Quantitative Temporal Logics over the Reals: PSPACE and below

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Abstract

In many cases, the addition of metric operators to qualitative temporal logics (TLs) increases the complexity of satisfiability by at least one exponential: while common qualitative TLs are complete for NP or PSPACE, their metric extensions are often EXPSPACE-complete or even undecidable. In this paper, we exhibit several metric extensions of qualitative TLs of the real line that are at most PSPACE-complete, and analyze the transition from NP to PSPACE for such logics. Our first result is that the logic obtained by extending since-until logic of the real line with the operators 'sometime within n time units in the past/future' is still PSPACE-complete. In contrast to existing results, we also capture the case where n is coded in binary and the finite variability assumption is not made. To establish containment in PSPACE, we use a novel reduction technique that can also be used to prove tight upper complexity bounds for many other metric TLs in which the numerical parameters to metric operators are coded in binary. We then consider metric TLs of the reals that do not offer any qualitative temporal operators. In such languages, the complexity turns out to depend on whether binary or unary coding of parameters is assumed: satisfiability is still PSPACE-complete under binary coding, but only NP-complete under unary coding.

1 Introduction

The classical approach to the specification and verification of reactive systems uses qualitative temporal logics (TLs) that are interpreted in the natural numbers [5,11,12]. When real-time properties play a crucial role in the description of the system behaviour, this rather abstract approach is no longer

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feasible since concrete distances between events cannot be described in qualitative TLs. Consequently, the basic logical tool for reasoning about real-time systems is provided by quantitative TLs, which are usually extensions of standard qualitative TLs with metric operators. To obtain a realistic model of time, such logics are usually interpreted in the real line [1–3,10,14] (although metric TLs of discrete time have also been proposed [8]). Unfortunately, moving from qualitative to quantitative logics is often accompanied by an increase in computational complexity of the satisfiability problem, both in discrete and continuous flows of time. The most important example demonstrating this effect is the PSpace-complete since-until logic of the real line [13], whose extension with a metric operator 'sometime in at least n but not more than m time units' (n and m coded in binary) becomes ExpSpace-complete if the case $n = m$ is not admitted, and even undecidable if it is [1,3,10]. It is well known that the complexity of this quantitative TL can be reduced to PSpace by enforcing that the lower parameter n to metric temporal operators is restricted to zero [1]. However, in contrast to the ExpSpace-completeness and undecidability results, this result has only been proved under the *finite variability assumption (FVA)* which states that no propositional variable changes its truth-value infinitely many times in any bounded interval. Additionally, in contrast to qualitative TLs, to the best of our knowledge there have been no attempts to obtain metric temporal logics that are only NP-complete by further restricting the language.

The purpose of this paper is to *investigate metric temporal logics of the real line that are at most* PSpace*-complete, and to investigate the transition from* NP *to* PSpace *for such logics*. In our analysis, we consider the case with and without FVA. Since the case without FVA is sometimes neglected in computer science, we first give some justification for why we believe that it is worth studying.

The FVA is used to capture the assumption that a system can change its state only finitely many times in a bounded time interval. While the FVA is an appropriate condition in classical control theory and most computer science applications [2], we believe that there are at least four reasons to study also the non-FVA case: first, qualitative temporal logic originated in philosophy and mathematics to study time itself, rather than the behavior of systems with discrete state changes as considered in most computer science applications. If quantitative TL is used for the former purpose, there is no obvious reason for adopting the FVA. Second, it has been argued that there are relevant real-time systems that may experience infinitely many state changes in bounded time intervals [9,4,6]. Third, even in a setting in which the intended models satisfy the FVA, reasoning without the FVA can be fruitfully employed: assume that a formula φ of a quantitative TL describes the specification of a real-time system. Further assume that φ has been found to be unsatisfiable under FVA, indicating that the described specification is not realizable. If an additional satisfiability check of φ *without* FVA gives a positive result, then the user obtains additional information on the source of the unrealizability of her specification: namely that it enforces an infinite number of state changes in a bounded time interval. Finally, in languages containing the operators 'since' and 'until' there exists a formula which expresses the FVA, see [10] and below. It follows that complexity upper bounds for the non-FVA case are more general than upper bounds in the FVA case and exhibit a uniform upper bound for both cases.

In this paper, we prove two main results. Our first result is that extending since-until logic of the real line with metric operators 'sometime in at most n time units in the past/future', n coded in binary, is PSPACE-complete even without FVA. To show this result, we propose a new method for polynomially reducing satisfiability in metric TLs where numerical parameters are coded in binary to satisfiability in the same logic with numbers coded in unary. The essence of this reduction is to introduce new propositional variables that serve as the bits of a binary counter which measures distances. For the metric TL mentioned above, we obtain a PSPACE upper bound since Hirshfeld and Rabinovich have shown that QTL, the same logic with numbers coded in unary, is PSPACE-complete without FVA [10]. This proof method can also be used to establish tight upper complexity bounds for many other metric temporal logics in which numerical parameters are coded in binary. To substantiate this claim, in the appendix we reprove EXPTIME-completeness of RTCTL (real-time computational tree logic), a metric extension of the branching time logic CTL proposed by Emerson et al. [8]. Whereas Emerson et al. use a tableau-based decision procedure to prove containment in EXPTIME of RTCTL, we reprove this result in a much simpler way by applying our reduction technique to polynomially reduce satisfiability in RTCTL to satisfiability in CTL.

Our second result is concerned with the transition from NP to PSPACE. We first sharpen the PSPACE lower bounds for metric temporal logics of the real line. In the current literature, such logics usually contain qualititative since-until logic as a proper fragment, and thus trivially inherit PSPACE-hardness [2,10,13]. We consider metric TLs with weaker qualitative operators and show that PSPACE-hardness can already be observed in the following three cases: (i) a future diamond and a future operator 'sometime in at most n time units', n coded in unary; (ii) *only* the future operator 'sometime in at most n time units', n coded in binary (i.e., no qualitative operators at all); (iii) only a metric version of the until operator for the interval $[0, 1]$.

Then we show that no further sharpening of the PSPACE lower bound is possible by proving that the quantitative TL with only the metric operator 'sometime within n time units', n coded in unary, is NP-complete, both with and without FVA. We thus obtain a quantitative counterpart of the result of Sistla and Zuck that qualitative TL with the future diamond as the only temporal operator is NP-complete on the real numbers [16]. To establish the upper bound, we devise an algorithm for satisfiability that first guesses a set of "types" (of polynomial cardinality), and then constructs and solves a system of rational linear inequalities over the real (or, equivalently, rational) numbers to deal with the metric operators. We give two separate algorithms for the case with and without FVA since, without the operators 'since' and 'until', there appears to be no "semantically transparent" reduction of the FVA case to the non-FVA case. When compared with PSpace-hardness of (ii) above, this result shows that the complexity of metric TLs without qualitative operators depends on the coding of numbers.

2 Preliminaries

We introduce the metric temporal language QTL of [10] which is closely related to the language MITL of [1]. Fix a countably infinite supply $p_0, p_1, \ldots$ of propositional variables. A QTL-formula is built according to the syntax rule

$$\varphi, \psi := p \;\mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \,\mathcal{S}\, \psi \mid \varphi \,\mathcal{U}\, \psi \mid \varphi \mathcal{S}^I \psi \mid \varphi \mathcal{U}^I \psi$$

with p ranging over the propositional variables and I ranging over intervals of the forms $(0, n)$, $(0, n]$, $[0, n)$, and $[0, n]$, where $n > 0$ is a natural number. The Boolean operators $\top$, $\vee$, $\rightarrow$, and $\leftrightarrow$ are defined as abbreviations in the usual way. Moreover, we introduce additional future modalities as abbreviations: $\Diamond_F^I \varphi = \top \,\mathcal{U}^I\, \varphi$, $\Box_F^I \varphi = \neg \Diamond_F^I \neg\varphi$, $\Diamond_F \varphi = \top \,\mathcal{U}\, \varphi$, $\Box_F \varphi = \neg \Diamond_F \neg \varphi$, and $\Box_F^+ \varphi = \varphi \wedge \Box_F \varphi$. Their past counterparts are defined analogously and have a subscript P.

Formulas of QTL are interpreted on the real line. A *model* $\mathfrak{M} = \langle \mathbb{R}, \mathfrak{V} \rangle$ is a pair consisting of the real numbers and a valuation $\mathfrak{V}$ mapping every propositional variable p to a set $\mathfrak{V}(p) \subseteq \mathbb{R}$. The satisfaction relation '$\models$' is defined inductively as follows, where for each time point $w \in \mathbb{R}$ and interval I of one of the above forms, we write $w + I$ to denote the set $\{w + x \mid x \in I\}$; $w - I$ is defined analogously.

$\mathfrak{M}, w \models p$	iff	$w \in \mathfrak{V}(p)$
$\mathfrak{M}, w \models \neg\varphi$	iff	$\mathfrak{M}, w \not\models \varphi$
$\mathfrak{M}, w \models \varphi \wedge \psi$	iff	$\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
$\mathfrak{M}, w \models \varphi \,\mathcal{U}\, \psi$	iff	there exists $u > w$ such that $\mathfrak{M}, u \models \psi$ and $\mathfrak{M}, v \models \varphi$ for all v such that $w < v < u$

$$
\begin{array}{lll}
\mathfrak{M}, w \models \varphi \,\mathcal{S}\, \psi & \text{iff} & \text{there exists } u < w \text{ such that } \mathfrak{M}, u \models \psi \text{ and } \mathfrak{M}, v \models \varphi \\
 & & \text{for all } v \text{ such that } u < v < w \\
\mathfrak{M}, w \models \varphi \,\mathcal{U}^I\, \psi & \text{iff} & \text{there exists } u \in w + I \text{ such that } \mathfrak{M}, u \models \psi \text{ and } \mathfrak{M}, v \models \varphi \\
 & & \text{for all } v \text{ such that } w < v < u \\
\mathfrak{M}, w \models \varphi \,\mathcal{S}^I\, \psi & \text{iff} & \text{there exists } u \in w - I \text{ such that } \mathfrak{M}, u \models \psi \text{ and } \mathfrak{M}, v \models \varphi \\
 & & \text{for all } v \text{ such that } u < v < w
\end{array}
$$

We will also write $w \models_{\mathfrak{V}} \varphi$ for $\langle \mathbb{R}, \mathfrak{V} \rangle, w \models \varphi$. A QTL-formula φ is satisfiable if there exists a model $\mathfrak{M}$ and a time point $w \in \mathbb{R}$ such that $\mathfrak{M}, w \models \varphi$. It is satisfiable under the *finite variability assumption (FVA)* if it is satisfiable in a model in which no propositional variable changes its truth-value infinitely many times in any bounded interval.

Our presentation of QTL deviates from the original one given by Hirshfeld and Rabinovich in [10], where only the metric operators $\Diamond_F^{(0,1)}$ and $\Diamond_P^{(0,1)}$ are admitted. If the numerical parameters of the metric operators are coded in unary, there exists an easy polynomial translation from Hirshfeld and Rabinovich's version of QTL to ours and vice versa. However, in this paper we also consider binary coding of numbers. If we want to emphasize this fact, we shall write QTL^b instead of QTL, and, likewise, QTL^u will denote unary coding of numbers.

It is worth noting that the results presented in this paper apply to formulas with *rational numbers* as parameters of metric operators $\mathcal{S}^I$ and $\mathcal{U}^I$ as well: such formulas can be translated (in polynomial time) to equi-satisfiable formulas containing only natural numbers as parameters by multiplying with the least common denominator of all (rational) parameters.

