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Quantitative Estimates for the Tensor Product (p,q)-Balázs-Szabados Operators and Associated Generalized Boolean Sum Operators

Esma Yıldız Özkan[a]

[a] *Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey*

Abstract. In this study, we give some approximation results for the tensor product of (p,q)-Balázs-Szabados operators associated generalized Boolean sum (GBS) operators. Firstly, we introduce tensor product (p,q)-Balázs-Szabados operators and give an uniform convergence theorem of these operators on compact rectangular regions with an illustrative example. Then we estimate the approximation for the tensor product (p,q)-Balázs-Szabados operators in terms of the complete modulus of continuity, the partial modulus of continuity, Lipschitz functions and Petree's K-functional corresponding to the second modulus of continuity. After that, we introduce the GBS operators associated the tensor product (p,q)-Balázs-Szabados operators. Finally, we improve the rate of smoothness by the mixed modulus of smoothness and Lipschitz class of Bögel continuous functions for the GBS operators.

1. Introduction and some auxiliary results

In approximation theory, q-type generalization of Bernstein polynomials was firstly introduced by Lupaş[19]. Later, Phillips[22] introduced an another modification of Bernstein polynomials. The rapid development of q-calculus has led to research the new generalization of Bernstein type operators involving q-integers.The details on q-calculus can be found in [17].

Mursaleen et al.[20] applied (p,q)-calculus in approximation theory and introduced (p,q)-analogue of Bernstein operators. Hence q-calculus has been extended to (p,q)-calculus in approximation theory. The references [1, 2, 15, 21, 27] can be given as recent studies on the approximation of some operators by (p,q)-integers.

We begin by recalling certain notation of (p,q)-calculus. Let $0 < q < p \leq 1$. For each nonnegative integer $n, k, n \geq k \geq 0$, the (p,q)-integer $[n]_{p,q}$, the (p,q)- factorial $[n]_{p,q}!$ and the (p,q)-binomial coefficients $\left[\begin{array}{c} n \\ k \end{array} \right]_{p,q}$ are defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q},$$

$$[n]_{p,q}! := \begin{cases} [n]_{p,q} [n-1]_{p,q} ... [2]_{p,q} [1]_{p,q}, & if\, n \geq 1 \\ 1, & if\, n = 0 \end{cases},$$

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Email address: esmayildiz@gazi.edu.tr (Esma Yıldız Özkan)

and

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} := \frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}.$$

Note that if we take $p = 1$ in above notations, they reduce to q-analogues. Further, we have

$$(ax+by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} a^{n-k} b^k x^{n-k} y^k,$$

$$(ax+by)^n_{p,q} = (ax+by)(pax+qby)\left(p^2ax+q^2by\right)...\left(p^{n-1}ax+q^{n-1}by\right).$$

K.Balázs [8] defined the Bernstein type rational functions. In [9], K.Balázs and J.Szabados obtained best possible estimate under more restrictive conditions, in which both the weight and the order of convergence would be better than [8].

q-form of these operators was given by O. Doğru[14]. Also, some approximation results of q-Balázs-Szabados operators on compact disks and polydisks can be found in [16, 24, 25].

(p,q)-analogue of Balázs-Szabados operators is defined by

$$R_n^{p,q}(f;x) = \frac{1}{(1+a_nx)_{p,q}} \sum_{k=0}^{n} f\left(\frac{[k]_{p,q}}{q^{k-1}b_n}\right) p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} (a_nx)^k,$$

where $f:\mathbb{R}_+\to\mathbb{R}$ is a function, $x\in\mathbb{R}_+=[0,\infty)$, $a_n=[n]^{\beta-1}_{p,q}$, $b_n=[n]^{\beta}_{p,q}$ are sequences for all $n\in\mathbb{N}$ such that $0<q<p\le 1$ and $0<\beta\le\frac{2}{3}$[26].

We have the following equalities for (p,q)-analogue of Balázs-Szabados operators:

$$R_n^{p,q}(1;x) = 1,$$

$$R_n^{p,q}(t;x) = \frac{x}{p^{n-1}+q^{n-1}a_nx},$$

$$R_n^{p,q}\left(t^2;x\right) = \frac{x}{b_n\left(p^{n-1}+q^{n-1}a_nx\right)} + \frac{\frac{p}{q}\frac{[n-1]_{p,q}}{[n]_{p,q}}x^2}{\prod\limits_{j=1}^{2}\left(p^{n-j}+q^{n-j}a_nx\right)},$$

$$R_n^{p,q}(t-x;x) = \frac{\left(1-p^{n-1}\right)x - q^{n-1}a_nx^2}{p^{n-1}+q^{n-1}a_nx},$$

$$R_n^{p,q}\left((t-x)^2;x\right) = \frac{x}{b_n\left(p^{n-1}+q^{n-1}a_nx\right)} + \frac{\left(\frac{p}{q}\frac{[n-1]_{p,q}}{[n]_{p,q}}-2p^{n-2}+p^{2n-3}\right)x^2}{\prod\limits_{j=1}^{2}\left(p^{n-j}+q^{n-j}a_nx\right)} + \frac{q^{n-2}\left(-2+p^{n-2}(p+q)\right)a_nx^3}{\prod\limits_{j=1}^{2}\left(p^{n-j}+q^{n-j}a_nx\right)} + \frac{q^{2n-3}a_n^2x^4}{\prod\limits_{j=1}^{2}\left(p^{n-j}+q^{n-j}a_nx\right)}.$$

Approximation properties of the (p,q)-analogue of Balázs-Szabados operators were investigated in [26].

