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Quantitative Models of Magnetic and Electric Fields in the Magnetosphere

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Abstract

In order to represent the magnetic field $\underline{B}$ in the magnetosphere various auxiliary functions can be used: the current density $\underline{j}$, the scalar potential γ, toroidal and poloidal potentials Ψ_1 and Ψ_2 and Euler potentials (α, β) -- or else, the components of $\underline{B}$ may be expanded directly, with constraints ensuring the vanishing of $\nabla \cdot \underline{B}$. The most versatile among the linear representations is the one based on (Ψ_1, Ψ_2); it has seen relatively little use in the past but appears to be the most promising one for future work. Euler potentials are non-linear and can only be recommended for cases where their special properties are utilized, e.g. the representation of electric potentials when $E_{\parallel} = 0$. Other classifications of models include simple "testbed" models vs. "comprehensive" ones and analytical vs. numerical representations. The electric field $\underline{E}$ in the magnetosphere is generally assumed to vary only slowly and to be orthogonal to $\underline{B}$, allowing the use of a scalar potential $\phi(\alpha, \beta)$ which may be deduced from observations in the ionosphere, from the shape of the plasmapause or (as McIlwain has done) from particle observations in synchronous orbit. A simple model of ϕ is discussed and general implications are described.

This talk is meant to be a review of technical points - of methods and ideas - involved in the construction of quantitative models of magnetic and electric fields in the magnetosphere.

Because time is limited, I shall not devote my talk to the cataloging and comparison of existing methods: I have a review article available which does just that for models of the magnetic field and you are welcome to take a copy with you, to read on the plane home. There are also available some copies of a somewhat more restricted piece of work on electric fields.

Instead, I would like to use the time to bring a bit of order to the profusion of models - to classify the wide variety of models according to mathematical type, representation and application. When we use a model our choice generally depends on the application for which it is intended and this classification, I hope, will make it clearer what is available.

(Figure 1)

Let me start with the m a g n e t i c f i e l d . One basic classification depends on the auxiliary functions which are used for representing the field.

The f i r s t t w o of the representations shown are based on the current density **j** , which is generally introduced in one of two ways. F i r s t , there exist cases where **j** is an observed quantity - say, the tail's current sheet, field aligned currents or the ring current as deduced from particle populations. Akasofu and Chapman, for instance, carried out extensive work on ring current fields based on this approach. Of course, what you get is then a model of what your theory predicts the field to be, not necessarily a representation of **B** as observed.

S e c o n d l y , you can express **j** in some general way and fit the expansion coefficients so that the observed field **B** is represented as closely as possible: this is Bill Olson's approach. By using **j** one can assure that the divergence of **B** in the derived model vanishes, although at first sight this does not appear much of an advantage, since

Mathematical Representations of $\underline{B}$

(1) $$\underline{B} = \frac{\mu_0}{4\pi} \int \frac{\underline{j} \times \underline{r}}{r^3} \, dV$$ Biot-Savart Law

(2) $$\underline{A} = \frac{\mu_0}{4\pi} \int (\underline{j}/r) \, dV$$ Vector Potential

(3) $$\underline{B} = - \nabla \gamma$$ Scalar Potential

(4) $$\underline{B} = \nabla \times \psi_1 \underline{r} + \nabla \times \nabla \times \psi_2 \underline{r}$$ Toroidal and Poloidal Components

$$= \nabla \psi_1 \times \underline{r} + \nabla (\partial / \partial r) \psi_2 r - \underline{r} \nabla^2 \psi_2$$

(5) $$\underline{B} = \nabla \alpha \times \nabla \beta$$ Euler Potentials

(6) $$B_i = \sum_{k,m,n} a_{ikmn} x^k y^m z^{n-k-m}$$ Direct Representation (with constraints)

(for example)

Figure 1

$\underline{j}$ also has to be divergence-free. The real advantage is that while $\underline{j}$ may be a rather discontinuous current density distribution - current filaments or what not - the resulting $\underline{B}$ is rather smooth, since it is obtained by integration.

Next, you have the representation by a s c a l a r p o t e n t i a l γ : this is only good for curl-free fields, but it is the preferred method for representing the main geomagnetic field which originates in the earth's core. Usually γ is expanded in spherical harmonics and the number of terms can go up to a 100 or so, depending on the accuracy which you want.

