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RELLICH INEQUALITIES ON FINSLER-HADAMARD MANIFOLDS

Alexandru Kristály and Dušan Repovš

Abstract

In this paper we are dealing with improved Rellich inequalities on Finsler-Hadamard manifolds with vanishing mean covariation where the remainder terms are expressed by means of the flag curvature. By exploiting various arguments from Finsler geometry we show that more weighty curvature implies more powerful improvements. The sharpness of the involved constants are also studied.

Keywords: Rellich inequality, Finsler-Hadamard manifold, Finsler-Laplace operator, curvature.
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1 Introduction and main results

The Hardy inequality

$$\int_{\mathbb{R}^n} |\nabla u|^2 \mathrm{d}x \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{u^2}{|x|^2} \mathrm{d}x,\ \forall u \in C_0^\infty(\mathbb{R}^n),$$

plays a central role in the study of singular elliptic problems, $n \geq 3$, where the constant $\frac{(n-2)^2}{4}$ is sharp but not achieved. The second-order Hardy inequalities are referred as *Rellich inequalities* whose most familiar forms can be stated as follows; given $n \geq 5$, one has

$$\int_{\mathbb{R}^n} (\Delta u)^2 \mathrm{d}x \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{u^2}{|x|^4} \mathrm{d}x, \forall u \in C_0^\infty(\mathbb{R}^n), \tag{1.1}$$

$$\int_{\mathbb{R}^n} (\Delta u)^2 \mathrm{d}x \geq \frac{n^2}{4} \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{|x|^2} \mathrm{d}x,\ \forall u \in C_0^\infty(\mathbb{R}^n), \tag{1.2}$$

where both constants $\frac{n^2(n-4)^2}{16}$ and $\frac{n^2}{4}$ are sharp, but are never achieved. Hereafter, Δ, ∇, $|\cdot|$ and $\mathrm{d}x$ denote the classical Laplace operator, the Euclidean gradient, the Euclidean norm and the Lebesgue measure on $\mathbb{R}^n$, respectively. Due to the lack of extremal functions in the Rellich inequalities, various improvements of (1.1) and (1.2) can be found in the literature; see e.g. Ghoussoub and Moradifam [6], Tertikas and Zographopoulos [13], and references therein.

Hardy and Rellich inequalities have also been studied on *curved spaces*. As far as we know, Carron [4] first studied Hardy inequalities on complete, non-compact Riemannian manifolds. Motivated by [4], Kombe and Özaydin [7, 8], and Yang, Su and Kong [15] presented various Brezis-Vazquez-type improvements of Hardy and Rellich inequalities on complete, non-compact Riemannian manifolds. Recently, Kristály [9] proved Hardy inequalities on reversible Finsler manifolds where the improvements are given in terms of the curvature.

The purpose of our paper is to describe improved Rellich inequalities on *Finsler-Hadamard manifolds* (i.e., complete, simply connected Finsler manifolds with non-positive flag curvature) where the remainder terms involve the flag curvature. Two facts should be highlighted:

- We prove that Rellich inequalities on Finsler-Hadamard manifolds are better improved once the flag curvature is more powerful. These phenomena can be considered as second-order versions of the result described in [9].
- Since Rellich inequalities on Finsler manifolds involve the highly nonlinear Finsler-Laplace operator $\boldsymbol{\Delta}$, expected properties usually fail (which trivially hold on the 'linear' Riemannian context). Although our results are also genuinely new in the Riemannian framework, we prefer to present them in the context of Finsler geometry. In this manner, we emphasize the deep connection between geometric and analytic phenomena which are behind of second-order Sobolev-type inequalities on Finsler manifolds, providing a new bridge between Finsler geometry and PDEs. This fact is interesting in its own right as well from the point of view of applications, see Antonelli, Ingarden and Matsumoto [1].

In order to present the nature of our results, we need some notations and notions, see §2.

Let (M,F) be an $n-$dimensional complete reversible Finsler manifold ($n \geq 5$), $d_F : M \times M \to \mathbb{R}$ being the natural distance function generated by the Finsler metric F, and let $F^* : T^*M \to [0,\infty)$ be the polar transform of F. Let $Du(x) \in T_x^*M$, $\boldsymbol{\nabla} u(x) \in T_xM$ and $\boldsymbol{\Delta} u(x)$ be the derivative, gradient and Finsler-Laplace operator of u at $x \in M$, respectively. Let $\mathrm{d}V_F(x)$ be the Busemann-Hausdorff measure on (M,F) and for a fixed $x_0 \in M$, let us denote $d(x) := d_F(x_0,x)$.

Let $G_F : C_0^\infty(M) \to \mathbb{R}$ be defined by

$$G_F(u) = \int_M \left[u(x)^2 \boldsymbol{\Delta}(d(x)^{-2}) - d(x)^{-2}\boldsymbol{\Delta}(u(x)^2)\right] \mathrm{d}V_F(x),$$

which gives the 'Green-deflection' of u with respect to the Finsler metric F; for a generic Finsler manifold (M,F), the function G_F does not vanish. However, $G_F \equiv 0$ whenever (M,F) is Riemannian due to Green's identity. Finally, we introduce the following class of functions

$$C_{0,F}^\infty(M) = \{u \in C_0^\infty(M) : G_F(u) = 0\}.$$

A simple consequence of our main results (see Theorems 3.1 & 3.2) can be stated as follows.

Theorem 1.1 *Let (M,F) be an $n-$dimensional reversible Finsler-Hadamard manifold with vanishing mean covariation, and suppose the flag curvature on (M,F) is bounded above by $c \leq 0$.*

(a) *If $n \geq 5$, then for every $u \in C_{0,F}^\infty(M)$ one has*

$$\begin{aligned} \int_M (\boldsymbol{\Delta} u)^2 \mathrm{d}V_F(x) \quad \geq \quad & \frac{n^2(n-4)^2}{16}\int_M \frac{u^2}{d(x)^4}\mathrm{d}V_F(x) \\ & + \frac{3|c|n(n-1)(n-2)(n-4)}{4}\int_M \frac{u^2}{(\pi^2+|c|d(x)^2)d(x)^2}\mathrm{d}V_F(x), \end{aligned}$$

and the constant $\frac{n^2(n-4)^2}{16}$ is sharp.