2. Construction of tensor product operators

Now, we define tensor product (p,q)-Balázs-Szabados operators as follows:

$$R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y)=\sum_{k=0}^{n_1}\sum_{j=0}^{n_2}v_{n_1,k}(x;p_1,q_1)\,s_{n_2,m}(y;p_2,q_2)\,f\left(\frac{[k]_{p_1,q_1}}{q_1^{k-1}b_n},\frac{[j]_{p_2,q_2}}{q_2^{j-1}d_n}\right)$$

where $f:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}$ is a continuous function, $(x,y)\in\mathbb{R}_+\times\mathbb{R}_+$ and $a_{n_1}=[n_1]_{p_1,q_1}^{\beta_1-1}$, $b_{n_1}=[n_1]_{p_1,q_1}^{\beta_1}$, $c_{n_2}=[n_2]_{p_2,q_2}^{\beta_2-1}$, $d_{n_2}=[n_2]_{p_2,q_2}^{\beta_2}$ are sequences for all $n_1,n_2\in\mathbb{N}$ such that $0<q_1<p_1\le 1$, $0<q_2<p_2\le 1$, $0<\beta_1\le\frac{2}{3}$ and $0<\beta_2\le\frac{2}{3}$. And also,

$$v_{n_1,k}(x;p_1,q_1):=\frac{p_1^{\frac{(n_1-k)(n_1-k-1)}{2}}q_1^{\frac{k(k-1)}{2}}\left[\begin{array}{c}n_1\\k\end{array}\right]_{p_1,q_1}(a_{n_1}x)^k}{(1+a_nx)_{p_1,q_1}^{n_1}}$$

and

$$s_{n_2,m}(y;p_2,q_2):=\frac{p_2^{\frac{(n_2-j)(n_2-j-1)}{2}}q_2^{\frac{j(j-1)}{2}}\left[\begin{array}{c}n_2\\j\end{array}\right]_{p_2,q_2}(c_{n_2}y)^j}{(1+c_ny)_{p_2,q_2}^{n_2}}.$$

Notice that, the operator $R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}:C(\mathbb{R}_+\times\mathbb{R}_+)\to C(\mathbb{R}_+\times\mathbb{R}_+)$ is the tensorial product of $xR_{n_1}^{(p_1,q_1)}$ and $yR_{n_2}^{(p_2,q_2)}$, i.e. $R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}=xR_{n_1}^{(p_1,q_1)}\circ yR_{n_2}^{(p_2,q_2)}$, where

$$xR_{n_1}^{(p_1,q_1)}=\sum_{k=0}^{n_1}v_{n_1,k}(x;p_1,q_1)\,f\left(\frac{[k]_{p_1,q_1}}{q_1^{k-1}b_n},y\right)$$

and

$$yR_{n_2}^{(p_2,q_2)}=\sum_{j=0}^{n_2}s_{n_2,m}(y;p_2,q_2)\,f\left(x,\frac{[j]_{p_2,q_2}}{q_2^{j-1}d_n}\right).$$

Lemma 2.1. *Let $e_{ij}(t,s)=t^is^j$ for $i,j=0,1,2$ be the test functions. We have the following equalities:*

$$R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{00};x,y)=1,$$

$$R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{10};x,y)=\frac{x}{p_1^{n_1-1}+q_1^{n_1-1}a_nx},$$

$$R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{01};x,y)=\frac{y}{p_2^{n_2-1}+q_2^{n_2-1}c_ny},$$

$$R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{20};x,y)=\frac{x}{b_{n_1}\left(p_1^{n_1-1}+q_1^{n_1-1}a_{n_1}x\right)}+\frac{\frac{p_1}{q_1}\frac{[n_1-1]_{p_1,q_1}}{[n_1]_{p_1,q_1}}x^2}{\prod_{s=1}^{2}\left(p_1^{n_1-s}+q^{n_1-s}a_{n_1}x\right)},$$

$$R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{02};x,y)=\frac{y}{d_{n_2}\left(p_2^{n_2-1}+q_2^{n_2-1}c_{n_2}y\right)}+\frac{\frac{p_2}{q_2}\frac{[n_2-1]_{p_2,q_2}}{[n_2]_{p_2,q_2}}y^2}{\prod_{s=1}^{2}\left(p_2^{n_2-s}+q_2^{n_2-s}c_{n_2}y\right)}.$$

Remark 2.2. *By applying Lemma 2.1, we have*

$$R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{10}-x;x,y)=\frac{\left(1-p_1^{n_1-1}\right)x-q_1^{n_1-1}a_{n_1}x^2}{p_1^{n_1-1}+q_1^{n_1-1}a_{n_1}x},$$

$$R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{01}-y;x,y)=\frac{\left(1-p_2^{n_2-1}\right)y-q_2^{n_2-1}a_{n_2}y^2}{p_2^{n_2-1}+q_2^{n_2-1}a_{n_2}y},$$

$$\begin{aligned}R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((e_{10}-x)^2;x,y\right)=&\frac{x}{b_{n_1}\left(p_1^{n_1-1}+q_1^{n_1-1}a_{n_1}x\right)}\\&+\frac{\left(\frac{p_1}{q_1}\frac{[n_1-1]_{p_1,q_1}}{[n_1]_{p_1,q_1}}-2p_1^{n_1-2}+p_1^{2n_1-3}\right)x^2}{\prod_{j=1}^{2}\left(p_1^{n_1-j}+q_1^{n_1-j}a_{n_1}x\right)}\\&+\frac{q_1^{n_1-2}\left(-2+p_1^{n_1-2}(p_1+q_1)\right)a_{n_1}x^3}{\prod_{j=1}^{2}\left(p_1^{n_1-j}+q_1^{n-j}a_{n_1}x\right)}\\&+\frac{q_1^{2n_1-3}a_{n_1}^2x^4}{\prod_{j=1}^{2}\left(p_1^{n_1-j}+q_1^{n_1-j}a_{n_1}x\right)},\end{aligned}$$