We close this section with a brief discussion of the relation between temporal logics with FVA and those without. The main observation is that, in any temporal logic that includes the since and until operators, satisfiability of a formula φ with FVA can be polynomially reduced to satisfiability without FVA [10]: let

$$
\begin{aligned}
\delta &= \bigwedge_{p \text{ used in } \varphi} (\Box_P \delta_p \wedge \delta_p \wedge \Box_F \delta_p), \text{ where} \\
\delta_p &= \big((p \,\mathcal{U}\, p) \vee (\neg p \,\mathcal{U}\, \neg p)\big) \wedge \big((p \,\mathcal{S}\, p) \vee (\neg p \,\mathcal{S}\, \neg p)\big).
\end{aligned}
$$

It is not hard to verify that φ is satisfiable with FVA iff $\varphi \wedge \delta$ is satisfiable without FVA. Note that the length of $\varphi \wedge \delta$ is polynomial in the length of φ. To the best of our knowledge, there is no polynomial reduction of this type for the language without since or until.

3 QTL^b is PSpace-complete without FVA

The purpose of this section is to prove that QTL^b-satisfiability without FVA is decidable in PSpace. This result is already known for QTL^u without FVA [10] and QTL^u and QTL^b with FVA [1]. Note that, due to polynomial reducibility of QTL^b with FVA to QTL^b without FVA, those three results follow from the new upper complexity bound for QTL^b without FVA. We first observe that our result indeed improves upon the existing ones by showing that QTL^b is exponentially more succinct than QTL^u.

Theorem 1 *Let ψ be a QTL-formula with numbers coded in unary that is equivalent to $\Box_F^{[0,n]}p$. Then ψ has length at least n.*

PROOF. Suppose by contradiction that there exists a QTL-formula ψ with numbers coded in unary such that ψ is equivalent to $\Box_F^{[0,n]}p$, for some $n \geq 1$, and the length of ψ is strictly smaller than n. We may assume that ψ contains no other propositional letters than p: replacing such letters in a formula equivalent to $\Box_F^{[0,n]}p$ with $\top$ is an equivalence-preserving operation.

For $n \geq 1$, set $\mathfrak{V}_n(p) := [-n, n]$ and $\mathfrak{M}_n := \langle \mathbb{R}, \mathfrak{V}_n \rangle$. Then $\mathfrak{M}_n, 0 \models \Box_F^{[0,n]}p$ and $\mathfrak{M}_n, 1 \not\models \Box_F^{[0,n]}p$. The former implies $\mathfrak{M}_n, 0 \models \psi$. Our aim is to derive a contradiction by showing that $\mathfrak{M}_n, 1 \models \psi$.

It is straightforward to prove the following by induction on the length $|\chi|$ of χ: for every subformula χ of ψ and all real numbers x, y from the interval $[-(n - |\chi|), n - |\chi|]$, we have

$$\mathfrak{M}_n, x \models \chi \text{ iff } \mathfrak{M}_n, y \models \chi.$$

Since the length of ψ is smaller than n, it follows that, in $\mathfrak{M}_n$, the points 0 and 1 satisfy the same subformulas of ψ. In particular, $\mathfrak{M}_n, 1 \models \psi$. $\Box$

We now establish the main result of this section.

Theorem 2 *Satisfiability in QTL with numbers coded in binary is* PSpace-*complete without FVA.*

Since (qualitative) since-until logic on the real line is PSpace-hard [13], it suffices to prove the upper bound. For simplicity, we prove the upper bound for the future fragment of QTL, i.e., we omit past operators. The proofs are easily extended to the general case. Within the future fragment, we consider only the metric operators $\Diamond_F^{(0,1)}$, $\Diamond_F^{(0,1]}$, $\Diamond_F^{[0,1)}$, and $\Diamond_F^{[0,n]}$. This can be done w.l.o.g. due to the following observations:

First, satisfiability in QTL^b can be reduced to satisfiability in QTL^b without the metric operators $\psi_1 \mathcal{U}^I \psi_2$: to decide satisfiability of a QTL^b-formula φ, introduce a new propositional variable p_{ψ_2} for every ψ_2 which occurs in a subformula of the form $\psi_1 \mathcal{U}^I \psi_2$ of φ. Inductively define a translation on QTL^b-formulas such that, for any subformula χ of φ, χ^p to denotes the result of replacing all outermost subformulas $\psi_1 \mathcal{U}^I \psi_2$ of χ by $\psi_1^p \mathcal{U} p_{\psi_2} \wedge \Diamond_F^I p_{\psi_2}$. Then φ is satisfiable iff

$$\varphi^p \wedge \Box_F^+ \Big[\bigwedge_{\psi_1 \mathcal{U}^I \psi_2 \in sub(\varphi)} (p_{\psi_2} \leftrightarrow \psi_2^p) \Big]$$

is satisfiable, where $sub(\varphi)$ denotes the set of subformulas of φ. Note that the length of the obtained formula is polynomial in the length of φ. Second, for any interval I of the form $(0,n)$, $(0,n]$, or $[0,n)$ with $n > 1$, $\Diamond_F^I \varphi$ is equivalent to $\Diamond_F^J \Diamond_F^{[0,n-1]} \varphi$, where J is obtained from I by replacing the upper interval bound n by 1.

In the following, we reduce satisfiability of QTL^b-formulas to the satisfiability of QTL-formulas in which all upper interval bounds have value 1. As the coding of numbers is not an issue in the latter logic, we obtain a PSpace upper bound from the result of [10] that QTL^u-satisfiability in models without FVA is decidable in PSpace.

Let φ be a QTL-formula meeting the restrictions laid out above. Let k be the greatest number occurring as a parameter to a metric operator in φ, $n_c = \lceil \log_2(k+2) \rceil$, and $\chi_1, \ldots, \chi_\ell$ the subformulas of φ that occur as an argument to a metric operator of the form $\Diamond_F^{[0,n]}$ with $n > 1$. We reserve, for $1 \leq i \leq \ell$, fresh propositional variables x_i, y_i, and $c^i_{n_c-1}, \ldots, c^i_0$ that do not occur in φ. The sequences $c^i_{n_c-1}, \ldots, c^i_0$ of propositional variables will be used to implement binary counters, one for each χ_i. Intuitively, the i-th counter measures the distance to the "nearest" future occurrence of the formula χ_i, rounded to the next larger natural number. A counter value greater than or equal to $k+1$ is a special case indicating that the nearest occurrence of χ_i is too far away to be of any relevance. The variables x_i and y_i will serve as markers on the real line with the following meaning: x_i holds in a point iff there is a natural number n such that χ_i holds at distance n in the future, but not in between; similarly, y_i holds iff there is a natural number n such that χ_i does not hold in the future at any distance up to (and including) n, but χ_i is true at points that converge (from the future) against the future point with distance n. In the following, we call the structure imposed on the real line by the markers x_i and y_i the (one-dimensional) "grid" for χ_i.

To implement the counters, we introduce the following auxiliary formulas: for $1 \leq i \leq \ell$, let

- $(C_i = m)$ be a formula saying that, at the current point, the value of the i-th counter is m, for $0 \le m < 2^{n_c}$. Obviously, there are exponentially many such formulas, but we will use only polynomially many of them in the reduction.

- $(C_i \le m)$ is a formula saying that, at the current point, the value of the i-th counter does not exceed m, for $0 \le m < 2^{n_c}$.
- $\bigcirc\varphi := \neg(x_i \vee y_i)\,\mathcal{U}((x_i \vee y_i) \wedge \varphi)$ says that, at the next grid point of the grid for χ_i, φ is satisfied.

To deal with effects of convergence, it is convenient to introduce an additional abbreviation. The formula $rc(\psi) := \neg(\neg\psi\,\mathcal{U}\,\top) \wedge \neg\psi$ describes convergence of ψ-points from the future against a point where ψ does not hold. We now inductively define a translation of QTL^b-formulas to QTL-formulas in which all upper interval bounds have value 1:

$$\begin{aligned}
p^* &:= p \\
(\neg\psi)^* &:= \neg\psi^* \\
(\psi_1 \wedge \psi_2)^* &:= \psi_1^* \wedge \psi_2^* \\
(\psi_1\,\mathcal{U}\,\psi_2)^* &:= \psi_1^*\,\mathcal{U}\,\psi_2^* \\
(\Diamond_F^I\psi)^* &:= \Diamond_F^I\psi^* \\
(\Diamond_F^{[0,n]}\chi_i)^* &:= (C_i \le n-1) \vee \big((C_i = n) \wedge \neg y_i\big)
\end{aligned}$$

Here, I ranges over intervals $(0,1]$, $(0,1)$, and $[0,1)$. It remains to enforce the existence of the grids for each χ_i and the behavior of the counters as described above. This is done with the following auxiliary formulas, for $1 \le i \le \ell$:

$$\vartheta_1^i := (C_i = 0) \leftrightarrow \big(\chi_i^* \vee rc(\chi_i^*)\big)$$

$$\vartheta_2^i := x_i \leftrightarrow \Big[\chi_i^* \vee \big(\Box_F^{(0,1)}(\neg\chi_i^* \wedge \neg x_i \wedge \neg y_i) \wedge \Diamond_F^{(0,1]} x_i \wedge \Diamond_F \chi_i^*\big)\Big]$$

$$\vartheta_3^i := y_i \leftrightarrow \Big[rc(\chi_i^*) \vee \big(\Box_F^{(0,1)}(\neg\chi_i^* \wedge \neg x_i \wedge \neg y_i) \wedge \Diamond_F^{(0,1]} y_i \wedge \Diamond_F rc(\chi_i^*)\big)\Big]$$

$$\begin{aligned}
\vartheta_4^i := \neg(C_i = 0) \wedge \Diamond_F^{(0,1]}(x_i \vee y_i) \rightarrow \\
\Big(\bigvee_{t=0..n_c-1} \Big(c_t^i \wedge \bigcirc\neg c_t^i \wedge \bigwedge_{\ell=0..t-1} (\neg c_\ell^i \wedge \bigcirc c_\ell^i) \wedge \bigwedge_{\ell=t+1..n_c-1} (c_\ell^i \leftrightarrow \bigcirc c_\ell^i) \Big) \\
\vee \bigwedge_{\ell=0..n_c-1} (c_\ell^i \wedge \bigcirc c_\ell^i) \Big)
\end{aligned}$$

$$\vartheta_5^i := \neg\Diamond_F^{[0,1)}(x_i \vee y_i) \rightarrow (C_i = 2^{n_c} - 1)$$

Intuitively, ϑ_1^i initializes the counter, ϑ_2^i and ϑ_3^i ensure that the grid points x_i and y_i behave as described above, ϑ_4^i increments the counter when traveling into the past and ensures that the counter stays in maximal value once it is reached, and ϑ^i_5 ensures that, when traveling into the future, the counter is set to the maximal value after the last occurrence of χ_i^*. Let ϑ^i be the conjunction of ϑ^i_1 to ϑ^i_5. The following finishes the reduction.

Lemma 3 *φ is satisfiable iff $\Box_F^+(\vartheta^1 \wedge \cdots \wedge \vartheta^\ell) \wedge \varphi^*$ is satisfiable.*

PROOF. "$\Leftarrow$": Let $\mathfrak{V}$ be a valuation and $w \in \mathbb{R}$ a time point such that $w \models_{\mathfrak{V}} \Box_F^+(\vartheta^1 \wedge \cdots \wedge \vartheta^\ell) \wedge \varphi^*$. We show, by induction, for all time points $v \in \mathbb{R}$ and every subformula χ of φ:

$$v \models_{\mathfrak{V}} \chi \;\text{ iff }\; v \models_{\mathfrak{V}} \chi^* \tag{†}$$

Clearly, $w \models_{\mathfrak{V}} \varphi$ follows. The cases for propositional variables, $\neg$, $\wedge$, $\mathcal{U}$, and $\Diamond_F^I$, where I ranges over intervals $(0,1)$, $(0,1]$, and $[0,1)$, are trivial and omitted here. Consider the remaining case $\chi = \Diamond_F^{[0,n]}\chi_i$.

For the direction from right to left, suppose $v \models_{\mathfrak{V}} (\Diamond_F^{[0,n]}\chi_i)^*$, i.e.,

$$v \models_{\mathfrak{V}} (C_i \leq n-1) \vee \Big((C_i = n) \wedge \neg y_i\Big).$$

We define a time point $u \in \mathbb{R}$, distinguishing two cases:

(i) $v \models_{\mathfrak{V}} x_i \vee y_i$. Set $u = v$.
(ii) $v \not\models_{\mathfrak{V}} x_i \vee y_i$. Let $u \in v + (0,1)$ be minimal such that $u \models_{\mathfrak{V}} x_i \vee y_i$.

