UDC 517.5

H. Gonska (Univ. Duisburg-Essen, Germany),
R. Păltănea (Transilvania Univ., Braşov, Romania)

QUANTITATIVE CONVERGENCE THEOREMS FOR A CLASS OF BERNSTEIN – DURRMEYER OPERATORS PRESERVING LINEAR FUNCTIONS

ТЕОРЕМИ ПРО КІЛЬКІСНУ ЗБІЖНІСТЬ ДЛЯ ОДНОГО КЛАСУ ОПЕРАТОРІВ БЕРНШТЕЙНА – ДУРРМЕЙЄРА, ЯКІ ЗБЕРІГАЮТЬ ЛІНІЙНІ ФУНКЦІЇ

We supplement recent results on a class of Bernstein – Durrmeyer operators preserving linear functions. This is done by discussing two limiting cases and proving quantitative Voronovskaya-type assertions involving the first and second order moduli of smoothness. The results generalize and improve earlier statements for Bernstein and genuine Bernstein – Durrmeyer operators.

Отримані нещодавно результати щодо одного класу операторів Бернштейна – Дуррмейєра, які зберігають лінійні функції, доповнено шляхом вивчення двох граничних випадків та доведення кількісних тверджень типу Вороновської, що містять модулі гладкості першого та другого порядків. Результати узагальнюють та покращують попередні твердження для операторів Бернштейна та справжніх операторів Бернштейна – Дуррмейєра.

1. Introduction. In the present paper we continue our research on a class of one parameter operators U_n^ρ of Bernstein – Durrmeyer type which preserve linear functions and constitute a link between the so-called "genuine Bernstein – Durrmeyer operators"U_n and the classical Bernstein operators B_n. A predecessor of this paper (see [1]) will appear soon in the Czechoslovak Mathematical Journal. Investigation on the operators in question started in a 2007 note by the second author (see [2]). In both articles more pertinent references can be found. We recall some basic facts.

Denote by $L_B[0,1]$ the space of bounded Lebesgue integrable functions on $[0,1]$ and by Π_n, the space of polynomials of degree at most $n \in \mathbb{N}_0$. The following definition was first given in [2].

Definition 1.1. *Let $\rho > 0$ and $n \in \mathbb{N}$, $n \geq 1$. Define the operator $U_n^\rho \colon L_B[0,1] \to \to \Pi_n$ for $f \in L_B[0,1]$ and $x \in [0,1]$ by*

$$U_n^\rho(f,x) := \sum_{k=0}^{n} F_{n,k}^\rho(f) \cdot p_{n,k}(x) :=$$

$$:= \sum_{k=1}^{n-1} \left(\int_0^1 f(t)\mu_{n,k}^\rho(t)dt \right) p_{n,k}(x) + f(0)(1-x)^n + f(1)x^n.$$

The fundamental functions $p_{n,k}$ are defined by

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \qquad 0 \leq k \leq n, \quad k,\, n \in \mathbb{N}, \quad x \in [0,1].$$

Moreover, for $1 \leq k \leq n-1$,

$$\mu_{n,k}^{\rho}(t) := \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)}$$

and

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x,\, y > 0,$$

is Euler's Beta function. For $\rho = 1$ we obtain

$$U_n(f,x) = (n-1)\sum_{k=1}^{n-1}\left(\int_0^1 f(t)p_{n-2,k-1}(t)dt\right)p_{n,k}(x)+$$

$$+(1-x)^n f(0) + x^n f(1), \quad f \in \mathcal{L}_B[0,1],$$

while, for $\rho \to \infty$, for each $f \in C[0,1]$, the sequence $U_n^{\rho}(f,x)$ uniformly converges to the Bernstein polynomial

$$B_n(f,x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{n,k}(x).$$

For several further properties of the U_n^{ρ} the reader is referred to our recent article [1] from which we will also use some results in the present note.

Here we supplement the results from [1]. In particular, we discuss the case $\rho \to 0$, consider iterates of U_n^{ρ} and prove two quantitative Voronovskaya-type assertions, thus generalizing and improving corresponding earlier results for U_n and B_n. Details will be given below.