(b) *If $n \geq 9$, then for every $u \in C_{0,F}^{\infty}(M)$ one has*

$$\int_M (\boldsymbol{\Delta} u)^2 \mathrm{d}V_F(x) \geq \frac{n^2}{4}\int_M \frac{F^*(x,Du(x))^2}{d(x)^2}\mathrm{d}V_F(x), + \frac{3|c|n(n-1)(n-4)^2}{8}\int_M \frac{u^2}{(\pi^2+|c|d(x)^2)d(x)^2}\mathrm{d}V_F(x),$$

and the constant $\frac{n^2}{4}$ is sharp.

Remark 1.1 (i) When the flag curvature on (M,F) becomes more powerful (i.e., $|c|$ is large), the Rellich inequalities in Theorem 1.1 is also better improved.

(ii) Theorem 1.1 is also new for Cartan-type Riemannian manifolds; indeed, these spaces belong to the class of Cartan-Finsler manifolds with vanishing mean covariation and $C_{0,F}^{\infty}(M) = C_0^{\infty}(M)$.

In Section 2 we shall recall some elements from Finsler geometry, namely the flag curvature, Laplace and volume comparisons, differentials. In Section 3 we shall prove our main results (see Theorems 3.1 & 3.2), while in Section 4 we shall present some concluding remarks.

2 Preliminaries

Let M be a connected $n-$dimensional C^∞ manifold and $TM = \bigcup_{x\in M} T_xM$ its tangent bundle. The pair (M,F) is called a reversible *Finsler manifold* if the continuous function $F : TM \to [0,\infty)$ satisfies the following conditions

(a) $F \in C^\infty(TM \setminus \{0\})$;

(b) $F(x,ty) = |t|F(x,y)$ for all $t \in \mathbb{R}$ and $(x,y) \in TM$;

(c) $g_{ij}(x,y) := [\frac{1}{2}F^2(x,y)]_{y^iy^j}$ is positive definite for all $(x,y) \in TM \setminus \{0\}$.

If $g_{ij}(x) = g_{ij}(x,y)$ is independent of y then (M,F) is called *Riemannian manifold*. A *Minkowski space* consists of a finite dimensional vector space V and a Minkowski norm which induces a Finsler metric on V by translation, i.e., $F(x,y)$ is independent of the base point x; in such cases we often write $F(y)$ instead of $F(x,y)$. While there is a unique Euclidean space (up to isometry), there are infinitely many (isometrically different) Minkowski spaces.

We consider the *polar transform* of F, defined for every $(x,\xi) \in T^*M$ by

$$F^*(x,\xi) = \sup_{y\in T_xM\setminus\{0\}} \frac{\xi(y)}{F(x,y)}. \tag{2.1}$$

Note that for every $x \in M$, the function $F^*(x,\cdot)$ is a Minkowski norm on T_x^*M. Since $F^*(x,\cdot)^2$ is twice differentiable on $T_x^*M \setminus \{0\}$, we consider the matrix

$$g_{ij}^*(x,\xi) := [\frac{1}{2}F^*(x,\xi)^2]_{\xi^i\xi^j}$$

for every $\xi = \sum_{i=1}^n \xi^i \mathrm{d}x^i \in T_x^*M \setminus \{0\}$ in a local coordinate system (x^i).

Let π^*TM be the pull-back bundle of the tangent bundle TM generated by the natural projection $\pi : TM \setminus \{0\} \to M$, see Bao, Chern and Shen [2, p. 28]. The vectors of the pull-back bundle π^*TM are denoted by $(v;w)$ with $(x,y) = v \in TM \setminus \{0\}$ and $w \in T_xM$. For simplicity, let $\partial_i|_v = (v; \partial/\partial x^i|_x)$ be the natural local basis for π^*TM, where $v \in T_xM$. One can introduce the *fundamental tensor* g on π^*TM by

$$g^v := g(\partial_i|_v, \partial_i|_v) = g_{ij}(x,y), \tag{2.2}$$

where $v = y^i(\partial/\partial x^i)|_x$. Unlike the Levi-Civita connection in the Riemannian case, there is no unique natural connection in the Finsler geometry. Among all natural connections on the pull-back bundle π^*TM, we choose a torsion free and almost metric-compatible linear connection on π^*TM, the so-called *Chern connection*, see Bao, Chern and Shen [2, Theorem 2.4.1]. The coefficients of the Chern connection are denoted by Γ^i_{jk}, which replace the well known Christoffel symbols from Riemannian geometry. A Finsler manifold is said to be of *Berwald type* if the coefficients $\Gamma^k_{ij}(x, y)$ in natural coordinates are independent of y. It is clear that Riemannian manifolds and (locally) Minkowski spaces are Berwald spaces. The Chern connection induces in a natural manner on π^*TM the *curvature tensor* R, see Bao, Chern and Shen [2, Chapter 3]. By means of the connection, we also have the *covariant derivative* $D_v u$ of a vector field u in the direction $v \in T_xM$. Note that $v \mapsto D_v u$ is not linear. A vector field $u = u(t)$ along a curve σ is said to be *parallel* if $D_{\dot\sigma} u = 0$. A C^∞ curve $\sigma : [0, a] \to M$ is called a *geodesic* if $D_{\dot\sigma}\dot\sigma = 0$. Geodesics are considered to be parametrized proportionally to their arc-length. The Finsler manifold is said to be *complete* if every geodesic segment can be extended to $\mathbb{R}$.

Let $u, v \in T_xM$ be two non-collinear vectors and $\mathcal{S} = \mathrm{span}\{u, v\} \subset T_xM$. By means of the curvature tensor R, the *flag curvature* of the flag $\{\mathcal{S}, v\}$ is then defined by

$$K(\mathcal{S}; v) = \frac{g^v(R(U, V)V, U)}{g^v(V, V)g^v(U, U) - g^v(U, V)^2}, \tag{2.3}$$

where $U = (v; u), V = (v; v) \in \pi^*TM$. If for some $c \in \mathbb{R}$ we have $K(\mathcal{S}; v) \le c$ for every choice of U and V, we say that the flag curvature is bounded from above by c and we write $\mathbf{K} \le c$. (M, F) is called a *Finsler-Hadamard* manifold if it is complete, simply connected and $\mathbf{K} \le 0$. If (M, F) is Riemannian, the flag curvature reduces to the well known sectional curvature.