Note that, in (ii), the required u exists: by definition of n_c, we have $n < 2^{n_c} - 1$ and thus $v \models_{\mathfrak{V}} (\Diamond_F^{[0,n]}\chi_i)^*$ implies $v \models_{\mathfrak{V}} \Diamond_F^{[0,1)}(x_i \vee y_i)$ by ϑ^i_5. Hence, there exists $u \in v + (0,1)$ such that $u \models_{\mathfrak{V}} x_i \vee y_i$. By ϑ^i_2 and ϑ^i_3, there exists a minimal such u. For $m \geq 1$, let c_m denote the natural number such that $u + m \models_{\mathfrak{V}} (C_i = c_m)$. Our aim is to show that one of the following holds:

(a) $u + c_0 \models_{\mathfrak{V}} \chi_i^*$ and $u \models_{\mathfrak{V}} x_i$;
(b) $u + c_0 \models_{\mathfrak{V}} rc(\chi_i^*)$ and $u \models_{\mathfrak{V}} y_i$.

For suppose that this has been shown. Then we obtain $v \models_{\mathfrak{V}} \Diamond_F^{[0,n]}\chi_i^*$, which can be seen by distinguishing the following four subcases, and thus get $v \models_{\mathfrak{V}} \Diamond_F^{[0,n]}\chi_i$ by induction hypothesis as desired.

- Cases (i) and (a). Since $v \models_{\mathfrak{V}} (\Diamond_F^{[0,n]}\chi_i)^*$ and $v = u$, we have $c_0 \leq n$. Thus, $u + c_0 \models_{\mathfrak{V}} \chi_i^*$ yields $v \models_{\mathfrak{V}} \Diamond_F^{[0,n]}\chi_i^*$.
- Cases (i) and (b). Then $u + c_0 \models_{\mathfrak{V}} rc(\chi_i^*)$ implies that we can find a time point $v' \in u + (c_0, c_0+1)$ such that $v' \models_{\mathfrak{V}} \chi_i^*$. Since $v \models_{\mathfrak{V}} (\Diamond_F^{[0,n]}\chi_i)^*$, $v = u$, and $u \models_{\mathfrak{V}} y_i$, we have $c_0 < n$. Thus, $v \models_{\mathfrak{V}} \Diamond_F^{[0,n]}\chi_i^*$.
- Cases (ii) and (a). Since $v \not\models_{\mathfrak{V}} x_i \vee y_i$, ϑ^i_1 to ϑ^i_3 yield that $v \models_{\mathfrak{V}} \neg(C_i = 0)$. Together with the existence of u with $u \models_{\mathfrak{V}} (C_i = c_0)$ and by ϑ^i_4, it follows

that $v \models_{\mathfrak{Y}} (C_i = c_0 + 1)$. Since $v \models_{\mathfrak{Y}} (\Diamond_F^{[0,n]} \chi_i)^*$, this yields $c_0 < n$. Thus $u + c_0 \models_{\mathfrak{Y}} \chi_i^*$ and the choice of u yield $v \models_{\mathfrak{Y}} \Diamond_F^{[0,n]} \chi_i^*$.

- Cases (ii) and (b). Then $u + c_0 \models_{\mathfrak{Y}} rc(\chi_i^*)$ implies that we can find a time point $v' \in u + (c_0, c_0 + 1)$ such that $v' \models_{\mathfrak{Y}} \chi_i^*$. As in the third subcase, we can show that $c_0 < n$. Thus $v' \models_{\mathfrak{Y}} \chi_i^*$ and the choice of u and v' yield $v \models_{\mathfrak{Y}} \Diamond_F^{[0,n]} \chi_i^*$.

It thus remains to show that one of (a) and (b) holds. To this end, we show by induction on m that, for $m \leq c_0$, we have

(1) $u + m \models_{\mathfrak{Y}} x_i \vee y_i$;
(2) $c_m = c_0 - m$;
(3) if $m < c_0$, then $v' \not\models_{\mathfrak{Y}} \chi_i^* \vee rc(\chi_i^*)$ for all $v' \in [u + m, u + m + 1)$.

For the induction start, let $m = 0$. Point 1 holds by choice of u and Point 2 is trivial. For Point 3, first assume that $u \models_{\mathfrak{Y}} \chi_i^* \vee rc(\chi_i^*)$. Then $c_0 = 0$ by ϑ_1^i, which is a contradiction to the precondition of (iii). It thus remains to show that $v' \not\models_{\mathfrak{Y}} \chi_i^* \vee rc(\chi_i^*)$ for all $v' \in (u, u+1)$. This is an immediate consequence of ϑ_2^i and ϑ_3^i together with the facts that $u \models x_i \vee y_i$ and $u \not\models_{\mathfrak{Y}} \chi_i^* \vee rc(\chi_i^*)$. For the induction step, assume that (1) to (3) have been shown up to and including $m < c_0$.

- Point 1. By induction, $u + m \models_{\mathfrak{Y}} x_i \vee y_i$ and $u + m \not\models_{\mathfrak{Y}} \chi_i^* \vee rc(\chi_i^*)$. Thus, we have $u + m + 1 \models_{\mathfrak{Y}} x_1 \vee y_i$ by ϑ_2^i and ϑ_3^i.
- Point 2. By induction, we have $c_m = c_0 - m$ which implies $u + m \models_{\mathfrak{Y}} \neg(C_i = 0)$ by $m < c_0$. Since Point 1 additionally gives us $u + m + 1 \models_{\mathfrak{Y}} x_i \vee y_i$, ϑ_4^i yields $c_m = c_{m+1} + 1$ and from Point 2 of the induction hypothesis we obtain $c_{m+1} = c_0 - (m + 1)$.
- Point 3. Assume $m + 1 < c_0$. Point 2 gives us $c_{m+1} = c_0 - (m + 1)$. We thus have $u + (m+1) \models_{\mathfrak{Y}} \neg(C_i = 0)$. Thus, ϑ_1^i implies $u + (m+1) \not\models_{\mathfrak{Y}} \chi_i^* \vee rc(\chi_i^*)$. It thus remains to show that $v' \not\models_{\mathfrak{Y}} \chi_i^* \vee rc(\chi_i^*)$ for all $v' \in (u + (m + 1), u + (m + 1) + 1)$. This is an immediate consequence of ϑ_2^i and ϑ_3^i together with the facts that $u + (m+1) \models x_i \vee y_i$ by Point 1 and $u + (m+1) \not\models_{\mathfrak{Y}} \chi_i^* \vee rc(\chi_i^*)$.

In particular, we have shown that $u + c_0 \models_{\mathfrak{Y}} (C_i = 0)$. Thus, $u + c_0 \models_{\mathfrak{Y}} \chi_i^* \vee rc(\chi_i^*)$ by ϑ_1^i. We have two sub-cases: first, $u + c_0 \models_{\mathfrak{Y}} \chi_i^*$. By ϑ_2^i, we have $u + m \models_{\mathfrak{Y}} x_i$ for all $m \leq c_0$, and thus Case (a) from above holds. The second case is $u + c_0 \models_{\mathfrak{Y}} rc(\chi_i^*)$. Then ϑ_3^i yields $u + m \models_{\mathfrak{Y}} y_i$ for all $m \leq c_0$ and Case (b) from above holds.

For the direction from left to right of (†), suppose $v \models_{\mathfrak{Y}} \Diamond_F^{[0,n]} \chi_i$. By the semantics, there is a $u \in w + [0, n]$ such that $u \models_{\mathfrak{Y}} \chi_i$. If there is a smallest such position u, then

(a) let u denote this position, otherwise

(b) let u be the smallest position such that $u \models_{\mathfrak{V}} rc(\chi_i)$.

In Case (b), we clearly have $u < w + n$. The induction hypothesis yields (a) $u \models_{\mathfrak{V}} \chi_i^*$ or (b) $u \models_{\mathfrak{V}} rc(\chi_i^*)$. With ϑ_1^i, we get $u \models_{\mathfrak{V}} (C_i = 0)$. Together with $v' \not\models_{\mathfrak{V}} \chi_i^*$ for each $v' \in (v, u)$, it follows from ϑ_2^i and ϑ_3^i that (a) $v'' \models_{\mathfrak{V}} x_i \wedge \neg y_i$ or (b) $v'' \models_{\mathfrak{V}} y_i \wedge \neg x_i$ for all v'' such that $v'' = u - j$ for some natural number $j \leq u - v$. In particular,

$$\text{(a) } v \models_{\mathfrak{V}} x_i \wedge \neg y_i \text{ or (b) } v \models_{\mathfrak{V}} y_i \wedge \neg x_i. \qquad (*)$$

Next, $\{w \in [v, u] \mid w \models_{\mathfrak{V}} \chi_i^* \vee rc(\chi_i^*)\} = \{u\}$, ϑ_1^i, and ϑ_4^i yield $v' \models_{\mathfrak{V}} (C_i = j)$ for every natural number $j \leq u - v$ and all $v' \in [u - j, u - j + 1)$. Since u was chosen such that (a) $u \in v + [0, n]$ or (b) $u \in v + [0, n)$, we obtain (a) $v \models_{\mathfrak{V}} (C_i \leq n)$ or (b) $v \models_{\mathfrak{V}} (C_i < n)$. Together with $(*)$, this yields $v \models_{\mathfrak{V}} (C_i \leq n - 1) \vee \big((C_i = n) \wedge \neg y_i\big)$ as required.

"$\Rightarrow$": Suppose φ is satisfiable, i.e., there is a valuation $\mathfrak{V}$ and a time point $w \in \mathbb{R}$ such that $w \models_{\mathfrak{V}} \varphi$. For $1 \leq i \leq \ell$, set

$$S_i := \{v \in \mathbb{R} \mid \exists u > v : u \models_{\mathfrak{V}} \chi_i\}$$

and let, for each $v \in S$, v_{χ_i} denote the smallest time point such that $v_{\chi_i} \geq v$ and $v_{\chi_i} \models_{\mathfrak{V}} \chi_i \vee rc(\chi_i)$. We extend $\mathfrak{V}$ to the additional propositional letters x_i, y_i, and c_t^i used in φ^* as follows:

(1) $v \in \mathfrak{V}(x_i)$ iff $v \in S_i$, $v_{\chi_i} - v$ is an integer, and $v_{\chi_i} \models_{\mathfrak{V}} \chi_i$;
(2) $v \in \mathfrak{V}(y_i)$ iff $v \in S_i$, $v_{\chi_i} - v$ is an integer, and $v_{\chi_i} \models_{\mathfrak{V}} rc(\chi_i)$;
(3) $v \in \mathfrak{V}(c_t^i)$ iff $v \notin S$, the t-th bit of the number $\lceil v_{\chi_i} - v \rceil$ is one, or this number exceeds the value $2^{n_c} - 2$.

It is not hard to verify that $w \models_{\mathfrak{V}} \Box_F^+(\vartheta^1 \wedge \cdots \wedge \vartheta^\ell)$. In order to show that $w \models_{\mathfrak{V}} \varphi^*$, the following can be proved by structural induction on ψ: for all $v \in \mathbb{R}$ and all subformulas ψ of φ,

$$v \models_{\mathfrak{V}} \psi \text{ iff } v \models_{\mathfrak{V}} \psi^*$$

Details are left to the reader. $\Box$

4 From NP to PSpace

Qualitative since-until logic on the real line is PSpace-complete, and thus not computationally simpler than QTL^b. However, several fragments are only NP-complete, an important example being the qualitative TL with temporal operators 'eventually in the future' [16]. In this section, we explore the transition from NP to PSPACE for fragments of *quantitative* temporal logics of the real line, i.e., for QTL and its fragments. We start with determining several weak, but still PSPACE-hard fragments of QTL. Observe that two of the fragments are purely quantitative, i.e., they do not admit qualitative temporal operators at all.