2. Previous results on moments. In what follows we write $e_j(t) = t^j$, $t \in [0,1]$, for $j \geq 0$. Two basic properties of the functionals $F_{n,k}^{\rho}$ are the following:

$$F_{n,k}^{\rho}(e_0) = 1, \qquad F_{n,k}^{\rho}(e_1) = \frac{k}{n}, \quad 0 \leq k \leq n.$$

This implies

$$U_n^{\rho}(e_0) = e_0, \qquad U_n^{\rho}(e_1) = e_1,$$

i.e., the operators U_n^{ρ} preserve linear function. Clearly this fact has an impact on the moments and the Voronovskaya-type theorem. In the following we use the definition $\Psi(t) := t(1-t)$, $t \in [0,1]$.

In [1] we proved the following formulae for the moments of U_n^{ρ}.

Theorem 2.1. *For $x, y \in [0,1]$, we have*

$$U_n^{\rho}(e_0,x) = 1, \qquad U_n^{\rho}(e_1 - ye_0, x) = x - y,$$

and, for $r \geq 1$,

$$U_n^{\rho}((e_1 - ye_0)^{r+1}, x) = \frac{\rho\Psi(x)}{n\rho + r}\Big(U_n^{\rho}((e_1 - ye_0)^r, x)\Big)_x' +$$

$$+\frac{(1-2y)r + n\rho(x-y)}{n\rho + r}U_n^{\rho}((e_1 - ye_0)^r, x)+$$

$$+\frac{r\Psi(y)}{n\rho + r}U_n^{\rho}((e_1 - ye_0)^{r-1}, x).$$

For brevity we use $M_r(x) := M_{n,r}(x) := M^{\rho}_{n,r}(x) := U^{\rho}_n((e_1 - xe_0)^r, x)$, $n \geq 1$, $r \geq 0$, $x \in [0,1]$, in what follows. It is immediate that

$$(M_{n,r}(x))' = \left(U^{\rho}_n((e_1 - ye_0)^r, x)\right)'_x\Big|_{y=x} - rM_{r-1}(x). \tag{2.1}$$

Using (2.1) and putting $y = x$ in Theorem 2.1 we obtain the following recursion for the central moments.

***Corollary* 2.1.**

$$M_{n,0}(x) = 1, \qquad M_{n,1}(x) = 0,$$

and, for $r \geq 1$,

$$M_{n,r+1}(x) = \frac{r(\rho+1)\Psi(x)}{n\rho+r}M_{n,r-1}(x) + \frac{r(1-2x)}{n\rho+r}M_{n,r}(x) + \frac{\rho\Psi(x)}{n\rho+r}(M_{n,r}(x))'.$$

In particular:

$$M_{n,2}(x) = \frac{(\rho+1)\Psi(x)}{n\rho+1},$$

$$M_{n,3}(x) = \frac{(\rho+1)(\rho+2)\Psi(x)\Psi'(x)}{(n\rho+1)(n\rho+2)},$$

$$M_{n,4}(x) = \frac{3\rho(\rho+1)^2\Psi^2(x)n}{(n\rho+1)(n\rho+2)(n\rho+3)}+$$

$$+\frac{-6(\rho+1)(\rho^2+3\rho+3)\Psi^2(x)+(\rho+1)(\rho+2)(\rho+3)\Psi(x)}{(n\rho+1)(n\rho+2)(n\rho+3)}.$$

3. The case $0 < \rho < 1$ revisited. Note that the above equalities for U^{ρ}_n are true for $0 < \rho$. It is thus of interest to describe the behavior of U^{ρ}_n as $\rho \to 0$. We show first that, for any fixed $n \geq 1$, $U^{\rho}_n(f;x)$ uniformly converges with a certain speed to the first Bernstein polynomial of f, i.e., to the linear function

$$B_1(f;x) = f(0)(1-x) + f(1)\cdot x.$$

To this end we use the following result which essentially comes from the first author's dissertation (cf. [3, p. 117]); see also the proof of Theorem 2.1 in [4].

