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QUANTIZATION FOR A NONLINEAR DIRAC EQUATION

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Abstract. We study solutions of certain nonlinear Dirac-type equations on Riemann spin surfaces. We first improve an energy identity theorem for a sequence of such solutions with uniformly bounded energy in the case of a fixed domain. Then, we prove the corresponding energy identity in the case that the equations have constant coefficients and the domains possibly degenerate to a spin surface with only Neveu-Schwarz type nodes.

1. Introduction

Let M be a closed Riemann surface with a fixed spin structure. Let ΣM be the spinor bundle over M with a hermitian metric $\langle\cdot,\cdot\rangle_{\Sigma M}$ and a compatible spin connection ∇. Let $\partial\!\!\!/$ be the Dirac operator defined on $\Gamma(\Sigma M)$, i.e., $\partial\!\!\!/ := e_1\cdot\nabla_{e_1} + e_2\cdot\nabla_{e_2}$ for a local orthonormal frame $\{e_1, e_2\}$ of TM.

We consider the following nonlinear Dirac-type equation on M:

$$\partial\!\!\!/\psi = H_{jkl}\langle\psi^j,\psi^k\rangle\psi^l, \tag{1.1}$$

where $\psi = (\psi^1,\psi^2,...,\psi^d), \psi^i \in \Gamma(\Sigma M)$ and $H_{jkl} = (H^1_{jkl}, H^2_{jkl}, ..., H^d_{jkl}) \in C^\infty(M,\mathbb{C}^d)$.

Nonlinear Dirac equations of the form (1.1) appear naturally in geometry and physics. Firstly, consider the Dirac-harmonic map (ϕ,ψ) with curvature term introduced by Chen-Jost-Wang [7, 8], which was derived from the nonlinear supersymmetric σ-model of quantum field theory, then the nonlinear Dirac equation for the spinor filed ψ reduces to (1.1) with H being real valued, when ϕ is a constant map. Secondly, the generalized Weierstrass representation indicates that solutions to some Dirac equations of the form (1.1) can be used to express surfaces immersed in $\mathbb{R}^3$, $\mathbb{R}^4$ and some three-dimensional Lie groups: $SU(2)$, Nil, Sol, $\widetilde{SL}_2$ (see e.g. [16]). Thirdly, Ammann-Humbert considered a similar Dirac equation to study the first conformal Dirac eigenvalue [3].

In order to discuss some analytic aspects of the equation (1.1), we recall that the energy of $\psi \in \Gamma(\Sigma M)$ on a domain $\Omega \subset M$ is defined by

$$E(\psi,\Omega) = \int_\Omega |\psi|^4 dvol, \tag{1.2}$$

where $|\psi| := \langle\psi,\psi\rangle^{\frac{1}{2}}$. Note that (1.1) and (1.2) are conformally invariant.

Chen-Jost-Wang [8] developed the basic geometric analysis tools for blow-up analysis of the solutions of (1.1) and proved an energy identity for a sequence of smooth solutions on a fixed domain with small uniform energy bound. For the energy identities of two dimensional harmonic maps, Pseudo-holomorphic curves, we refer to [10, 14, 15, 18, 9]. For the regularity issue of (1.1), we refer to Wang [17], where any weak solution to (1.1) was shown to be smooth.

In this article, we will prove the energy identity without assuming the small uniform energy bound. More precisely, we have the following:

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Theorem 1.1. *Let M be a closed Riemann surface with a fixed spin structure, and suppose that ψ_n is a sequence of smooth solutions of* (1.1) *on M with uniformly bounded energy $E(\psi_n) = \int_M |\psi_n|^4 \leq \Lambda < \infty$. Then there exist finitely many blow-up points $\{x_1, x_2, ..., x_I\}$, a solution ψ on M to* (1.1) *and finitely many solutions $\xi^{i,l}$ on S^2 of* (1.1) *with $H \equiv H(x_i)$, $i = 1, 2, ..., I; l = 1, 2, ..., L_i$, such that, after selection of a subsequence, ψ_n converges in C^∞_{loc} to ψ on $M \setminus \{x_1, x_2, ..., x_I\}$ and the following holds:*

$$\lim_{n\to\infty} E(\psi_n) = E(\psi) + \sum_{i=1}^{I} \sum_{l=1}^{L_i} E(\xi^{i,l}). \tag{1.3}$$

Furthermore, we prove that the corresponding energy identity holds in the case that the domain converge to a possibly noncompact Riemann spin surface with all punctures (if there are any) of Neveu-Schwarz type.