Theorem 4 *Satisfiability (with and without FVA) is* PSPACE*-hard for the fragments of QTL whose only temporal operators are:*

(i) $\Diamond_F$ *and* $\Diamond_F^{[0,n]}$ *with* $n > 0$ *coded in unary;*
(ii) $\Diamond_F^{[0,n]}$ *with* $n > 0$ *coded in binary;*
(iii) $\mathcal{U}^{[0,1]}$.

PROOF. Since the proof uses standard techniques, it is only sketched here. Details are easily filled in. To show Point (i), we reduce satisfiability in LTL, i.e., qualitative temporal logic of the natural numbers with the only temporal operators and $\Diamond_F$,which is PSPACE-hard [15]. The main idea of the reduction is to represent the discrete natural numbers on the real line by alternating between intervals that make a propositional variable a true and intervals that make $\neg a$ true. Intuitively, the former represent the time points of discrete time. The length of the a-intervals and the $\neg a$-intervals is between 2 and 3. These requirements are formalized by the formula $\vartheta = \vartheta_1 \wedge \vartheta_2 \wedge \vartheta_3$:

$$\begin{aligned}
\vartheta_1 &= \Box_F^{[0,2]} a, \\
\vartheta_2 &= \Box_F^{+}(a \rightarrow \Diamond_F^{[0,3]} \Box_F^{[0,2]} \neg a), \\
\vartheta_3 &= \Box_F^{+}(\neg a \rightarrow \Diamond_F^{[0,3]} \Box_F^{[0,2]} a).
\end{aligned}$$

Note that models of ϑ can also contain a and $\neg a$-intervals of length smaller than 1. These small intervals are located between successive a and $\neg a$-intervals of length at least 2. However, their presence does not interfere with the reduction. Inductively define a translation $(\cdot)^*$ as follows:

$$\begin{aligned}
p^* &:= p \\
(\neg\psi)^* &:= \neg\psi^* \\
(\psi_1 \wedge \psi_2)^* &:= \psi_1^* \wedge \psi_2^* \\
(\psi)^* &:= \Diamond_F^{[0,3]} \Big(\Box_F^{[0,2]} \neg a \wedge \Diamond_F^{[0,3]} \Box_F^{[0,2]} (\psi^* \wedge a) \Big) \\
(\Diamond_F \psi)^* &:= \Diamond_F \Big(\Box_F^{[0,2]} \neg a \wedge \Diamond_F \Box_F^{[0,2]} (\psi^* \wedge a) \Big)
\end{aligned}$$

Additionally, a formula ϑ' is needed to take care of uniformity, i.e., to make sure that the same propositional variables hold in all time points of an interval that makes a true:

$$\vartheta' = \Box_F^+ \bigwedge_{p \text{ used in } \varphi} \Big(\big(p \wedge a \to \Box_F^{[0,2]}(a \to p) \big) \wedge \big(\neg p \wedge a \to \Box_F^{[0,2]}(a \to \neg p) \big) \Big).$$

Now, φ is satisfiable over the natural numbers iff $\varphi^* \wedge \vartheta \wedge \vartheta'$ is satisfiable over the real numbers with FVA iff it is satisfiable over the real numbers without FVA.

A similar reduction can be used to prove (ii). Notice that satisfiability in LTL is already PSPACE-hard if the natural numbers are replaced by a finite strict linear order (an initial segment of the natural numbers). Moreover, any formula φ which is satisfiable in a finite strict linear order is also satisfiable in a finite strict linear order of length not exceeding $2^{|\varphi|}$. Based on this observation, using the operator $\Diamond_F^{[0,n]}$, $n > 0$ coded in binary, instead of $\Diamond_F$, we can reduce satisfiability of an LTL-formula φ in such a finite strict linear order to satisfiability over the real line (with and without FVA).

Finally, (iii) can be proved by reducing satisfiability over the real line in $\mathrm{QTL}_{\mathcal{U}}$, the QTL-fragment with only temporal operator $\mathcal{U}$, which is known to be PSPACE-hard without FVA [13], to satisfiability of formulas with the operator $\mathcal{U}^{[0,1]}$ over the interval $(0,1)$. The idea of the reduction is to embed the whole real line into the interval $(0,1)$: given a formula φ of $\mathrm{QTL}_{\mathcal{U}}$, fix a fresh propositional variable a that does not occur in φ. Define a translation $(\cdot)^*$ that recursively replaces every subformula of φ of the form $\psi_1 \,\mathcal{U}\, \psi_2$ with $\psi_1 \,\mathcal{U}^{[0,1]}(a \wedge \psi_2)$. Then φ is satisfiable iff $\varphi^* \wedge a \wedge \big(a\,\mathcal{U}^{[0,1]}(\Box_F^{[0,1]} \neg a)\big)$ is. For the FVA case, we note that the PSPACE-hardness proof for $\mathrm{QTL}_{\mathcal{U}}$ does not depend on variables changing their value an infinite number of times in any bounded interval. □

We now exhibit a purely quantitative temporal logic of the real line for which satisfiability is NP-complete: the fragment of QTL with only the quantitative diamond and numbers coded in unary, with and without FVA. This logic may appear rather weak since it does not allow to make statements about *all* time points. Still, it is useful for reasoning about the behavior of systems up to a previously fixed time point. Note that our NP-completeness result shows that Points (i) and (ii) of Theorem 4 are optimal in the following sense: in Point (i) we cannot drop $\Diamond_F$, and in Point (ii) we cannot switch to unary coding.

Theorem 5 *In the fragment of QTL with temporal operators $\Diamond_F^I$ and $\Diamond_P^I$, where I is of the form $(0,n)$, $[0,n)$, $[0,n]$, or $(0,n]$, and $n > 0$ coded in unary, satisfiability is decidable in* NP*, both, with and without FVA.*

The lower bound is immediate from propositional logic and thus we only have to prove the upper bound. Since numbers are coded in unary, we may restrict our attention to temporal operators whose upper interval bound is 1. In the proof, we only consider the temporal operator $\Diamond_F^{[0,1]}$; an extension to past operators and open intervals is straightforward.

Let φ be a formula whose satisfiability is to be decided. We introduce some convenient abbreviations: m_φ denotes the nesting depth of operators $\Diamond_F^{[0,1]}$ in φ (henceforth called *diamond depth*), $n_\varphi = 2 \times |\varphi|^3 + |\varphi|^2$, and $r_\varphi = |\varphi| \times n_\varphi$. Denote by $cl(\varphi)$ the closure of the set of subformulas of φ under single negation. A *type t for φ* is a subset of $cl(\varphi)$ such that (i) $\neg\psi \in t$ iff $\psi \notin t$ for all $\neg\psi \in cl(\varphi)$, and (ii) $\psi_1 \wedge \psi_2 \in t$ iff $\psi_1, \psi_2 \in t$ for all $\psi_1 \wedge \psi_2 \in cl(\varphi)$. For a model $\langle \mathbb{R}, \mathfrak{V} \rangle$ and time point $w \in \mathbb{R}$, set

$$t(w) = \{\psi \in cl(\varphi) \mid w \models_{\mathfrak{V}} \psi\},$$
$$t^{\Diamond}(w) = \{\Diamond_F^{[0,1]}\psi \in cl(\varphi) \mid w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi\}.$$

Notice that $t(w)$ is a type for φ. First, we devise an algorithm for satisfiability without FVA. To begin with, we show that satisfiability of φ implies satisfiability of φ in a "homogeneous" model, whose most important property is that the number of realized types is polynomial in the length of φ.

Lemma 6 *Let φ be satisfiable without FVA. Then there is a sequence $x_0, \ldots, x_{n_\varphi}$ in $\mathbb{R}$ such that $0 = x_0 < x_1 < \cdots < x_{n_\varphi} = m_\varphi$, and a valuation $\mathfrak{V}$ such that $\langle \mathbb{R}, \mathfrak{V} \rangle, 0 \models \varphi$ and*

- $|\{t(w) \mid 0 \leq w \leq m_\varphi\}| \leq r_\varphi$*;*
- *for every n with $0 \leq n < n_\varphi$ and each type t for φ, the set*

$$\{w \in \mathbb{R} \mid x_n < w < x_{n+1} \text{ and } w \models_{\mathfrak{V}} t\}$$

  is either empty or dense in the interval (x_n, x_{n+1}).

PROOF. Consider a model $\mathfrak{M} = \langle \mathbb{R}, \mathfrak{V}' \rangle$ with $\mathfrak{M}, 0 \models \varphi$. Observe first that the truth of φ in 0 does not depend on the value of propositional variables after m_φ. Therefore, we can assume that $w \models_{\mathfrak{V}'} \neg p$ for every $w > m_\varphi$ and propositional variable p. Moreover, the semantics of $\Diamond_F^{[0,1]}$ yields:

(∗) for any $\Diamond_F^{[0,1]}\psi \in cl(\varphi)$, the set $\{w \in \mathbb{R} \mid 0 \leq w \leq m_\varphi \text{ and } w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi\}$ is a union of intervals of length at least 1 and at most two intervals of length smaller than 1.

The two possibly shorter intervals are the one starting at 0 and the one ending at m_φ. Using (∗), we can show that there is a sequence $y_0, \ldots, y_k$ in $\mathbb{R}$ for some $k \leq 2 \times |\varphi|^2 + |\varphi|$ such that

- $0 = y_0 < \cdots < y_k = m_\varphi$ and

- $t^\diamond(w) = t^\diamond(w')$ whenever $y_i < w < w' < y_{i+1}$ for any $i < k$.

To see this, take a formula $\diamond_F^{[0,1]}\psi \in cl(\varphi)$. The *toggle points* for $\diamond_F^{[0,1]}\psi$ in the interval $[0, m_\varphi]$ are those time points x such that either (i) there is a $y > x$ such that the truth value of $\diamond_F^{[0,1]}\psi$ at x is different from the truth value of $\diamond_F^{[0,1]}\psi$ at all points z with $x < z < y$ or (ii) there is a $y < x$ such that the truth value of $\diamond_F^{[0,1]}\psi$ at x is different from the truth value of $\diamond_F^{[0,1]}\psi$ at all points z with $y < z < x$. By $(*)$, there are at most $2 \times m_\varphi + 1 < 2 \times |\varphi| + 1$ toggle points for each formula $\diamond_F^{[0,1]}\psi$ in $[0, m_\varphi]$, and thus at most $2 \times |\varphi|^2 + |\varphi|$ toggle points altogether in this interval. These points form the required sequence $y_0, \ldots, y_k$.

We convert this sequence into the desired sequence $x_0, \ldots, x_{n_\varphi}$ by arranging the elements of the set

$$\{y_0, \ldots, y_k\} \cup \bigcup_{\substack{i<k \\ 1 \leq j < m_\varphi}} \{y_i + j \mid y_i + j < m_\varphi\}$$

in ascending order according to the ordering relation '$<$' on $\mathbb{R}$, possibly introducing arbitrary intermediate points from the interval $[m_\varphi - 1, m_\varphi]$ to obtain a sequence of length $n_\varphi + 1$. This construction ensures that if $x_i = x$ and $x_i \leq m_\varphi - 1$, then $x_j = x + 1$ for some $j > i$.