Theorem 3.1. *Let* $L\colon C[0,1] \to C[0,1]$ *be a positive linear operator reproducing linear functions. Then for* $f \in C[0,1]$ *and* $x \in [0,1]$ *the following inequality holds*:

$$\left|L(f;x) - B_1(f;x)\right| \leq \frac{9}{4}\omega_2\left(f; \sqrt{L(e_1\cdot(e_0-e_1);x)}\right).$$

Proof. For $g \in C^2[0,1]$ arbitrary we have

$$\left|L(f;x) - B_1(f;x)\right| \leq |(L-B_1)(f-g;x)| + |(L-B_1)(g;x)| \leq$$

$$\leq (\|L\| + \|B_1\|)\|f-g\|_\infty + |(L-B_1)(g;x)| =$$

$$= 2\|f-g\|_\infty + |(L-B_1)(g;x)|.$$

Since both L and B_1 reproduce linear functions, we have

$$L(B_1 g) = B_1(B_1 g) = B_1 g \in \Pi_1,$$

giving

$$\big|(L - B_1)(g;x)\big| = |L(g;x) - B_1(g;x) - L(B_1 g;x) + B_1(B_1 g;x)| =$$

$$= |L(g - B_1 g;x)| \le L(|g - B_1 g|;x) \le$$

$$\le \frac{1}{2}\|g''\|_\infty L(e_1(e_0 - e_1);x).$$

Thus

$$\big|L(f;x) - B_1(f;x)\big| \le 2\|f - g\|_\infty + \frac{1}{2}\|g''\|_\infty \, L(e_1(e_0 - e_1);x).$$

We now use Lemma 2 in [5] (also published in [6]) showing that for $0 < h \le \dfrac{1}{2}$ fixed and any $\epsilon > 0$ there is a polynomial $p = p(h,\epsilon)$ such that

$$\|f - p\|_\infty \le \frac{3}{4}\omega_2(f;h) + \epsilon,$$

and

$$\|p''\|_\infty \le \frac{3}{2h^2}\omega_2(f;h).$$

In the above we take $g = p$ and arrive at

$$\big|L(f;x) - B_1(f;x)\big| \le$$

$$\le \frac{3}{2}\omega_2(f;h) + 2\epsilon + \frac{3}{4h^2}L(e_1(e_0 - e_1);x)\omega_2(f;h).$$

If $L(e_1(e_0 - e_1);x) = 0$, then $\big|L(f;x) - B_1(f;x)\big| \le \dfrac{3}{2}\omega_2(f;h) + 2\epsilon$ for h and ϵ arbitrarily small. Hence in this case $\big|L(f;x) - B_1(f;x)\big| = 0$, and the inequality of the theorem is true.

Otherwise we take $h = \sqrt{L(e_1(e_0 - e_1);x)}$ and let ϵ tend to zero. This shows that the inequality is true for all cases of $x \in [0,1]$.

Theorem 3.1 is proved.

It is now easy to derive the following theorem.

Theorem 3.2. *For U_n^ρ, $0 < \rho < \infty$, $n \ge 1$, we have*

$$\big|U_n^\rho(f;x) - B_1(f;x)\big| \le \frac{9}{4}\omega_2\left(f;\sqrt{\frac{n\rho - \rho}{n\rho + 1}\cdot\psi(x)}\right).$$

In particular, for any fixed n, we have

$$\lim_{\rho\to 0} U_n^\rho f = B_1 f$$

uniformly.

Proof. It is only necessary to observe that

$$U_n^\rho(e_1(e_0-e_1);x)=U_n^\rho(e_1;x)-U_n^\rho(e_2;x)=$$

$$=x-x^2-\left[U_n^\rho(e_2;x)-x^2\right]=x(1-x)-M_{n,2}^\rho(x)=$$

$$=x(1-x)-\frac{\rho+1}{n\rho+1}x(1-x)=\frac{n\rho-\rho}{n\rho+1}\psi(x).$$

Theorem 3.2 is proved.

It is also interesting to describe the convergence to $f(x)$ for $0<\rho<1$ fixed. In fact, there is uniform convergence for $n\to\infty$, however, getting slower and slower as ρ approaches 0.

Using Corollary 2.2.1 on p. 31 of the second author's book [7], we have the following theorem.

Theorem 3.3. *For* $0<\rho<\infty$, $n\geq 1$, $f\in C[0,1]$ *and* $x\in[0,1]$ *there holds*

$$\left|U_n^\rho(f;x)-f(x)\right|\leq\left(1+\frac{1}{2}\frac{n(\rho+1)}{n\rho+1}\right)\omega_2\left(f;\sqrt{\frac{x(1-x)}{n}}\right).$$

Proof. The inequality is trivially true if $x\in\{0,1\}$. Otherwise we put $h=$ $=\sqrt{\dfrac{x(1-x)}{n}}$ in the theorem cited and arrive immediately at the upper bound claimed.

Theorem 3.3 is proved.

