\frac12}\\ &\qquad + (t-s)^{\frac{H_1}{2}-\frac14}r^{\frac{H_1}{2}-\frac14}+(t-s)^{\frac{H_1}{2}-\frac14}(s-r)^{\frac{H_1}{2}-\frac14}\big)^{1/H_0}\Big\}^{2H_0}\\ &< +\infty.\end{aligned}$$

Therefore,

$$\int_{\substack{0<r<s<t\\0<\theta<s'<t}} drdsd\theta ds'\int_{[0,t]^2}dr'd\theta'\gamma_0(s-s')\gamma_0(r-r')\gamma_0(\theta-\theta')\mathcal{A}_0 \lesssim R.$$

For the case $\gamma_0=\delta_0$, the expression for $\mathcal{B}_0$ reduces to

$$\begin{aligned}\mathcal{B}_0 &= (t-r)^{\frac{H_1}{2}-\frac14}(t-\theta)^{\frac{H_1}{2}-\frac14}\big[(s-r)^{\frac{H_1}{2}-\frac14}r^{\frac{H_1}{2}-\frac14}+(s-r)^{H_1-\frac12}\\ &\qquad + (t-s)^{\frac{H_1}{2}-\frac14}r^{\frac{H_1}{2}-\frac14}+(t-s)^{\frac{H_1}{2}-\frac14}(s-r)^{\frac{H_1}{2}-\frac14}\big]\\ &\quad\times\big[(s-\theta)^{\frac{H_1}{2}-\frac14}\theta^{\frac{H_1}{2}-\frac14}+(s-\theta)^{H_1-\frac12}+(t-s)^{\frac{H_1}{2}-\frac14}\theta^{\frac{H_1}{2}-\frac14}\\ &\qquad + (t-s)^{\frac{H_1}{2}-\frac14}(s-\theta)^{\frac{H_1}{2}-\frac14}\big].\end{aligned}$$

and for the same reason as above,

$$\int_{\substack{0<r<s<t\\0<\theta<s<t}} drdsd\theta\mathcal{A}_0 \lesssim R\int_{\substack{0<r<s<t\\0<\theta<s<t}} drdsd\theta\mathcal{B}_0 \lesssim R.$$

The remaining three cases

$$\begin{cases} 0 < s < r < t \text{ and } 0 < \theta < s' < t \\ 0 < r < s < t \text{ and } 0 < s' < \theta < t \\ 0 < s < r < t \text{ and } 0 < s' < \theta < t \end{cases}$$

can be estimated in an almost same way and finally we can get the same upper bounds. That is, we obtain the desired bound $\mathcal{A} \lesssim R$ and hence conclude the proof of (1.12) under Hypothesis 2. □

4.2 Proof of some auxiliary results

In this subsection, we introduce some auxiliary results and provide the proof of Lemma 4.2 and 4.3 in Section 4.2.2.

4.2.1 Estimates for fixed Wiener chaoses

Fix $0 < s < t < \infty$ and $x, y \in \mathbb{R}$. For any $n = 1, 2, \dots$, let $g^1_{s,t,x,n}$ and $g^2_{s,y,t,x,n}$ be functions on $[s,t]^n \times \mathbb{R}^n$ given by

$$g^1_{s,t,x,n}(\mathbf{s}_n, \mathbf{y}_n) = n! f_{t,x,n}(\mathbf{s}_n, \mathbf{y}_n) \mathbf{1}_{[s,t]^n}(\mathbf{s}_n), \tag{4.9}$$

and

$$g^2_{s,y,t,x,n}(\mathbf{s}_n, \mathbf{y}_n) = p_{s_{\sigma(1)}-s}(y_{\sigma(1)} - y) g^1_{s,t,x,n}(\mathbf{s}_n, \mathbf{y}_n), \tag{4.10}$$

where $f_{t,x,n}$ is defined as in (2.6) and σ, because of the indicator function $\mathbf{1}_{[s,t]^n}$, is now a permutation on $\{1, \dots, n\}$ such that $s < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$.

In Section 4.2.2 below, we will see that g^1 and g^2 are closely related to the chaos coefficients of the Mallivain derivatives of u. In fact, the next lemmas, which give some estimates for g^1_n and g^2_n, are essential to the proofs of Lemmas 4.2 and 4.3.

Lemma 4.4. *Let $0 < s < t < \infty$ and let $x \in \mathbb{R}$. Fix a positive integer n, and let g^1_n be given as in* (4.9)*. Then, the following equalities hold.*

$$\Big(\int_{[s,t]^n} d\mathbf{s}_n \| g^1_{s,t,x,n}(\mathbf{s}_n, \bullet) \|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes n}} \Big)^{2H_0} \le C_2(n, t-s) \tag{4.11}$$

and

$$\Big(\int_{[s,t]^n} d\mathbf{s}_n \| g^1_{s,t,x+x',n}(\mathbf{s}_n, \bullet) - g^1_{s,t,x,n}(\mathbf{s}_n, \bullet) \|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes n}} \Big)^{2H_0} \le C_2(n, t-s) N_{t-s}(x')^2, \tag{4.12}$$

where N_t is defined as in (2.16) *and $C_2(n,t) = c_1 c_2^n \Gamma((2H_0 + H_1 - 1)n + 1) t^{(2H_0+H_1-1)n}$ for all positive integers n and real numbers $t > 0$ with some constants c_1, c_2 depending on H_0 and H_1.*

Proof. Fix $s < s_1 < s_2 < \cdots < s_n < t$. Denote by $\widehat{g}^1_n$ the Fourier transformation of g^1_n with respect to the spatial arguments. Following the idea in [14, Theorem 3.4], we can show that

$$\begin{aligned} \| g^1_{s,t,x,n}(\mathbf{s}_n, \bullet) \|^2_{\mathcal{H}_1^{\otimes n}} &= c^n_{H_1} \int_{\mathbb{R}^n} d\boldsymbol{\xi}_n \prod_{j=1}^n |\xi_j|^{1-2H_1} |\widehat{g}^1_{s,t,x,n}(\mathbf{s}_n, \boldsymbol{\xi}_n)|^2 \\ &\le c^n_{H_1} \sum_{\boldsymbol{\alpha}_k \in D_n} \int_{\mathbb{R}^n} d\boldsymbol{\eta}_n |\eta_1|^{1-2H_1} \prod_{j=1}^n |\eta_j|^{\alpha_j} \times \prod_{j=1}^{n-1} e^{-(s_{j+1}-s_j)|\eta_j|^2} \times e^{-(t-s_n)|\eta_n|^2} \\ &\le c_1 c_2^n \sum_{\boldsymbol{\alpha}_n \in D_n} (s_2 - s_1)^{-\frac{2-2H_1+\alpha_1}{2}} \times \prod_{j=2}^n (s_{j+1} - s_j)^{-\frac{1+\alpha_j}{2}}, \end{aligned} \tag{4.13}$$

where D_n is a collection of multi-indexes $\boldsymbol{\alpha}_k = (\alpha_1, \dots, \alpha_n)$ with

$$\alpha_1, \alpha_n \in \{0, 1-2H_1\},\ \alpha_i \in \{0, 1-2H_1, 2(1-2H_1)\},\ \forall i = 2, \dots, n-1, \tag{4.14}$$

and

$$\sum_{i=1}^{n} \alpha_i = (n-1)(1-2H_1). \tag{4.15}$$

Applying [15, Lemma 4.5], and the fact that

$$\Big(\sum_{i=1}^{m} x_i\Big)^{\gamma} \le m^{\gamma} \sum_{i=1}^{m} x_i^{\gamma}$$

for all $m = 1, 2, \dots$, and $x_1, \dots, x_m, \gamma > 0$, we get from (4.13) that

$$\begin{aligned}\int_{[s,t]^n} d\mathbf{s}_n \|g^1_{s,t,x,n}(\mathbf{s}_n, \bullet)\|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes n}} =& n! \int_{\mathbb{T}_n^{s,t}} d\mathbf{s}_n \|g^1_{s,t,x,n}(\mathbf{s}_n, \bullet)\|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes n}} \\ \le& c_1 c_2^n n! (t-s)^{\frac{2H_0+H_1-1}{2H_0} n} \Gamma\Big(\frac{2H_0+H_1-1}{2H_0} n + 1\Big)^{-1}. \end{aligned} \tag{4.16}$$

Inequality (4.11) is thus a consequence of inequality (4.16) and the next inequality

$$\Gamma\Big(\frac{2H_0+H_1-1}{2H_0} n + 1\Big)^{-2H_0} \le c_1 c_2^n \Gamma\big((2H_0+H_1-1)n+1\big)^{-1}. \tag{4.17}$$