$$\begin{aligned}R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((e_{01}-y)^2;x,y\right)=&\frac{y}{d_{n_2}\left(p_2^{n_2-1}+q_2^{n_2-1}c_{n_2}x\right)}\\&+\frac{\left(\frac{p_2}{q2}\frac{[n_2-1]_{p_2,q_2}}{[n_2]_{p_2,q_2}}-2p_2^{n_2-2}+p_2^{2n_2-3}\right)y^2}{\prod_{j=1}^{2}\left(p_2^{n_2-j}+q_2^{n_2-j}c_{n_2}y\right)}\\&+\frac{q_2^{n_2-2}\left(-2+p_2^{n_2-2}(p_2+q_2)\right)c_{n_2}y^3}{\prod_{j=1}^{2}\left(p_2^{n_2-j}+q_2^{n_2-j}c_{n_2}y\right)}\\&+\frac{q_2^{2n_2-3}c_{n_2}^2y^4}{\prod_{j=1}^{2}\left(p_2^{n_2-j}+q_2^{n_2-j}c_{n_2}y\right)}.\end{aligned}$$

We consider the tensor product (p,q)-Balázs-Szabados operators and their GBS operators. In this study, we give the approximation properties for the tensor product (p,q)-Balázs-Szabados operators and their GBS operators.

Let $I=I_1\times I_2$ such that $I_i=[0,r_i]\,,r_i>0,i=1,2$ and $C(I)$ be the space of all real valued continuous functions f on I with the norm

$$\|f\|=\sup\left\{\left|f(x,y)\right|:(x,y)\in I\right\}.$$

In order to obtain to uniform convergence of the tensor product operators $R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}$, we take the sequences $(p_{1,n_1})\,,(q_{1,n_1})\,,(p_{2,n_2})$ and (q_{2,n_2}) satisfying $q_{1,n_1},q_{2,n_2}\in(0,1)$ and $p_{1,n_1}\in(q_{1,n_1},1]$, $p_{2,n_2}\in(q_{2,n_2},1]$ such that

$$\lim_{n_1\to\infty}p_{1,n_1}=\lim_{n_1\to\infty}(p_{1,n_1})^{n_1}=\lim_{n_1\to\infty}q_{1,n_1}=1, \tag{1}$$

$$\lim_{n_1\to\infty}(q_{1,n_1})^{n_1}=l_1,0<l_1<1, \tag{2}$$

and

$$\lim_{n_2\to\infty}p_{2,n_2}=\lim_{n_2\to\infty}(p_{2,n_2})^{n_2}=\lim_{n_2\to\infty}q_{2,n_2}=1, \tag{3}$$

$$\lim_{n_2\to\infty}(q_{2,n_2})^{n_2}=l_2,0<l_2<1. \tag{4}$$

For example, the sequences $(p_{1,n_1}) = \left(1 - \frac{1}{(n_1)^2}\right), (q_{1,n_1}) = \left(1 - \frac{1}{n_1}\right), (p_{2,n_2}) = \left(1 - \frac{1}{(n_2)^2}\right)$ and $(q_{2,n_2}) = \left(1 - \frac{1}{n_2}\right)$ satisfy the conditions (1-4) for all $n_1, n_2 \in \mathbb{N}$.

Under the conditions (1-4), we have

$$\lim_{n_1\to\infty} a_{n_1} = \lim_{n_1\to\infty} \frac{1}{b_{n_1}} = 0,$$

and

$$\lim_{n_2\to\infty} c_{n_2} = \lim_{n_2\to\infty} \frac{1}{d_{n_2}} = 0.$$

Throughout the paper, in all theorems, $\delta_{n_1}(x)$ and $\delta_{n_2}(y)$ will be denoted by

$$\delta_{n_1}(x) := \left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((e_{10} - x)^2 ; x\right)\right)^{1/2}$$

and

$$\delta_{n_2}(y) := \left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((e_{01} - y)^2 ; y\right)\right)^{1/2},$$

which are given as in Remark 2.2.

Theorem 2.3. *Let be the sequences $(p_{1,n_1}), (q_{1,n_1}), (p_{2,n_2})$ and (q_{2,n_2}) satisfying the conditions (1-4). Then the tensor product operators $R_{n_1,n_2}^{(p_{1,n_1},q_{1,n_1},p_{2,n_2},q_{2,n_2})}(f;x,y)$ converge uniformly to f on I, for all $f \in C(I)$.*

Proof. From Lemma 2.1, taking into account Volkov's theorem in [28] (also see in [4], p.245), the theorem can be easily proved, so we will omit the proof. □

In the following illustrative example, it can be seen clearly the convergence of the operators $R_{n_1,n_2}^{(p_{1,n_1},q_{1,n_1},p_{2,n_2},q_{2,n_2})}(f;x,y)$ to a certain function $f(x,y)$ on the unit square:

Example 2.4. *Let $I = [0,1] \times [0,1]$. For $n_1, n_2 = 15$ and different values of p_1, q_1, p_2, q_2, the convergence of $R_{15,15}^{(p_1,q_1,p_2,q_2)}(f;x,y)$ to $f(x,y) = xy^2 - x^2y - \sin(xy)$ on I is illustrated in Figure 1, Figure 2, Figure 3 and Figure 4. The list of figure captions is given in Table 1.*