Remark **3.1.** The inequality of Theorem 3.3 can also be derived from Theorem 5.2, formula (5.3), case $r=0$ in [1]. For a similar inequality see Theorem 2.3, inequality (2.11) in [2].

In view of $\dfrac{n(\rho+1)}{n\rho+1}\nearrow\dfrac{\rho+1}{\rho}$ for $n\to\infty$, the constant in front of $\omega_2(f;\ldots)$ becomes arbitrarily large for ρ close to 0. However, uniform convergence is still warranted for $n\to\infty$. We have seen before that the situation is different for n fixed and $\rho\to 0$.

Remark **3.2.** The linear function B_1f is also the uniform limit of over-iterated operator images $[U_n^\rho]^m f$, if $m\to\infty$. Here $n\geq 1$ and $0<\rho<\infty$ are fixed. In fact, using Corollary 2.4 in [4] it is easy to see that

$$\left|[U_n^\rho]^m(f;x)-B_1(f;x)\right|\leq$$

$$\leq\frac{9}{4}\omega_2\left(f;\sqrt{\left(1-\frac{\rho+1}{n\rho+1}\right)^m\psi(x)}\right),\quad f\in C[0,1],\quad x\in[0,1].$$

For $n\geq 1$, $0<\rho<\infty$, one has $0\leq 1-\dfrac{\rho+1}{n\rho+1}<1$, and this implies uniform convergence as $m\to\infty$.

4. Quantitative Voronovskaya theorem with first order modulus. In the present section we prove a quantitative Voronoskaja theorem using the least concave majorant of the first order modulus of continuity. This will be based upon the following general theorem.

Theorem 4.1 (see [8]). *Let* $q\in\mathbb{N}_0$, $f\in C^q[0,1]$ *and* $L\colon C[0,1]\to C[0,1]$ *be a positive linear operator. Then*

$$\left| L(f,x) - \sum_{r=0}^{q} L((e_1-x)^r,x)\frac{f^{(r)}(x)}{r!} \right| \le$$

$$\le \frac{L(|e_1-x|^q,x)}{q!}\tilde{\omega}\left(f^{(q)}, \frac{1}{q+1}\frac{L(|e_1-x|^{q+1},x)}{L(|e_1-x|^q,x)}\right).$$

Here $\tilde{\omega}$ is the least concave majorant of the first order modulus of continuity.

We use the above theorem for $q=2$, also recalling that $U_n^\rho(e_i)=e_i$, $i=0,1$. This leads to the following theorem.

Theorem 4.2. *For U_n^ρ, given as above, $f \in C^2[0,1]$, $n \ge 2$ and $x \in (0,1)$ we have*

$$\left| \frac{n\rho+1}{\rho+1}[U_n^\rho(f,x)-f(x)] - \frac{1}{2}\Psi(x)f''(x) \right| \le$$

$$\le \frac{1}{2}\Psi(x)\tilde{\omega}\left(f'', \frac{1}{3}\sqrt{m_\rho}\sqrt{\frac{\rho^2}{(n\rho+1)^2}+\frac{\rho\Psi(x)}{n\rho+1}}\right), \qquad (4.1)$$

where $m_\rho = \max\left\{\frac{5\rho^2+13\rho+12}{\rho^2}, \left(\frac{7\rho^2+3\rho+2}{\rho(\rho+1)}\right)^2\right\}$.

Proof. The general estimate from Theorem 4.1 reduces to

$$\left| U_n^\rho(f,x) - f(x) - \frac{1}{2}U_n^\rho((e_1-x)^2,x)f''(x) \right| \le$$

$$\le \frac{1}{2}U_n^\rho((e_1-x)^2,x)\,\tilde{\omega}\left(f'', \frac{1}{3}\frac{U_n^\rho(|e_1-x|^3,x)}{U_n^\rho((e_1-x)^2,x)}\right).$$

Using the above representation of the second moment this turns into

$$\left| U_n^\rho(f,x) - f(x) - \frac{1}{2}\frac{(\rho+1)\Psi(x)}{n\rho+1}f''(x) \right| \le$$

$$\le \frac{1}{2}\frac{(\rho+1)\Psi(x)}{n\rho+1}\,\tilde{\omega}\left(f'', \frac{1}{3}\frac{U_n^\rho(|e_1-x|^3,x)}{U_n^\rho((e_1-x)^2,x)}\right). \qquad (4.2)$$

We now consider two cases.