The next method uses two functions Ψ_1 and Ψ_2 which, as far as I know, have no names, and I'll therefore call them here t o r o i d a l and p o l o i d a l p o t e n t i a l s , respectively, since the terms in which they appear are called the toroidal and poloidal components of $\underline{B}$. This very powerful representation - it is equivalent to the use of spherical vector harmonics - was introduced into dynamo theory by Walter Elsasser about 30 years ago and is well-known to astronomers, but not, apparently, to those engaged in magnetospheric physics. It deserves more attention from us and I will have something to say about this later on.

To give you some intuitive feeling for what these functions mean, notice from the second line in item (4) that the toroidal field is perpendicular to $\underline{r}$: it thus represents field lines circling the origin in some manner, like field lines which circle a wire in which an electric current flows.

The poloidal component, on the other hand, resembles what you find in magnetospheric models. The dipole field, for instance, is poloidal; more generally, you will note that if Ψ_2 is harmonic, the poloidal component is curl-free and, in fact, all curl-free fields can be thus represented. The representation is unique - that is, any part of $\underline{B}$ has to be either poloidal or toroidal, there remains no ambiguity.

The preceding 4 representations all form a single group: they are all <u>linear</u> and can therefore be superposed as we see fit: we could, for instance, combine a main field represented by γ with a tail field given by $\underline{j}$ - this is done in the Mead-Williams model - and improve the fit by adding expansions of ψ_1 and ψ_2. By contrast the <u>next</u> method on the list - E u l e r p o t e n t i a l s - is not linear, since in using it you multiply derivatives of α by those of β .

Because of this non-linearity one cannot in this case add up contributions - instead, α and β have to be calculated from the beginning for the total field. This is a great inconvenience, so unless you have a very good reason - or work with the dipole field, where α and β are simple - it may be better to use a different representation.

The a d v a n t a g e of Euler potentials over other methods is that they give an explicit analytical representation of magnetic field lines. Whenever the physics of the situation demands such a representation, they tend to be extremely useful: later on, when electric fields are discussed, we shall see one example of this.

Finally, $\underline{B}$ can be expanded in a general analytical or numerical way.

(Figure 2)

One problem here is in ensuring the vanishing of $\nabla\cdot\underline{B}$. The magnetospheric models of Mead and Fairfield, for instance, expand the components of $\underline{B}$ in powers of x, y and z, as shown in the figure, and they ensure the vanishing of $\nabla\cdot\underline{B}$ by the addition of linear constraints, which are taken into account (when the coefficients are derived) by the method of Lagrangian multipliers.

Notice, however, that the same result could be obtained more neatly if we used toroidal and poloidal potentials and expanded t h e m in powers of (x, y, z). If you do this, then $\psi_1\underline{r}$ and $\psi_2\underline{r}$ are sums

Direct Representations of $\underline{B}$

G.H. Mead and D.H. Fairfield, "A Quantitative Magnetospheric Model Derived from Spacecraft Magnetometer Data", JGR 80, 523, February 1975, use the representation

$$B_i = \sum_{k,m,n} a_{ikmn} x^k y^m z^{n-k-m} \qquad n \leq 2$$

Linear constraints assure the vanishing of $\nabla \cdot \underline{B}$ and terms are omitted to preserve symmetry.

However, if

$$\underline{B} = \nabla \times \Psi_1 \underline{r} + \nabla \times \nabla \times \Psi_2 \underline{r}$$

$$\Psi_i = \sum_{k,m,n} a_{ikmn} x^k y^m z^{n-k-m}$$

then, since

$$\underline{r} = \hat{\underline{x}}\, x + \hat{\underline{y}}\, y + \hat{\underline{z}}\, z$$

the same expansion results with no need for constraints.

For better control at large r it helps to modify the expansion to

$$\Psi_i = \sum_{k,m,n} a_{ikmn} x^k y^m z^{n-k-m} e^{-(r/r_o)}$$

Figure 2

of the unit vectors in the (x, y, z) directions multiplied by polynomials in (x, y, z), and if you take the curl or double curl you are still left with expressions of the same sort.