3. Rate of Convergence

For $f \in C(I)$, the complete modulus of continuity for the bivariate case is defined as

$$\omega(f;\delta_1,\delta_2) = \sup\left\{\left|f(t,s) - f(x,y)\right| : |t-x| \le \delta_1,\ |s-y| \le \delta_2\right\},$$

for all $(t,s), (x,y) \in I$, $\delta_1 > 0$, $\delta_2 > 0$. Further, $\omega(f;\delta_1,\delta_2)$ satisfies the following properties

$$\omega(f;\delta_1,\delta_2) \to 0 \text{ if } \delta_1 \to 0, \delta_2 \to 0,$$

$$\left|f(t,s) - f(x,y)\right| \le \omega(f;\delta_1,\delta_2)\left(1 + \frac{|t-x|}{\delta_1}\right)\left(1 + \frac{|s-y|}{\delta_2}\right). \tag{5}$$

Also, the partial modulus of continuity with respect to x and y are given by

$$\omega^{(1)}(f;\delta) = \sup\left\{\left|f(x_1,y) - f(x_2,y)\right| : y \in I_2 \text{ and } |x_1 - x_2| \le \delta\right\}$$

and

$$\omega^{(2)}(f;\delta) = \sup\left\{\left|f(x,y_1) - f(x,y_2)\right| : x \in I_1 \text{ and } |y_1 - y_2| \le \delta\right\}.$$

It is clear that they satisfy the properties of the usual modulus of continuity. The details of the modulus of continuity for the bivariate case can be found in [5]. Now, we can give the following estimates for the bivariate operator in terms of the complete modulus of continuity and the partial modulus of continuity.

Theorem 3.1. *Let $f \in C(I)$. Then for all $(x,y) \in I$, it holds the following inequality*

$$\left|R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f(x,y)\right| \leq 4\omega\left(f,\delta_{n_1}(x),\delta_{n_2}(y)\right).$$

Proof. Using the linearity of the operators and considering (5), we have

$$\begin{aligned}\left|R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f(x,y)\right| &\leq R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\left|f(t,s) - f(x,y)\right|;x,y\right)\\ &\leq \omega\left(f,\delta_{n_1}(x),\delta_{n_2}(y)\right)\left\{R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(1;x,y)\right.\\ &\quad\left.+\frac{1}{\left(\delta_{n_1}(x)\right)^{1/2}}R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(|t-x|;x,y\right)\right\}\\ &\quad\times\left\{R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(1;x,y)\right.\\ &\quad\left.+\frac{1}{\left(\delta_{n_2}(y)\right)^{1/2}}\left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\left|s-y\right|;x,y\right)\right)^{1/2}\right\}\end{aligned}$$

Applying the Cauchy-Schwarz inequality, and considering Remark 2.2, we get the desired result. □

Theorem 3.2. *Let $f \in C(I)$. Then for all $(x,y) \in I$, it holds the following inequality*

$$\left|R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f(x,y)\right| \leq 2\left\{\omega^{(1)}\left(f;\delta_{n_1}(x)\right) + \omega^{(2)}\left(f;\delta_{n_2}(y)\right)\right\}.$$

Proof. Using the linearity of the operators and considering the definition of partial modulus of continuity and using the Cauchy-Schwarz inequality, we can write

$$\begin{aligned}\left|R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f(x,y)\right| &\leq R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\left|f(t,s) - f(x,y)\right|;x,y\right)\\ &\leq R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\left|f(t,s) - f(t,y)\right|;x,y\right)\\ &\quad+R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\left|f(t,y) - f(x,y)\right|;x,y\right)\\ &\leq R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\omega_2\left(f;\left|s-y\right|\right);x,y\right)\\ &\quad+R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\omega_1\left(f;|t-x|\right);x,y\right)\\ &\leq \omega_2(f;\delta_2)\left(1+\frac{1}{\delta_2}R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\left|s-y\right|;x,y\right)\right)\\ &\quad+\omega_1(f;\delta_1)\left(1+\frac{1}{\delta_1}R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(|t-x|;x,y\right)\right)\\ &\leq \omega_2(f;\delta_2)\left(1+\frac{1}{\delta_2}R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((s-y)^2;x,y\right)\right)\\ &\quad+\omega_1(f;\delta_1)\left(1+\frac{1}{\delta_1}R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((t-x)^2;x,y\right)\right).\end{aligned}$$

Taking $\delta_1 = \delta_{n_1}(x)$ and $\delta_2 = \delta_{n_2}(y)$, we get the desired result. □

The Lipschitz class $Lip_M(\alpha_1,\alpha_2)$ for the bivariate case is defined by:

$$f \in Lip_M(\alpha_1,\alpha_2) \text{ iff } \left|f(t,s) - f(x,y)\right| \leq M|t-x|^{\alpha_1}\left|s-y\right|^{\alpha_2} \text{ for } f \in C(I),$$

where $0 < \alpha_1, \alpha_2 \leq 1$, $(t,s),(x,y) \in I$ are arbitrary.

Now, we give the following estimate for the tensor product operators in terms of the Lipschitz functions.

Theorem 3.3. *Let $f \in Lip_M(\alpha_1, \alpha_2)$. Then for all $(x, y) \in I$, it holds the following inequality*

$$\left|R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f; x, y) - f(x, y)\right| \leq M(\delta_{n_1}(x))^{\alpha_1}(\delta_{n_2}(y))^{\alpha_2},$$

where $M > 0$, $0 < \alpha_1, \alpha_2 \leq 1$.