***Case* 4.1.** $x \in \left[\frac{\rho}{n\rho+1}, 1-\frac{\rho}{n\rho+1}\right]$.

Using the Cauchy – Schwarz inequality we first observe that

$$\frac{U_n^\rho(|e_1-x|^3,x)}{U_n^\rho((e_1-x)^2,x)} \le \sqrt{\frac{U_n^\rho((e_1-x)^4,x)}{U_n^\rho((e_1-x)^2,x)}} = \sqrt{\frac{M_4(x)}{M_2(x)}}.$$

The above representations of the two moments show that

$$\frac{M_4(x)}{M_2(x)} = \frac{(3\rho(\rho+1)n - 6(\rho^2+3\rho+3))\Psi(x) + (\rho+2)(\rho+3)}{(n\rho+2)(n\rho+3)} \le$$

$$\leq \frac{1}{(n\rho+2)(n\rho+3)}\left[3\rho(\rho+1)n+\frac{(\rho+2)(\rho+3)}{\Psi(x)}\right]\Psi(x).$$

In the above interval we have

$$\Psi(x)\geq \frac{\rho}{n\rho+1}\left(1-\frac{\rho}{n\rho+1}\right)=\frac{\rho[(n-1)\rho+1]}{(n\rho+1)^2}.$$

Therefore

$$\frac{M_4(x)}{M_2(x)}\leq \frac{\Psi(x)}{(n\rho+2)(n\rho+3)}\left[3\rho(\rho+1)n+\frac{(\rho+2)(\rho+3)(n\rho+1)^2}{\rho(\rho n+1-\rho)}\right]\leq$$

$$\leq \frac{\rho\Psi(x)}{n\rho+1}\left[\frac{3(\rho+1)}{\rho}+2\frac{(\rho+2)(\rho+3)}{\rho^2}\right]=\frac{\rho\Psi(x)}{n\rho+1}c_\rho,$$

where $c_\rho=\dfrac{5\rho^2+13\rho+12}{\rho^2}$.

***Case* 4.2.** $x\in\left[0,\dfrac{\rho}{n\rho+1}\right]\cup\left[1-\dfrac{\rho}{n\rho+1},1\right]$.

We only consider the left interval; for the right one the statement follows by symmetry. We have successively

$$U_n^\rho(|e_1-x|^3,x)=U_n^\rho(2(e_1-x)^2(x-e_1)_+,x)+U_n^\rho((e_1-x)^3,x)=$$

$$=2\sum_{k=1}^{n-1}\left(\int\limits_0^x(x-t)^3\mu_{n,k}^\rho(t)dt\right)p_{n,k}(x)+2(1-x)^nx^3+M_3(x)\leq$$

$$\leq 2x^3\sum_{k=1}^{n-1}\left(\int\limits_0^1\mu_{n,k}^\rho(t)dt\right)p_{n,k}(x)+2x^2\Psi(x)+M_3(x)=$$

$$=2x^2\Psi(x)\Big(1+\frac{1}{1-x}\Big)+\frac{(\rho+1)(\rho+2)\Psi(x)\Psi'(x)}{(n\rho+1)(n\rho+2)}\leq$$

$$\leq \frac{\rho^2\Psi(x)}{(n\rho+1)^2}\left[2+2\frac{n\rho+1}{(n-1)\rho+1}+\frac{(\rho+1)(\rho+2)}{\rho^2}\right]\leq$$

$$\leq \frac{\rho^2\Psi(x)}{(n\rho+1)^2}d_\rho,$$

where $d_\rho=6+\dfrac{(\rho+1)(\rho+2)}{\rho^2}$ (the latter being correct for $n\geq 2$). Hence

$$\frac{U_n^\rho(|e_1-x|^3,x)}{U_n^\rho((e_1-x)^2,x)}\leq \frac{\rho^2\Psi(x)d_\rho}{(n\rho+1)^2}\frac{n\rho+1}{(\rho+1)\Psi(x)}=\sqrt{\frac{\rho^2}{(n\rho+1)^2}\left(\frac{\rho d_\rho}{\rho+1}\right)^2}.$$

From the above two cases it follows that the r.h.s. in (4.2) is bounded from above by

$$\frac{1}{2}\frac{(\rho+1)\Psi(x)}{n\rho+1}\tilde{\omega}\left(f'',\frac{1}{3}\sqrt{m_\rho}\sqrt{\frac{\rho\Psi(x)}{n\rho+1}+\frac{\rho^2}{(n\rho+1)^2}}\right),$$

where $m_\rho = \max\left\{c_\rho, \left(\dfrac{\rho d_\rho}{\rho+1}\right)^2\right\}$. Multiplying both sides by $\dfrac{n\rho+1}{\rho+1}$ gives the final result.