Inequality (4.17) can be proved as a corollary of Lemma 2.9 (i). Denote by $H = \frac{2H_0+H_1-1}{2H_0}$. Thanks to Lemma 2.9 (i), we have

$$\Gamma(Hn+1)^{-2H_0} \le \Big(\sqrt{2\pi}(Hn+1)^{Hn+\frac{1}{2}} e^{-(Hn+1)}\Big)^{-2H_0} = A_1 \times A_2, \tag{4.18}$$

where

$$A_1 = (2\pi)^{-\frac{1}{2}} \big(2H_0Hn + 2H_0\big)^{-2H_0Hn - \frac{1}{2}} e^{(2H_0Hn+1) - \frac{1}{12(2H_0Hn+1)}}$$

and

$$A_2 = (2\pi)^{\frac{1}{2}-H_0} (2H_0)^{2H_0Hn - H_0} (2H_0Hn + 2H_0)^{\frac{1}{2}-H_0} e^{(2H_0-1) + \frac{1}{12(2H_0Hn+1)}} \le c_1 c_2^n, \tag{4.19}$$

with some constants c_1 and c_2 depending on H_0 and H_1. Notice that $H_0 \ge \frac{1}{2}$. It follows that $(2H_0Hn+2H_0)^{-2H_0Hn-\frac{1}{2}} \le (2H_0Hn+1)^{-2H_0Hn-\frac{1}{2}}$, and thus due to Lemma 2.9 (i) again,

$$A_1 \le (2\pi)^{-\frac{1}{2}} \big(2H_0Hn+1\big)^{-2H_0Hn-\frac{1}{2}} e^{(2H_0Hn+1) - \frac{1}{12(2H_0Hn+1)}} \le \Gamma\big(2H_0Hn+1\big)^{-1}. \tag{4.20}$$

Then, inequality (4.17) follows from (4.18)–(4.20). The proof of (4.11) is complete.

The proof of (4.12) is quite similar. Fix $s < s_1 < \cdots < s_n < t$. By the Fourier transformation, we can write

$$\begin{aligned} J_n :=& \|g^1_{s,t,x+x',n}(\mathbf{s}_n, \bullet) - g^1_{s,t,x,n}(\mathbf{s}_n, \bullet)\|^2_{\mathcal{H}_1^{\otimes n}} \\ =& c_{H_1}^n \int_{\mathbb{R}^n} d\boldsymbol{\xi}_n \prod_{j=1}^{n} |\xi_j|^{1-2H_1} |\widehat{g}^1_{s,t,x+x',n}(\mathbf{s}_n, \boldsymbol{\xi}_n) - \widehat{g}^1_{s,t,x,n}(\mathbf{s}_n, \boldsymbol{\xi}_n)|^2 \\ \le& c_{H_1}^n \sum_{\boldsymbol{\alpha} \in D_n} \int_{\mathbb{R}^n} d\boldsymbol{\eta}_n |\eta_1|^{1-2H_1} \prod_{j=1}^{n} |\eta_j|^{\alpha_j} \times \prod_{j=1}^{n} e^{-(s_{j+1}-s_j)|\eta_j|^2} |e^{-i(x+x')\eta_n} - e^{-ix\eta_n}|^2, \end{aligned}$$

where D_n is a set of multi-indexes defined as in (4.14) and (4.15). Using the elementary calculus, we can show that for all $x \in \mathbb{R}$,

$$|e^{-ix} - 1| \le |x|\mathbf{1}_{\{|x|\le 1\}} + 2 \times \mathbf{1}_{\{|x|>1\}}, \tag{4.21}$$

and thus,

$$\begin{aligned} J_n \le & c_1 c_2^n \sum_{\boldsymbol{\alpha}\in D_n} \Big[|x'|^2 \int_{\mathbb{R}^n} d\boldsymbol{\eta}_n |\eta_1|^{1-2H_1} \prod_{j=1}^n |\eta_j|^{\alpha_j} |\eta_n|^2 \prod_{j=1}^n e^{-(s_{j+1}-s_j)|\eta_j|^2} \mathbf{1}_{\{|x'\eta_n|\le 1\}} \\ & + \int_{\mathbb{R}^n} d\boldsymbol{\eta}_n |\eta_1|^{1-2H_1} \prod_{j=1}^n |\eta_j|^{\alpha_j} \times \prod_{j=1}^n e^{-(s_{j+1}-s_j)|\eta_j|^2} \mathbf{1}_{\{|x'\eta_n|>1\}} \Big] \\ \le & c_1 c_2^n \sum_{\boldsymbol{\alpha}_n\in D_n} (s_2 - s_1)^{-\frac{2-2H_1+\alpha_1}{2}} \times \prod_{j=2}^{n-1} (s_{j+1} - s_j)^{-\frac{1+\alpha_j}{2}} \\ & \times \Big[|x'|^2 \int_0^{|x'|^{-1}} d\eta_n |\eta_n|^{2+\alpha_n} e^{-(t-s_n)\eta_n^2} + \int_{|x'|^{-1}}^{\infty} d\eta_n |\eta_n|^{\alpha_n} e^{-(t-s_n)\eta_n^2} \Big]. \end{aligned}$$

Preforming the changing of variable $\sqrt{t-s_n}\eta_n = \eta$, we can write

$$\begin{aligned} & \int_0^{|x'|^{-1}} d\eta_n |\eta_n|^{2+\alpha_n} e^{-(t-s_n)\eta_n^2} \\ & = (t-s_n)^{-\frac{3}{2}-\frac{1}{2}\alpha_n} \int_0^{\frac{\sqrt{t-s_n}}{|x'|}} d\eta |\eta|^{2+\alpha_n} e^{-\eta^2} \\ & \le (t-s_n)^{-\frac{3}{2}-\frac{1}{2}\alpha_n} \Big(\int_0^{\infty} d\eta |\eta|^{2+\alpha_n} e^{-\eta^2} \mathbf{1}_{\{|x'|\le\sqrt{t-s_n}\}} \\ & \quad + \int_0^{\frac{\sqrt{t-s_n}}{|x'|}} d\eta |\eta|^{2+\alpha_n} \mathbf{1}_{\{|x'|>\sqrt{t-s_n}\}} \Big) \\ & \le c_1 (t-s_n)^{-\frac{3}{2}-\frac{1}{2}\alpha_n} \big(\mathbf{1}_{\{|x'|\le\sqrt{t-s_n}\}} + (t-s_n)^{\frac{3+\alpha_n}{2}} |x'|^{-3-\alpha_n} \mathbf{1}_{\{|x'|>\sqrt{t-s_n}\}} \big) \\ & \le c_1 \Big((t-s_n)^{-\frac{1}{4}-\frac{1}{2}\alpha_n - H_0} |x'|^{-\frac{5}{2}+2H_0} \mathbf{1}_{\{|x'|\le\sqrt{t-s_n}\}} \\ & \qquad + (t-s_n)^{-\frac{1+\alpha_n}{2}} |x'|^{-2} \mathbf{1}_{\{|x'|>\sqrt{t-s_n}\}} \Big). \end{aligned}$$

Similarly, we can also show that

$$\begin{aligned} & \int_{|x'|^{-1}}^{\infty} d\eta_n |\eta_n|^{\alpha_n} e^{-(t-s_n)\eta_n^2} \\ & = (t-s_n)^{-\frac{1}{2}-\frac{1}{2}\alpha_n} \int_{\frac{\sqrt{t-s_n}}{|x'|}}^{\infty} d\eta_n |\eta_n|^{\alpha_n} e^{-\eta_n^2} \\ & \le c_1 (t-s_n)^{-\frac{1}{2}-\frac{1}{2}\alpha_n} \bigg(\Big(\frac{|x'|}{\sqrt{t-s_n}} \Big)^2 \mathbf{1}_{\{|x'|\le\sqrt{t-s_n}\}} + \mathbf{1}_{\{|x'|>\sqrt{t-s_n}\}} \bigg) \\ & \le c_1 \Big((t-s_n)^{-\frac{1}{4}-\frac{1}{2}\alpha_n - H_0} |x'|^{-\frac{1}{2}+2H_0} \mathbf{1}_{\{|x'|\le\sqrt{t-s_n}\}} \\ & \qquad + (t-s_n)^{-\frac{1+\alpha_n}{2}} \mathbf{1}_{\{|x'|>\sqrt{t-s_n}\}} \Big). \end{aligned}$$