This approach not only eliminates the need for constraint equations but also makes it easy to generalize the method. The Mead-Fairfield expansion stops at quadratic terms - those with $n = 2$ - because higher powers are hard to control near the boundary and besides, the constraints become non-linear. However, with Ψ_1 and Ψ_2 you can add an exponential term which limits the expansion terms at large distances, giving a model similar to the one devised by Olson but with strict control over $\nabla \cdot \underline{B}$.

Some time ago I have developed a computer program which implements this method and it seems to work quite well. If anyone here is interested, I will be glad to discuss it in private later on. Other modifications to this approach could also be devised: because toroidal and poloidal potentials are such versatile tools, I expect them to be important in future development of quantitative magnetospheric models.

(Figure 3)

With so many methods of representation available, many different models can be - and have been - constructed. They seem to fall into two main classes. There are "t e s t b e d m o d e l s" which aim at simplicity: you use them in theoretical work when you want to investigate effects involving some qualitative properties of the field <u>without</u> dragging in too much complexity. For instance, if you wish to develop a theory of effects due to the South Atlantic anomaly, you might be satisfied - at least at first - with the eccentric dipole.

On the other hand - and of more interest to this meeting - there exist "c o m p r e h e n s i v e m o d e l s" (some people here may call them "quantitative models", although strictly speaking <u>all</u> models discussed here are quantitative) - which try to represent observations as accurately as possible. The procedure by which such models are derived usually involves some mathematical representation

Types of Geomagnetic Models

Class of Models	Specific Models	Specific Applications
Testbed Models	Dipole field	General - simplest approximation
	Eccentric dipole	South Atlantic anomaly
	Image dipole Mead's 3-term model Mead-Williams 3-term (α, β)	Simple model of distorted field and dayside boundary
	2-dimensional models of tail field	Particle motion in plasma sheet
Comprehensive Models	Main field γ	Study of internal field and of field near surface of earth
	Main field (α, β)	Conjugate points
	Olson and Pfitzer model Mead and Fairfield model Generalization by Ψ_1 and Ψ_2	General use. Fitting of satellite data. Correlation with tilt, K_p, sector etc.
	Magnetospheric (α, β) (in future?)	Mapping of $\underline{E}$

Figure 3

which contains a number of unknown coefficients, and the values of the coefficients which best fit the observed data are derived by least squares fitting. Such models perform several useful functions:

(1) They average out fluctuations in the data.
(2) They help relate observations of particles etc. to the "real" magnetosphere.
(3) They enable one to extract from large data sets the average behavior of the magnetosphere - how it changes with K_p , with the tilt angle of the dipole axis, with the interplanetary field and its sectors, with solar wind pressure, and so forth.

A word of caution is however appropriate: such models do not provide data where none is available. It is the nature of models to bridge over regions of sparse data, or to extend to distances beyond those for which data exists, and the model is then no more than a mathematical interpolation or extrapolation. This is especially important to remember with models of the electric field, like McIlwain's - and even if the author there warns all users that the model is only valid in a limited region, there exists great temptation to follow it beyond its limits.

Time does not allow me to go into other details, but there exists one more division of models which should be discussed, namely of analytical vs. numerical.

In all representations given in figure 1 the functions representing the field may be given either analytically or numerically. So far almost all models have been analytical, simply because even a sparse numerical grid introduces a tremendous jump in the number of coefficients handled. Yet we might be approaching the limit of practical accuracy in analytical representations.

One simple remedy is to use different representations for different regions and splice them together where they meet. This might be a useful

The facing page is left blank so that figures will remain matched to the text thing to do in the tail region - one really should not expect the same expansion to describe the tail and the main magnetosphere, since the physical processes in the two regions are quite different. Ultimately, I suspect, there exists a stage at which it pays to express the difference between the average field and our "best" analytical model by a coarse numerical grid - since the difference would be small, the grid would not require great accuracy. At the present time, however, the dispersion of our observational data is so large that analytical models satisfy all our needs.

Let me now switch over to the electric field $\underline{E}$ where things are in a much worse shape, mainly because of the lack of data.