Theorem 1.2. *Let $(M_n, c_n, \mathfrak{S}_n)$ be a sequence of closed Riemann spin surfaces of genus $g > 1$ with complex structures c_n and spin structures $\mathfrak{S}_n$. Assume that $(M_n, c_n, \mathfrak{S}_n)$ converges to a possibly noncompact Riemann spin surface $(M, c, \mathfrak{S})$ with only Neveu-Schwarz type punctures (if there are any). Let ψ_n be a sequence of smooth solutions of* (1.1) *on M_n with $H \equiv const$ and with uniformly bounded energy $E(\psi_n, M_n) \leq \Lambda < \infty$. Then there exist a solution ψ of* (1.1) *on $(\overline{M}, \overline{c}, \overline{\mathfrak{S}})$, where $(\overline{M}, \overline{c}, \overline{\mathfrak{S}})$ is the normalization of $(M, c, \mathfrak{S})$ and finitely many solutions ξ^k of* (1.1) *on S^2, $k = 1, 2, ..., K$, such that, after selection of a subsequence, the following holds:*

$$\lim_{n\to\infty} E(\psi_n) = E(\psi) + \sum_{k=1}^{K} E(\xi^k). \tag{1.4}$$

We remark that, in the simple case of $d = 1$ and $H \equiv 1$, the equation (1.1) becomes

$$\partial\!\!\!/ \psi = |\psi|^2 \psi. \tag{1.5}$$

It is well known that any solution to (1.5) represents a branched conformal immersion in $\mathbb{R}^3$ with constant mean curvature $H \equiv 1$ (see c.f. [16, 1]) and hence the concentrated energy in (1.3) and (1.4) can be explicitly quantized, i.e., in multiples of 4π.

2. Preliminaries

We collect some basic analytic properties for solutions of (1.1) proved in [8].

Theorem 2.1. *Let D be the unit disk. There exists a constant $\epsilon_0 > 0$ such that*

(1) (*ϵ-regularity*) *Let ψ be a smooth solution of* (1.1) *satisfying*

$$E(\psi, D) = \int_D |\psi|^4 < \epsilon_0.$$

Then, we have

$$||\psi||_{\widetilde{D},k,p} \leq C||\psi||_{D,0,4},$$

$\forall \widetilde{D} \subset\subset D$, $p > 1$ and $k \in \mathbb{Z}_+$, where $C = C(\widetilde{D}, k, p) > 0$ is a constant.

(2) (*Singularity removability*) *Let ψ be a smooth solution of* (1.1) *defined on $D \setminus \{0\}$ with the nontrivial spin structure. If*

$$E(\psi, D) = \int_D |\psi|^4 < \infty,$$

then ψ extends to a smooth solution of (1.1) *on the whole D.*

(3) *For any nontrivial solution ψ of* (1.1) *on S^2, we have*

$$E(\psi) = \int_{S^2} |\psi|^4 \geq \epsilon_0.$$

Remark 2.1. Theorem 2.1 was proved in [8] for equation (1.1) with real valued H as well as certain complex valued H (Section 5. in [8]). It is easy to check that the results hold true also in the case of general complex valued H.

For the notion of the nontriviality of a spin structure on an annulus or a cylinder, we refer to [2, 3, 4]. Following the terminology introduced by Jarvis-Kimura-Vaintrob [11], the puncture $\{0\}$ in (2) of Theorem 2.1 is said to be of Neveu-Schwarz type. If $D \setminus \{0\}$ is equipped with the trivial spin structure, then the puncture $\{0\}$ is said to be of Ramond type. See [21] for similar discussions.

Applying the analytic properties in Theorem 2.1, Chen-Jost-Wang [8] proved the following:

Theorem 2.2. *Let M be a closed Riemann surface with a fixed spin structure, and suppose that ψ_n is a sequence of smooth solutions of* (1.1) *on M with real valued H and with uniformly bounded energy $E(\psi_n) = \int_M |\psi_n|^4 \leq \Lambda < \infty$, and assume that ψ_n weakly converges to some ψ in $L^4(\Sigma M)$. Then the blow-up set*

$$S := \bigcap_{r>0} \left\{ x \in M \,|\, \liminf_{n\to\infty} \int_{D(x,r)} |\psi_n|^4 \geq \epsilon_0 \right\}$$

is a finite set of points $\{x_1, x_2, ..., x_I\}$, where ϵ_0 is as in Theorem 2.1. Furthermore, there exists a constant $c_0 > 0$ depending only on M such that if

$$\sup_{M,i,j,k,l} |H^i_{jkl}| \sqrt{\Lambda} < c_0, \tag{2.1}$$

then there are finitely many solutions of (1.1) *on S^2: $\xi^{i,l}, i = 1, 2, ..., I; l = 1, 2, ..., L_i$, after selection of a subsequence, ψ_n converges in C^∞_{loc} to ψ on $M \setminus \{x_1, x_2, ..., x_I\}$ and the following holds:*

$$\lim_{n\to\infty} E(\psi_n) = E(\psi) + \sum_{i=1}^{I} \sum_{l=1}^{L_i} E(\xi^{i,l}). \tag{2.2}$$

3. Energy identity

In this section, we will prove Theorem 1.1 and Theorem 1.2.