Proof. Let $f \in Lip_M(\alpha_1, \alpha_2)$. From definition of the Lipschitz functions, we can write

$$\begin{aligned}\left|R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f; x, y) - f(x, y)\right| &\leq R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\left|f(t,s) - f(x,y)\right|; x, y\right) \\ &\leq MR_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(|t-x|^{\alpha_1}\left|s-y\right|^{\alpha_2}; x, y\right) \\ &\leq MR_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(|t-x|^{\alpha_1}; x, y\right) \\ &\quad \times R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\left|s-y\right|^{\alpha_2}; x, y\right).\end{aligned}$$

Applying the Hölder's inequality with $u_1 = \frac{2}{\alpha_1}, v_1 = \frac{2}{2-\alpha_1}, u_2 = \frac{2}{\alpha_2}$ and $v_2 = \frac{2}{2-\alpha_2}$, respectively, we get

$$\begin{aligned}\left|R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f; x, y) - f(x, y)\right| &\leq M\left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((t-x)^2; x, y\right)\right)^{\alpha_1/2} \\ &\quad \times\left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(1; x, y)\right)^{(2-\alpha_1)/2} \\ &\quad \times\left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((s-y)^2; x, y\right)\right)^{\alpha_2/2} \\ &\quad \times\left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(1; x, y)\right)^{(2-\alpha_2)/2} \\ &\leq M(\delta_{n_1}(x))^{\alpha_1}(\delta_{n_2}(y))^{\alpha_2},\end{aligned}$$

which completes the proof. □

Let $C^{(2)}(I)$ be the space of all functions $f \in C(I)$ such that $\frac{\partial^i f}{\partial x^i}, \frac{\partial^i f}{\partial y^i}$ for $i = 1, 2$ belong to $C(I)$. The norm on the space $C^{(2)}(I)$ is defined by

$$\left\|f\right\|_{C^{(2)}(I)} = \left\|f\right\|_{C(I)} + \sum_{i=1}^{2}\left(\left\|\frac{\partial^i f}{\partial x^i}\right\|_{C(I)} + \left\|\frac{\partial^i f}{\partial y^i}\right\|_{C(I)}\right).$$

The Petree's K-functional for the functions $f \in C(I)$ is defined by

$$K(f; \delta) = \inf_{g \in C^{(2)}(I)}\left\{\left\|f - g\right\|_{C(I)} + \delta\left\|g\right\|_{C^{(2)}(I)}\right\}$$

for all $\delta > 0$. It holds the following inequality

$$K(f; \delta) \leq M_1\left\{\overline{\omega}_2\left(f; \sqrt{\delta}\right) + \min(1, \delta)\left\|f\right\|_{C(I)}\right\}, \tag{6}$$

for all $\delta > 0$, where the constant M is independent of δ and f, and $\overline{\omega}_2\left(f; \sqrt{\delta}\right)$ is the second order complete modulus of continuity.(see [13] p.192).

We can give an estimate for the tensor product operators in terms of the Petree's K-functional for the functions $f \in C(I)$.

Theorem 3.4. *If $f \in C(I)$, then we have*

$$\begin{aligned}\left|R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f; x, y) - f(x, y)\right| &\leq M\left\{\overline{\omega}_2\left(f; \sqrt{\mu_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y)}\right)\right. \\ &\quad \left. + \min\left\{1, \mu_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y)\right\}\left\|f\right\|_{C(I)}\right\} \\ &\quad +\omega\left(f; \sqrt{\varphi_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y)}\right),\end{aligned}$$

where

$$\begin{aligned}\mu_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y) \quad : \quad &= R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{10}-x;x,y) + R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((e_{10}-x)^2;x,y\right) \\ &+R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{01}-y;x,y) + R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((e_{01}-y)^2;x,y\right),\end{aligned}$$

$$\varphi_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y) := R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((e_{10}-x)^2;x,y\right) + R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((e_{01}-y)^2;x,y\right).$$

Proof. We define the following auxiliary operator

$$\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) = R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f\left(\theta_{n_1}^{(p_1,q_1)}(x), \theta_{n_2}^{(p_2,q_2)}(y)\right) + f(x,y).$$

By Lemma 2.1, we obtain

$$\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{10};x,y) = x,$$

$$\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{01};x,y) = y,$$

which imply

$$\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{10}-x;x,y) = 0,$$

$$\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{01}-y;x,y) = 0.$$

Let $h \in C^{(2)}(I)$ and $t \in I_1, s \in I_2$. Using the Taylor theorem, we can write

$$\begin{aligned}h(t,s) - h(x,y) \quad &= \quad h(t,y) - h(x,y) + h(t,s) - h(t,y) \\ &= \quad \frac{\partial h(x,y)}{\partial x}(t-x) + \int_x^t (t-\xi)\frac{\partial^2 h(\xi,y)}{\partial \xi^2}d\xi \\ &\quad + \frac{\partial h(x,y)}{\partial y}(s-y) + \int_y^s (s-\eta)\frac{\partial^2 h(x,\eta)}{\partial \eta^2}d\eta.\end{aligned}$$

Taking

$$\theta_{n_1}^{(p_1,q_1)}(x) := R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{10};x,y),$$

and

$$\theta_{n_2}^{(p_2,q_2)}(y) := R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{01};x,y),$$

and also applying the operator $\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}$ to the last equality,we get

$$\begin{aligned}\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(h;x,y)-h(x,y) &= \widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\int\limits_x^t (t-\xi)\frac{\partial^2 h(\xi,y)}{\partial\xi^2}d\xi;x,y\right)\\ &\quad+\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\int\limits_y^s (s-\eta)\frac{\partial^2 h(x,\eta)}{\partial\eta^2}d\eta;x,y\right)\\ &= R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\int\limits_x^t (t-\xi)\frac{\partial^2 h(\xi,y)}{\partial\xi^2}d\xi;x,y\right)\\ &\quad+\int\limits_x^{\theta_{n_1}^{(p_1,q_1)}(x)}\left(\theta_{n_1}^{(p_1,q_1)}(x)-\xi\right)\frac{\partial^2 h(\xi,y)}{\partial\xi^2}d\xi\\ &\quad+R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\int\limits_y^s (s-\eta)\frac{\partial^2 h(x,\eta)}{\partial\eta^2}d\eta;x,y\right)\\ &\quad+\int\limits_y^{\theta_{n_2}^{(p_2,q_2)}(y)}\left(\theta_{n_2}^{(p_2,q_2)}(y)-\eta\right)\left(\frac{\partial^2 h(x,\eta)}{\partial\eta^2}\right)d\eta.\end{aligned}$$

