Open access • Journal Article • DOI:10.1007/S00023-018-0651-0

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Sven Bachmann, Alex Bols, Wojciech De Roeck, Martin Fraas ...+1 more authors

Institutions: University of British Columbia, Katholieke Universiteit Leuven, Virginia Tech

Published on: 20 Feb 2018 - Annales Henri Poincaré (Springer International Publishing)

Topics: Torus, Curvature and Quantum

QUANTIZATION OF CONDUCTANCE IN GAPPED INTERACTING SYSTEMS

SVEN BACHMANN, ALEX BOLS, WOJCIECH DE ROECK, AND MARTIN FRAAS

Abstract. We provide a short proof of the quantisation of the Hall conductance for gapped interacting quantum lattice systems on the two-dimensional torus. This is not new and should be seen as an adaptation of the proof of [1], simplified by making the stronger assumption that the Hamiltonian remains gapped when threading the torus with fluxes. We argue why this assumption is very plausible. The conductance is given by Berry's curvature and our key auxiliary result is that the curvature is asymptotically constant across the torus of fluxes.

1. Setup and Results

1.1. **Preamble.** It is now common lore that the remarkable precision of the plateaus appearing in Hall measurements at low temperatures is explained by linking the Hall conductance with a topological invariant. For translationally invariant, non-interacting systems, it is the Chern number of the ground state bundle over the Brillouin zone, and the connection can be established by a direct computation. In interacting systems, the Brillouin zone is replaced by a torus associated with fluxes threading the system. In [2] Avron and Seiler prove quantisation of the Hall conductance in this framework with an assumption regarding constancy (in the flux parameter) of the adiabatic curvature of the ground state bundle. Thouless, Niu and Wu give in [3] a similar argument. In both cases, the assumption can be replaced by an averaging over the flux torus. The averaging is also analogous to the adiabatic insertion of a quantum unit of flux in Laughling's argument [4]. Furthermore, Thouless and Niu argue in [5] for a certain locality of the Hall conductance, of which the argument in the present article is reminiscent. Proving these assumptions, or bypassing them, was considered an open problem [6], and was resolved only thirty years later by Hastings and Michalakis in [1] by relying on a crucial locality estimate.

The present paper gives a streamlined and expository version of the proof in [1], presenting also a result in the thermodynamic limit. Our version is shorter, at the cost of making a stronger assumption. Indeed, we assume that the gap remains open for the system threaded with fluxes. This is a prerequisite to even speak about the adiabatic curvature on the torus of fluxes, cf. the framework discussed above. It is quite remarkable that [1] does not need this assumption by bypassing the notion of adiabatic curvature. A recent related work of Giuliani, Mastropietro and Porta [7] yields a similar result, namely the quantisation of the Hall conductance for interacting electrons in the thermodynamic limit, but restricted to weak interactions. They also bypass the geometric picture in favour of Ward identities and constructive quantum field theory.

1.2. **Quantum lattice systems.** We consider a two dimensional discrete torus Γ with L^2 sites, which we identify with a square $[0, L) \times [0, L) \cap \mathbb{Z}^2$ whose edges are glued together. For simplicity we assume that L is even. A finite-dimensional Hilbert space $\mathcal{H}_x$ is associated to each site of the torus and for a subset X of the torus we define $\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x$. The evolution of the system is

Date: November 20, 2017.

governed by a finite range Hamiltonian

$$H = \sum_{X \subset \Gamma} \Phi(X), \quad \Phi(X) \in \mathcal{B}(\mathcal{H}_X)$$

that is assumed to be *gapped*, see below. By finite range, we mean that

$$\Phi(X) = 0 \quad \text{whenever} \quad \operatorname{diam}(X) > R.$$

As usual, we identify operators acting on a subset X with their trivial extension to Γ by

$$A \in \mathcal{B}(\mathcal{H}_X) \quad \longleftrightarrow \quad A \otimes \mathbb{I}_{\Gamma \setminus X} \in \mathcal{B}(\mathcal{H}_\Gamma). \tag{1.1}$$

For a subset X we define

$$H^X := \sum_{Y \cap X \neq \emptyset} \Phi(Y), \qquad H_X := \sum_{Y \subset X} \Phi(Y),$$

so that $H = H^X + H_{\Gamma \setminus X}$.