First, we recall the following lemma proved in [6] (see [8] for a different proof):

Lemma 3.1. *Let ψ be a solution of*

$$\partial\!\!\!/ \psi = f$$

on the unit disk D, with $\psi|_{\partial D} = \varphi$, and $f \in L^p(D), \varphi \in W^{1,p}(\partial D)$ for some $p > 1$, then

$$||\psi||_{D,1,p} \leq C(||f||_{D,0,p} + ||\varphi||_{\partial D,1,p}),$$

where $C = C(p) > 0$ is a constant.

Next, inspired by the proof of Theorem 4.2 in [8], we give the following lemma:

Lemma 3.2. *Let ψ be a smooth solution of* (1.1) *on the annulus $A_{r_1,r_2} := \{x \in \mathbb{R}^2 | r_1 \leq |x| \leq r_2\}$, where $0 < r_1 < 2r_1 < r_2/2 < r_2 < 1$ and assume that*

$$\sup_{A_{r_1,r_2},i,j,k,l} |H^i_{jkl}| \leq h_0 < \infty.$$

Then we have

$$
\begin{aligned}
\Big(\int_{A_{2r_1,r_2/2}} |\psi|^4\Big)^{\frac{1}{4}} &\leq C_0\Big(\int_{A_{r_1,r_2}} |\psi|^4\Big)^{\frac{1}{2}}\Big(\int_{A_{r_1,r_2}} |\psi|^4\Big)^{\frac{1}{4}} \\
&\quad + C\Big(\int_{A_{r_1,2r_1}} |\psi|^4\Big)^{\frac{1}{4}} + C\Big(\int_{A_{r_2/2,r_2}} |\psi|^4\Big)^{\frac{1}{4}}, \qquad (3.1)\\
\Big(\int_{A_{2r_1,r_2/2}} |\nabla\psi|^{\frac{4}{3}}\Big)^{\frac{3}{4}} &\leq C_0\Big(\int_{A_{r_1,r_2}} |\psi|^4\Big)^{\frac{1}{2}}\Big(\int_{A_{r_1,r_2}} |\psi|^4\Big)^{\frac{1}{4}} \\
&\quad + C\Big(\int_{A_{r_1,2r_1}} |\psi|^4\Big)^{\frac{1}{4}} + C\Big(\int_{A_{r_2/2,r_2}} |\psi|^4\Big)^{\frac{1}{4}}, \qquad (3.2)
\end{aligned}
$$

where C_0, C are positive constants that do not depend on r_1, r_2 and $C_0 = C_0(h_0)$ depends on h_0.

Proof. We will prove this lemma using some arguments from [8]. Let D be the unit disk. Choose a cut-off function $\eta \in [0,1]$ on D satisfying

$$\eta \in C_0^\infty(A_{r_1,r_2}) \quad ; \qquad \eta \equiv 1 \text{ in } A_{2r_1,r_2/2} \qquad (3.3)$$

$$|\nabla\eta| \leq 4/r_1 \text{ in } A_{r_1,2r_1} \quad ; \qquad |\nabla\eta| \leq 4/r_2 \text{ in } A_{r_2/2,r_2}. \qquad (3.4)$$

Then by the equation (1.1) and Lemma 3.1, we have

$$
\begin{aligned}
\|\eta\psi\|_{D,1,4/3} &\leq C\|\partial\!\!\!/(\eta\psi)\|_{D,0,4/3} \\
&\leq C\|\eta\partial\!\!\!/\psi\|_{D,0,4/3} + C\||\nabla\eta||\psi|\|_{D,0,4/3} \\
&\leq Ch_0\|\psi\|^2_{A_{r_1,r_2},0,4}\|\eta\psi\|_{D,0,4} + C\||\nabla\eta||\psi|\|_{D,0,4/3}. \qquad (3.5)
\end{aligned}
$$

It follows from (3.4) and Cauchy inequality that

$$
\begin{aligned}
\||\nabla\eta||\psi|\|_{D,0,4/3} &\leq \||\nabla\eta||\psi|\|_{A_{r_1,2r_1},0,4/3} + \||\nabla\eta||\psi|\|_{A_{r_2/2,r_2},0,4/3} \\
&\leq C\|\psi\|_{A_{r_1,2r_1},0,4} + C\|\psi\|_{A_{r_2/2,r_2},0,4}. \qquad (3.6)
\end{aligned}
$$