Let $\sigma : [0, r] \to M$ be a piecewise C^∞ curve. The value $L_F(\sigma) = \int_0^r F(\sigma(t), \dot\sigma(t))\,\mathrm{d}t$ denotes the *integral length* of σ. For $x_1, x_2 \in M$, denote by $\Lambda(x_1, x_2)$ the set of all piecewise C^∞ curves $\sigma : [0, r] \to M$ such that $\sigma(0) = x_1$ and $\sigma(r) = x_2$. Define the *distance function* $d_F : M \times M \to [0, \infty)$ by

$$d_F(x_1, x_2) = \inf_{\sigma \in \Lambda(x_1, x_2)} L_F(\sigma). \tag{2.4}$$

Clearly, d_F satisfies all properties of the metric (i.e., $d_F(x_1, x_2) = 0$ if and only if $x_1 = x_2$, d_F is symmetric, and it satisfies the triangle inequality). The open *metric ball* with center $x_0 \in M$ and radius $\rho > 0$ is defined by $B(x_0, \rho) = \{x \in M : d_F(x_0, x) < \rho\}$.

Let $\{\partial/\partial x^i\}_{i=1,...,n}$ be a local basis for the tangent bundle TM, and let $\{\mathrm{d}x^i\}_{i=1,...,n}$ be its dual basis for T^*M. Let $B_x(1) = \{y = (y^i) : F(x, y^i\partial/\partial x^i) < 1\}$ be the unit tangent ball at T_xM. The *Busemann-Hausdorff volume form* $\mathrm{d}V_F$ on (M, F) is defined by

$$\mathrm{d}V_F(x) = \sigma_F(x)\mathrm{d}x^1 \wedge ... \wedge \mathrm{d}x^n, \tag{2.5}$$

where $\sigma_F(x) = \frac{\omega_n}{\mathrm{Vol}(B_x(1))}$. Hereafter, ω_n will denote the volume of the unit $n-$dimensional ball and $\mathrm{Vol}(S)$ the Euclidean volume of the set $S \subset \mathbb{R}^n$. The *Finslerian-volume* of a bounded open set $S \subset M$ is defined as $\mathrm{Vol}_F(S) = \int_S \mathrm{d}V_F(x)$. In general, one has that for every $x \in M$,

$$\lim_{\rho \to 0^+} \frac{\mathrm{Vol}_F(B(x, \rho))}{\omega_n \rho^n} = 1. \tag{2.6}$$

When $(\mathbb{R}^n, F)$ is a Minkowski space, then by virtue of (2.5), $\mathrm{Vol}_F(B(x, \rho)) = \omega_n\rho^n$ for every $\rho > 0$ and $x \in \mathbb{R}^n$.

The *Legendre transform* $J^*: T^*M \to TM$ associates to each element $\xi \in T_x^*M$ the unique maximizer on T_xM of the map $y \mapsto \xi(y) - \frac{1}{2}F^2(x,y)$. This element can also be interpreted as the unique vector $y \in T_xM$ with the following properties

$$F(x,y) = F^*(x,\xi) \text{ and } \xi(y) = F(x,y)F^*(x,\xi). \tag{2.7}$$

In a similar manner we can define the Legendre transform $J: TM \to T^*M$. In particular, $J^* = J^{-1}$ on T_x^*M and if $\xi = \sum_{i=1}^n \xi^i \mathrm{d}x^i \in T_x^*M$ and $y = \sum_{i=1}^n y^i(\partial/\partial x^i) \in T_xM$, then one has

$$J(x,y) = \sum_{i=1}^n \frac{\partial}{\partial y^i}\left(\frac{1}{2}F(x,y)^2\right)\frac{\partial}{\partial x^i} \text{ and } J^*(x,\xi) = \sum_{i=1}^n \frac{\partial}{\partial \xi^i}\left(\frac{1}{2}F^*(x,\xi)^2\right)\frac{\partial}{\partial x^i}. \tag{2.8}$$

Let $u: M \to \mathbb{R}$ be a differentiable function in the distributional sense. The *gradient* of u is defined by

$$\boldsymbol{\nabla} u(x) = J^*(x, Du(x)), \tag{2.9}$$

where $Du(x) \in T_x^*M$ denotes the (distributional) *derivative* of u at $x \in M$. In general, $u \mapsto \boldsymbol{\nabla} u$ is not linear.

Let $x_0 \in M$ be fixed. From now on when no confusion arises, we shall introduce the abbreviation

$$d(x) = d_F(x_0, x). \tag{2.10}$$

Due to Ohta and Sturm [10] and by relation (2.7), one has

$$F(x, \boldsymbol{\nabla} d(x)) = F^*(x, Dd(x)) = Dd(x)(\boldsymbol{\nabla} d(x)) = 1 \text{ for a.e. } x \in M. \tag{2.11}$$

In fact, relations from (2.11) are valid for every $x \in M \setminus (\{x_0\} \cup \mathrm{Cut}(x_0))$, where $\mathrm{Cut}(x_0)$ denotes the cut locus of x_0, see Bao, Chern and Shen [2, Chapter 8]. Note that $\mathrm{Cut}(x_0)$ has null Lebesgue (thus Hausdorff) measure for every $x_0 \in M$.

Let X be a vector field on M. In a local coordinate system (x^i), by virtue of (2.5), the *divergence* is defined by $\mathrm{div}(X) = \frac{1}{\sigma_F}\frac{\partial}{\partial x^i}(\sigma_F X^i)$. The *Finsler-Laplace operator*

$$\boldsymbol{\Delta} u = \mathrm{div}(\boldsymbol{\nabla} u)$$

acts on $W^{1,2}_{\mathrm{loc}}(M)$ and for every $v \in C_0^\infty(M)$, we have

$$\int_M v \boldsymbol{\Delta} u \mathrm{d}V_F(x) = -\int_M Dv(\boldsymbol{\nabla} u)\mathrm{d}V_F(x), \tag{2.12}$$

see Ohta and Sturm [10] and Shen [12]. In the Riemannian case, the Finsler-Laplace operator reduces to the Laplace-Beltrami operator, see Bonanno, G. Molica Bisci, V. Rădulescu [3].

Let $\{e_i\}_{i=1,\dots,n}$ be a basis for T_xM and $g_{ij}^v = g^v(e_i, e_j)$. The *mean distortion* $\mu: TM \setminus \{0\} \to (0,\infty)$ is defined by $\mu(v) = \frac{\sqrt{\det(g_{ij}^v)}}{\sigma_F}$. The *mean covariation* $\mathbf{S}: TM \setminus \{0\} \to \mathbb{R}$ is defined by

$$\mathbf{S}(x,v) = \frac{d}{dt}(\ln \mu(\dot\sigma_v(t)))\big|_{t=0},$$

where σ_v is the geodesic such that $\sigma_v(0) = x$ and $\dot\sigma_v(0) = v$. We say that (M,F) has *vanishing mean covariation* if $\mathbf{S}(x,v) = 0$ for every $(x,v) \in TM$, and we denote this by $\mathbf{S} = 0$. We recall that any Berwald space has vanishing mean covariation, see Shen [11].