Therefore,

$$\begin{aligned} J_n \le & c_1 c_2^n \sum_{\boldsymbol{\alpha}_n\in D_n} (s_2 - s_1)^{-\frac{2-2H_1+\alpha_1}{2}} \times \prod_{j=2}^{n-1} (s_{j+1} - s_j)^{-\frac{1+\alpha_j}{2}} \\ & \times \big((t-s_n)^{-\frac{1}{4}-\frac{1}{2}\alpha_n - H_0} |x'|^{-\frac{1}{2}+2H_0} \mathbf{1}_{\{|x'|\le\sqrt{t-s_n}\}} + (t-s_n)^{-\frac{1+\alpha_n}{2}} \mathbf{1}_{\{|x'|>\sqrt{t-s_n}\}} \big). \end{aligned}$$

By using [15, Lemma 4.5] again, we get the following inequality,

$$\Big(n!\int_{\mathbb{T}_n^{s,t}} d\mathbf{s}_n J_n^{\frac{1}{2H_0}}\Big)^{2H_0} \le c_1 c_2^n\Big[n!\Gamma\Big(\frac{2H_0+H_1-1}{2H_0}n+1\Big)^{-1}(t-s)^{\frac{2H_0+H_1-1}{2H_0}n} \times \big(|x'|^{\frac{-1+4H_0}{4H_0}}(t-s)^{\frac{1-4H_0}{8H_0}}\mathbf{1}_{\{|x'|\le\sqrt{t-s}\}}+\mathbf{1}_{\{|x'|>\sqrt{t-s}\}}\big)\Big]^{2H_0}. \tag{4.22}$$

Hence, inequality (4.12) follows from inequality (4.22) and Lemma 2.9 (i). The proof of Lemma 4.4 is competed. □

Lemma 4.5. *Let* $0<s<t<\infty$ *and let* $x,y\in\mathbb{R}$. *For any* $n=1,2,\dots$, *let* g_n *be given in* (4.10). *Then, the following equalities hold.*

$$\Big(\int_{[s,t]^n} d\mathbf{s}_n \|g^2_{s,y,t,x,n}(\mathbf{s}_n,\bullet)\|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes n}}\Big)^{2H_0} \le C_2(n,t-s)p_{t-s}(x-y)^2, \tag{4.23}$$

$$\Big(\int_{[s,t]^n} d\mathbf{s}_n \|g^2_{s,y+y',t,x,n}(\mathbf{s}_n,\bullet)-g^2_{s,y,t,x,n}(\mathbf{s}_n,\bullet)\|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes n}}\Big)^{2H_0} \le C_2(n,t-s)\big(\big|\Delta_{t-s}(y-x,y')\big|+p_{t-s}(x-y)N_{t-s}(y')\big)^2, \tag{4.24}$$

$$\Big(\int_{[s,t]^n} d\mathbf{s}_n \|g^2_{s,y,t,x+x',n}(\mathbf{s}_n,\bullet)-g^2_{s,y,t,x,n}(\mathbf{s}_n,\bullet)\|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes n}}\Big)^{2H_0} \le C_2(n,t-s)\big(\big|\Delta_{t-s}(x-y,x')\big|+p_{t-s}(x-y)N_{t-s}(x')\big)^2, \tag{4.25}$$

and

$$\Big(\int_{[s,t]^n} d\mathbf{s}_n \|g^2_{s,y+y',t,x+x',n}(\mathbf{s}_n,\bullet)-g^2_{s,y,t,x+x',n}(\mathbf{s}_n,\bullet) - g^2_{s,y+y',t,x,n}(\mathbf{s}_n,\bullet)+g^2_{s,y,t,x,n}(\mathbf{s}_n,\bullet)\|^{\frac{1}{H_0}}_{\mathcal{H}^{\otimes n}}\Big)^{2H_0} \le C_2(n,t-s)\big(\big|R_{t-s}(x-y,x',y')\big|+\big|\Delta_{t-s}(x-y,x')\big|N_{t-s}(y') + \big|\Delta_{t-s}(y-x,y')\big|N_{t-s}(x')+p_{t-s}(x-y)N_{t-s}(x')N_{t-s}(y')\big)^2, \tag{4.26}$$

where Δ_t, R_t *and* N_t *are defined as in* (2.14)–(2.16) *and* $C_2(n,t-s)$ *are the same as in Lemma 4.4.*

Remark 4.6. In what follows, one may find some structures that are almost the same as in Section 3.2. However, because of the rough dependence in space, the maximal principle is not valid under Hypothesis 2. Therefore, we provide the estimates via a different approach, which involves more careful computations.

Proof of Lemma 4.5. We divide the proof of this lemma into three steps. In Step 1, we prove inequality (4.23), and then inequalities (4.24) and (4.25) in Step 2. Finally, the proof of inequality (4.26) is left in Step 3.

Step 1. Fix $s<s_1<\cdots<s_n<t$. Taking into account formulas (3.15) and (3.16), we can write

$$\widehat{g}^2_{s,y,t,x,n}(\mathbf{s}_n,\boldsymbol{\xi}_n) = p_{t-s}(x-y)\exp\Big(-iy\frac{t-s_1}{t-s}\sum_{k_1=1}^n\Big(\xi_{k_1}\prod_{k_2=2}^{k_1}\frac{t-s_{k_2}}{t-s_{k_2-1}}\Big)\Big) \times\exp\Big[-ix\sum_{j=1}^n\frac{s_j-s_{j-1}}{t-s_{j-1}}\sum_{k_1=j}^n\Big(\xi_{k_1}\prod_{k_2=j+1}^{k_1}\frac{t-s_{k_2}}{t-s_{k_2-1}}\Big)\Big] \times\prod_{j=1}^n\exp\Big[-\frac{(t-s_j)(s_j-s_{j-1})}{2(t-s_{j-1})}\Big[\sum_{k_1=j}^n\Big(\xi_{k_1}\prod_{k_2=j+1}^{k_1}\frac{t-s_{k_2}}{t-s_{k_2-1}}\Big)\Big]^2\Big], \tag{4.27}$$

where by convention $s_0 = s$ and $y_0 = y$. Recall that the maximal inequality is not applicable in this situation and we need to apply another method. Notice that the spectral measure $\mu(d\xi)$ of Hilbert space $\mathcal{H}_1$ has a density $|\xi|^{1-2H_1}$ under Hypothesis 2. This allows us to perform a change of variables. For any $j = 1, \dots, n$, let

$$\eta_j = \sum_{k_1=j}^{n} \xi_{k_1} \prod_{k_2=j+1}^{k_1} \frac{t - s_{k_2}}{t - s_{k_2-1}}.$$

Then, it is clear that $\xi_n = \eta_n$ and

$$\xi_j = \eta_j - \Big(\frac{t - s_{j+1}}{t - s_j}\Big)\eta_{j+1},$$

for all $j = k+2, \dots, n$. Denote by $\Sigma = \Sigma_n = \frac{\partial \boldsymbol{\xi}_n}{\partial \boldsymbol{\eta}_n}$ the Jacobian matrix of the transformation $\boldsymbol{\xi}_n \to \boldsymbol{\eta}_n$. Then, we have $\det(\Sigma) = 1$ and thus

$$\begin{aligned}
&\|g^2_{s,y,t,x,n}(\mathbf{s}_n, \bullet)\|^2_{\mathcal{H}_1^{\otimes n}} \\
=&c^n_{H_1} \int_{\mathbb{R}^n} d\boldsymbol{\xi}_n \prod_{i=1}^{n} |\xi_i|^{1-2H_1} |\widehat{g}^2_{s,y,t,x,n}(\mathbf{s}_n, \boldsymbol{\xi}_n)|^2 \\
=&c^n_{H_1} p_{t-s}(x-y)^2 \int_{\mathbb{R}^n} d\boldsymbol{\eta}_n \prod_{i=1}^{n-1} \Big|\eta_i - \frac{t - s_{i+1}}{t - s_i}\eta_{i+1}\Big|^{1-2H_1} |\eta_n|^{1-2H_1} \\
&\times \prod_{i=1}^{n} \exp\Big(-\frac{(t-s_i)(s_i - s_{i-1})}{(t - s_{i-1})}\eta_i^2\Big).
\end{aligned} \tag{4.28}$$