(Figure 4)

I do not have the time here to go into the history of this subject, which is fascinating, or the theory, which is controversial - let me just say that Alfvén's original speculation about a large-scale dawn-to-dusk electric field across the magnetosphere seems to be borne out. The same electric field probably also extends across the geomagnetic tail, while near the earth it must be modified by the addition of an extra component due to the earth's rotation (and perhaps some contribution from ionospheric motions).

In most applications it can be assumed that the magnetic field does not vary with time, so that $\underline{E}$ can be represented by a scalar potential ϕ. If conductivity along magnetic field lines is high then such lines will be electric equipotentials: the result is best expressed when the magnetic field is given in terms of Euler potentials (in fact, I know of no other way) and reduces to ϕ being a function of α and β alone.

For instance, the contribution of the co-rotation field to ϕ, in the case of an axisymmetrical model of the main magnetic field, is

Models of the Electric Field

For time-independent fields

$$\underline{E} = -\nabla\phi$$

If $\underline{E}\cdot\underline{B} = 0$ then

$$\phi = \phi(\alpha, \beta)$$

and ϕ in the magnetosphere is determined by its value in the ionosphere or (for closed field lines) in the equatorial plane.

If $\underline{B}$ is symmetric around the axis of rotation, the contribution of co-rotation to ϕ is

$$\phi_{cor.} = \alpha\,\omega\,R_e$$

where ω is the angular velocity of the earth's rotation and R_e is the earth's radius.

Figure 4

$$\phi = \alpha \omega R_e$$

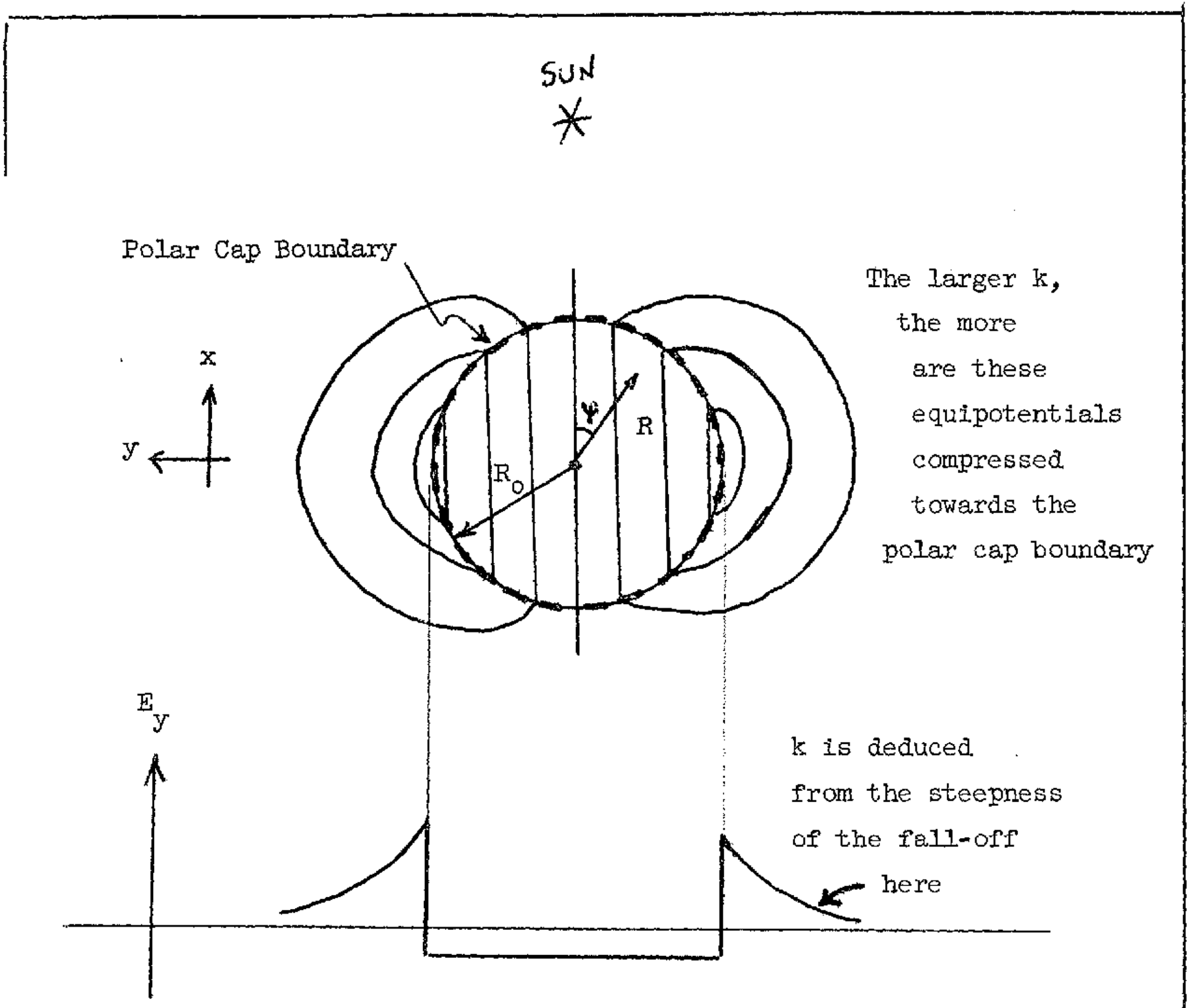
as shown in Figure 4. If the field's asymmetry <u>is</u> taken into account, the rotation of the earth leads to a finite $\partial \underline{B}/\partial t$ in the frame of reference of the magnetosphere and one cannot use ϕ alone any more. Ways do exist for handling this situation but I do not have the time to describe them.