To obtain a valuation $\mathfrak{V}$ as required by the lemma, fix a set T_i of types in $\mathfrak{M}$ for each $i < n_\varphi$ as follows: for each $\diamond_F^{[0,1]}\psi \in cl(\varphi)$, choose a $w \in (x_i, x_{i+1})$ with $\psi \in t(w)$ if such a w exists. Then, T_i is the set of types $t(w)$ of all points w chosen in this way. Clearly $|T_i| \leq |\varphi|$. For each $i < n_\varphi$, take a collection $(X_t^i)_{t \in T_i}$, of subsets of (x_i, x_{i+1}) which form a partitioning of (x_i, x_{i+1}) such that each X_t^i is dense in (x_i, x_{i+1}). Now define a valuation $\mathfrak{V}$ by setting, for every propositional variable p,

$$\mathfrak{V}(p) := (\mathfrak{V}'(p) \cap \{x_0, \ldots, x_{n_\varphi}\}) \cup \bigcup_{i<n_\varphi, t \in T_i} \{X_t^i \mid p \in t\}.$$

Let t_i, $i \leq n_\varphi$, be the type $\{\psi \in cl(\varphi) \mid x_i \models_{\mathfrak{V}'} \psi\}$ for φ realized in point x_i of the original model $\mathfrak{M}$. To show that $\mathfrak{V}$ is as required, it is sufficient to show for all $\psi \in cl(\varphi)$ and all $w \in [0, m_\varphi]$:

$$\begin{aligned} w \models_{\mathfrak{V}} \psi \;\text{ iff there is an } i \leq n_\varphi \text{ such that} \\ &\text{(a) } w = x_i \text{ and } \psi \in t_i, \text{ or} \\ &\text{(b) } w \in X_t^i \text{ and } \psi \in t \text{ for some } t \in T_i. \end{aligned}$$

Let ψ and w be as above. The proof is by induction on the structure of ψ. The cases for propositional variables, $\neg$, and $\wedge$ are left to the reader. Consider the case for $\Diamond_F^{[0,1]}$.

"⇒": Suppose $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$. Then there is a $w' \in w+[0,1]$ such that $w' \models_{\mathfrak{V}} \psi$. First assume that $m_\varphi - w \geq 1$. Then we distinguish four cases:

- $w = x_i$ for some $i < n_\varphi$ and $w' = x_j$ for some j with $i \leq j \leq n_\varphi$. The induction hypothesis yields $\psi \in t_j$. Then $x_j \models_{\mathfrak{V}'} \psi$. From $x_j - x_i \leq 1$, it follows that $x_i \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. Hence $\Diamond_F^{[0,1]}\psi \in t_i$.
- $w = x_i$ for some $i < n_\varphi$ and $w' \in X_t^j$ for some j with $i \leq j < n_\varphi$ and $t \in T_j$. The induction hypothesis yields $\psi \in t$. Then, by definition of T_j, there is a $w'' \in (x_j, x_{j+1})$ such that $w'' \models_{\mathfrak{V}'} \psi$. By definition of the sequence $x_0, \ldots, x_{n_\varphi}$, there is an ℓ with $i < \ell \leq n_\varphi$ such that $x_\ell = x_i + 1$. Then $x_{j+1} \leq x_\ell$ because $w' \in X_t^j \subseteq (x_j, x_{j+1})$. Thus $w'' - w < 1$ and it follows that $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. Hence $\Diamond_F^{[0,1]}\psi \in t_i$.
- $w \in X_t^i$ for some $i < n_\varphi - 1$ and $t \in T_i$, and $w' = x_j$ for some j with $i < j \leq n_\varphi$. The induction hypothesis yields $\psi \in t_j$. Then $x_j \models_{\mathfrak{V}'} \psi$. Now, from $x_j - w \leq 1$, it follows that $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. But then, by definition of the sequence $x_0, \ldots, x_{n_\varphi}$, it holds that $w'' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$ for all $w'' \in (x_i, x_{i+1})$. Therefore, $\Diamond_F^{[0,1]}\psi \in t'$ for each $t' \in T_i$.
- $w \in X_t^i$ for some $i < n_\varphi - 1$ and $t \in T_i$, and $w' \in X_{t'}^j$ for some j with $i \leq j < n_\varphi$ and $t' \in T_j$. The induction hypothesis yields $\psi \in t'$. Then, by definition of T_j, there is a $w'' \in (x_j, x_{j+1})$ such that $w'' \models_{\mathfrak{V}'} \psi$. By definition of the sequence $x_0, \ldots, x_{n_\varphi}$, there is an ℓ with $i < \ell \leq n_\varphi$ such that $x_\ell = x_{i+1} + 1$. Then $x_{j+1} \leq x_\ell$ because $w \in X_t^i \subseteq (x_i, x_{i+1})$ and $w' \in X_{t'}^j \subseteq (x_j, x_{j+1})$. Thus, there is a point $v \in (x_i, x_{i+1})$ such that $w'' - v \leq 1$. It follows that $v \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. But then, by definition of the sequence $x_0, \ldots, x_{n_\varphi}$, it holds that $v' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$ for all $v' \in (x_i, x_{i+1})$. Therefore, $\Diamond_F^{[0,1]}\psi \in t''$ for each $t'' \in T_i$.

Now let $m_\varphi - w < 1$. First assume $w' > m_\varphi$. Since $\mathfrak{V}$ and $\mathfrak{V}'$ are identical beyond m_φ, we have $w' \models_{\mathfrak{V}'} \psi$ and $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$ follows. If $w = x_i$ for some $i \leq n_\varphi$, then this yields $\Diamond_F^{[0,1]}\psi \in t_i$ as required. If $w \in X_t^i$ for some $i < n_\varphi$ and $t \in T_i$, then by definition of the sequence $x_0, \ldots, x_{n_\varphi}$, it holds that $v \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$ for all $v \in (x_i, x_{i+1})$. Therefore, $\Diamond_F^{[0,1]}\psi \in t$ as required. Now assume $w' \leq m_\varphi$. Then we can distinguish the same four subcases as above. In each of these cases, the proof is a slight variation of what was done above. We leave details to the reader.

"⇐": Let $i \leq n_\varphi$ such that

(a) $w = x_i$ and $\Diamond_F^{[0,1]}\psi \in t_i$. Then $x_i \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$, i.e., there is a $w' \in x_i + [0,1]$ such that $w' \models_{\mathfrak{V}'} \psi$. Distinguish three cases:
   - $w' = x_j$ for some j with $i \leq j \leq n_\varphi$. Then $\psi \in t_j$. The induction hypothesis in (a) yields $w' \models_{\mathfrak{V}} \psi$. From $w' - x_i \leq 1$, it follows that

$w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.

- $w' \in (x_j, x_{j+1})$ for some j with $i \leq j < n_\varphi$. By definition of T_j, there is a $t \in T_j$ such that $\psi \in t$. The induction hypothesis in (b) yields $w'' \models_{\mathfrak{V}} \psi$ for any $w'' \in X_t^j$. Since X_t^j is dense in the interval (x_j, x_{j+1}), there is such a w'' such that $w'' \leq w'$. Thus $w'' - x_i \leq 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.
- $w' > m_\varphi$. Then $w' \models_{\mathfrak{V}} \psi$ because $w' \models_{\mathfrak{V}'} \psi$ and the valuations $\mathfrak{V}$ and $\mathfrak{V}'$ coincide after m_φ. Now $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$ follows.

(b) $w \in X_t^i$ and $\Diamond_F^{[0,1]}\psi \in t$ for some $t \in T_i$. By definition of T_i, there is a $w' \in (x_i, x_{i+1})$ such that $w' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. But then, by definition of the sequence $x_0, \ldots, x_{n_\varphi}$, it holds that $w'' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$ for all $w'' \in (x_i, x_{i+1})$. In particular, $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. We derive that $v \models_{\mathfrak{V}'} \psi$ for some $v \in w + [0,1]$. Distinguish three cases:

- $v = x_j$ for some j with $i < j \leq n_\varphi$. Then $\psi \in t_j$. The induction hypothesis in (a) yields $x_j \models_{\mathfrak{V}} \psi$. From $x_j - w \leq 1$, it follows that $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.
- $v \in (x_j, x_{j+1})$ for some j with $i \leq j < n_\varphi$. By definition of T_j, there is a $t \in T_j$ such that $\psi \in t$. The induction hypothesis in (b) yields $v' \models_{\mathfrak{V}} \psi$ for all $v' \in X_t^j$. Fix a point $v' \in X_t^j$ such that $v' \leq v$ if $j > i$, and $v' \geq v$ otherwise. Such a v' exists since X_t^j is dense in the interval (x_j, x_{j+1}). Then $v' - w \leq 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.
- $v > m_\varphi$. Then $v \models_{\mathfrak{V}} \psi$ because $v \models_{\mathfrak{V}'} \psi$. Hence, $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.

□

Lemma 6 suggests the following idea for deciding in non-deterministic polynomial time whether a formula φ is satisfiable: guess a (polynomially bounded) set of types for φ to be realized in a homogeneous model, a sequence $v_0, \ldots, v_{n_\varphi}$ of variables, and construct a system of linear inequalities whose solution in $\mathbb{R}$ determines a sequence of points $x_0, \ldots, x_{n_\varphi}$ from which we can build a homogeneous model realizing the guessed types. More precisely, to decide the satisfiability of φ, we non-deterministically choose

- a set T of types for φ such that $|T| \leq r_\varphi$;
- a type $t_i \in T$ such that $\varphi \in t_0$, for every $i \leq n_\varphi$;
- a non-empty set of types $T_i \subseteq T$, for every $i < n_\varphi$.

Intuitively, the type t_i is to be realized at point x_i, and the types in T_i are those types realized in the interval (x_i, x_{i+1}). Then we take variables $v_0, \ldots, v_{n_\varphi}$ and check whether the system of inequalities given in Figure 1 has a solution in $\mathbb{R}$. The Inequalities 2 to 9 are only added if $i < n_\varphi$. To understand the inequalities (in particular 4 and 5), note that to obtain a model satisfying φ in 0 it is not required that the points x_i described by variable v_i realize exactly the type t_i. This is only required for those elements of t_i whose diamond depth is at

(1) $0 = v_0 < v_1 < \cdots < v_{n_\varphi} = m_\varphi$

(2) $v_j - v_i > 1$ if $\neg\Diamond_F^{[0,1]}\psi \in t_i$, $j \geq i$, and $\psi \in t_j$

(3) $v_j - v_i \geq 1$ if $\neg\Diamond_F^{[0,1]}\psi \in t_i$, $j \geq i$, and $\psi \in t$ for some $t \in T_j$

(4) $m_\varphi - v_i < 1$ if $\Diamond_F^{[0,1]}\psi \in t_i$, but there is no $j \geq i$ such that $\psi \in t_j$ or $\psi \in t$ for a $t \in T_j$

(5) $m_\varphi - v_i \leq 1$ if $\Diamond_F^{[0,1]}\psi \in t$ for some $t \in T_i$, there is no $j > i$ such that $\psi \in t_j$, and there is no $j \geq i$ such that $\psi \in t'$ for some $t' \in T_j$

(6) $v_j - v_i \leq 1$ if $\Diamond_F^{[0,1]}\psi \in t_i$ and $j \geq i$ is minimal such that $\psi \in t_j$ and, for every j' with $i \leq j' < j$, $\psi \notin t$ for any $t \in T_{j'}$

(7) $v_j - v_i < 1$ if $\Diamond_F^{[0,1]}\psi \in t_i$ and $j \geq i$ is minimal such that $\psi \in t$ for some $t \in T_j$ and there is no j' with $i \leq j' \leq j$ such that $\psi \in t_{j'}$

(8) $v_j - v_i \leq 1$ if $\Diamond_F^{[0,1]}\psi \in t$ for some $t \in T_i$, $\psi \notin t'$ for any $t' \in T_i$, and $j > i$ is minimal such that $\psi \in t_j$ or $\psi \in t'$ for some $t' \in T_j$

(9) $v_j - v_{i+1} \geq 1$ if $\neg\Diamond_F^{[0,1]}\psi \in t$ for some $t \in T_i$, and ($j \geq i$ and $\psi \in t'$ for some $t' \in T_j$) or ($j > i$ and $\psi \in t_j$)

Fig. 1. The system of inequalities.

most $\lfloor m_\varphi - x_i \rfloor$. Similarly, it is sufficient that points from (x_i, x_{i+1}) described by a type $t \in T_i$ realize those elements of t whose diamond depth is at most $\lfloor m_\varphi - x_i \rfloor$.