By using Remark 2.2, we have

$$\begin{aligned}\left|\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(h;x,y)-h(x,y)\right| &\le R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\int\limits_x^t |t-\xi|\left|\frac{\partial^2 h(\xi,y)}{\partial\xi^2}\right|d\xi;;x,y\right)\\ &\quad+\int\limits_x^{\theta_{n_1}^{(p_1,q_1)}(x)}\left|\theta_{n_1}^{(p_1,q_1)}(x)-\xi\right|\left|\frac{\partial^2 h(\xi,y)}{\partial\xi^2}\right|d\xi\\ &\quad+R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\int\limits_y^s |s-\eta|\left|\frac{\partial^2 h(x,\eta)}{\partial\eta^2}\right|d\eta;;x,y\right)\\ &\quad+\int\limits_y^{\theta_{n_2}^{(p_2,q_2)}(y)}\left|\theta_{n_2}^{(p_2,q_2)}(y)-\eta\right|\left|\frac{\partial^2 h(x,\eta)}{\partial\eta^2}\right|d\eta\\ &\le \left\{R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((e_{10}-x)^2;x,y\right)\right.\\ &\quad\left.+R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{10}-x;x,y)\right\}\|h\|_{C^{(2)}(I)}\\ &\quad+\left\{R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((e_{01}-y)^2;x,y\right)\right.\\ &\quad\left.+R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{01}-y;x,y)\right\}\|h\|_{C^{(2)}(I)},\end{aligned}$$

which imply

$$\left|\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(h;x,y)-h(x,y)\right| \le \mu_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y)\|h\|_{C^{(2)}(I)}. \tag{7}$$

On the other hand, we have

$$\left|\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(h;x,y)\right| \leq \left|R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(h;x,y)\right| + \left|h\left(\theta_{n_1}^{(p_1,q_1)}(x),\theta_{n_2}^{(p_2,q_2)}(y)\right)\right| + \left|h(x,y)\right|,$$

which implies

$$\left|\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(h;x,y)\right| \leq 3\left\|h\right\|_{C(I)}. \tag{8}$$

Considering (7) and (8), for $f \in C(I)$, we can write

$$\begin{aligned}
\left|R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f(x,y)\right| &\leq \Big|\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f(x,y) \\
&\quad + f\left(\theta_{n_1}^{(p_1,q_1)}(x),\theta_{n_2}^{(p_2,q_2)}(y)\right) - f(x,y)\Big| \\
&\leq \left|\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f-h;x,y)\right| \\
&\quad + \left|\widetilde{R}_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(h;x,y) - h(x,y)\right| \\
&\quad + \left|h(x,y) - f(x,y)\right| \\
&\quad + \left|f\left(\theta_{n_1}^{(p_1,q_1)}(x),\theta_{n_2}^{(p_2,q_2)}(y)\right) - f(x,y)\right| \\
&\leq 4\left\|f-h\right\|_{C(I)} + \mu_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y)\left\|h\right\|_{C^{(2)}(I)} \\
&\quad + \omega\left(f;\sqrt{\varphi_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y)}\right).
\end{aligned}$$

Taking the infimum on the right-hand side over all $h \in C^{(2)}(I)$ and using the inequality (6), we obtain

$$\begin{aligned}
\left|R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f;x,y) - f(x,y)\right| &\leq 4K\left(f;\sqrt{\mu_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y)}\right) \\
&\quad + \omega\left(f;\sqrt{\varphi_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y)}\right) \\
&\leq M\left\{\overline{\omega}_2\left(f;\sqrt{\mu_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y)}\right)\right. \\
&\quad \left. + \min\left\{1,\mu_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y)\right\}\left\|f\right\|_{C(I)}\right\} \\
&\quad + \omega\left(f;\sqrt{\varphi_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(x,y)}\right).
\end{aligned}$$

□

4. Construction of GBS Operators

Recently, the generalized Boolean sums of some tensor product operators have been introduced and studied their approximation properties(see [3, 18, 23]).

Now , we define the generalized Boolean sum (GBS) operators associated with tensor product (p,q)-Balázs-Szabados operators as follows:

$$G_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f(t,s);x,y) := R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f(t,y)+f(x,s)-f(t,s);x,y),$$

for all $(x,y) \in I$.

The generalized Boolean sum (GBS) operators are linear and positive operators defined from the space $C(I)$ on itself.

5. Rate of Convergence of GBS Operators

In [10] , Bögel defined Bögel-continuous and Bögel-bounded functions. Now, we recall some basic definitions and notations given by Bögel. Details can be found in [10–12].

Let X and Y be compact subset of $\mathbb{R}$. A function $f : X\times Y \to \mathbb{R}$ is called Bögel-continuous (B-continuous) function at $(x_0, y_0) \in X \times Y$ if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta_{(x,y)} f\left[x_0, y_0; x, y\right] = 0,$$

where $\Delta_{(x,y)} f\left[x_0, y_0; x, y\right]$ denotes the mixed difference defined by

$$\Delta_{(x,y)} f\left[x_0, y_0; x, y\right] = f(x, y) - f(x, y_0) - f(x_0, y) + f(x_0, y_0).$$

Let A is a subset of $\mathbb{R}^2$. The function $f : A \to \mathbb{R}$ is Bögel-bounded (B-bounded) function on A if there exists $M > 0$ such that $\left|\Delta_{(x,y)} f\left[t, s; x, y\right]\right| \le M$, for every $(x, y), (t, s) \in A$. If A is a compact subset of $\mathbb{R}^2$, then each B-continuous function is a B-bounded function.