We conclude this section by some important comparison results. Let $x_0 \in M$ be fixed and recall the notation introduced in (2.10). First, one has

$$\mathbf{\Delta} d(x) - \frac{n-1}{d(x)} = o(1) \text{ as } x \to x_0. \tag{2.13}$$

In order to have a global estimate for $\mathbf{\Delta} d(x)$, we consider for every $c \leq 0$ the function $\mathbf{ct}_c : (0, \infty) \to \mathbb{R}$ defined by

$$\mathbf{ct}_c(\rho) = \begin{cases} \frac{1}{\rho} & \text{if} \quad c = 0, \\ \sqrt{|c|}\coth(\sqrt{|c|}\rho) & \text{if} \quad c < 0. \end{cases}$$

Theorem 2.1 *Let (M, F) be an $n-$dimensional Finsler-Hadamard manifold with $\mathbf{S} = 0$ and $\mathbf{K} \leq c \leq 0$, and let $x_0 \in M$ be fixed. Then the following assertions hold:*

(a) (see [14, Theorem 5.1]) *For a.e. $x \in M$ one has $\mathbf{\Delta} d(x) \geq (n-1)\mathbf{ct}_c(d(x))$.*

(b) (see [14, Theorem 6.1]) *The function $\rho \mapsto \frac{\mathrm{Vol}_F(B(x,\rho))}{\rho^n}$ is non-decreasing, $\rho > 0$. In particular, by (2.6) we have*

$$\mathrm{Vol}_F(B(x,\rho)) \geq \omega_n \rho^n \textit{ for all } x \in M \textit{ and } \rho > 0.$$

3 Main results

Let $\mathbf{D}_c : [0, \infty) \to \mathbb{R}$ be the function defined by

$$\mathbf{D}_c(\rho) = \begin{cases} 0 & \text{if} \quad \rho = 0, \\ \rho\mathbf{ct}_c(\rho) - 1 & \text{if} \quad \rho > 0. \end{cases}$$

It is clear that $\mathbf{D}_c \geq 0$.

In order to establish our main results, we first need a quantitative Hardy inequality; see [9] for a particular form. For the reader's convenience we provide its proof.

Lemma 3.1 *Let (M, F) be an $n-$dimensional Finsler-Hadamard manifold with $\mathbf{S} = 0$ and let $\mathbf{K} \leq c \leq 0$, $x_0 \in M$ be fixed, and choose any $\alpha \in \mathbb{R}$ such that $n - 2 + \alpha > 0$. Then for every $u \in C_0^\infty(M)$ we have*

$$\begin{aligned} \int_M d(x)^\alpha F^*(x, Du(x))^2 \mathrm{d}V_F(x) \quad \geq \quad & \frac{(n-2+\alpha)^2}{4} \int_M d(x)^{\alpha-2} u(x)^2 \mathrm{d}V_F(x) \\ & + \frac{(n-2+\alpha)(n-1)}{2} \int_M d(x)^{\alpha-2} \mathbf{D}_c(d(x)) u(x)^2 \mathrm{d}V_F(x). \end{aligned}$$

Proof. By convexity and (2.8), one has

$$F^*(x, \xi_2)^2 \geq F^*(x, \xi_1)^2 + 2(\xi_2 - \xi_1)(J^*(x, \xi_1)), \ \forall \xi_1, \xi_2 \in T_x^* M. \tag{3.1}$$

Let $u \in C_0^\infty(M)$ be arbitrarily and choose $\tau = \frac{n-2+\alpha}{2} > 0$. Let $v(x) = d(x)^\tau u(x)$. Therefore, for $u(x) = d(x)^{-\tau} v(x)$ one has $Du(x) = -\tau d(x)^{-\tau-1} v(x) Dd(x) + d(x)^{-\tau} Dv(x)$. By inequality (3.1) applied for $\xi_2 = -Du(x)$ and $\xi_1 = \tau d(x)^{-\tau-1} v(x) Dd(x)$, the symmetry of $F^*(x, \cdot)$ implies that

$$\begin{aligned} F^*(x, Du(x))^2 \quad &= \quad F^*(x, -Du(x))^2 \\ &\geq \quad F^*(x, \tau d(x)^{-\tau-1} v(x) Dd(x))^2 - 2d(x)^{-\tau} Dv(x)(J^*(x, \tau d(x)^{-\tau-1} v(x) Dd(x))). \end{aligned}$$

Since $F^*(x, Dd(x)) = 1$ (see (2.11)), $J^*(x, Dd(x)) = \boldsymbol{\nabla} d(x)$ and $Dv(x) \in T_x^*M$, we obtain

$$F^*(x, Du(x))^2 \geq \tau^2 d(x)^{-2\tau-2} v(x)^2 - 2\tau d(x)^{-2\tau-1} v(x) Dv(x)(\boldsymbol{\nabla} d(x)).$$

Multiplying the latter inequality by $d(x)^\alpha$, and integrating over M, we obtain

$$\int_M d(x)^\alpha F^*(x, Du(x))^2 \mathrm{d}V_F(x) \geq \tau^2 \int_M d(x)^{\alpha-2\tau-2} v(x)^2 \mathrm{d}V_F(x) + R_0,$$

where

$$\begin{aligned}
R_0 &= -2\tau \int_M d(x)^{\alpha-2\tau-1} v(x) Dv(x)(\boldsymbol{\nabla} d(x)) \mathrm{d}V_F(x) \\
&= -\frac{\tau}{\alpha - 2\tau} \int_M D(v(x)^2)(\boldsymbol{\nabla}(d(x)^{\alpha-2\tau})) \mathrm{d}V_F(x) \\
&= \frac{\tau}{\alpha - 2\tau} \int_M v(x)^2 \boldsymbol{\Delta}(d(x)^{\alpha-2\tau}) \mathrm{d}V_F(x) && \text{(see (2.12))} \\
&= \tau \int_M u(x)^2 d(x)^{\alpha-2} \left[\alpha - 2\tau - 1 + d(x) \boldsymbol{\Delta} d(x)\right] \mathrm{d}V_F(x) \\
&\geq \tau(n-1) \int_M u(x)^2 d(x)^{\alpha-2} \left[d(x) \mathbf{ct}_c(d(x)) - 1\right] \mathrm{d}V_F(x), && \text{(see Theorem 2.1 (a))} \\
&= \tau(n-1) \int_M d(x)^{\alpha-2} \mathbf{D}_c(d(x)) u(x)^2 \mathrm{d}V_F(x),
\end{aligned}$$