If ϕ is expressed in terms of the field-line parameters α and β one only has to know its value at o n e p o i n t on each field line in order that ϕ be fully specified. Convenient choices for that point are <u>either</u> at the "roots" of the field line in the ionosphere <u>or</u> in the equatorial plane; as it turns out, these are also the two locations where most of the information about $\underline{E}$ is obtained.

(Figure 5)

The electric field in the upper ionosphere has been inferred from ionospheric currents, barium cloud drifts, auroral motions and direct observations from OGO 6 and Injun 5, from rockets and even from balloons, and all the evidence points to a two-celled electric field as shown in Figure 5 . What the figure shows is a schematic map of equipotentials in the polar cap, and below it you can see a sketch of how the dawn-dusk component of $\underline{E}$ varies during a pass over the middle of the polar cap.

If, in the map drawn here, one introduces plane polar coordinates (R, φ), then ϕ can be represented (very nearly) by the analytical functions given on the slide. These functions contain one adjustable parameter k which represents the steepness with which the electric field falls off just outside the polar cap boundary: from profiles of the polar electric field, similar to the one drawn in Figure 5 and obtained by Heppner on OGO 6 , one finds that $k \approx 4$. Note that it is the region o u t s i d e the polar cap that interests us most, since it corresponds to field lines which close inside about 10 R_e . Field lines connected to the polar cap are either open or lead into the tail and are much harder to include in a model, since their properties are not well known.



Inside polar cap $\quad \phi \approx \phi_o (R/R_o) \sin\varphi$

Outside polar cap $\quad \phi \approx \phi_o (R/R_o)^{-k} \sin\varphi$

(plus co-rotation)

Figure 5

(Figure 6)

You now translate your map into (α, β), add the co-rotation field and thus get a model valid for the entire volume threaded by your field lines. You can, for instance, map the electric field into the equatorial plane and it is interesting to note what happens: with $k = 2$ the field there - without co-rotation - is a constant field from dawn to dusk, with equipotentials stretched along the noon-midnight direction. With k less than 2 the equipotentials are pinched near earth while with k more than 2 - the actual case - they bulge out there. The sketches at the bottom of the figure show how it all looks when co-rotation is added.

The method outlined here is probably the most feasible for mapping out $\underline{E}$ in any detail: in 5 years or so, if the Electrodynamic Explorer satellite ever becomes reality, we ought to be getting quite detailed maps of the electric field in the polar ionosphere as functions of (α, β), and they can then be mapped into the equatorial plane or anywhere else.

The next figure (Figure 7) shows how the $k = 4$ equipotentials actually look in the equatorial plane.