Let denote by $C_b(A)$, the space of all real valued B-continuous functions defined on A with the norm $\left\|f\right\|_B = \sup\left\{\left|\Delta_{(x,y)} f\left[t, s; x, y\right]\right| : (x, y), (t, s) \in A\right\}$. And also, we denote with $C(A)$ and $B(A)$ the space of all real valued continuous and bounded functions on defined A, respectively. $C(A)$ and $B(A)$ are Banach spaces with the norm $\left\|f\right\| = \sup\left\{\left|f(x, y)\right| : (x, y) \in A\right\}$. It is known that $C(A) \subset C_b(A)$.

In this section, we estimate the degree of the approximation for GBS operators in terms of the mixed modulus of smoothness and the Lipschitz class for B-continuous functions.

The mixed modulus of smoothness of $f \in C_b(I)$ is defined by

$$\omega_{mixed}(f; \delta_1, \delta_2) := \sup\left\{\left|\Delta_{(x,y)} f\left[t, s; x, y\right]\right| : |t - x| \le \delta_1,\ \left|s - y\right| \le \delta_2\right\},$$

for all $(x, y), (t, s) \in I, \delta_1, \delta_2 \in (0, \infty)$. ω_{mixed} is well defined. The basic properties of mixed modulus of smoothness were obtained in [6] and [7], which are similar to properties of the usual modulus of continuity. The mixed modulus of smoothness satisfies the following property

$$\omega_{mixed}(f; \lambda_1\delta_1, \lambda_2\delta_2) \le (1 + \lambda_1)(1 + \lambda_2)\,\omega_{mixed}(f; \delta_1, \delta_2) \text{ for } \lambda_1, \lambda_2 > 0. \tag{9}$$

Theorem 5.1. *Let $f \in C_b(I)$. Then for all $(x, y) \in I$, it holds the following inequality*

$$\left|G_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f(t, s); x, y) - f(x, y)\right| \le 4\omega_{mixed}(f; \delta_{n_1}(x), \delta_{n_2}(y)).$$

Proof. Using the definition of mixed modulus of smoothness and the inequality in (9), we can write

$$\left|\Delta_{(x,y)} f\left[t, s; x, y\right]\right| \le \omega_{mixed}\left(f; |t - x|, \left|s - y\right|\right),$$

which implies

$$\left|\Delta_{(x,y)} f\left[t, s; x, y\right]\right| \le \left(1 + \frac{|t - x|}{\delta_1}\right)\left(1 + \frac{\left|s - y\right|}{\delta_2}\right)\omega_{mixed}(f; \delta_1, \delta_2), \tag{10}$$

for every $(x, y), (t, s) \in I$ and for any $\delta_1, \delta_2 > 0$.

From the definition of $\Delta_{(x,y)} f\left[x_0, y_0; x, y\right]$, we have

$$f(x, s) - f(t, y) - f(t, s) = f(x, y) - \Delta_{(x,y)} f\left[t, s; x, y\right].$$

Applying the operators $R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}$ to the last equation and considering the definition of GBS operator $G_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}$, we can write

$$G_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f(t,s);x,y) = f(x,y)R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(e_{00};x,y) - R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\Delta_{(x,y)}f[t,s;x,y];x,y\right).$$

Since $R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(1;x,y) = 1$, considering the inequality in (10) and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\left|G_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f(t,s);x,y) - f(x,y)\right| &\leq R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\left|\Delta_{(x,y)}f[t,s;x,y]\right|;x,y\right) \\
&\leq \Big\{R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(1;x,y) \\
&\quad+\frac{1}{\delta_1}\left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((t-x)^2;x,y\right)\right)^{1/2} \\
&\quad+\frac{1}{\delta_2}\left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((s-y)^2;x,y\right)\right)^{1/2} \\
&\quad+\frac{1}{\delta_1\delta_2}\left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((t-x)^2;x,y\right)\right)^{1/2} \\
&\quad\times\left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((s-y)^2;x,y\right)\right)^{1/2}\Big\} \\
&\quad\times\omega_{mixed}(f;\delta_1,\delta_2)
\end{aligned}$$

Choosing $\delta_1 = \delta_{n_1}(x)$ and $\delta_2 = \delta_{n_2}(y)$, we get the desired result. □

Now, we define the Lipschitz class for B-continuous functions.

The Lipschitz class $B - Lip_M(\alpha_1,\alpha_2)$ for $f \in C_b(I)$, is defined by

$$f \in B - Lip_M(\alpha_1,\alpha_2) \text{ iff } \left|\Delta_{(x,y)}f[t,s;x,y]\right| \leq M|t-x|^{\alpha_1}|s-y|^{\alpha_2},$$

where $0 < \alpha_1,\alpha_2 \leq 1$, $(t,s),(x,y) \in I$ are arbitrary.

Theorem 5.2. *Let $f \in B - Lip_M(\alpha_1,\alpha_2)$. Then for all $(x,y) \in I$, we have*

$$\left|G_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f(t,s);x,y) - f(x,y)\right| \leq M(\delta_{n_1}(x))^{\alpha_1}(\delta_{n_2}(y))^{\alpha_2},$$

where $M > 0$, $0 < \alpha_1,\alpha_2 \leq 1$.