In view of (3.3), we conclude from the Sobolev embedding theorem that

$$\|\psi\|_{A_{2r_1,r_2/2},0,4} + \|\psi\|_{A_{2r_1,r_2/2},1,4/3} \leq \|\eta\psi\|_{D,0,4} + \|\eta\psi\|_{D,1,4/3} \leq 2\|\eta\psi\|_{D,1,4/3}. \qquad (3.7)$$

Combining (3.5), (3.6) and (3.7) gives (3.1) and (3.2). □

Now, let us recall the conformal transformation between an annulus and a cylinder (c.f. [21]). Let (r,θ) be the polar coordinates of $\mathbb{R}^2$ centered at 0 and $h_{eucl} = dr^2 + r^2 d\theta^2$ be the Euclidean metric on $\mathbb{R}^2$. Equip the cylinder $\mathbb{R}^1 \times S^1$ with the metric $ds^2 = dt^2 + d\theta^2$, where $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Then the following map $f : \mathbb{R}^1 \times S^1 \to \mathbb{R}^2$

$$r = e^{-t}, \quad \theta = \theta, \quad (t,\theta) \in \mathbb{R}^1 \times S^1. \qquad (3.8)$$

is a conformal transformation. One can verify that

$$f^* h_{eucl} = e^{-2t} ds^2.$$

Given $r_1 > r_2$, then, the annulus $A_{r_1,r_2} := \{re^{i\theta} | r_2 \leq r \leq r_1\}$ is mapped to the cylinder $P_{t_1,t_2} := [t_1,t_2] \times S^1$, where $t_i = -\log r_i$, $i = 1,2$.

Let ψ be a solution of (1.1) defined on the annulus $A_{r_1,r_2} \subset \mathbb{R}^2$. Set

$$\Psi := e^{-\frac{t}{2}} f^*\psi.$$

Then by the conformal invariance of (1.1), Ψ is a solution of (1.1) defined on the cylinder $P_{t_1,t_2} \subset \mathbb{R}^1 \times S^1$.

Denote by $P_{T_1,T_2} = [T_1, T_2] \times S^1$ a cylinder with metric $ds^2 = dt^2 + d\theta^2$ and with the spin structure being nontrivial along the boundary curves. Then we have the following cylindrical version of Lemma 3.2:

Lemma 3.3. *Let Ψ be a smooth solution of* (1.1) *on P_{T_1,T_2}, where $T_2 - 1 > T_1 + 1 > 1$. Assume that*

$$\sup_{P_{T_1,T_2}, i,j,k,l} |H^i_{jkl}| \leq h_0 < \infty.$$

Then we have

$$\begin{aligned}(\int_{P_{T_1+1,T_2-1}} |\Psi|^4)^{\frac{1}{4}} &\leq C_0(\int_{P_{T_1,T_2}} |\Psi|^4)^{\frac{1}{2}}(\int_{P_{T_1,T_2}} |\Psi|^4)^{\frac{1}{4}} \\ &\quad + C(\int_{P_{T_2-1,T_2}} |\Psi|^4)^{\frac{1}{4}} + C(\int_{P_{T_1,T_1+1}} |\Psi|^4)^{\frac{1}{4}}, \end{aligned} \tag{3.9}$$

$$\begin{aligned}(\int_{P_{T_1+1,T_2-1}} |\nabla\Psi|^{\frac{4}{3}})^{\frac{3}{4}} &\leq C_0(\int_{P_{T_1,T_2}} |\Psi|^4)^{\frac{1}{2}}(\int_{P_{T_1,T_2}} |\Psi|^4)^{\frac{1}{4}} \\ &\quad + C(\int_{P_{T_2-1,T_2}} |\Psi|^4)^{\frac{1}{4}} + C(\int_{P_{T_1,T_1+1}} |\Psi|^4)^{\frac{1}{4}}, \end{aligned} \tag{3.10}$$

where C_0, C are positive constants that do not depend on T_1, T_2 and $C_0 = C_0(h_0)$ depends on h_0.

Proof. Applying the conformal transformation (3.8) to Lemma 3.2, then, (3.9), (3.10) are direct consequences of (3.1), (3.2). □

Lemma 3.4. *Given a cylinder P_{T_1-1,T_2+1} and assume that*

$$\sup_{P_{T_1-1,T_2+1}, i,j,k,l} |H^i_{jkl}| \leq h_0 < \infty.$$

Then there exists $\epsilon_1 = \epsilon_1(h_0) > 0$ such that if Ψ is a smooth solution of (1.1) *defined on P_{T_1-1,T_2+1} and*

$$\int_{P_{T_1-1,T_2+1}} |\Psi|^4 \leq \Lambda < \infty, \tag{3.11}$$

$$\omega := \sup_{t\in[T_1-1,T_2]} \int_{[t,t+1]\times S^1} |\Psi|^4 \leq \epsilon_1, \tag{3.12}$$

then

$$\int_{P_{T_1,T_2}} |\Psi|^4 + \int_{P_{T_1,T_2}} |\nabla\Psi|^{\frac{4}{3}} \leq C(h_0, \Lambda)\omega^{\frac{1}{3}}. \tag{3.13}$$

Here, $C(h_0, \Lambda)$ is a constant depending only on h_0 and Λ, but not on T_1, T_2.