Using the trivial inequality that $|a+b|^{1-2H_1} \le |a|^{1-2H_1} + |b|^{1-2H_1}$ for all $H_1 \in (0, \frac{1}{2})$ and $a, b \in \mathbb{R}$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} d\boldsymbol{\eta}_n \prod_{i=1}^{n-1} \Big|\eta_i - \frac{t - s_{i+1}}{t - s_i}\eta_{i+1}\Big|^{1-2H_1} |\eta_n|^{1-2H_1} \prod_{i=1}^{n} \exp\Big(-\frac{(t-s_i)(s_i - s_{i-1})}{(t - s_{i-1})}\eta_i^2\Big) \\
\le& \sum_{\boldsymbol{\beta}_{n-1} = (\beta_1, \dots, \beta_{n-1}) \in \{0,1\}^{n-1}} J_{\boldsymbol{\beta}_{n-1}},
\end{aligned} \tag{4.29}$$

where

$$\begin{aligned}
J_{\boldsymbol{\beta}_{n-1}} :=& \int_{\mathbb{R}^n} d\boldsymbol{\eta}_n \prod_{i=1}^{n-1} \Big(|\eta_i|^{\beta_i(1-2H_1)} \Big|\frac{t - s_{i+1}}{t - s_i}\eta_{i+1}\Big|^{(1-\beta_i)(1-2H_1)}\Big) \times |\eta_n|^{1-2H_1} \\
&\times \prod_{i=1}^{n} \exp\Big(-\frac{(t-s_i)(s_i - s_{i-1})}{(t - s_{i-1})}\eta_i^2\Big) \\
=& \int_{\mathbb{R}^n} d\boldsymbol{\eta}_n |\eta_1|^{\beta_1(1-2H_1)} \prod_{i=2}^{n-1} |\eta_i|^{(1-\beta_{i-1}+\beta_i)(1-2H_1)} |\eta_n|^{(2-\beta_{n-1})(1-2H_1)} \\
&\times \prod_{i=1}^{n-1} \Big(\frac{t - s_{i+1}}{t - s_i}\Big)^{(1-\beta_i)(1-2H_1)} \prod_{i=1}^{n} \exp\Big(-\frac{(t-s_i)(s_i - s_{i-1})}{(t - s_{i-1})}\eta_i^2\Big).
\end{aligned}$$

Fix $\boldsymbol{\beta}_{n-1} \in \{0,1\}^{n-1}$. Then, we can show that

$$\begin{aligned} J_{\boldsymbol{\beta}_{n-1}} \le & c_1 c_2^n \Big(\frac{(t-s)}{(t-s_1)(s_1-s)} \Big)^{\frac12 + \frac12 \beta_1 (1-2H_1)} \\ & \times \prod_{i=2}^{n-1} \Big(\frac{(t-s_{i-1})}{(t-s_i)(s_i - s_{i-1})} \Big)^{\frac12 + \frac12 (1-\beta_{i-1}+\beta_i)(1-2H_1)} \\ & \times \Big(\frac{(t-s_{n-1})}{(t-s_n)(s_n-s_{n-1})} \Big)^{\frac12 + \frac12 (2-\beta_{n-1})(1-2H_1)} \prod_{i=2}^{n} \Big(\frac{t-s_i}{t-s_{i-1}} \Big)^{(1-\beta_{i-1})(1-2H_1)}. \end{aligned}$$

After simplification, we get

$$\begin{aligned} J_{\boldsymbol{\beta}_{n-1}} \le & c_1 c_2^n (t-s)^{\frac12 + \frac12 \beta_1 (1-2H_1)} (t-s_1)^{\frac12 (\beta_2 - 1)(1-2H_1)} \prod_{i=2}^{n-2} (t-s_i)^{\frac12 (\beta_{i+1} - \beta_{i-1})(1-2H_1)} \\ & \times (t-s_{n-1})^{\frac12 (1-\beta_{n-2})(1-2H_1)} (t-s_n)^{-\frac12 - \frac12 \beta_{n-1}(1-2H_1)} (s_1 - s)^{-\frac12 - \frac12 \beta_1 (1-2H_1)} \\ & \times \prod_{i=2}^{n-1} (s_i - s_{i-1})^{-\frac12 - \frac12 (1-\beta_{i-1}+\beta_i)(1-2H_1)} (s_n - s_{n-1})^{-\frac12 - \frac12 (2-\beta_{n-1})(1-2H_1)}. \end{aligned} \quad (4.30)$$

Recalling that $s < s_1 < \cdots < s_n < t$, we can write, with the convention $s_0 = s$,

$$\begin{aligned} & (t-s)^{\frac12 + \frac12 \beta_1 (1-2H_1)} (t-s_1)^{\frac12 (\beta_2 - 1)(1-2H_1)} \prod_{i=2}^{n-2} (t-s_i)^{\frac12 (\beta_{i+1} - \beta_{i-1})(1-2H_1)} \\ & \times (t-s_{n-1})^{\frac12 (1-\beta_{n-2})(1-2H_1)} (t-s_n)^{-\frac12 - \frac12 \beta_{n-1}(1-2H_1)} \\ = & (t-s)^{\frac12} (t-s_1)^{-\frac12 (1-2H_1)} \prod_{i=1}^{n-1} \Big(\frac{t-s_{i-1}}{t-s_{i+1}} \Big)^{\frac12 \beta_i (1-2H_1)} (t-s_{n-1})^{\frac12 (1-2H_1)} (t-s_n)^{-\frac12} \\ \le & \Big(\frac{t-s}{t-s_n} \Big)^{\frac12} \Big(\frac{t-s_{n-1}}{t-s_1} \Big)^{\frac12 (1-2H_1)} \prod_{i=1}^{n-1} \Big(\frac{t-s_{i-1}}{t-s_{i+1}} \Big)^{\frac12 (1-2H_1)} = \Big(\frac{t-s}{t-s_n} \Big)^{1-H_1}. \end{aligned} \quad (4.31)$$

Therefore, combining (4.28)–(4.31), we have

$$\begin{aligned} \| g_n^2(\mathbf{s}_n, \bullet, s, y, t, x) \|^2_{\mathcal{H}_1^{\otimes n}} \le & c_1 c_2^n p_{t-s}(x-z)^2 (t-s)^{1-H_1} (t-s_n)^{H_1 - 1} \\ & \times \sum_{\boldsymbol{\beta}_{n-1} \in \{0,1\}^{n-1}} (s_1 - s)^{-\frac12 - \frac12 \beta_1 (1-2H_1)} \prod_{i=2}^{n-1} (s_i - s_{i-1})^{-\frac12 - \frac12 (1-\beta_{i-1}+\beta_i)(1-2H_1)} \\ & \times (s_n - s_{n-1})^{-\frac12 - \frac12 (2-\beta_{n-1})(1-2H_1)}. \end{aligned}$$

In the next step, we estimate the time integral of $\| g_n^2(\mathbf{s}_n, \bullet, s, y, t, x) \|^2_{\mathcal{H}_1^{\otimes n}}$. Fix $\boldsymbol{\beta}_{n-1} \in \{0,1\}^{n-1}$. Put

$$\alpha_1 = \frac{-\frac12 - \frac12 \beta_1 (1-2H_1)}{2H_0}, \quad \alpha_i = \frac{-\frac12 - \frac12 (1-\beta_{i-1}+\beta_i)(1-2H_1)}{2H_0}$$

for all $i = 2, \dots, n-1$, and

$$\alpha_n = \frac{-\frac12 - \frac12 (2-\beta_{n-1})(1-2H_1)}{2H_0}, \quad \text{and} \quad \alpha_{n+1} = \frac{H_1 - 1}{2H_0}.$$

Then, as a consequence of Hypothesis 2, one can show that $\alpha_i \in (-1, 0]$ for all

$i = 1, \dots, n+1$. This allows us to apply the beta function iteratively and get

$$\int_{\mathbb{T}_n^{s,t}} d\mathbf{s}_n \prod_{i=1}^{n+1} (s_i - s_{i-1})^{\alpha_i} = \frac{\prod_{i=1}^{n+1} \Gamma(\alpha_i + 1)(t-s)^{\sum_{i=1}^{n+1} \alpha_i + n}}{\Gamma(\sum_{i=1}^{n+1} \alpha_i + n + 1)}$$
$$\leq c_1 c_2^n \frac{(t-s)^{\sum_{i=1}^{n+1} \alpha_i + n}}{\Gamma(\sum_{i=1}^{n+1} \alpha_i + n + 1)},$$

where by convention convention $s_{n+1} = t$ and positive constants c_1, c_2 depend on H_0 and H_1. The above inequality is true for all $\alpha_i \in (-1, 1)$, $i = 1, \dots, n+1$, and [15, Lemma 4.5] is a special case when $\alpha_{n+1} = 0$. It follows that

$$\int_{\mathbb{T}_n^{s,t}} d\mathbf{s}_n \prod_{i=1}^{n+1} (s_i - s_{i-1})^{\alpha_i} \leq c_1 c_2^n \frac{(t-s)^{\frac{2H_0+H_1-1}{2H_0} n + \frac{H_1-1}{2H_0}}}{\Gamma(\frac{2H_0+H_1-1}{2H_0} n + \frac{H_1-1}{2H_0} + 1)}.$$

Therefore,

$$\int_{[s,t]^n} d\mathbf{s}_n \|g_n^2(\mathbf{s}_n, \bullet, s, y, t, x)\|_{\mathcal{H}_1^{\otimes n}}^{\frac{1}{H_0}} \leq \frac{c_1 c_2^n n! p_{t-s}(x-y)^{\frac{1}{H_0}} (t-s)^{\frac{2H_0+H_1-1}{2H_0} n}}{\Gamma(\frac{2H_0+H_1-1}{2H_0} n + \frac{H_1-1}{2H_0} + 1)}. \tag{4.32}$$

Thus, inequality (4.23) follows from inequality (4.32) and Lemma 2.9 (i) by a similar argument as in the proof of Lemma 4.4.