***Remark* 4.1.** The above multiplication by $\dfrac{n\rho+1}{\rho+1}$ is somewhat arbitrary. It is, for example, also possible to multiply by $n\rho+1$ or simply by n to arrive at slightly different inequalities.

***Remark* 4.2.** In case $\rho = 1$ the right-hand side of inequality (4.1) becomes

$$\frac{1}{2}\Psi(x)\tilde{\omega}\left(f'',\, 2\sqrt{\frac{1}{(n+1)^2}+\frac{\Psi(x)}{n+1}}\right),$$

improving the term

$$\frac{1}{2}\Psi(x)\tilde{\omega}\left(f'', 4\sqrt{\frac{1}{(n+1)^2}+\frac{\Psi(x)}{n+1}}\right),$$

which is obtained using the approach in [8].

5. Quantitative Voronovskaya theorem with second order modulus. In this section we replace the quantity $\tilde{\omega}_1(f'';...)$ by the second order modulus of smoothness $\omega_2(f;\delta)$ given by

$$\sup\Big\{|f(x-h)-2f(x)+f(x+h)|,\quad a+h\le x\le b-h,\quad 0\le h\le\delta\Big\},$$

for $0\le\delta\le\dfrac{1}{2}$. The general underlying inequality was recently given in [5] and reads as follows.

Theorem 5.1. *Let $L\colon C[0,1]\to C[0,1]$ be a positive linear operator such that $Le_i = e_i$, $i = 0,1$. If $f\in C^2[0,1]$, then for any $0<h\le\dfrac{1}{2}$ the following inequality holds*:

$$\left|L(f;x)-f(x)-\frac{1}{2}L((e_1-x)^2;x)f''(x)\right|\le$$

$$\le L((e_1-x)^2;x)\times$$

$$\times\left\{\frac{5}{6h}\frac{|L((e_1-x)^3;x)|}{L((e_1-x)^2;x)}\omega_1(f'';h)+\left(\frac{3}{4}+\frac{1}{16h^2}\frac{L((e_1-x)^4;x)}{L((e_1-x)^2;x)}\right)\omega_2(f'';h)\right\}.$$

We will use the theorem in its following form.

***Corollary* 5.1.** *Putting $h=\sqrt{\dfrac{L((e_1-x)^4;x)}{L((e_1-x)^2;x)}}$ and assuming that $h>0$, the inequality in the theorem becomes:*

$$\left|L(f;x)-f(x)-\frac{1}{2}L((e_1-x)^2;x)f''(x)\right|\le$$

$$\le L((e_1-x)^2;x)\left\{\frac{5}{6}\frac{|L((e_1-x)^3;x)|}{\sqrt{L((e_1-x)^2L((e_1-x)^4;x)}}\times\right.$$

$$\times\omega_1\left(f'';\sqrt{\frac{L((e_1-x)^4;x)}{L((e_1-x)^2;x)}}\right)+\frac{13}{16}\omega_2\left(f'';\sqrt{\frac{L((e_1-x)^4;x)}{L((e_1-x)^2;x)}}\right)\Bigg\}.$$

In the case of U_n^ρ we temporarily write $X = \Psi(x)$, $X' = \Psi'(x)$ and

$$A := \frac{[M_3(x)]^2}{M_2(x)M_4(x)}, \qquad B := \frac{M_4(x)}{M_2(x)}.$$

Explicitly,

$$A = \frac{(\rho+2)^2(X')^2(n\rho+3)}{(n\rho+2)\{[3\rho(\rho+1)n-6(\rho^2+3\rho+3]X+(\rho+2)(\rho+3)\}},$$

$$B = \frac{3[\rho(\rho+1)n-2(\rho^2+3\rho+3)]X+(\rho+2)(\rho+3)}{(n\rho+2)(n\rho+3)}.$$

Inserting the information available on U_n^ρ now into the inequality of Corollary 5.1 and multiplying both sides by $n\rho+1$ leads to the assertion of the following theorem.