Proof. Let $\epsilon_1 = \min\{\frac{1}{8C_0^2}, 1\}$, where $C_0 > 0$ is the constant in Lemma 3.3. Then by assumption (3.12), we have

$$\sup_{t\in[T_1-1,T_2]} \int_{[t,t+1]\times S^1} |\Psi|^4 \leq \epsilon_1 \leq \frac{1}{8C_0^2}. \tag{3.14}$$

Note that $\mu(t) := \int_{[T_1,t]\times S^1} |\Psi|^4$ is a continuous and nondecreasing function defined on $[T_1, T_2]$ and the energy of Ψ over P_{T_1-1,T_2+1} is bounded by Λ. With similar arguments as in [19]

(Theorem 3.5, p. 134), we can separate P_{T_1,T_2} into finitely many parts as follows (c.f. [21], Lemma 3.3)

$$P_{T_1,T_2} = \bigcup_{n=1}^{N_0} P^n, P^n := [T^{n-1}, T^n] \times S^1, T^0 = T_1, T^{N_0} = T_2$$

such that N_0 is an integer no larger than $[8C_0^2\Lambda] + 1$, and the following hold:

$$E(\Psi; P^n) \leq \frac{1}{4C_0^2}, \quad n = 1, 2, ..., N_0. \tag{3.15}$$

Applying Lemma 3.3 to each part P^n gives

$$\begin{aligned}(\int_{P^n} |\Psi|^4)^{\frac{1}{4}} \quad \leq \quad & C_0(\int_{[T^{n-1}-1,T^n+1]\times S^1} |\Psi|^4)^{\frac{1}{2}}(\int_{[T^{n-1}-1,T^n+1]\times S^1} |\Psi|^4)^{\frac{1}{4}} \\ & + C(\int_{[T^{n-1}-1,T^{n-1}]\times S^1} |\Psi|^4)^{\frac{1}{4}} + C(\int_{[T^n,T^n+1]\times S^1} |\Psi|^4)^{\frac{1}{4}} \end{aligned} \tag{3.16}$$

It follows from the definition of ω (see (3.12)) that

$$(\int_{P^n} |\Psi|^4)^{\frac{1}{4}} \leq C_0((\int_{P^n} |\Psi|^4)^{\frac{1}{2}} + \omega^{\frac{1}{2}} + \omega^{\frac{1}{2}})((\int_{P^n} |\Psi|^4)^{\frac{1}{4}} + \omega^{\frac{1}{4}} + \omega^{\frac{1}{4}}) + \omega^{\frac{1}{4}} + \omega^{\frac{1}{4}}. \tag{3.17}$$

By the energy bound (3.11), we have

$$(\int_{P^n} |\Psi|^4)^{\frac{1}{4}} \leq C_0(\int_{P^n} |\Psi|^4)^{\frac{1}{2}}(\int_{P^n} |\Psi|^4)^{\frac{1}{4}} + C(h_0, \Lambda)(\omega^{\frac{1}{4}} + \omega^{\frac{1}{2}} + \omega^{\frac{3}{4}}). \tag{3.18}$$

Here $C(h_0, \Lambda)$ depends on h_0 and Λ. From (3.15), we can rewrite (3.18) as follows:

$$(\int_{P^n} |\Psi|^4)^{\frac{1}{4}} \leq C(h_0, \Lambda)(\omega^{\frac{1}{4}} + \omega^{\frac{1}{2}} + \omega^{\frac{3}{4}}). \tag{3.19}$$

Since $\epsilon_1 \leq 1$, by assumption (3.12), we get

$$\omega := \sup_{t\in[T_1-1,T_2]} \int_{[t,t+1]\times S^1} |\Psi|^4 \leq \epsilon_1 \leq 1.$$

Hence, we conclude from (3.19) that

$$(\int_{P^n} |\Psi|^4)^{\frac{1}{4}} \leq C(h_0, \Lambda)(\omega^{\frac{1}{4}} + \omega^{\frac{1}{2}} + \omega^{\frac{3}{4}}) \leq C(h_0, \Lambda)\omega^{\frac{1}{4}}.$$