The algorithm runs in non-deterministic polynomial time and returns 'φ is satisfiable' if there is a solution to this system of inequalities, and 'φ is not satisfiable' otherwise. By considering the contrapositive, it is easily seen that φ is unsatisfiable if the algorithm answers 'no': if φ has a model, then by Lemma 6 it also has a homogeneous model, and this model suggests a choice of types such that the corresponding system of inequalities is satisfiable. Conversely, if the algorithm returns 'yes', we can construct a homogeneous model:

Lemma 7 *If the algorithm returns 'φ is satisfiable', then φ is satisfiable.*

PROOF. Suppose there are types t_i, $i \leq n_\varphi$, and sets of types T_i, $i < n_\varphi$, such that there is a solution $x_0, \ldots, x_{n_\varphi}$ for the corresponding system of inequalities.

For each $i < n_\varphi$, take a partitioning $(X^i_t)_{t \in T_i}$ of (x_i, x_{i+1}) such that each X^i_t is dense in (x_i, x_{i+1}). Now define a valuation $\mathfrak{V}$ by putting, for every propositional variable p,

$$\mathfrak{V}(p) := \bigcup_{i \leq n_\varphi} \Big(\{x_i \mid p \in t_i\} \cup \bigcup_{i < n_\varphi, t \in T_i} \{X^i_t \mid p \in t\} \Big).$$

It is now straightforward to prove that, for all $k \leq m_\varphi$, all $\psi \in cl(\varphi)$ with diamond depth bounded by k, and all $w \in [0, m_\varphi - k]$, we have

$$\begin{aligned} w \models_{\mathfrak{V}} \psi \;\text{ iff }\; &\text{there is an } i \leq n_\varphi \text{ such that} \\ &\qquad \text{(a) } w = x_i \text{ and } \psi \in t_i, \text{ or} \\ &\qquad \text{(b) } w \in X^i_t \text{ and } \psi \in t \text{ for some } t \in T_i. \end{aligned}$$

It is an immediate consequence that $0 \models_{\mathfrak{V}} \varphi$. □

We now come to the proof of Theorem 5 with FVA. Again, the first step is to show that if φ is satisfiable under FVA, then it is satisfiable in a homogeneous model (this time with FVA) in which only polynomially many types are realized:

Lemma 8 *Suppose φ is satisfiable with FVA. Then there exists a sequence $z_0, \ldots, z_{r_\varphi}$ in $\mathbb{R}$ such that $0 = z_0 < z_1 < \cdots < z_{r_\varphi} = m_\varphi$, and a valuation $\mathfrak{V}$ such that*

- $\langle \mathbb{R}, \mathfrak{V} \rangle, 0 \models \varphi$;
- *for all n with $0 \leq n < r_\varphi$, all $\psi \in cl(\varphi)$, and all $z_n < w < w' < z_{n+1}$, we have $w \models_{\mathfrak{V}} \psi$ iff $w' \models_{\mathfrak{V}} \psi$.*

Notice the difference to Lemma 6: in each interval (z_i, z_{i+1}), $i < r_\varphi$, only a single type for φ is realized; in contrast, homogeneous models as described by Lemma 6 allow polynomially many types in every such interval.

PROOF. Consider a model $\mathfrak{M} = \langle \mathbb{R}, \mathfrak{V}' \rangle$ with FVA satisfying φ in 0. As in Lemma 6, we may assume that $w \models_{\mathfrak{V}'} \neg p$ for every $w > m_\varphi$ and propositional variable p.

Construct a sequence $0 = y_0 < y_1 < \cdots < y_k = m_\varphi$, $k \leq 2 \times |\varphi|^2 + |\varphi|$, of toggle points as in Lemma 6. Then the sequence $x_0, \ldots, x_{n_\varphi}$ is obtained by arranging the elements of the set

$$\{y_0, \ldots, y_k\} \cup \bigcup_{\substack{i<k \\ 1 \leq j < m_\varphi}} \{y_i + j \mid y_i + j < m_\varphi\} \cup \bigcup_{\substack{i \leq k \\ 1 \leq j < m_\varphi}} \{y_i - j \mid y_i - j > 0\}$$

in ascending order according to the ordering relation '<' on $\mathbb{R}$ (where we possibly have to add new intermediate points to obtain a sequence of length $n_\varphi + 1$). Let

$$\sigma = \min\{x_{i+1} - x_i \mid 0 \leq i < n_\varphi\},$$

and set, for $i < n_\varphi$, $\sigma_i = \frac{1}{|\varphi|^{i+1}} \times \sigma$. The sequence

$$0 = z_0 < z_1 < \cdots < z_{r_\varphi} = m_\varphi$$

is obtained by adding to the sequence $x_0, \ldots, x_{n_\varphi}$ the points

$$y_i^j = x_i + \frac{j}{|\varphi|} \times \sigma_i,$$

for all $i < n_\varphi$ and $j \leq |\varphi|$. For $i < n_\varphi$, denote by t^{-i} the type t which is realized in some interval of the form (x_i, y). Note that such an interval exists since we are in a model with FVA. Also, denote by t^{+i} the type which is realized in some interval of the form (y, x_{i+1}). Now, for $i < n_\varphi$, take for each $\diamond_F^{[0,1]}\psi \in cl(\varphi)$ such that there exists $w \in (x_i, x_{i+1})$ with $\psi \in t(w)$ such a type $t(w)$ and denote the collection of selected types plus the types t^{-i} and t^{+i} by T_i. Notice that $|T_i| \leq |\varphi|$. Let $t_0^i, \ldots, t_{|\varphi|-1}^i$ be an ordering of the types in T_i such that $t_0^i = t^{-i}$ (if T_i has cardinality $< |\varphi|$, then take some t from T_i more than once in this ordering.) Define a valuation $\mathfrak{V}$ by setting, for every propositional variable p,

$$\mathfrak{V}(p) = \{x_i \mid i \leq n_\varphi, x_i \models_{\mathfrak{V}'} p\} \cup \bigcup_{i<n_\varphi, j<|\varphi|} \{(y_i^j, y_i^{j+1}] \mid p \in t_j^i\} \cup \bigcup_{i<n_\varphi} \{(y_i^{|\varphi|}, x_{i+1}) \mid p \in t^{+i}\}.$$

To show that $\mathfrak{V}$ is as required, it suffices to show by induction for all $\psi \in cl(\varphi)$ and all $w \in [0, m_\varphi]$:

$w \models_{\mathfrak{V}} \psi \Leftrightarrow$ there is an $i \leq n_\varphi$ such that

(a) $w = x_i$ and $x_i \models_{\mathfrak{V}'} \psi$, or

(b) $w \in (y_i^\ell, y_i^{\ell+1}]$ and $\psi \in t_\ell^i$ for some $\ell < |\varphi|$, or

(c) $w \in (y_i^{|\varphi|}, x_{i+1})$ and $\psi \in t^{+i}$.

Let ψ and w be as above. The proof is by induction on the structure of ψ. The cases for propositional variables, $\neg$, and $\wedge$ are left to the reader. Consider the case for $\diamond_F^{[0,1]}$.
"$\Rightarrow$": Suppose $w \models_{\mathfrak{V}} \diamond_F^{[0,1]}\psi$. Then there is a $w' \in w + [0,1]$ such that $w' \models_{\mathfrak{V}} \psi$. Similarly to the proof of Lemma 6 above we assume first that $m_\varphi - w \geq 1$ and distinguish four cases:

- $w = x_i$ for some $i < n_\varphi$ and $w' = x_j$ for some j with $i \leq j \leq n_\varphi$. The induction hypothesis in (a) yields $x_j \models_{\mathfrak{Y}'} \psi$. From $x_j - x_i \leq 1$, it follows that $x_i \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$.
- $w = x_i$ for some $i < n_\varphi$ and $w' \in (x_j, x_{j+1})$ for some j with $i \leq j < n_\varphi$. If $w' \in (y_j^\ell, y_j^{\ell+1}]$ for some $\ell < |\varphi|$, then the induction hypothesis in (b) yields $\psi \in t_\ell^j$. Otherwise, i.e., if $w' \in (y_j^{|\varphi|}, x_{j+1})$, it holds that $\psi \in t^{+j}$ by the induction hypothesis in (c). Since t_ℓ^j, $t^{+j} \in T_j$, it follows by definition of T_j that there is a $w'' \in (x_j, x_{j+1})$ such that $w'' \models_{\mathfrak{Y}'} \psi$. By definition of the sequence $x_0, \ldots, x_{n_\varphi}$, there is an i' with $i < i' \leq n_\varphi$ such that $x_{i'} = x_i + 1$. But then $x_{j+1} \leq x_{i'}$; otherwise $x_j \geq x_{i'}$ and thus $w' \in (x_j, x_{j+1})$ contradicts $w' \in w + [0,1]$. Now from $w'' - w < 1$, it follows that $w \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$.
- $w \in (x_i, x_{i+1})$ for some $i < n_\varphi - 1$, and $w' = x_j$ for some j with $i < j \leq n_\varphi$. The induction hypothesis in (a) yields $x_j \models_{\mathfrak{Y}'} \psi$. From $x_j - w \leq 1$, it follows that $w \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$. But then, by definition of the sequence $x_0, \ldots, x_{n_\varphi}$, it holds that $w'' \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$ for all $w'' \in (x_i, x_{i+1})$. Therefore, $\Diamond_F^{[0,1]}\psi \in t'$ for each $t' \in T_i$. Hence, $\Diamond_F^{[0,1]}\psi \in t_\ell^i$ if $w \in (y_i^\ell, y_i^{\ell+1}]$ for some $\ell < |\varphi|$, and $\Diamond_F^{[0,1]}\psi \in t^{+i}$ if $w \in (y_i^{|\varphi|}, x_{i+1})$.
- $w \in (x_i, x_{i+1})$ for some $i < n_\varphi - 1$, and $w' \in (x_j, x_{j+1})$ for some j with $i \leq j < n_\varphi$. If $w' \in (y_j^\ell, y_j^{\ell+1}]$ for some $\ell < |\varphi|$, then the induction hypothesis in (b) yields $\psi \in t_\ell^j$. Otherwise, i.e., if $w' \in (y_j^{|\varphi|}, x_{j+1})$, it holds that $\psi \in t^{+j}$ by the induction hypothesis in (c). Since t_ℓ^j, $t^{+j} \in T_j$, it follows by definition of T_j that there is a $w'' \in (x_j, x_{j+1})$ such that $w'' \models_{\mathfrak{Y}'} \psi$. By definition of the sequence $x_0, \ldots, x_{n_\varphi}$, there is an $i' > i + 1$ such that $x_{i'} = x_{i+1} + 1$. But then $x_{j+1} \leq x_{i'}$; otherwise $x_j \geq x_{i'}$ and thus $w' \in (x_j, x_{j+1})$ contradicts $w' \in w + [0,1]$. Thus, there is a point $v \in (x_i, x_{i+1})$ such that $w'' - v \leq 1$. It follows that $v \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$. But then, by definition of the sequence $x_0, \ldots, x_{n_\varphi}$, it holds that $v' \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$ for all $v' \in (x_i, x_{i+1})$. Therefore, $\Diamond_F^{[0,1]}\psi \in t'$ for any $t' \in T_i$. Hence $\Diamond_F^{[0,1]}\psi \in t_\ell^i$ if $w \in (y_i^\ell, y_i^{\ell+1}]$ for some $\ell < |\varphi|$, and $\Diamond_F^{[0,1]}\psi \in t^{+i}$ if $w \in (y_i^{|\varphi|}, x_{i+1})$.

Now let $m_\varphi - w < 1$. Using the fact that $w'' \models_{\mathfrak{Y}} \neg p$ and $w'' \models_{\mathfrak{Y}'} \neg p$ for all $w'' > m_\varphi$ and all propositional variables p, the inductive proof can be carried out in almost the same way as above. We leave details to the reader.