Step 2. Fix $s < s_1 < \cdots < s_n < t$. Let

$$J_n := \|g_n^2(\mathbf{s}_n, \bullet, s, y + y', t, x) - g_n^2(\mathbf{s}_n, \bullet, s, y, t, x)\|_{\mathcal{H}_1^{\otimes n}}^2.$$

Taking into account formula (4.27), we obtain the next equality analogously to (4.28),

$$\begin{aligned} J_n =& c_{H_1}^n \int_{\mathbb{R}^n} d\boldsymbol{\eta}_n \Big| p_{t-s}(x-y-y') \exp\Big(-i(y+y')\frac{t-s_1}{t-s}\eta_1\Big) \\ &- p_{t-s}(x-y) \exp\Big(-iy\frac{t-s_1}{t-s}\eta_1\Big)\Big|^2 \prod_{i=1}^{n-1} |\eta_i - \frac{t-s_{i+1}}{t-s_i}\eta_{i+1}|^{1-2H_1} |\eta_n|^{1-2H_1} \\ &\times \prod_{i=1}^{n} \exp\Big(-\frac{(t-s_i)(s_i-s_{i-1})}{(t-s_{i-1})}\eta_i^2\Big) \leq c_1 c_2^n (G_1 + G_2), \end{aligned} \tag{4.33}$$

where

$$\begin{aligned} G_1 =& |p_{t-s}(x-y-y') - p_{t-s}(x-y)|^2 \int_{\mathbb{R}^n} d\boldsymbol{\eta}_n \prod_{i=1}^{n-1} |\eta_i - \frac{t-s_{i+1}}{t-s_i}\eta_{i+1}|^{1-2H_1} |\eta_n|^{1-2H_1} \\ &\times \prod_{i=1}^{n} \exp\Big(-\frac{(t-s_i)(s_i-s_{i-1})}{(t-s_{i-1})}\eta_i^2\Big), \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} G_2 =& p_{t-s}(x-y)^2 \int_{\mathbb{R}^n} d\boldsymbol{\eta}_n \Big| \exp\Big(-i(y+y')\frac{t-s_1}{t-s}\eta_1\Big) - \exp\Big(-iy\frac{t-s_1}{t-s}\eta_1\Big)\Big|^2 \\ &\times \prod_{i=1}^{n-1} \Big|\eta_i - \frac{t-s_{i+1}}{t-s_i}\eta_{i+1}\Big|^{1-2H_1} |\eta_n|^{1-2H_1} \prod_{i=1}^{n} \exp\Big(-\frac{(t-s_i)(s_i-s_{i-1})}{(t-s_{i-1})}\eta_i^2\Big). \end{aligned} \tag{4.35}$$

Using inequalities (4.28) and (4.32), and Lemma 2.9, we can write

$$\Big(n! \int_{\mathbb{T}_n^{s,t}} d\mathbf{s}_n G_1^{\frac{1}{2H_0}}\Big)^{2H_0} \leq C_2(n, t-s)\big(p_{t-s}(x-y-y') - p_{t-s}(x-y)\big)^2. \tag{4.36}$$

To estimate G_2, we apply inequality (4.21) and get

$$G_2 \leq p_{t-s}(x-y)^2 \int_{\mathbb{R}^n} d\boldsymbol{\eta}_n \Big(|y'|^2 \Big| \frac{t-s_1}{t-s}\eta_1 \Big|^2 \mathbf{1}_{\{|\frac{t-s_1}{t-s}\eta_1 y'|\leq 1\}} + 2 \times \mathbf{1}_{\{|\frac{t-s_1}{t-s}\eta_1 y'|>1\}} \Big) \times \prod_{i=1}^{n-1} \Big|\eta_i - \frac{t-s_{i+1}}{t-s_i}\eta_{i+1}\Big|^{1-2H_1} |\eta_n|^{1-2H_1} \prod_{i=1}^{n} \exp\Big(- \frac{(t-s_i)(s_i-s_{i-1})}{(t-s_{i-1})}\eta_i^2 \Big).$$

Using the same idea as in (4.29) and (4.30), we deduce that

$$G_2 \leq c_1 c_2^n p_{t-s}(x-y)^2 (G_{21} + G_{22}), \tag{4.37}$$

where, with $\beta_n = 1$,

$$\begin{aligned} G_{21} =& |y'|^2 \sum_{\boldsymbol{\beta}_{n-1} \in \{0,1\}^{n-1}} \Big[\int_{\mathbb{R}} d\eta_1 |\eta_1|^{2+\beta_1(1-2H_1)} \mathbf{1}_{\{|\frac{t-s_1}{t-s}\eta_1 y'|\leq 1\}} \exp\Big(- \frac{(t-s_1)(s_1-s)}{t-s}\eta_1^2 \Big) \Big] \\ &\times (t-s)^{-2} (t-s_1)^{\frac{5}{2}+\frac{1}{2}(\beta_1+\beta_2-1)(1-2H_1)} \prod_{i=2}^{n-1} (t-s_i)^{\frac{1}{2}(\beta_{i+1}-\beta_{i-1})(1-2H_1)} \\ &\times (t-s_n)^{-\frac{1}{2}+\frac{1}{2}\beta_{n-1}(1-2H_1)} \prod_{i=2}^{n} (s_i - s_{i-1})^{-\frac{1}{2}-\frac{1}{2}(1-\beta_{i-1}+\beta_i)(1-2H_1)} \end{aligned}$$

and

$$\begin{aligned} G_{22} =& \sum_{\boldsymbol{\beta}_{n-1} \in \{0,1\}^{n-1}} \Big[\int_{\mathbb{R}} d\eta_1 |\eta_1|^{\beta_1(1-2H_1)} \mathbf{1}_{\{|\frac{t-s_1}{t-s}\eta_1 y'|>1\}} \exp\Big(- \frac{(t-s_1)(s_1-s)}{t-s}\eta_1^2 \Big) \Big] \\ &\times (t-s_1)^{\frac{1}{2}+\frac{1}{2}(\beta_1+\beta_2-1)(1-2H_1)} \prod_{i=2}^{n-1} (t-s_i)^{\frac{1}{2}(\beta_{i+1}-\beta_{i-1})(1-2H_1)} \\ &\times (t-s_n)^{-\frac{1}{2}+\frac{1}{2}\beta_{n-1}(1-2H_1)} \prod_{i=2}^{n} (s_i - s_{i-1})^{-\frac{1}{2}-\frac{1}{2}(1-\beta_{i-1}+\beta_i)(1-2H_1)}. \end{aligned}$$

Preforming the change of variable $(\frac{(t-s_1)(s_1-s)}{t-s})^{\frac{1}{2}}\eta_1 = \eta$, we can show that

$$\begin{aligned} &\int_{\mathbb{R}} d\eta_1 |\eta_1|^{2+\beta_1(1-2H_1)} \mathbf{1}_{\{|\frac{t-s_1}{t-s}\eta_1 y'|\leq 1\}} \exp\Big(- \frac{(t-s_1)(s_1-s)}{t-s}\eta_1^2 \Big) \\ =& 2\Big(\frac{t-s}{(t-s_1)(s_1-s)} \Big)^{\frac{3}{2}+\frac{1}{2}\beta_1(1-2H_1)} \int_0^{\big(\frac{(t-s)(s_1-s)}{t-s_1}\big)^{\frac{1}{2}}|y'|^{-1}} d\eta |\eta|^{2+\beta_1(1-2H_1)} e^{-\eta^2} \\ \leq& 2\Big(\frac{t-s}{(t-s_1)(s_1-s)} \Big)^{\frac{3}{2}+\frac{1}{2}\beta_1(1-2H_1)} \Big(\int_0^{\infty} d\eta |\eta|^{2+\beta_1(1-2H_1)} e^{-\eta^2} \mathbf{1}_{\{|y'|\leq\sqrt{s-s_1}\}} \\ &+ \int_0^{\big(\frac{(t-s)(s_1-s)}{t-s_1}\big)^{\frac{1}{2}}|y'|^{-1}} d\eta |\eta|^{2+\beta_1(1-2H_1)} \mathbf{1}_{\{|y'|>\sqrt{s_1-s}\}} \Big) \\ \leq& c_1 \Big[\Big(\frac{t-s}{t-s_1} \Big)^{\frac{3}{2}+\frac{1}{2}\beta_1(1-2H_1)} (s_1-s)^{-\frac{3}{2}-\frac{1}{2}\beta_1(1-2H_1)} \mathbf{1}_{\{|y'|\leq\sqrt{s_1-s}\}} \\ &+ \Big(\frac{t-s}{t-s_1} \Big)^{3+\beta_1(1-2H_1)} |y'|^{-3-\beta_1(1-2H_1)} \mathbf{1}_{\{|y'|>\sqrt{s_1-s}\}} \Big]. \end{aligned}$$