With similar arguments, we have (by (3.10) in Lemma 3.3)

$$(\int_{P^n} |\nabla\Psi|^{\frac{4}{3}})^{\frac{3}{4}} \leq C(h_0, \Lambda)\omega^{\frac{1}{4}}.$$

Summing up the above estimates on P^n gives

$$\int_{P_{T_1,T_2}} |\Psi|^4 = \sum_{n=1}^{N_0} \int_{P^n} |\Psi|^4 \leq C(h_0, \Lambda)N_0\omega \leq C(h_0, \Lambda)\omega^{\frac{1}{3}} \tag{3.20}$$

and

$$\int_{P_{T_1,T_2}} |\nabla\Psi|^{\frac{4}{3}} = \sum_{n=1}^{N_0} \int_{P^n} |\nabla\Psi|^{\frac{4}{3}} \leq C(h_0, \Lambda)N_0\omega^{\frac{1}{3}} \leq C(h_0, \Lambda)\omega^{\frac{1}{3}}. \tag{3.21}$$

(3.13) follows immediately from combining (3.20) and (3.21). □

Applying Lemma 3.4, we show Theorem 1.1.

Proof of Theorem 1.1: The uniform energy bound $E(\psi_n) = \int_M |\psi_n|^4 \le \Lambda < \infty$ implies that ψ_n weakly subconverges to some ψ in $L^4(\Sigma M)$. By a standard covering argument and ϵ-regularity, there exist finitely many blow-up points $\{x_1, x_2, ..., x_I\}$ such that, after passing to subsequences, ψ_n converges in C^∞_{loc} to ψ on $M \setminus \{x_1, x_2, ..., x_I\}$. It follows from the smoothness of ψ_n and the singularity removability that ψ extends to a smooth solution of (1.1) on M.

To prove the energy identity (1.3), we only need to consider the case that $I = 1$ and $L_1 = 1$, because the general case can be reduced to the simplest case by induction. Following the arguments and notations as in the proof of Theorem 4.2 in [8] (see Theorem 3.6 in [5] for similar arguments), we only need to show that

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{n\to\infty} E(\Psi_n, P_{T_0,T_n}) = 0, \tag{3.22}$$

where $P_{T_0,T_n} = [T_0, T_n] \times S^1, T_0 := |\log\delta|, T_n := |\log \lambda_n R|, \delta > 0, R > 0$. Here, Ψ_n are induced from the solutions ψ_n on anuli near the blow-up point under a conformal transformation (c.f. Theorem 4.2 in [8]) and hence Ψ_n are smooth solutions of (1.1) on P_{T_0-1,T_n+1} with corresponding $\widetilde{H}$ satisfying

$$\max_{n,i,j,k,l}\left\{|\widetilde{H}^i_{jkl}|(x) : x \in P_{T_0-1,T_n+1}\right\} \le \max_{i,j,k,l}\left\{|H^i_{jkl}|(x) : x \in M\right\} \le C < +\infty.$$

Moreover, through a standard argument by contradiction, one can prove that

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{n\to\infty}\sup_{t\in[T_0-1,T_n]}\int_{[t,t+1]\times S^1} |\Psi_n|^4 = 0,$$

On the other hand, we have

$$\int_{P_{T_0-1,T_n+1}} |\Psi_n|^4 \le E(\psi_n, M_n) \le \Lambda < \infty.$$

Then we can apply Lemma 3.4 to conclude that

$$\lim_{R\to\infty}\lim_{\delta\to 0}\lim_{n\to\infty}\Big(\int_{P_{T_0,T_n}} |\Psi|^4 + \int_{P_{T_0,T_n}} |\nabla\Psi|^{\frac{4}{3}}\Big) = 0. \tag{3.23}$$

In particular, (3.22) holds. This completes the proof. □

Now, we consider a sequence of smooth solutions of (1.1) on long spin cylinders under certain assumptions and give the following proposition, which is analogous to the cases of harmonic maps and Dirac-harmonic maps (c.f. Proposition 3.1 in [20] and Proposition 3.1 in [21]). The scheme of the proof is similar to the neck analysis for certain approximate harmonic maps by Ding-Tian [9].