Theorem 5.2. *For U_n^ρ, given as above, $f \in C^2[0,1]$, $n \geq 2$ and $x \in [0,1]$, the following inequality holds*:

$$\left|(n\rho+1)[U_n^\rho(f,x)-f(x)]-\frac{\rho+1}{2}\psi(x)f''(x)\right| \leq$$

$$\leq (\rho+1)\psi(x)\left\{\frac{5}{6}\sqrt{A}\omega_1\left(f'';\sqrt{B}\right)+\frac{13}{16}\omega_2(f'';\sqrt{B})\right\}.$$

Moreover,

(i) *if $f \in C^3[0,1]$, then*

$$\left|(n\rho+1)[U_n^\rho(f,x)-f(x)]-\frac{\rho+1}{2}\psi(x)f''(x)\right| = \psi(x)O\left(\frac{1}{\sqrt{n}}\right)\|f'''\|,$$

and

(ii) *if $f \in C^4[0,1]$, then*

$$\left|(n\rho+1)\big[U_n^\rho(f,x)-f(x)\big]-\frac{\rho+1}{2}\psi(x)f''(x)\right| = \psi(x)O\left(\frac{1}{n}\right)\|f^{(4)}\|.$$

In both cases $O(\ldots)$ is independent of f and x.

***Corollary* 5.2.** *For $\rho = 1$ this coincides with the inequalities in Theorem* 5 *of* [5].

Proceeding as before, i.e., inserting the information on U_n^ρ into the inequality of Corollary 5.1, but multiplying both sides by n (instead of $n\rho+1$), implies the following corollary.

***Corollary* 5.3.**

$$\lim_{\rho\to\infty}\left|n\big[U_n^\rho(f;x)-f(x)\big]-\frac{1}{2}\frac{n(\rho+1)}{n\rho+1}\psi(x)f''(x)\right| \leq$$

$$\leq \lim_{\rho\to\infty}\frac{n(\rho+1)}{n\rho+1}\psi(x)\left\{\frac{5}{6}\sqrt{A}\omega_1(f'';\sqrt{B})+\frac{13}{16}\omega_2(f'';\sqrt{B})\right\}.$$

In other words, for $n \geq 2$ we have

$$\left| n[B_n(f;x) - f(x)] - \frac{1}{2}\psi(x) f''(x) \right| \leq$$

$$\leq \psi(x) \left\{ \frac{5}{6} \frac{|\psi'(x)|}{\sqrt{3(n-2)\psi(x)+1}} \omega_1 \left(f''; \sqrt{\frac{3(n-2)\psi(x)+1}{n^2}} \right) + \right.$$

$$\left. + \frac{13}{16} \omega_2 \left(f''; \sqrt{\frac{3(n-2)\psi(x)+1}{n^2}} \right) \right\}.$$

This is the same inequality as the one given in Theorem 4 *of* [5].

1. *Gonska H., Păltănea R.* Simultaneous approximation by a class of Bernstein – Durrmeyer operators preserving linear functions // Czech. Math. J. (to appear).
2. *Păltănea R.* A class of Durrmeyer type operators preserving linear functions // Ann. Tiberiu Popoviciu Sem. Funct. Equat. Approxim. and Convex. (Cluj-Napoca). – 2007. – **5**. – P. 109 – 117.
3. *Gonska H.* Quantitative Aussagen zur Approximation durch positive lineare Operatoren: Doctor. Dissertation. – Univ. Duisburg, 1979. – 190 p.
4. *Gonska H., Kacsó D., Piţul P.* The degree of convergence of over-iterated positive linear operators // J. Appl. Funct. Anal. – 2006. – **1**. – P. 403 – 423.
5. *Gonska H., Raşa I.* A Voronovskaya estimate with second order modulus of smoothness // Proc. Math. Inequal. (Sibiu / Romania, Sept. 2008) (to appear).
6. *Gonska H., Raşa I.* Four notes on Voronovskaya's theorem // Schriftenr. Fachbereichs Math. Univ. Duisburg-Essen. – 2009.
7. *Păltănea R.* Approximation theory using positive linear operators. – Boston: Birkhäuser, 2004.
8. *Gonska H.* On the degree of approximation in Voronovskaja's theorem // Stud. Univ. Babeş-Bolyai. Math. – 2007. – **52**. – P. 103 – 115.

Received 12.06.09