It follows that $G_{21} \le c_1 c_2^n (G_{21}^1 + G_{21}^2)$, where

$$\begin{aligned} G_{21}^1 = &\sum_{\boldsymbol{\beta}_{n-1} \in \{0,1\}^{n-1}} |y'|^2 (s_1 - s)^{-\frac{3}{2} - \frac{1}{2}\beta_1(1-2H_1)} \mathbf{1}_{\{|y'| \le \sqrt{s_1 - s}\}} (t-s)^{-\frac{1}{2} + \frac{1}{2}\beta_1(1-2H_1)} \\ &\times (t - s_1)^{1 + \frac{1}{2}(\beta_2 - 1)(1-2H_1)} \prod_{i=2}^{n-1} (t - s_i)^{\frac{1}{2}(\beta_{i+1} - \beta_{i-1})(1-2H_1)} (t - s_n)^{-\frac{1}{2} + \frac{1}{2}\beta_{n-1}(1-2H_1)} \\ &\times \prod_{i=2}^{n} (s_i - s_{i-1})^{-\frac{1}{2} - \frac{1}{2}(1 - \beta_{i-1} + \beta_i)(1-2H_1)} \end{aligned}$$

and

$$\begin{aligned} G_{21}^2 = &\sum_{\boldsymbol{\beta}_{n-1} \in \{0,1\}^{n-1}} |y'|^{-1-\beta_1(1-2H_1)} \mathbf{1}_{\{|y'| > \sqrt{s_1 - s}\}} (t-s)^{1+\beta_1(1-2H_1)} \\ &\times (t - s_1)^{-\frac{1}{2} + \frac{1}{2}(\beta_2 - \beta_1 - 1)(1-2H_1)} \prod_{i=2}^{n-1} (t - s_i)^{\frac{1}{2}(\beta_{i+1} - \beta_{i-1})(1-2H_1)} \\ &\times (t - s_n)^{-\frac{1}{2} + \frac{1}{2}\beta_{n-1}(1-2H_1)} \prod_{i=2}^{n} (s_i - s_{i-1})^{-\frac{1}{2} - \frac{1}{2}(1 - \beta_{i-1} + \beta_i)(1-2H_1)}. \end{aligned}$$

Notice that on the set $\{|y'| \le \sqrt{s_1 - s}\}$,

$$|y'|^2 (s_1 - s)^{-\frac{3}{2} - \frac{1}{2}\beta_1(1-2H_1)} \le |y'|^{2H_0 - \frac{1}{2}} (s_1 - s)^{-\frac{1}{4} - \frac{1}{2}\beta_1(1-2H_1) - H_0}.$$

Combining this fact with inequalities (4.31) and (4.32), and Lemma 2.9, we get

$$\begin{aligned} &\Big(n! \int_{\mathbb{T}_n^{s,t}} d\mathbf{s}_n \big| G_{21}^1 \big|^{\frac{1}{2H_0}} \Big)^{2H_0} \\ &\le c_1 c_2^n \sum_{\beta_1 \in \{0,1\}} |y'|^{2H_0 - \frac{1}{2}} \bigg[\int_{[s,t]^n} d\mathbf{s}_n \Big((s_1 - s)^{-\frac{1}{4} - \frac{1}{2}\beta_1(1-2H_1) - H_0} \\ &\quad \times \frac{t - s_1}{t - s} \prod_{i=2}^{n} (s_i - s_{i-1})^{-\frac{1}{2} - \frac{1}{2}(1 - \beta_{i-1} + \beta_i)(1-2H_1)} \Big)^{\frac{1}{2H_0}} \bigg]^{2H_0} \mathbf{1}_{\{|y'| \le \sqrt{t-s}\}} \\ &\le C_2(n, t-s) (t-s)^{\frac{1}{4} - H_0} |y'|^{2H_0 - \frac{1}{2}} \mathbf{1}_{\{|y'| \le \sqrt{t-s}\}}. \end{aligned}$$

Following the same arguments, we can also deduce that

$$\begin{aligned} &\Big(n! \int_{\mathbb{T}_n^{s,t}} d\mathbf{s}_n \big| G_{21}^2 \big|^{\frac{1}{2H_0}} \Big)^{2H_0} \\ &\le C_2(n, t-s) \sum_{\beta_1 \in \{0,1\}} (t-s)^{\frac{1}{2} - 2H_0 + \frac{1}{2}\beta_1(1-2H_1)} |y'|^{-1-\beta_1(1-2H_1)} \big[(t-s) \wedge |y'|^2 \big]^{2H_0} \\ &\le C_2(n, t-s) \big((t-s)^{\frac{1}{4} - H_0} |y'|^{2H_0 - \frac{1}{2}} \mathbf{1}_{\{|y'| \le \sqrt{t-s}\}} + \mathbf{1}_{\{|y'| > \sqrt{t-s}\}} \big) \end{aligned}$$

and

$$\begin{aligned} &\Big(n! \int_{\mathbb{T}_n^{s,t}} d\mathbf{s}_n G_{22}^{\frac{1}{2H_0}} \Big)^{2H_0} \\ &\le C_2(n, t-s) \big((t-s)^{\frac{1}{4} - H_0} |y'|^{2H_0 - \frac{1}{2}} \mathbf{1}_{\{|y'| \le \sqrt{t-s}\}} + \mathbf{1}_{\{|y'| > \sqrt{t-s}\}} \big). \end{aligned} \tag{4.38}$$

Therefore, inequality (4.24) is a consequence of inequalities (4.33) and (4.36)–(4.38).

Inequality (4.25) is just another version of (4.24), if one make the change of variable $s_i = t + s - u_i$ for all $i = 1, \dots, n$. Thus we can conclude that (4.25) holds true.

Step 3. Due to formulas (4.27) and (4.28), we can write

$$\begin{aligned}
&\big\|g_n^2(\mathbf{s}_n,\bullet,s,y+y',t,x+x')-g_n^2(\mathbf{s}_n,\bullet,s,y,t,x+x')\\
&\quad-g_n^2(\mathbf{s}_n,\bullet,s,y+y',t,x)+g_n^2(\mathbf{s}_n,\bullet,s,y,t,x)\big\|^2_{\mathcal{H}_1^n}\\
=&c_{H_1}^n\int_{\mathbb{R}^n}d\boldsymbol{\eta}_n\big|h(x+x',y+y')-h(x+x',y)-h(x,y+y')+h(x,y)\big|^2\\
&\times\prod_{i=1}^{n-1}|\eta_i-\frac{t-s_{i+1}}{t-s_i}\eta_{i+1}|^{1-2H_1}|\eta_n|^{1-2H_1}\prod_{i=1}^{n}\exp\Big(-\frac{(t-s_i)(s_i-s_{i-1})}{(t-s_{i-1})}\eta_i^2\Big),
\end{aligned}\tag{4.39}$$

where $h(x,y)=h_1(x,y)h_2(y)h_3(x)$, with

$$h_1(x,y)=p_{t-s}(x-y),\quad h_2(y)=\exp\left(-iy\frac{t-s_1}{t-s}\eta_1\right)$$

and

$$h_3(x)=\exp\left(-ix\sum_{j=1}^{n}\frac{s_j-s_{j-1}}{t-s_{j-1}}\eta_j\right).$$