"$\Leftarrow$": Let $i \leq n_\varphi$ and suppose one of the cases (a), (b), or (c) is satisfied. First, consider Case (a), i.e., suppose $w = x_i$ and $x_i \models_{\mathfrak{Y}'} \Diamond_F^{[0,1]}\psi$. Then there is a $w' \in x_i + [0,1]$ such that $w' \models_{\mathfrak{Y}'} \psi$. Distinguish three cases:

- $w' = x_j$ for some j with $i \leq j \leq n_\varphi$. The induction hypothesis in (a) yields $x_j \models_{\mathfrak{Y}} \psi$. From $x_j - x_i \leq 1$, it follows that $x_i \models_{\mathfrak{Y}} \Diamond_F^{[0,1]}\psi$.
- $w' \in (x_j, x_{j+1})$ for some j with $i \leq j < n_\varphi$. By definition of T_j, there is an $\ell < |\varphi|$ such that $t_\ell^j \in T_j$ and $\psi \in t_\ell^j$. Then the induction hypothesis in (b) yields $w'' \models_{\mathfrak{Y}} \psi$ for all $w'' \in (y_j^\ell, y_j^{\ell+1}]$. Fix such a w''. By definition

of the sequence $x_0, \ldots, x_\varphi$, there is an $i' > i$ such that $x_{i'} = x_i + 1$. But then $x_{j+1} \leq x_{i'}$; otherwise $x_j \geq x_{i'}$ and thus $w' \in (x_j, x_{j+1})$ contradicts $w' \in x_i + [0,1]$. Now, from $w'' - x_i \leq 1$, it follows that $x_i \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.

- $w' > m_\varphi$. $\mathfrak{V}$ and $\mathfrak{V}'$ coincide beyond m_φ. Hence, $w' \models_{\mathfrak{V}} \psi$ and it follows that $x_i \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.

Consider Case (b), i.e., suppose $w \in (y_i^\ell, y_i^{\ell+1}]$ and $\Diamond_F^{[0,1]}\psi \in t_\ell^i$ for some $\ell < |\varphi|$. By definition of T_i, there is a $w' \in (x_i, x_{i+1})$ such that $w' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. Then it follows by definition of the sequence $x_0, \ldots, x_{n_\varphi}$ that $w'' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$ for any $w'' \in (x_i, x_{i+1})$. In particular, $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. Then $v \models_{\mathfrak{V}'} \psi$ for some $v \in w + [0,1]$. Distinguish four cases:

- $v = x_j$ for some j with $i < j \leq n_\varphi$. The induction hypothesis in (a) yields $v \models_{\mathfrak{V}} \psi$. From $v - w \leq 1$, it follows that $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.
- $v \in (x_i, x_{i+1})$. By definition of T_i, there is a $t \in T_i$ such that $\psi \in t$. Distinguish two subcases: First, suppose that $\psi \in t_{\ell'}^i$ for some $\ell' \geq \ell$, or $\psi \in t^{+i}$. The induction hypothesis in (b) or (c) yields $v' \models_{\mathfrak{V}} \psi$ for all $v' \in (y_j^{\ell'}, y_j^{\ell'+1}]$, or all $v' \in (y_i^{|\varphi|}, x_{i+1})$, respectively. Then there is such a v' such that $v' - w < 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.
  Second, suppose there is no $\ell' \geq \ell$ such that $\psi \in t_{\ell'}^i$, and $\psi \notin t^{+i}$. Note that this implies $\ell > 0$. Since $\psi \notin t^{+i}$, there is an interval of the form (y, x_{i+1}) such that $y' \not\models_{\mathfrak{V}'} \psi$ for all $y' \in (y, x_{i+1})$. Take such a y'. Since $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$, it follows by definition of the sequence $x_0, \ldots, x_{n_\varphi}$ that $y' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]}\psi$. Then there is a $v' \in y' + [0,1]$ such that $v' \models_{\mathfrak{V}'} \psi$ and $v' \geq x_{i+1}$. By definition of the sequence $x_0, \ldots, x_{n_\varphi}$, there is an i' such that $x_{i'} = x_i + 1$. Consider only the case where $v' \in (x_j, x_{j+1})$ where $j = i'$; the other cases are straightforward. Note that there is no such $j > i'$. For suppose otherwise, it holds that $x_i + 1 < x_j < x_{i+1} + 1$. But then, by definition of the sequence $x_0, \ldots, x_{n_\varphi}$, there is a j' such that $x_{j'} = x_j - 1$. Thus $x_i < x_{j'} < x_{i+1}$; a contradiction. So, consider the case $j = i'$, i.e., $x_j = x_i + 1$. By definition of T_j, there is an $\ell' < |\varphi|$ such that $t_{\ell'}^j \in T_j$ and $\psi \in t_{\ell'}^j$. Then the induction hypothesis in (b) yields $v'' \models_{\mathfrak{V}} \psi$ for all $v'' \in (y_j^{\ell'}, y_j^{\ell'+1}]$. Take such a v''. From $\ell > 0$ and $\sigma_j \leq \frac{1}{|\varphi|} \times \sigma_i$ by definition of σ_j, it follows that $y_i^\ell + 1 \geq x_j + \sigma_j$. Then $y_j^{\ell'+1} - y_i^\ell < 1$ and thus $v'' - w < 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]}\psi$.
- $v \in (x_j, x_{j+1})$ for some j with $i < j < n_\varphi$. By definition of the sequence $x_0, \ldots, x_{n_\varphi}$, there is an i' such that $x_{i'} = x_i + 1$. Consider only the case where $j = i'$; the other cases are straightforward. Note that there is no such $j > i'$. For suppose otherwise, it holds that $x_i + 1 < x_j < x_{i+1} + 1$. But then, by definition of the sequence $x_0, \ldots, x_{n_\varphi}$, there is a j' such that $x_{j'} = x_j - 1$. Thus $x_i < x_{j'} < x_{i+1}$; a contradiction. So, consider the case $j = i'$, i.e., $x_j = x_i + 1$. Now, distinguish three subcases (recall that $w \in (y_i^\ell, y_i^{\ell+1}]$):
  - · $\ell = 0$ and $w' \models_{\mathfrak{V}'} \psi$ for some w' with $x_i < w' \leq x_j$. Then it is easy to see that there is a $v'' \geq w$ such that $v'' \models_{\mathfrak{V}} \psi$ and $v'' - w \leq 1$. Hence

$w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$.

- $\ell = 0$ and $w' \not\models_{\mathfrak{V}'} \psi$ for all w' with $x_i < w' \leq x_j$. Since $w \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$, it follows by definition of the sequence $x_0, \ldots, x_{n_\varphi}$ that $w'' \models_{\mathfrak{V}'} \Diamond_F^{[0,1]} \psi$ for all w'' with $x_i < w'' < w$. Take such a w''. Then there is a $v'' \in w'' + [0,1]$ such that $v'' \models_{\mathfrak{V}'} \psi$ and $v'' > x_j$. This implies that $\psi \in t^{-j} = t_0^j$. Then the induction hypothesis in (b) yields $v' \models_{\mathfrak{V}} \psi$ for all $v' \in (y_j^0, y_j^1]$. Clearly, there is such a v' such that $w - v' \leq 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$.
- $1 \leq \ell < |\varphi|$. Since $v \models_{\mathfrak{V}'} \psi$ and $v \in (x_j, x_{j+1})$, there is, by definition of T_j, an $\ell' < |\varphi|$ such that $t_{\ell'}^j \in T_j$ and $\psi \in t_{\ell'}^j$. The induction hypothesis in (b) yields $v' \models_{\mathfrak{V}} \psi$ for all $v' \in (y_j^{\ell'}, y_j^{\ell'+1}]$. Take such a v'. From $\sigma_j \leq \frac{1}{|\varphi|} \times \sigma_i$ by definition of σ_j, it follows that $y_i^\ell + 1 \geq x_j + \sigma_j$ and thus $v' - w < 1$. Hence $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$.

- $v > m_\varphi$. Then $v \models_{\mathfrak{V}} \psi$ if, and only if, $v \models_{\mathfrak{V}'} \psi$. Hence, $w \models_{\mathfrak{V}} \Diamond_F^{[0,1]} \psi$.

Case (c) with $w \in (y_i^{|\varphi|}, x_{i+1})$ and $\psi \in t^{+i}$ is similar to (b) and left to the reader. $\square$

Using Lemma 8, we can adapt the (non-deterministic polynomial time) algorithm for satisfiability without FVA to the FVA case: given a formula φ whose satisfiability with FVA is to be decided, we non-deterministically choose

- a type t_0 such that $\varphi \in t_0$;
- a type t_i, for every $1 \leq i \leq r_\varphi$;
- a type t_i', for every $i < r_\varphi$.

Intuitively, the type t_i is to be realized at point z_i, and the type t_i' in the interval (z_i, z_{i+1}). It remains to determine a set of rational linear inequalities which represent the truth conditions in models of the form described in Lemma 8. To this end, we modify the system of inequalities given in Figure 1 as follows: in Inequality (1) replace n_φ with r_φ and in (3) to (9) replace the set of types T_j, $j < r_\varphi$, with the singleton set $\{t_j'\}$, respectively. We obtain a modified system of inequalities in the variables $v_0, \ldots, v_{r_\phi}$. Then we check whether this system has a solution in $\mathbb{R}$. The algorithm returns 'φ is satisfiable' if there is a solution to this system of inequalities, and 'φ is not satisfiable' otherwise. Correctness of this modified decision procedure can be shown similarly to the case without FVA. We leave this exercise to the reader.

5 Conclusion

We have presented two complexity results for quantitative temporal logics over the real line: first, we have used a rather general method for reducing quantitative logics with binary coding of parameters to quantitative logics with unary coding of parameters to show that satisfiability in QTL^b without FVA is decidable in PSPACE. This result implies the known result that satisfiability in QTL^b with FVA is decidable in PSPACE. In the appendix, we have shown another application of this method by reproving that satisfiability in RTCTL is decidable in EXPTIME. The second complexity result determines the transition from NP to PSPACE for fragments of QTL. By a reduction to solvability of linear inequalities over the rationals, it is shown that the fragment of QTL with temporal operators 'sometime within n time units', with n coded in unary, is a maximal natural fragment of QTL whose satisfiability is still in NP. This result is proved with and without FVA.

An interesting open question is whether these results hold for other dense flows of time as well. For example, in various proofs we used the fact that the real line is Dedekind-complete. Thus, it is not obvious whether the complexity results presented here hold as well for the rational numbers.

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A ExpTime-completeness of RTCTL Reproved

We demonstrate the generality of the reduction technique proposed in Section 3 by applying it to a metric temporal logic that is based on discrete branching time. This logic is Real-Time Computational Tree Logic (RTCTL), which has been introduced by Emerson et al. as the extension of well-known CTL with a metric version of the until operator [8]. Since we are concerned with a discrete time framework, adding such an operator does not increase the expressive power. However, RTCTL is exponentially more succinct than CTL since the arguments to the metric operator are coded in binary. Nevertheless, Emerson and colleagues show that satisfiability in RTCTL is ExpTime-complete, and thus not more complex than satisfiability in CTL. In their proof, a tableau-based decision procedure is used for the upper bound. We reprove this upper bound in a simpler way using our reduction techniques. A similar (but easier) reduction can be used to show that the corresponding extension of the logic LTL based on discrete linear time is in PSpace. For the sake of completeness, we first introduce the syntax and semantics of RTCTL.

Definition 9 (RTCTL Syntax) *Let $p_0, p_1, \dots$ be a countably infinite supply of propositional variables.* RTCTL *formulas are built according to the syntax rule*

$$\varphi := p_i \ \mid \neg\varphi \mid \varphi \wedge \psi \mid E\bigcirc\varphi \mid E(\psi\,\mathcal{U}\,\varphi) \mid A(\psi\,\mathcal{U}\,\varphi) \mid E(\psi\mathcal{U}^{\leq k}\varphi) \mid A(\psi\mathcal{U}^{\leq k}\varphi)$$

where k denotes a natural number that is coded in binary. A CTL-formula *is an RTCTL formula that does not use the metric version of the until operator.*

The abbreviations $\top$, $\bot$, $\vee$, $\rightarrow$, and $\leftrightarrow$ are defined as usual. Moreover, we abbreviate $A\bigcirc\varphi = \neg E\bigcirc\neg\varphi$ and $A\Box\varphi = \neg E(\top\,\mathcal{U}\,\neg\varphi)$.