Proof. From the definition of GBS operator $G_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}$ and by the linearity of $R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}$ and by our hypothesis, we can write

$$\begin{aligned}
\left|G_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f(t,s);x,y) - f(x,y)\right| &\leq R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(\left|\Delta_{(x,y)}f[t,s;x,y]\right|;x,y\right) \\
&\leq MR_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(|t-x|^{\alpha_1}|s-y|^{\alpha_2};x,y\right) \\
&= R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(|t-x|^{\alpha_1};x,y\right) \\
&\quad\times R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left(|s-y|^{\alpha_2};x,y\right).
\end{aligned}$$

Now, using the Hölder's inequality with $u_1 = \frac{2}{\alpha_1}, v_1 = \frac{2}{2-\alpha_1}, u_2 = \frac{2}{\alpha_2}$ and $v_2 = \frac{2}{2-\alpha_2}$, respectively, we have

$$\left|G_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}(f(t,s);x,y) - f(x,y)\right| \leq \left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((t-x)^2;x\right)\right)^{\alpha_1/2}\left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((s-y)^2;y\right)\right)^{\alpha_2/2}$$

Replacing $\delta_{n_1}(x) = \left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((t-x)^2;x\right)\right)^{1/2}$ and $\delta_{n_2}(y) = \left(R_{n_1,n_2}^{(p_1,q_1,p_2,q_2)}\left((s-y)^2;y\right)\right)^{1/2}$, we get the desired result. □

Table 1: The List of Figure Captions

Colour	Figure 1	Figure 2	Figure 3	Figure 4
Red	f	f	f	f
Navy blue	$R_{15,15}^{(0.99,0.99,0.8,0.8)}$	$R_{15,15}^{(0.95,0.95,0.8,0.8)}$	$R_{15,15}^{(0.9,0.9,0.8,0.8)}$	$R_{15,15}^{(0.85,0.85,0.8,0.8)}$
Yellow	$R_{15,15}^{(0.99,0.99,0.7,0.7)}$	$R_{15,15}^{(0.95,0.95,0.7,0.7)}$	$R_{15,15}^{(0.9,0.9,0.7,0.7)}$	$R_{15,15}^{(0.85,0.85,0.7,0.7)}$
Pink	$R_{15,15}^{(0.99,0.99,0.6,0.6)}$	$R_{15,15}^{(0.95,0.95,0.6,0.6)}$	$R_{15,15}^{(0.9,0.9,0.6,0.6)}$	$R_{15,15}^{(0.85,0.85,0.6,0.6)}$

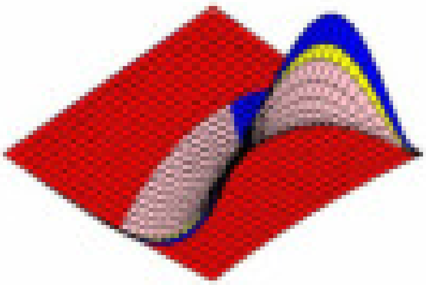

Figure 1:

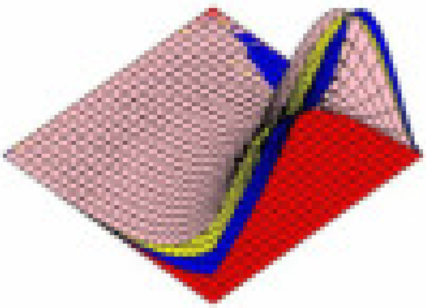

Figure 2:

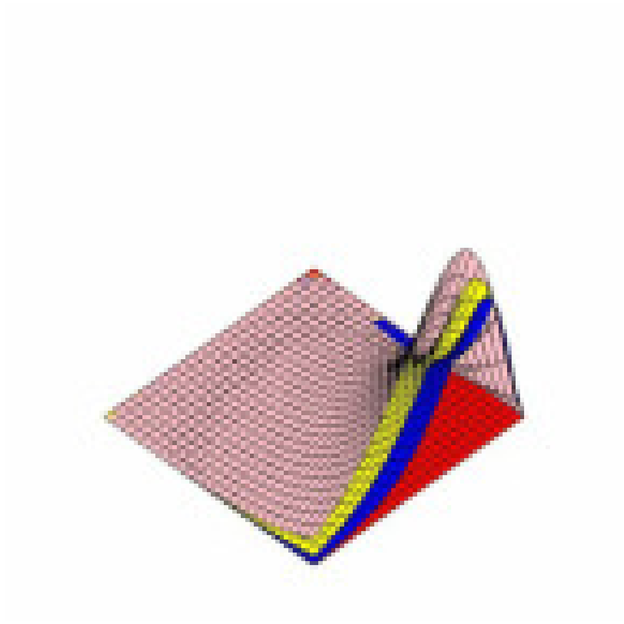

Figure 3:

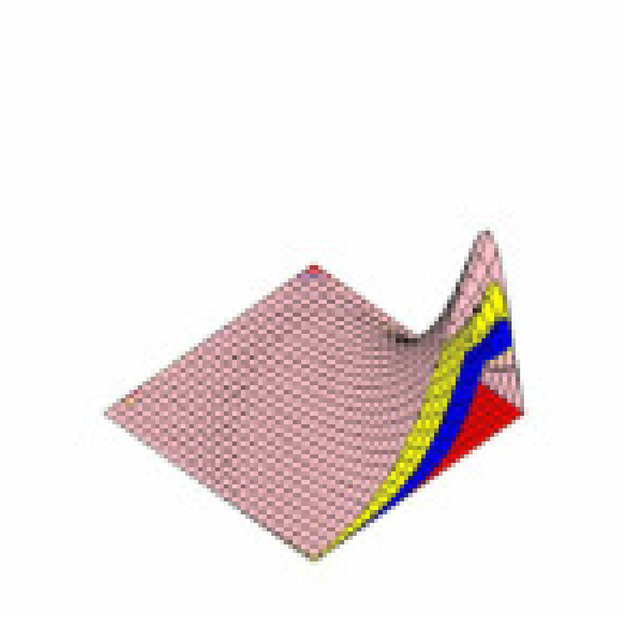

Figure 4:

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