Notice that we can bound the rectangular increment as follows:

$$\begin{aligned}
&\big|h(x+x',y+y')-h(x+x',y)-h(x,y+y')+h(x,y)\big|\\
\le&|h_1(x+x',y+y')-h_1(x+x',y)-h_1(x,y+y')+h_1(x,y)||h_2(y+y')||h_3(x+x')|\\
&+|h_1(x+x',y)-h_1(x,y)||h_2(y+y')-h_2(y)||h_3(x+x')|\\
&+|h_1(x,y+y')-h(x,y)||h_2(y+y')||h_3(x+x')-h_3(x)|\\
&+|h_1(x,y)||h_2(y+y')-h_2(y)||h_3(x+x')-h_3(x)|:=\hbar_1+\hbar_2+\hbar_3+\hbar_4.
\end{aligned}$$

Following similar arguments as in Step 2, we can estimate the expressions

$$\begin{aligned}
&\Bigg[\int_{\mathbb{T}_n^{s,t}}d\mathbf{s}_n\Bigg(\int_{\mathbb{R}^n}d\boldsymbol{\eta}_n\hbar_k^2\prod_{i=1}^{n-1}|\eta_i-\frac{t-s_{i+1}}{t-s_i}\eta_{i+1}|^{1-2H_1}|\eta_n|^{1-2H_1}\\
&\times\prod_{i=1}^{k_2}\exp\Big(-\frac{(t-s_i)(s_i-s_{i-1})}{(t-s_{i-1})}\eta_i^2\Big)\Bigg)^{\frac{1}{2H_0}}\Bigg]^{2H_0}
\end{aligned}$$

for all $k=1,\dots,4$ and obtain inequality (4.26). The proof of this lemma is complete. ☐

4.2.2 Proof of Lemmas 4.2 and 4.3

Having Lemmas 4.4 and 4.5, we are ready to present the proof of Lemmas 4.2 and 4.3. By [29, Proposition 1.2.7], we can write the chaos expansion for the Malliavin derivatives of u as follows. Fix $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, then for all $s,s_1,s_2\in[s,t]$ and $y,y_1,y_2\in\mathbb{R}$,

$$D_{s,y}u(t,x)=\sum_{n=0}^{\infty}I_n\big((n+1)f_{t,x,n+1}(\bullet,s,y)\big)\tag{4.40}$$

and

$$D^2_{r,z,s,y}u(t,x)=\sum_{n=0}^{\infty}I_n\big((n+2)(n+1)f_{t,x,n+2}(\bullet,r,s,z,y)\big),\tag{4.41}$$

where $f_{t,x,n}$ is defined as in (2.6).

Proof of Lemma 4.2. Suppose that $p=2$. By the chaos expansion (4.40) of $D_{s,y}u(t,x)$, in order to prove inequality (4.3), we need to estimate the following expression

$$\|I_n((n+1)f_{t,x,n+1}(\bullet,s,y))\|_2^2=(n+1)^2n!\|f_{t,x,n+1}(\bullet,s,y)\|^2_{\mathfrak{H}^{\otimes n}}.$$

Due to the embedding inequality (2.10), we know that

$$\|f_{t,x,n+1}(\bullet,s,y)\|^2_{\mathfrak{H}^{\otimes n}}\le c^n_{H_0}\Big(\int_{[0,t]^n}d\mathbf{s}_n\|f_{t,x,n+1}(\mathbf{s}_n,\bullet,s,y)\|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes n}}\Big)^{2H_0}.$$

Notice that we can decompose the integral region in time as follows,

$$[0,t]^n=\bigcup_{k=0}^n\bigcup_{\sigma\in\Sigma_n}\{\mathbf{s}_n\in[0,t]^n,0<s_{\sigma(1)}<\cdots<s_{\sigma(k)}<s<s_{\sigma(k+1)}<\cdots<s_{\sigma(n)}<t\}\bigcup\mathcal{N},$$

where Σ_n denotes a set of permutations on $\{1,\dots,n\}$ and $\mathcal{N}$ is a subset included in $[0,t]^n$ of zero Lebesgue measure. On the other hand, freezing $0<s_1<\cdots<s_k<s<s_{k+1}\cdots<s_n<t$, we have

$$f_{t,x,n+1}(\mathbf{s}_n,\mathbf{y}_n,s,y)=\frac{1}{(n+1)!}g^1_{0,s,y,k}(\mathbf{s}_k,\mathbf{y}_k)g^2_{s,y,t,x,n-k}(\mathbf{s}_{k:n},\mathbf{y}_{k:n})\tag{4.42}$$

where g^1_k and g^2_{n-k} are defined as in (4.9) and (4.10). It follows that

$$\begin{aligned}&\|f_{t,x,n+1}(\bullet,s,y)\|^2_{\mathfrak{H}^{\otimes n}}\\&\le c^n_{H_0}\Big(\sum_{k=0}^n\binom{n}{k}\int_{[0,s]^k}d\mathbf{s}_k\int_{[s,t]^{n-k}}d\mathbf{s}_{k:n}\|f_{t,x,n+1}(\mathbf{s}_n,\bullet,s,y)\|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes n}}\Big)^{2H_0}\\\le&\frac{c_1c_2^n}{[(n+1)!]^2}\sum_{k=0}^n\binom{n}{k}^{2H_0}\Big(\int_{[0,s]^k}d\mathbf{s}_k\|g^1_{0,s,y,k}(\mathbf{s}_k,\bullet)\|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes k}}\Big)^{2H_0}\\&\times\Big(\int_{[s,t]^{n-k}}d\mathbf{s}_{k:n}\|g^2_{s,y,t,x,n-k}(\mathbf{s}_{k:n},\bullet)\|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes(n-k)}}\Big)^{2H_0}\end{aligned}$$

As a consequence of Lemmas 2.9 (i), 4.4, 4.5, we deduce that

$$(n+1)^2n!\|f_{t,x,n+1}(\bullet,s,y)\|^2_{\mathfrak{H}^{\otimes n}}\le\frac{c_1c_2^nt^{(2H_0+H_1-1)n}p_{t-s}(x-y)^2}{\Gamma(H_1n+1)}.\tag{4.43}$$

Finally, it follows from the asymptotic bound of the Mittag-Leffler function (see Lemma 2.9 (ii)) that

$$\begin{aligned}\|D_{r,z}u(t,x)\|_2^2=&\sum_{n=0}^\infty\mathbb{E}\big|I_n\big((n+1)f_{t,x,n+1}(\bullet,r,z)\big)\big|^2\\=&\sum_{n=0}^\infty(n+1)^2n!\|f_{t,x,n+1}(\bullet,s,y)\|^2_{\mathfrak{H}^{\otimes n}}\\\le&p_{t-r}(x-z)^2\sum_{n=0}^\infty\frac{c_1c_2^nt^{(2H_0+H_1-1)n}}{\Gamma(H_1n+1)}\\\le&c_1\exp\Big(c_2t^{\frac{2H_0+H_1-1}{2H_1}}\Big)p_{t-r}(x-z)^2.\end{aligned}$$

This proves inequality (4.3) in the case $p=2$. For $p>2$, using (2.3), we can write

$$\begin{aligned}\|D_{r,z}u(t,x)\|_p^2 &\le \Big(\sum_{n=0}^{\infty}\big\|I_n\big((n+1)f_{t,x,n+1}(\bullet,r,z)\big)\big\|_p\Big)^2\\ &\le \Big(\sum_{n=0}^{\infty}(p-1)^{\frac{n}{2}}\big\|I_n\big((n+1)f_{t,x,n+1}(\bullet,r,z)\big)\big\|_2\Big)^2\\ &\le \Big(c_1p_{t-s}(x-y)\sum_{n=0}^{\infty}\frac{c_2^n(p-1)^{\frac{n}{2}}t^{\frac{2H_0+H_1-1}{2}n}}{\Gamma(H_1n+1)^{\frac{1}{2}}}\Big)^2 \le C_1(t)p_{t-s}(x-y)^2.\end{aligned}$$

The proof of inequality (4.3) is complete.