which completes the proof. □

For every $x \in M$ and $y \in T_xM$, $\xi \in T_x^*M$, we introduce the function

$$K_F(x, y, \xi) = \xi(y) - J(x, y)(J^*(x, \xi)). \tag{3.2}$$

For $\alpha \in \mathbb{R}$ with $n - 4 + \alpha > 0$ we introduce the *Green-deflection* function $G_F^\alpha : C_0^\infty(M) \to \mathbb{R}$ defined by

$$G_F^\alpha(u) = \int_M K_F\left(x, \boldsymbol{\nabla}(u(x))^2, D(d(x)^{\alpha-2})\right) \mathrm{d}V_F(x).$$

The layer cake representation and the fact that $n - 4 + \alpha > 0$ imply that the function G_F^α is well defined. Moreover, by definition of K_F and relations (2.9) and (2.12) one has

$$G_F^\alpha(u) = \int_M \left[u(x)^2 \boldsymbol{\Delta}(d(x)^{\alpha-2}) - d(x)^{\alpha-2} \boldsymbol{\Delta}(u(x)^2)\right] \mathrm{d}V_F(x). \tag{3.3}$$

It is now clear that $G_F^\alpha \equiv 0$ whenever (M, F) is Riemannian due to Green's identity. In fact, the latter statement also holds by the following observation.

Proposition 3.1 *$K_F \equiv 0$ if and only if (M, F) is Riemannian.*

Proof. If (M, F) is Riemannian then $g(x, y) = a(x)$, where $a(x)$ is a symmetric and positive-definite matrix and by Riesz representation, one can identify T_xM and T_x^*M. Moreover, $J(x, y) = a(x)y$ and $J^*(x, \xi) = a(x)^{-1}\xi$. Consequently, we have

$$K_F(x, y, \xi) = \xi(y) - J(x, y)(J^*(x, \xi)) = \xi(y) - a(x)y(a(x)^{-1}\xi) = 0.$$

Conversely, we assume that $K_F \equiv 0$, i.e., $\xi(y) - J(x,y)(J^*(x,\xi)) = 0$ for every $x \in M$, $y \in T_xM$ and $\xi \in T_x^*M$. For an arbitrary $z \in T_xM$ replace $\xi = J(x,z) \in T_x^*M$ into the preceding relation to obtain $J(x,z)(y) = J(x,y)(z)$. In particular, $J(x,\cdot)$ is linear; by virtue of (2.8) it implies that $F(x,\cdot)^2$ comes from an inner product on T_xM. □

Let us consider the following set of functions

$$C_{0,F,\alpha}^\infty(M) = \{u \in C_0^\infty(M) : G_F^\alpha(u) = 0\}.$$

By Proposition 3.1, $C_{0,F,\alpha}^\infty(M) = C_0^\infty(M)$ whenever (M,F) is Riemannian. However, in the generic Finsler context the role of $C_{0,F,\alpha}^\infty(M)$ seems to be indispensable for the study of Rellich inequalities.

We are in position to state our first main result.

Theorem 3.1 (Rellich inequality I) *Let (M,F) be an $n-$dimensional Finsler-Hadamard manifold with $\mathbf{S} = 0$ and $\mathbf{K} \leq c \leq 0$, let $x_0 \in M$ be fixed, and choose any $\alpha \in \mathbb{R}$ such that $n-4+\alpha > 0$ and $\alpha < 2$. Then for every $u \in C_{0,F,\alpha}^\infty(M)$ we have*

$$\begin{aligned}\int_M d(x)^\alpha (\mathbf{\Delta} u(x))^2 \mathrm{d}V_F(x) \quad &\geq \quad \frac{(n-4+\alpha)^2(n-\alpha)^2}{16}\int_M d(x)^{\alpha-4}u(x)^2\mathrm{d}V_F(x) \\ &+\frac{(n-4+\alpha)(n-\alpha)(n-2)(n-1)}{4}\int_M d(x)^{\alpha-4}\mathbf{D}_c(d(x))u(x)^2\mathrm{d}V_F(x).\end{aligned}$$

Moreover, the constant $\frac{(n-4+\alpha)^2(n-\alpha)^2}{16}$ is sharp.

Proof. Throughout the proof, we shall consider $\gamma = \frac{n-4+\alpha}{2} > 0$. Since $\alpha < 2$, a simple calculation and Theorem 2.1(a) yield

$$\begin{aligned}\mathbf{\Delta}(d(x)^{\alpha-2}) \quad &= \quad (\alpha-2)[\alpha-3+d(x)\mathbf{\Delta}(d(x))]d(x)^{\alpha-4} \\ &\leq \quad (\alpha-2)[\alpha-3+(n-1)d(x)\mathbf{ct}_c(d(x))]d(x)^{\alpha-4} \\ &= \quad (\alpha-2)\left[2\gamma+(n-1)\mathbf{D}_c(d(x))\right]d(x)^{\alpha-4}.\end{aligned}$$

Let us fix $u \in C_{0,F,\alpha}^\infty(M)$. Multiplying the above inequality by u^2, we see that

$$\int_M \mathbf{\Delta}(d(x)^{\alpha-2})u(x)^2\mathrm{d}V_F(x) \leq (\alpha-2)\int_M [2\gamma+(n-1)\mathbf{D}_c(d(x))]d(x)^{\alpha-4}u(x)^2\mathrm{d}V_F(x). \tag{3.4}$$

Note that

$$\mathbf{\Delta}(u(x)^2) = 2\mathrm{div}(u\mathbf{\nabla}(u(x)) = 2F^*(x,Du(x))^2 + 2u\mathbf{\Delta}(u(x)).$$

Multiplying the latter relation by $d^{\alpha-2}$ and integrating over M, we obtain

$$\int_M d(x)^{\alpha-2}\mathbf{\Delta}(u(x)^2)\mathrm{d}V_F(x) = 2\int_M d(x)^{\alpha-2}F^*(x,Du(x))^2\mathrm{d}V_F(x) + 2\int_M d(x)^{\alpha-2}u\mathbf{\Delta}(u(x))\mathrm{d}V_F(x).$$