The closed contour marks the boundary at which the co-rotation field becomes dominant and this seems to correspond to the plasmapause. Volland (JGR 78, 171, 1973) used the observed shape of the plasmapause and also obtained $k \approx 4$ for quiet times, which seems to support this approach. For disturbed times he got $k \approx 2.73$ and in addition there was a slight rotation of the pattern, so that it was no longer symmetric with respect to the noon-midnight meridian. This rotation shifts the bulge of the plasmapause towards midnight, as has been observed. Earlier calculations of this kind, by Vasyliunas and by Nagata and Kokubun, are also cited by Volland.

In dipole field, using spherical coordinates (r, θ, φ)

$$R \approx \text{constant} \cdot \alpha^{1/2}$$

$$\varphi = \beta / R_e$$

If α_o corresponds to R_o (= to polar cap boundary)
then

$$- \phi_o (R/R_o) \sin\varphi \rightarrow - \phi_o (\alpha/\alpha_o)^{1/2} \sin\varphi \quad (|\alpha| < |\alpha_o|)$$

$$- \phi_o (R_o/R)^k \sin\varphi \rightarrow - \phi_o (\alpha_o/\alpha)^{k/2} \sin\varphi \quad (|\alpha| > |\alpha_o|)$$

To this one has to add $- \alpha \omega R_e$ due to co-rotation.

To map into equatorial plane, note that there $\alpha \approx \frac{\text{constant}}{r}$

Result:

$k < 2$ $\qquad$ $k = 2$ $\qquad$ $k > 2$

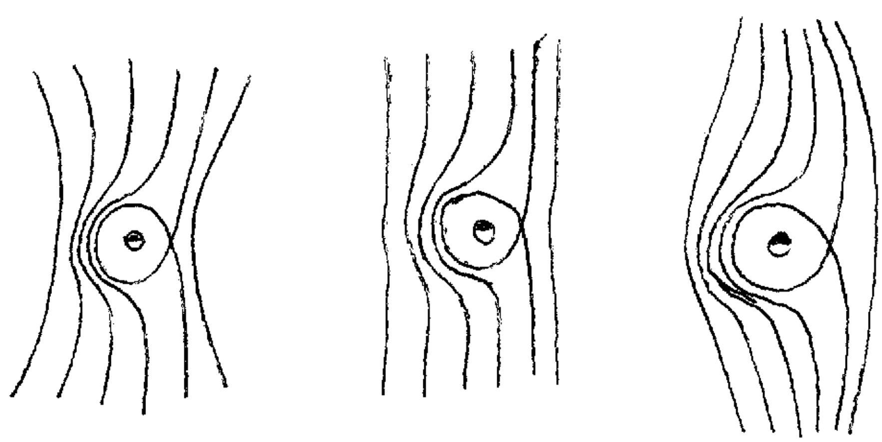


Figure 6

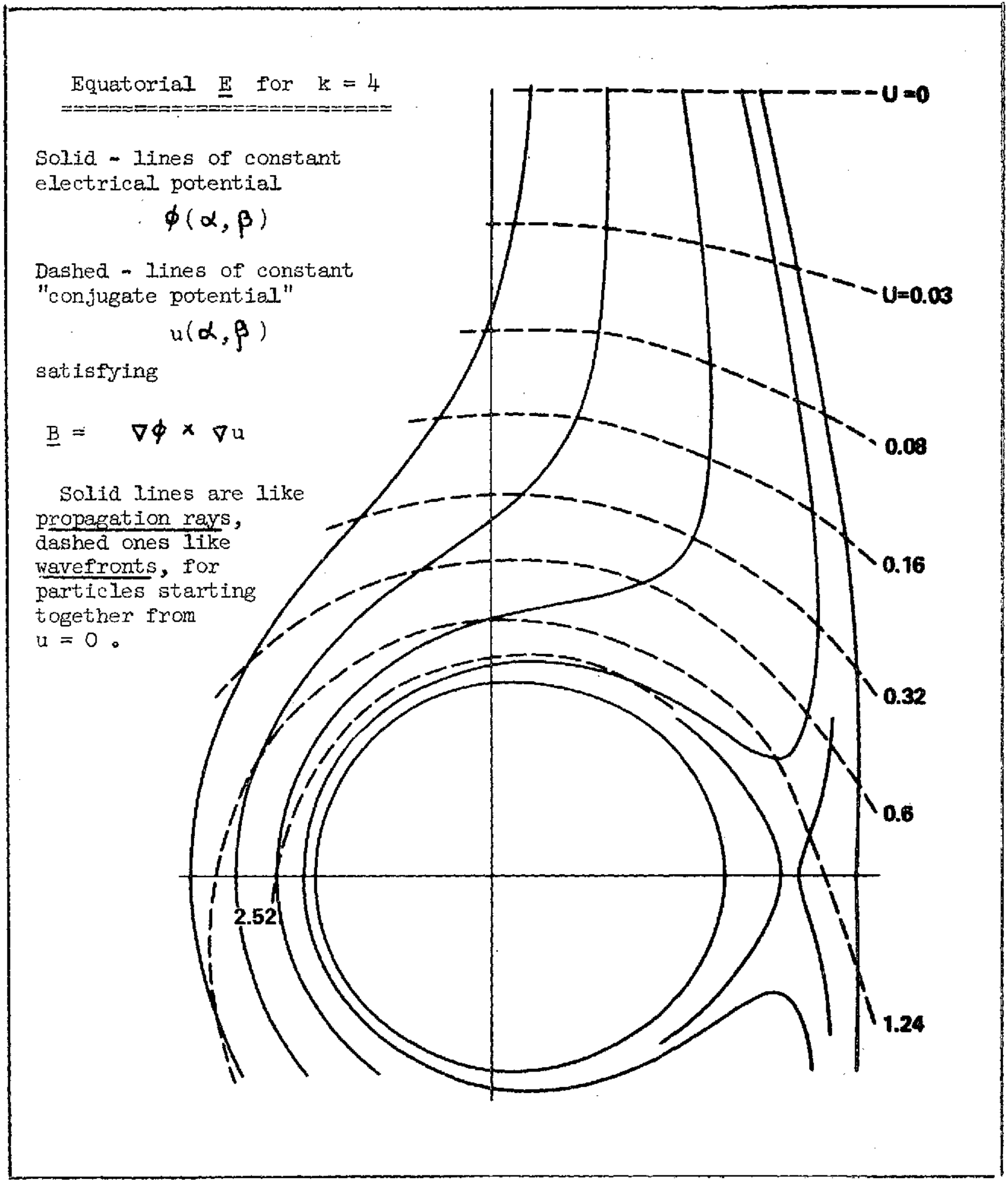

Equatorial E for k = 4
Solid - lines of constant electrical potential
$\phi(\alpha, \beta)$
Dashed - lines of constant "conjugate potential"
$u(\alpha, \beta)$
satisfying
$\underline{B} = \nabla\phi \times \nabla u$
Solid lines are like propagation rays, dashed ones like wavefronts, for particles starting together from u = 0 .
U =0
U=0.03
0.08
0.16
0.32
0.6
1.24
2.52

Figure 7

In the equatorial plane $\underline{E}$ itself is too weak to be measured directly, but McIlwain has constructed some rather detailed models of $\underline{E}$ based on observations by ATS-5 in synchronous orbit. His data come from enhanced fluxes of low energy particles - especially protons - injected during substorms, and he assumed that all particles were impulsively injected at the inner edge of the plasma sheet, at a single instant. He also assumed that the electric field did not vary in time and proceeded to express its potential by means of a general mathematical expansion: the coefficients of this expansion were adjusted until they fit as closely as was possible the observed particle spectra and the times at which they were observed.

This method claims to give $\underline{E}$ within the range of 5 to 10 R_e, although it is difficult to assess its accuracy. I hope that later in the session we will have the opportunity to hear more about it.

Ultimately, in models of both $\underline{B}$ and $\underline{E}$, we are going to run against the limit imposed by the variability of these fields. The variation of $\underline{E}$ is especially pronounced and has been explored by Chen, Grebowsky and others: they deduce it from variations of the plasmapause, which lead to "tails" and/or "islands" of plasma isolated from the main body of the plasmasphere.

There still remains a lot here that's not only poorly mapped but also poorly understood. I hope that within the next 5 years we will obtain at least good models of the average magnetic and electric fields in the magnetosphere: after that we might try what happens on shorter scales of space and time.

Thank you.