Proposition 3.1. *Let Ψ_n be a sequence of smooth solutions of* (1.1) *defined on P_n, where $P_n = [T_n^1, T_n^2]\times S^1$ equipped with the nontrivial spin structure. Suppose that there is a constant $C > 0$ such that*

$$\sup_{P_n,i,j,k,l} |H^i_{jkl}| \le C < +\infty.$$

Assume that:

(1)
$$1 \ll T_n^1 \ll T_n^2, \tag{3.24}$$

(2)
$$E(\Psi_n, P_n) \leq \Lambda < \infty, \tag{3.25}$$
(3)
$$\lim_{n\to\infty} \omega(\Psi_n, P_{T_n^1, T_n^1+R}) = \lim_{n\to\infty} \omega(\Psi_n, P_{T_n^2-R, T_n^2}) = 0, \quad \forall R \geq 1, \tag{3.26}$$
where
$$\omega(\Psi, P_{T_1,T_2}) := \sup_{t\in[T_1,T_2-1]} \int_{[t,t+1]\times S^1} |\Psi|^4.$$

Then there are finitely many solutions of (1.1) *on* S^2*:* $\zeta^{j,l}$*,* $l = 1, 2, ..., L_j$*;* $j = 1, 2, ..., K$*, such that after selection of a subsequence of* (Ψ_n, P_n)*, the following holds:*
$$\lim_{n\to\infty} E(\Psi_n, P_n) = \sum_{j=1}^{K} \sum_{l=1}^{L_j} E(\zeta^{j,l}). \tag{3.27}$$

Proof. In view of Theorem 1.1 and Theorem 2.1, with similar arguments as in [20] (Proposition 3.1), we can decompose P_n into neck domains $\cup_{i=0}^{K} I_n^i$ and bubble domains $\cup_{j=1}^{K} J_n^j$ (take subsequences if necessary):
$$P_n = \cup_{i=0}^{K} I_n^i \cup_{j=1}^{K} J_n^j, \tag{3.28}$$
where K is independent of n. Furthermore, we have

(1) For each i, $\lim_{n\to\infty} \omega(\Psi_n, I_n^i) = 0$.

(2) For each j, there are finitely many solutions of (1.1) on S^2: $\zeta^{j,l}, l = 1, 2, ..., L_j$, such that:
$$\lim_{n\to\infty} E(\Psi_n, J_n^j) = \sum_{l=1}^{L_j} E(\zeta^{j,l}). \tag{3.29}$$

Note that, here, some bubbles (solutions of (1.1) on $\mathbb{R} \times S^1$) corresponding to collapsing homotopically nontrivial simple closed curves on P_n can possibly appear. Therefore, in order to apply the singularity removability result - Theorem 2.1 (2), the nontriviality of the spin structures along P_n should be required (see Proposition 3.1 in [21] for similar discussions).

We need to verify that, in the limit, the necks $\Psi_n : I_n^i \to N, i = 0, 1, ..., K$ contain no energy. It is not difficult to verify that, after passing to subsequences, the local energy of Ψ_n over a small neighborhood of the two boundary components of I_n^i can be arbitrary small. Then, applying Lemma 3.4 gives
$$\sum_{i=0}^{K} E(\Psi_n, I_n^i) \leq C(\Lambda) \sum_{i=0}^{K} (\omega(\Psi_n, I_n^i))^{\frac{1}{3}} \to 0, \quad n \to \infty. \tag{3.30}$$
(3.27) follows from combining (3.29) and (3.30). □

Now, we shall use Proposition 3.1 to prove Theorem 1.2.

Proof of Theorem 1.2: Recall that any closed surface of genus $g > 1$ is of general type (c.f. [20]). For each n, let h_n be the hyperbolic metric on M_n compatible with the complex structure c_n. As discussed in [20, 21], we can assume that (M_n, h_n, c_n) converges to a hyperbolic Riemann surface (M, h, c) by collapsing a possibly empty collection of finitely many pairwise disjoint simple closed geodesics $\{\gamma_n^j, j \in J\}$ on M_n. Note that $0 \leq |J| \leq 3g-3$. For each j, the geodesics γ_n^j degenerate into a pair of punctures $(\mathcal{E}^{j,1}, \mathcal{E}^{j,2})$ and $l_n^j := \text{length}(\gamma_n^j) \to 0$ as $n \to \infty$. Let P_n^j be the standard cylindrical collar about γ_n^j (c.f. [20]), namely
$$P_n^j = \left[\frac{2\pi}{l_n^j} \arctan(\sinh(\frac{l_n^j}{2})), \frac{2\pi}{l_n^j}(\pi - \arctan(\sinh(\frac{l_n^j}{2})))\right] \times S^1$$

with metric $ds^2 = (\frac{l_n^j}{2\pi \sin \frac{l_n^j t}{2\pi}})^2(dt^2 + d\theta^2)$. Let $\tau_n : M \to M_n \setminus \cup_{j\in J}\gamma_n^j$ be the corresponding diffeomorphisms realizing the convergence (c.f. [20]). Let $(\overline{M}, \overline{c})$ be the normalization of (M, c).