Subtracting the latter relation by (3.4), one gets that

$$\begin{aligned}G_F^\alpha(u) \quad &\leq \quad (\alpha-2)\int_M [2\gamma+(n-1)\mathbf{D}_c(d(x))]d(x)^{\alpha-4}u(x)^2\mathrm{d}V_F(x) \\ &\quad -2\int_M d(x)^{\alpha-2}F^*(x,Du(x))^2\mathrm{d}V_F(x) - 2\int_M d(x)^{\alpha-2}u\mathbf{\Delta}(u(x))\mathrm{d}V_F(x).\end{aligned}$$

Since $u \in C^\infty_{0,F,\alpha}(M)$, then $G^\alpha_F(u) = 0$ and we obtain that

$$-\int_M d(x)^{\alpha-2} u\mathbf{\Delta}(u(x))\mathrm{d}V_F(x) \geq \frac{2-\alpha}{2}\int_M [2\gamma + (n-1)\mathbf{D}_c(d(x))] d(x)^{\alpha-4} u(x)^2 \mathrm{d}V_F(x) + \int_M d(x)^{\alpha-2} F^*(x, Du(x))^2 \mathrm{d}V_F(x). \tag{3.5}$$

For the latter term we apply the Hardy inequality (Lemma 3.1), and obtain

$$\int_M d(x)^{\alpha-2} F^*(x, Du(x))^2 \mathrm{d}V_F(x) \geq \gamma^2 \int_M d(x)^{\alpha-4} u(x)^2 \mathrm{d}V_F(x) + \gamma(n-1)\int_M d(x)^{\alpha-4}\mathbf{D}_c(d(x)) u(x)^2 \mathrm{d}V_F(x). \tag{3.6}$$

Combining these inequalities, a trivial rearrangement now yields

$$-\int_M d(x)^{\alpha-2} u\mathbf{\Delta}(u(x))\mathrm{d}V_F(x) \geq \frac{\gamma(n-\alpha)}{2}\int_M d(x)^{\alpha-4} u(x)^2 \mathrm{d}V_F(x) + \frac{(n-1)(n-2)}{2}\int_M d(x)^{\alpha-4}\mathbf{D}_c(d(x)) u(x)^2 \mathrm{d}V_F(x).$$

The Hölder inequality for the left hand side of the above inequality gives that

$$\left(\int_M d(x)^\alpha (\mathbf{\Delta} u(x))^2 \mathrm{d}V_F(x)\right)^{\frac{1}{2}} \cdot \left(\int_M d(x)^{\alpha-4} u(x)^2 \mathrm{d}V_F(x)\right)^{\frac{1}{2}} \geq \int_M d(x)^{\alpha-2} |u\mathbf{\Delta}(u(x))| \mathrm{d}V_F(x). \tag{3.7}$$

The last inequalities and a simple estimate show that

$$\int_M d(x)^\alpha (\mathbf{\Delta} u(x))^2 \mathrm{d}V_F(x) \geq \frac{\gamma^2(n-\alpha)^2}{4}\int_M d(x)^{\alpha-4} u(x)^2 \mathrm{d}V_F(x) + \frac{\gamma(n-\alpha)(n-2)(n-1)}{2}\int_M d(x)^{\alpha-4}\mathbf{D}_c(d(x)) u(x)^2 \mathrm{d}V_F(x),$$

which completes the proof of Rellich inequality I.

Now, we shall prove that in the Rellich inequality I the constant $\tilde{C} := \frac{\gamma^2(n-\alpha)^2}{4}$ is sharp. Clearly, it is enough to prove that

$$\tilde{C} = \inf_{u \in C^\infty_{0,F,\alpha}(M)\setminus\{0\}} \frac{\int_M d(x)^\alpha (\mathbf{\Delta} u(x))^2 \mathrm{d}V_F(x)}{\int_M d(x)^{\alpha-4} u(x)^2 \mathrm{d}V_F(x)}. \tag{3.8}$$

First, it follows by (2.13) that there exists $0 < r_0 < \frac{n-\alpha}{2}$ such that

$$\left|\mathbf{\Delta} d(x) - \frac{n-1}{d(x)}\right| \leq 1 \text{ for a.e. } x \in B(x_0, r_0).$$

In particular, one has

$$|-\gamma - 1 + d(x)\mathbf{\Delta} d(x)| \leq \frac{n-\alpha}{2} + d(x) \quad \text{for a.e. } x \in B(x_0, r_0). \tag{3.9}$$

Let us fix numbers $r, R \in \mathbb{R}$ such that $0 < r < R < r_0$ and a smooth cutoff function $\psi : M \to [0,1]$ with $\mathrm{supp}(\psi) = \overline{B(x_0,R)}$ and $\psi(x) = 1$ for $x \in B(x_0, r)$. For every $0 < \varepsilon < r$, let

$$u_\varepsilon(x) = (\max\{\varepsilon, d(x)\})^{-\gamma}, \ x \in M. \tag{3.10}$$

Note that ψu_ε can be approximated by elements from $C_0^\infty(M)$ and since both functions ψ and u_ε are $d(x)-$radial, it follows by the representation (3.3) of G_F^α that $G_F^\alpha(\psi u_\varepsilon) = 0$, therefore, $\psi u_\varepsilon \in C_{0,F,\alpha}^\infty(M)$ for every $0 < \varepsilon < r$.

One the one hand, by relation (3.9) one has

$$\begin{aligned} I_1(\varepsilon) &:= \int_M d(x)^\alpha (\mathbf{\Delta}(\psi(x)u_\varepsilon(x)))^2 \mathrm{d}V_F(x) \\ &= \int_{B(x_0,r)\setminus B(x_0,\varepsilon)} d(x)^\alpha (\mathbf{\Delta}(d(x)^{-\gamma}))^2 \mathrm{d}V_F(x) \\ &\quad + \int_{B(x_0,R)\setminus B(x_0,r)} d(x)^\alpha (\mathbf{\Delta}(\psi(x)d(x)^{-\gamma}))^2 \mathrm{d}V_F(x) \\ &= \gamma^2 \int_{B(x_0,r)\setminus B(x_0,\varepsilon)} d(x)^{\alpha-2\gamma-4}[-\gamma - 1 + d(x)\mathbf{\Delta} d(x)]^2 \mathrm{d}V_F(x) + c(\alpha, r, R) \\ &\leq \gamma^2 \left(\frac{n-\alpha}{2} + r\right)^2 \int_{B(x_0,r)\setminus B(x_0,\varepsilon)} d(x)^{\alpha-2\gamma-4} \mathrm{d}V_F(x) + c(\alpha, r, R) \\ &= \gamma^2 \left(\frac{n-\alpha}{2} + r\right)^2 \tilde{I}(\varepsilon) + c(\alpha, r, R), \end{aligned}$$