A *model* $\mathfrak{M} = \langle S, R, \mathfrak{V}\rangle$ is a triple consisting of a set of states S, a binary relation $R \subseteq S \times S$, and a valuation $\mathfrak{V}$ mapping every propositional variable p to a subset $\mathfrak{V}(p)$ of S. W.l.o.g., we assume that the graph (S, R) is a tree since every model can be unwound into a tree. Moreover, we assume that for every state, there is an R-successor. Given a state $w \in S$, a *w-fullpath* is an infinite sequence $u_0u_1 \cdots \in S^\omega$ of states such that $u_0 = w$ and $(u_i, u_{i+1}) \in R$ for all positions $i \geq 0$.

Definition 10 (RTCTL Semantics) *Let $\mathfrak{M} = \langle S, R, \mathfrak{V}\rangle$ be a model. Define the truth-relation '$\models$' of RTCTL inductively as follows: for all states $w \in S$,*

- $\mathfrak{M}, w \models p$ *iff* $w \in \mathfrak{V}(p)$*, for all propositional variables p;*
- $\mathfrak{M}, w \models \neg\varphi$ *iff* $\mathfrak{M}, w \not\models \varphi$*;*
- $\mathfrak{M}, w \models \psi \wedge \varphi$ *iff* $\mathfrak{M}, w \models \psi$ *and* $\mathfrak{M}, w \models \varphi$*;*
- $\mathfrak{M}, w \models E\bigcirc\varphi$ *iff there exists an R-successor v of w such that* $\mathfrak{M}, v \models \varphi$*;*

- $\mathfrak{M}, w \models E(\psi \,\mathcal{U}\, \varphi)$ *iff there exists a w-fullpath $u_0u_1\cdots$ and a position $i \geq 0$ such that $\mathfrak{M}, u_i \models \varphi$ and $\mathfrak{M}, u_j \models \psi$ for all positions $j < i$;*
- $\mathfrak{M}, w \models A(\psi \,\mathcal{U}\, \varphi)$ *iff for all w-fullpaths $u_0u_1\cdots$, there is a position $i \geq 0$ such that $\mathfrak{M}, u_i \models \varphi$ and $\mathfrak{M}, u_j \models \psi$ for all positions $j < i$;*
- $\mathfrak{M}, w \models E(\psi \mathcal{U}^{\leq k} \varphi)$ *iff there exists a w-fullpath $u_0u_1\cdots$ and a position $i \leq k$ such that $\mathfrak{M}, u_i \models \varphi$ and $\mathfrak{M}, u_j \models \psi$ for all positions $j < i$;*
- $\mathfrak{M}, w \models A(\psi \mathcal{U}^{\leq k} \varphi)$ *iff for all w-fullpaths $u_0u_1\cdots$, there is a position $i \leq k$ such that $\mathfrak{M}, u_i \models \varphi$ and $\mathfrak{M}, u_j \models \psi$ for all positions $j < i$.*

Our aim is to prove the following result:

Theorem 11 *Satisfiability in RTCTL is* ExpTime*-complete.*

The lower bound is an immediate consequence of the fact that CTL is a fragment of RTCTL, and the former is ExpTime-hard [7]. We prove a matching upper bound by a polynomial reduction to satisfiability of CTL, which is known to be in ExpTime [7].

The reduction is similar to the reduction presented in Section 3. In particular, the main idea is to replace subformulas of the forms $E(\psi \mathcal{U}^{\leq k} \varphi)$ and $A(\psi \mathcal{U}^{\leq k} \varphi)$ with binary counters that are implemented using propositional variables to represent the bits. However, there are also two significant differences: first, RTCTL is interpreted in discrete models, and thus it is not necessary to construct a 'grid' using variables x_i and y_i to measure the distance 'exactly one' as in the reduction for QTL. Second, RTCTL models are not linear, and therefore we cannot simply increment the value of a distance-measuring counter when going to a predecessor state. Instead, the value at this predecessor state is determined by incrementing the least or greatest counter value of its successor nodes, depending on whether we are simulating a formula $E(\psi \mathcal{U}^{\leq k} \varphi)$ or $A(\psi \mathcal{U}^{\leq k} \varphi)$. For identifying the least and greatest counter value among the successors, we use a marking procedure based on additional propositional variables. Before we describe this marking in detail, we fix a some notation.

Let φ be an RTCTL-formula whose satisfiability is to be decided. As an upper bound for the number of counter bits needed, let $n_c = \lceil \log_2(k+1) \rceil$ where k is the largest natural number occurring as a parameter to an until operator in φ. For simplicity, we assume w.l.o.g. that φ contains at least one subformula of the form $E(\psi \mathcal{U}^{\leq k} \varphi')$ and at least one subformula of the form $A(\psi \mathcal{U}^{\leq k} \varphi')$. Now, let $\chi_0, \ldots, \chi_{\ell'}$ be an enumeration of all subformulas of φ that are of the form $E(\psi \mathcal{U}^{\leq k} \varphi')$, and let $\chi_{\ell'+1}, \ldots, \chi_\ell$ be an enumeration of all subformulas of φ that are of the form $A(\psi \mathcal{U}^{\leq k} \varphi')$. If $\chi_i = Q(\psi \mathcal{U}^{\leq k} \varphi')$, $Q \in \{E, A\}$, for some $i \leq \ell$, we use ψ_i to denote ψ and φ_i to denote φ'. For the reduction, we use the following propositional variables:

- the bits of the i-th counter, $i \leq \ell$, are represented using propositional variables $c^i_{n_c-1}, \ldots, c^i_0$;

- to mark the bits of the i-th counter, $i \leq \ell$, we use propositional variables $m^i_{n_c-1}, \ldots, m^i_0$.

The marking procedure for finding the greatest counter value among the successors of a node can be described as follows. We mark bits of the counter in successor nodes by proceeding from the most ($n_c - 1$-st) to the least (0-th) significant bit. If s' is a successor of a node s and $i < n_c$, then s' is called *i-active* if the bits $n_c - 1, \ldots, i+1$ of s' are all marked, and *active* if all bits of s' are marked. Now, the i-th bit of s' is marked if and only if one of the following conditions is true:

(1) all i-active successors of s agree on the value of the i-th bit and s' is i-active;
(2) the i-active successors of s do not agree on the value of the i-th bit, s' is i-active, and the i-th bit of s' is one.

The result of this marking is that only those successors are active whose counter value is highest among all the successors of s. A corresponding marking scheme for finding the lowest value is obtained by changing the last part of the second condition to 'the i-th bit of s' is zero'. The marking of the i-th counter, $i \leq \ell$, can be implemented using the following formula ϑ^i_1. It marks highest values for $i \leq \ell'$ and smallest values for $i > \ell'$ (recall that ℓ' is such that $\chi_0, \ldots, \chi_{\ell'}$ are existentially path-quantified while $\chi_{\ell'+1}, \ldots, \chi_\ell$ are universally quantified). In the formula $(i \leq \ell')$ abbreviates $\top$ if $i \leq \ell'$, and $\bot$ otherwise. Moreover, act^i_t abbreviates $\bigwedge_{t<j<n_c} m^i_j$.

$$
\begin{aligned}
\vartheta^i_1 := \bigwedge_{t=0..n_c-1} \Big(& \big((A\bigcirc(act^i_t \to c^i_t) \vee A\bigcirc(act^i_t \to \neg c^i_t)) \\
& \qquad\qquad \to A\bigcirc(m^i_t \leftrightarrow act^i_t)\big) \wedge \\
& \big((E\bigcirc(act^i_t \wedge c^i_t) \wedge E\bigcirc(act^i_t \wedge \neg c^i_t) \wedge (i \leq \ell')) \\
& \qquad\qquad \to A\bigcirc(m^i_t \leftrightarrow (\neg c^i_t \wedge act^i_t))\big) \wedge \\
& \big((E\bigcirc(act^i_t \wedge c^i_t) \wedge E\bigcirc(act^i_t \wedge \neg c^i_t) \wedge (i > \ell')) \\
& \qquad\qquad \to A\bigcirc(m^i_t \leftrightarrow (c^i_t \wedge act^i_t))\big) \Big)
\end{aligned}
$$

We now inductively define a translation $(\cdot)^*$ of subformulas of φ to CTL-formulas, where the formula $(C_i \leq n)$ is defined as in Section 3:

$$
\begin{aligned}
p^* &:= p \\
(\neg\psi)^* &:= \neg\psi^* \\
(\psi_1 \wedge \psi_2)^* &:= \psi_1^* \wedge \psi_2^* \\
(E\bigcirc\psi)^* &:= E\bigcirc\psi^*
\end{aligned}
$$

$$(E(\psi_1 \mathcal{U}\, \psi_2))^* := E(\psi_1^* \mathcal{U}\, \psi_2^*)$$

$$(A(\psi_1 \mathcal{U}\, \psi_2))^* := A(\psi_1^* \mathcal{U}\, \psi_2^*)$$

$$(E(\psi_1 \mathcal{U}^{\leq k} \psi_2))^* := (C_i \leq k) \text{ if } \chi_i = E(\psi_1 \mathcal{U}^{\leq k} \psi_2)$$

$$(A(\psi_1 \mathcal{U}^{\leq k} \psi_2))^* := (C_i \leq k) \text{ if } \chi_i = A(\psi_1 \mathcal{U}^{\leq k} \psi_2)$$

It remains to properly update the counters, which is done by the following formulas, for $i \leq \ell$, where the formulas $(C_i \leq n)$ and $(C_i = n)$ are defined as in Section 3. Recall that $\chi_i = E(\psi_i \mathcal{U}^{\leq k} \varphi_i)$ if $i \leq \ell'$, and $\chi_i = A(\psi_i \mathcal{U}^{\leq k} \varphi_i)$ if $\ell' < i < \ell$. We use act^i to denote $m^i_{n_c-1} \wedge \cdots \wedge m^i_0$.

$$\vartheta_2^i := (C_i = 0) \leftrightarrow \varphi_i^*$$

$$\begin{aligned} \lambda := \; & \neg\psi_i^* \vee \\ & \Big((i \leq \ell') \wedge A\bigcirc(C_i = 2^{n_c} - 1)\Big) \vee \\ & \Big((i > \ell') \wedge E\bigcirc(C_i = 2^{n_c} - 1)\Big) \end{aligned}$$

$$\begin{aligned} \vartheta_3^i := \; & \Big((\neg\varphi_i^* \wedge \lambda) \rightarrow (C_i = 2^{n_c} - 1)\Big) \wedge \\ & \Bigg((\neg\varphi_i^* \wedge \neg\lambda) \rightarrow \bigvee_{t=0..n_c-1} \Bigg(c_t^i \wedge E\bigcirc(act^i \wedge \neg c_t^i) \wedge \bigwedge_{r=0..t-1} \Big(\neg c_r^i \wedge E\bigcirc(act^i \wedge c_r^i)\Big) \wedge \\ & \qquad \bigwedge_{t<r<n_c} \Big(c_r^i \leftrightarrow E\bigcirc(act^i \wedge c_r^i)\Big) \Bigg)\Bigg) \end{aligned}$$

Intuitively, ϑ_2^i initializes the counter and ϑ_3^i ensures that the counter of a node is obtained by incrementing the counter value of its active successors. Similarly to the reduction for QTL, the value $2^{n_c} - 1$ of the i-th counter is used to express that, on all paths (for existential path quantification) or some path (for universal path quantification), the formula φ_i is too far to be of any relevance. The value $2^{n_c} - 1$ is also used to indicate that ψ_i is false at some point on the way to the next occurrence of φ_i.

It is left to the reader to prove the following lemma, which finishes the reduction.

Lemma 12 *φ is satisfiable iff $\varphi^* \wedge \bigwedge_{i \leq \ell} A\Box(\vartheta_1^i \wedge \vartheta_2^i \wedge \vartheta_3^i)$ is satisfiable.*