Moreover, by taking subsequences, we can assume that τ_n is compatible with the spin structures $\mathfrak{S}_n$, namely, the pull-back spin structure on the limit surface M is fixed. We denote it by $\mathfrak{S}$. In particular, for each j, $\mathfrak{S}$ is nontrivial or trivial along the pair of punctures $(\mathcal{E}^{j,1}, \mathcal{E}^{j,2})$ if and only if $\mathfrak{S}_n$ is nontrivial or trivial along the geodesic γ_n^j for all n. By assumption, all punctures of the limit spin surface $(M, \mathfrak{S})$ are of Neveu-Schwarz type. It is equivalent to say that the spin structure $\mathfrak{S}$ on M is nontrivial around all punctures of M. Thus, $\mathfrak{S}$ extends to some spin structure $\overline{\mathfrak{S}}$ on $\overline{M}$ (c.f. [2, 4, 21]).

As in [21] (see [13] for a more detailed explanation), by pulling back the geometric data via the diffeomorphisms τ_n, we can fix the spinor bundle ΣM and think of the hyperbolic metrics and the compatible complex structures (h_n, c_n) as all living on the limit surface M and converging in C^∞_{loc} to (h, c). Let ∇_n be the connection on ΣM coming from h_n and ∇ the connection on ΣM coming from h. Then, we can consider ψ_n as a sequence of solutions of (1.1) defined on $(M, h_n, c_n, \mathfrak{S})$ with respect to (c_n, ∇_n).

Note that all estimates in Theorem 2.1 and Theorem 1.1 are uniform for the metrics h_n and the complex structures c_n. With similar arguments as in [20] (Theorem 1.1) and [21] (Theorem 1.1, Theorem 1.2), we can apply Theorem 1.1 and Theorem 2.1 to prove that there exist finitely many blow-up points $\{x_1, x_2, ..., x_I\}$ which are away from the punctures $\{(\mathcal{E}^{j,1}, \mathcal{E}^{j,2}), j \in J\}$ and finitely many smooth solutions of (1.1) on S^2: $\xi^{i,l}$, $l = 1, 2, ..., L_i$, near the i-th blow-up point x_i; a smooth solution ψ of (1.1) on $(\overline{M}, \overline{c}, \overline{\mathfrak{S}})$, such that, after selection of a subsequence, the following holds:

$$\lim_{n\to\infty} E(\psi_n) = E(\psi) + \sum_{i=1}^{I}\sum_{l=1}^{L_i} E(\xi^{i,l}) + \sum_{j\in J}\lim_{\delta\to 0}\lim_{n\to\infty} E(\psi_n, P_n^{j,\delta}), \tag{3.31}$$

where $P_n^{j,\delta}$ is the δ-subcollars of P_n^j, for $\delta \in \left[\frac{l_n^j}{2}, \operatorname{arcsinh}(1)\right]$ (see the proof of Theorem 1.1 in [20]), namely,

$$P_n^{j,\delta} := [T_n^{1,j,\delta}, T_n^{2,j,\delta}] \times S^1 \subseteq P_n^j,$$

where

$$T_n^{1,j,\delta} = \frac{2\pi}{l_n^j}\arcsin\left(\frac{\sinh(\frac{l_n^j}{2})}{\sinh\delta}\right), \quad T_n^{2,j,\delta} = \frac{2\pi^2}{l_n^j} - \frac{2\pi}{l_n^j}\arcsin\left(\frac{\sinh(\frac{l_n^j}{2})}{\sinh\delta}\right).$$

In fact, for each fixed n and each fixed $\delta \in \left[\frac{l_n^j}{2}, \operatorname{arcsinh}(1)\right]$, $P_n^{j,\delta}$ is exactly the j-th component of the δ-thin part of the hyperbolic surface (M_n, h_n).

To capture the concentrated energy at the punctures, i.e.,

$$\sum_{j\in J}\lim_{\delta\to 0}\lim_{n\to\infty} E(\psi_n, P_n^{j,\delta}),$$

we shall apply Proposition 3.1. By conformal invariance of the equation (1.1) and the energy functional (1.2), we equip P_n^j with the Euclidean metric. Then applying similar arguments as in [20] (Theorem 1.1) and [21] (Theorem 1.2), we can use Proposition 3.1 to show that there exist finitely many smooth solutions of (1.1) on S^2: $\zeta^{j,k}, k = 1, 2, ..., K_j, j \in J$, such that, after selection of a subsequence of (ψ_n, M_n), we have

$$\lim_{\delta\to 0}\lim_{n\to\infty} E(\psi_n, P_n^{j,\delta}) = \sum_{k=1}^{K_j} E(\zeta^{j,k}), \quad j \in J. \tag{3.32}$$

Finally, combining (3.31) and (3.32) gives the energy identity (1.4). Thus, we have finished the proof. □

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