where

$$\tilde{I}(\varepsilon) = \int_{B(x_0,r)\setminus B(x_0,\varepsilon)} d(x)^{\alpha-2\gamma-4} \mathrm{d}V_F(x) = \int_{B(x_0,r)\setminus B(x_0,\varepsilon)} d(x)^{-n} \mathrm{d}V_F(x)$$

and

$$c(\alpha, r, R) = \int_{B(x_0,R)\setminus B(x_0,r)} d(x)^\alpha (\mathbf{\Delta}(\psi(x)d(x)^{-\gamma}))^2 \mathrm{d}V_F(x).$$

Clearly, $c(\alpha, r, R)$ is finite. On the other hand,

$$\begin{aligned} I_2(\varepsilon) &:= \int_M d(x)^{\alpha-4} \psi(x)^2 u_\varepsilon(x)^2 \mathrm{d}V_F(x) \\ &\geq \int_{B(x_0,r)\setminus B(x_0,\varepsilon)} d(x)^{\alpha-4-2\gamma} \mathrm{d}V_F(x) \\ &= \tilde{I}(\varepsilon). \end{aligned}$$

By applying the layer cake representation and the volume comparison (see Theorem 2.1 (b)), we deduce that

$$\begin{aligned} \tilde{I}(\varepsilon) &= \int_{B(x_0,r)\setminus B(x_0,\varepsilon)} d(x)^{-n} \mathrm{d}V_F(x) = \int_{r^{-n}}^{\varepsilon^{-n}} \mathrm{Vol}_F(B(x_0, \rho^{-\frac{1}{n}})) \mathrm{d}\rho \\ &\geq \omega_n \int_{r^{-n}}^{\varepsilon^{-n}} \rho^{-1} \mathrm{d}\rho \\ &= n\omega_n(\ln r - \ln \varepsilon). \end{aligned}$$

In particular, $\lim_{\varepsilon\to 0^+}\tilde{I}(\varepsilon)=+\infty$. Therefore, it follows that

$$\begin{aligned}\tilde{C} &\leq \inf_{u\in C^\infty_{0,F,\alpha}(M)\setminus\{0\}}\frac{\int_M d(x)^\alpha(\mathbf{\Delta}u(x))^2\mathrm{d}V_F(x)}{\int_M d(x)^{\alpha-4}u(x)^2\mathrm{d}V_F(x)}\\ &\leq \lim_{\varepsilon\to 0^+}\frac{I_1(\varepsilon)}{I_2(\varepsilon)}\\ &\leq \lim_{\varepsilon\to 0^+}\frac{\gamma^2\left(\frac{n-\alpha}{2}+r\right)^2\tilde{I}(\varepsilon)+c(\alpha,r,R)}{\tilde{I}(\varepsilon)}\\ &= \gamma^2\left(\frac{n-\alpha}{2}+r\right)^2.\end{aligned}$$

Since $r>0$ is arbitrary, we can take $r\to 0^+$, which completes the proof of (3.8). □

Our second main result connects first to second order terms and it can be stated as follows.

Theorem 3.2 (Rellich inequality II) *Let (M,F) be an $n-$dimensional Finsler-Hadamard manifold with $\mathbf{S}=0$ and $\mathbf{K}\leq c\leq 0$, let $x_0\in M$ be fixed, and choose any $\alpha\in\mathbb{R}$ such that $n-8+3\alpha>0$ and $\alpha<2$. Then for every $u\in C^\infty_{0,F,\alpha}(M)$ we have*

$$\begin{aligned}\int_M d(x)^\alpha(\mathbf{\Delta}u(x))^2\mathrm{d}V_F(x) &\geq \frac{(n-\alpha)^2}{4}\int_M d(x)^{\alpha-2}F^*(x,Du(x))^2\mathrm{d}V_F(x)\\ &\quad+\frac{(n-4+\alpha)^2(n-\alpha)(n-1)}{8}\int_M d(x)^{\alpha-4}\mathbf{D}_c(d(x))u(x)^2\mathrm{d}V_F(x).\end{aligned}$$

Moreover, the constant $\frac{(n-\alpha)^2}{4}$ is sharp.

Proof. We shall keep the notations and shall invoke some of the arguments from the proof of Theorem 3.1. Let $u\in C^\infty_{0,F,\alpha}(M)$. By applying the arithmetic-geometric mean inequality to the left hand side of (3.7), it follows that

$$2\int_M d(x)^{\alpha-2}|u\mathbf{\Delta}(u(x))|\mathrm{d}V_F(x)\leq \tilde{C}^{-\frac{1}{2}}\int_M d(x)^\alpha(\mathbf{\Delta}u(x))^2\mathrm{d}V_F(x)+\tilde{C}^{\frac{1}{2}}\int_M d(x)^{\alpha-4}u(x)^2\mathrm{d}V_F(x).$$

Combining this inequality with (3.5), we see that

$$\begin{aligned}2\int_M d(x)^{\alpha-2}F^*(x,Du(x))^2\mathrm{d}V_F(x) &\leq \tilde{C}^{-\frac{1}{2}}\int_M d(x)^\alpha(\mathbf{\Delta}u(x))^2\mathrm{d}V_F(x)\\ &\quad+\left(\tilde{C}^{\frac{1}{2}}-2(2-\alpha)\gamma\right)\int_M d(x)^{\alpha-4}u(x)^2\mathrm{d}V_F(x)\\ &\quad-(2-\alpha)(n-1)\int_M d(x)^{\alpha-4}\mathbf{D}_c(d(x))u(x)^2dV_F(x).\end{aligned}$$

Since $\tilde{C}^{\frac{1}{2}}-2(2-\alpha)\gamma=\frac{(n-8+3\alpha)\gamma}{2}>0$, by applying Rellich inequality I to the second integrand on the right hand side of the above inequality, a reorganization of the expressions implies that

$$\begin{aligned}2\int_M d(x)^{\alpha-2}F^*(x,Du(x))^2\mathrm{d}V_F(x) &\leq \frac{8}{(n-\alpha)^2}\int_M d(x)^\alpha(\mathbf{\Delta}u(x))^2\mathrm{d}V_F(x)\\ &\quad-\frac{(n-4+\alpha)^2(n-1)}{n-\alpha}\int_M d(x)^{\alpha-4}\mathbf{D}_c(d(x))u(x)^2dV_F(x).\end{aligned}$$

Once we multiply this inequality by $\frac{(n-\alpha)^2}{8}$, we obtain the Rellich inequality II.