In the next step, we provide the proof of inequality (4.4). Firstly, by chaos expansion (4.40) and embedding inequality (2.10), we have

$$\begin{aligned}&\|D_{s,y+y'}u(t,x)-D_{s,y}u(t,x)\|_2^2\\ &\le \sum_{n=0}^{\infty}\frac{c_1c_2^n}{n!}\sum_{k=0}^{n}\binom{n}{k}^{2H_0}\Big[\Big(\int_{[s,t]^{n-k}}d\mathbf{s}_{k:n}\|g^2_{s,y,t,x,n-k}(\mathbf{s}_{k:n},\bullet)\|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes(n-k)}}\\ &\quad\times\int_{[0,s]^k}d\mathbf{s}_k\|g^1_{0,s,y+y',k}(\mathbf{s}_k,\bullet)-g^1_{0,s,y,k}(\mathbf{s}_k,\bullet)\|^{\frac{1}{H_0}}_{\mathcal{H}^{\otimes k}}\Big)^{2H_0}\\ &\quad+\Big(\int_{[0,s]^{n-k}}d\mathbf{s}_{k:n}\|g^2_{s,y+y',t,x,n-k}(\mathbf{s}_{k:n},\bullet)-g^2_{s,y,t,x,n-k}(\mathbf{s}_{k:n},\bullet)\|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes(n-k)}}\Big)^{2H_0}\\ &\quad\times\Big(\int_{[0,s]^k}d\mathbf{s}_k\sup_{z\in\mathbb{R}}\|g^1_{0,s,z,k}(\mathbf{s}_k,\bullet)\|^{\frac{1}{H_0}}_{\mathcal{H}_1^{\otimes k}}\Big)^{2H_0}\Big].\end{aligned}\tag{4.44}$$

As a consequence of Lemmas 4.4, 4.5 and 2.9 (ii), we obtain inequality (4.4) for $p=2$ and thus for all $p\ge 2$ due to inequality (2.3). The proof of this lemma is complete. □

Proof of Lemma 4.3. It suffices to show this lemma for $p=2$. Denote by LHS the left hand side of (4.5). Then, by the chaos expansion (4.41), we get the following inequality, in the same way as for (4.44),

$$\begin{aligned}LHS \le&\sum_{n=0}^{\infty}\frac{c_1c_2^n}{n!}\sum_{k_2=0}^{n}\sum_{k_1=0}^{k_2}\binom{n}{k_2}^{2H_0}\binom{k_2}{k_1}^{2H_0}\\ &\times\Big(\int_{[0,r]^{k_1}}d\mathbf{s}_{k_1}\int_{[r,s]^{k_2-k_1}}d\mathbf{s}_{k_1:k_2}\int_{[s,t]^{n-k_2}}d\mathbf{s}_{k_2:n}(K_1+K_2+K_3+K_4)^{\frac{1}{H_0}}\Big)^{2H_0},\end{aligned}\tag{4.45}$$

where

$$\begin{aligned}K_1=&\|g^2_{s,y,t,x,n-k_2}(\mathbf{s}_{k_2:n},\bullet)\|_{\mathcal{H}_1^{\otimes(n-k_2)}}\|g^1_{0,r,z+z',k_1}(\mathbf{s}_{k_1},\bullet)-g^1_{0,r,z,k_1}(\mathbf{s}_{k_1},\bullet)\|_{\mathcal{H}_1^{\otimes k_1}}\\ &\times\|g^2_{r,z,s,y+y',k_2-k_1}(\mathbf{s}_{k_1:k_2},\bullet)-g^2_{r,z,s,y,k_2-k_1}(\mathbf{s}_{k_1:k_2},\bullet)\|_{\mathcal{H}_1^{\otimes(k_2-k_1)}},\end{aligned}$$

$$\begin{aligned}K_2=&\|g^2_{s,y,t,x,n-k_2}(\mathbf{s}_{k_2:n},\bullet)\|_{\mathcal{H}_1^{\otimes(n-k_2)}}\|g^1_{0,r,z+z',k_1}(\mathbf{s}_{k_1},\bullet)\|_{\mathcal{H}_1^{\otimes k_1}}\\ &\times\|g^2_{r,z+z',s,y+y',k_2-k_1}(\mathbf{s}_{k_1:k_2},\bullet)-g^2_{r,z+z',s,y,k_2-k_1}(\mathbf{s}_{k_1:k_2},\bullet)\\ &-g^2_{r,z,s,y+y',k_2-k_1}(\mathbf{s}_{k_1:k_2},\bullet)+g^2_{r,z,s,y,k_2-k_1}(\mathbf{s}_{k_1:k_2},\bullet)\|_{\mathcal{H}_1^{\otimes(k_2-k_1)}},\end{aligned}$$

$$\begin{aligned}K_3=&\|g^2_{s,y+y',t,x,n-k_2}(\mathbf{s}_{k_2:n},\bullet)-g^2_{s,y,t,x,n-k_2}(\mathbf{s}_{k_2:n},\bullet)\|_{\mathcal{H}_1^{\otimes(n-k_2)}}\\ &\times\|g^2_{r,z,s,y+y',k_2-k_1}(\mathbf{s}_{k_1:k_2},\bullet)\|_{\mathcal{H}_1^{\otimes(k_2-k_1)}}\|g^1_{0,r,z+z',k_1}(\mathbf{s}_{k_1},\bullet)-g^1_{0,r,z,k_1}(\mathbf{s}_{k_1},\bullet)\|_{\mathcal{H}_1^{\otimes k_1}},\end{aligned}$$

and

$$
\begin{aligned}
K_4 =&\|g^2_{s,y+y',t,x,n-k_2}(\mathbf{s}_{k_2:n},\bullet)-g^2_{s,y,t,x,n-k_2}(\mathbf{s}_{k_2:n},\bullet)\|_{\mathcal{H}_1^{\otimes(n-k_2)}}\|g^1_{0,r,z+z',k_1}(\mathbf{s}_{k_1},\bullet)\|_{\mathcal{H}_1^{\otimes k_1}}\\
&\times\|g^2_{r,z+z',s,y+y',k_2-k_1}(\mathbf{s}_{k_1:k_2},\bullet)-g^2_{r,z,s,y+y',k_2-k_1}(\mathbf{s}_{k_1:k_2},\bullet)\|_{\mathcal{H}_1^{\otimes(k_2-k_1)}}.
\end{aligned}
$$

By Lemmas 2.9 (i), 4.4, and 4.5, we get

$$
\begin{aligned}
&\Big(\int_{[0,r]^{k_1}}d\mathbf{s}_{k_1}\int_{[r,s]^{k_2-k_1}}d\mathbf{s}_{k_1:k_2}\int_{[s,t]^{n-k_2}}d\mathbf{s}_{k_2:n}K_1^{\frac{1}{H_0}}\Big)^{2H_0}\\
\le& C_2(n,t)p_{t-s}(x-y)^2N_r(z')^2\big(\big|\Delta_{s-r}(y-z,y')\big|+p_{s-r}(y-z)N_{s-r}(y')\big)^2,
\end{aligned}\tag{4.46}
$$

$$
\begin{aligned}
&\Big(\int_{[0,r]^{k_1}}d\mathbf{s}_{k_1}\int_{[r,s]^{k_2-k_1}}d\mathbf{s}_{k_1:k_2}\int_{[s,t]^{n-k_2}}d\mathbf{s}_{k_2:n}K_2^{\frac{1}{H_0}}\Big)^{2H_0}\\
\le& C_2(n,t)p_{t-s}(x-y)^2\big(\big|R_{s-r}(y-z,y',z')\big|+\big|\Delta_{s-r}(y-z,y')\big|N_{s-r}(z')\\
&+\big|\Delta_{s-r}(z-y,z')\big|N_{s-r}(y')+p_{s-r}(y-z)N_{s-r}(y')N_{s-r}(z')\big)^2,
\end{aligned}\tag{4.47}
$$

$$
\begin{aligned}
&\Big(\int_{[0,r]^{k_1}}d\mathbf{s}_{k_1}\int_{[r,s]^{k_2-k_1}}d\mathbf{s}_{k_1:k_2}\int_{[s,t]^{n-k_2}}d\mathbf{s}_{k_2:n}K_3^{\frac{1}{H_0}}\Big)^{2H_0}\\
\le& C_2(n,t)p_{s-r}(y+y'-z)^2N_r(z')\big|\big(\big|\Delta_{t-s}(y-x,y')\big|+p_{t-s}(x-y)^2N_{t-s}(y')\big)^2
\end{aligned}\tag{4.48}
$$

and

$$
\begin{aligned}
&\Big(\int_{[0,r]^{k_1}}d\mathbf{s}_{k_1}\int_{[r,s]^{k_2-k_1}}d\mathbf{s}_{k_1:k_2}\int_{[s,t]^{n-k_2}}d\mathbf{s}_{k_2:n}K_4^{\frac{1}{H_0}}\Big)^{2H_0}\\
\le& C_2(n,t)\big(\big|\Delta_{s-r}(z-y-y',z')\big|+p_{s-r}(y+y'-z)N_{s-r}(z')\big)^2\\
&\times\big(\big|\Delta_{t-s}(y-x,y')\big|+p_{t-s}(x-y)N_{t-s}(y')\big)^2.
\end{aligned}\tag{4.49}
$$

Therefore, inequality (4.5) is a consequence of inequalities (4.45)–(4.49), and Lemma 2.9. This completes the proof of Lemma 4.3. □

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