It remains to prove that in Rellich inequality II the constant $\frac{(n-\alpha)^2}{4}$ is sharp. By using the same functions as in the proof of Theorem 3.1, it follows by (2.11) that

$$\begin{aligned} I_3(\varepsilon) &:= \int_M d(x)^{\alpha-2} F^*(x, D(\psi u_\varepsilon)(x))^2 \mathrm{d}V_F(x) \\ &\geq \gamma^2 \int_{B(x_0,r)\setminus B(x_0,\varepsilon)} d(x)^{\alpha-4-2\gamma} \mathrm{d}V_F(x) \\ &= \gamma^2 \tilde{I}(\varepsilon). \end{aligned}$$

The rest of the proof is similar as for Theorem 3.1. □

Proof of Theorem 1.1. Take in Theorems 3.1 and 3.2 the value $\alpha = 0$. By considering the continued fraction representation of the function $\rho \mapsto \coth(\rho)$, one has

$$\rho \coth(\rho) - 1 \geq \frac{3\rho^2}{\pi^2 + \rho^2}, \ \forall \rho > 0,$$

and this concludes the proof. □

4 Concluding remarks and questions

Remark 4.1 [*Tour of Rellich inequalities*] The technical hypothesis $n - 8 + 3\alpha > 0$ is indispensable in the proof of Theorem 3.2. However, we believe an alternative proof should eliminate this assumption. Interestingly, Rellich inequalities I and II are *deducible from each other* via the Hardy inequality once the assumption $n-8+3\alpha > 0$ holds. First, we have seen that the proof of Theorem 3.2 is obtained from the statement of Theorem 3.1. Conversely, by Rellich inequality II and Hardy inequality (see relation (3.6)), we obtain

$$\begin{aligned} \int_M d(x)^\alpha (\mathbf{\Delta} u(x))^2 dV_F(x) &\geq \frac{(n-\alpha)^2}{4} \int_M d(x)^{\alpha-2} F^*(x, Du(x))^2 dV_F(x) \\ &\quad + \frac{(n-4+\alpha)^2(n-\alpha)(n-1)}{8} \int_M d(x)^{\alpha-4} \mathbf{D}_c(d(x)) u(x)^2 dV_F(x) \\ &\geq \frac{(n-\alpha)^2\gamma^2}{4} \int_M d(x)^{\alpha-4} u(x)^2 \mathrm{d}V_F(x) \\ &\quad + \left[\frac{(n-4+\alpha)^2(n-\alpha)(n-1)}{8} + \frac{(n-\alpha)^2}{4}\gamma(n-1) \right] \times \\ &\quad \times \int_M d(x)^{\alpha-4} \mathbf{D}_c(d(x)) u(x)^2 \mathrm{d}V_F(x) \\ &= \frac{(n-4+\alpha)^2(n-\alpha)^2}{16} \int_M d(x)^{\alpha-4} u(x)^2 dV_F(x) \\ &\quad + \frac{(n-4+\alpha)(n-\alpha)(n-2)(n-1)}{4} \int_M d(x)^{\alpha-4} \mathbf{D}_c(d(x)) u(x)^2 dV_F(x), \end{aligned}$$

which is precisely Rellich inequality I. In particular, the Euclidean Rellich inequalities (1.1) and (1.2) can be considered to be equivalent whenever $n \geq 9$.

Remark 4.2 [*Rigidity*] For a generic Finsler manifold (M, F) the vanishing of Green-deflection G_F (where the function K_F appears) played a crucial role in Rellich inequalities. As we have already pointed out in Proposition 3.1, $K_F \equiv 0$ if and only if (M, F) is Riemannian. On account of this characterization we believe that the *full* Rellich inequality holds, i.e.,

$$\frac{(n-4+\alpha)^2(n-\alpha)^2}{16} = \inf_{u \in C_0^\infty(M)\setminus\{0\}} \frac{\int_M d(x)^\alpha (\boldsymbol{\Delta} u(x))^2 \mathrm{d}V_F(x)}{\int_M d(x)^{\alpha-4} u(x)^2 \mathrm{d}V_F(x)},$$

if and only if (M, F) is Riemannian. Note that in Theorem 3.1 only the set of functions $C^\infty_{0,F,\alpha}(M)$ is considered while the latter relation is formulated for the *entire* space $C_0^\infty(M)$.

Remark 4.3 [*Mean value property vs.* $K_F \equiv 0$ *on Minkowski spaces*] Let $(M, F) = (\mathbb{R}^n, F)$ be a Minkowski space. Recently, Ferone and Kawohl [5, p. 252] proved the mean value property for $\boldsymbol{\Delta}$-harmonics whenever

$$\frac{\langle a, b\rangle}{F(a)F^*(b)} = \langle \nabla F(a), \nabla F^*(b)\rangle, \quad \forall a, b \in \mathbb{R}^n \setminus \{0\}. \tag{4.1}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^n$. Interestingly, one can show that (4.1) is *equivalent* to $K_F \equiv 0$, see relation (2.8). Therefore, according to Proposition 3.1, no proper non-Euclidean class of Minkowski norms can be delimited in [5] to verify the mean value property. In fact, we conjecture that the validity of the mean value property of $\boldsymbol{\Delta}$-harmonics on a Minkowski space $(\mathbb{R}^n, F)$ holds if and only if $(\mathbb{R}^n, F)$ is Euclidean. This problem will be studied in a forthcoming paper.

Remark 4.4 [*Nonreversible Finsler manifolds*] In order to avoid further technicalities, we focused our study only to reversible Finsler manifolds. However, by employing suitable modifications in the proofs, we can state Hardy and Rellich inequalities on not necessarily reversible Finsler manifolds.

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Alexandru Kristály
Institute of Applied Mathematics, Óbuda University, 1034 Budapest, Hungary
Email: alexandru.kristaly@yahoo.com

Dušan Repovš
Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana & Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia
Email: dusan.repovs@guest.arnes.si