We are interested in a charge transport. The charge at site x is given by a Hermitian operator Q_x that takes integer values, namely its spectrum is a finite subset of $\mathbb{Z}$. The total charge in a region X is then given by

$$Q_X = \sum_{x \in X} Q_x.$$

The charge is a locally conserved quantity:

$$[Q_X, \Phi(Y)] = 0, \quad \text{if} \quad Y \subset X \subset \Gamma. \tag{1.2}$$

As we will often have to deal with boundaries of spatial regions, we introduce the following sets

$$X^r = \{x \in \Gamma : \operatorname{dist}(x, X) \le r\}, \qquad X_r = \{x \in \Gamma : \operatorname{dist}(x, \Gamma \setminus X) \le r\}$$

and

$$\partial X(r) = X^r \cap X_r,$$

which corresponds to symmetric ribbon of width $2r$ around the boundary of X.

We shall denote $\partial X \equiv \partial X(R)$ since this is practically the only case of relevance. In particular, it follows from charge conservation and the fact that Φ has finite range R that $[Q_X, H] \in \mathcal{B}(\mathcal{H}_{\partial X})$.

For any $X \subset \Gamma$ and any operator A we write

$$\operatorname{tr}_X(A) := \frac{1}{\dim \mathcal{H}_X} \operatorname{Tr}_X(A)$$

for the normalized partial trace $\operatorname{tr}_X : \mathcal{B}(\mathcal{H}_\Gamma) \to \mathcal{B}(\mathcal{H}_{\Gamma \setminus X})$ with respect to the set X.

Remark. Whereas the above setting is phrased in terms of a quantum spin system and on rectangular lattice, this is not necessary. One can equally well consider fermions on the lattice and other types of lattices.

In the fermionic picture, the algebras of observables $\mathcal{B}(\mathcal{H}_X)$ are replaced by the algebra of canonical anticommutation relations built upon $l^2(X; \mathbb{C}^N)$. The anticommutation properties of fermionic observables require one further restriction and one change to keep the crucial locality properties of a quantum spin system. First of all, the interactions $\Phi(X)$ and charges Q_x must be even in the fermionic creation/annihilation operators. Secondly, the partial trace must be replaced by another projection $\mathbb{E}_X : l^2(\Gamma; \mathbb{C}^N) \to l^2(X; \mathbb{C}^N)$. See Section 4 in [8] for details. With this, the Lieb-Robinson bound and its corollaries carry over to lattice fermion systems, see [8, 9].

The advantage of this extension is that there are natural examples that fit our scheme, most notably the (second quantized) Haldane and Harper models with a small interaction term added to them, see [10, 11].

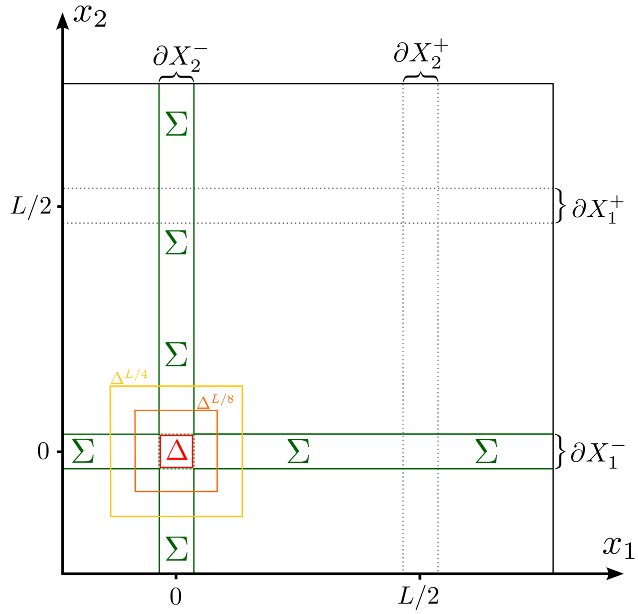

FIGURE 1. The spatial regions relevant for the analysis. The change in the twist Hamiltonian $\tilde{H}(\phi)$ with respect to the flux ϕ is supported in Σ. The associated adiabatic curvature $[\partial_1 \tilde{P}(\phi), \partial_2 \tilde{P}(\phi)]$ is supported in a neighbourhood of Δ. The adiabatic curvatures for the twist Hamiltonian and the twist-antitwist Hamiltonian coincide up to $\mathcal{O}(L^{-\infty})$ term in the region $\Delta^{L/4}$.

1.3. **Hamiltonians with fluxes.** We consider regions $X_1 = \{(x_1, x_2) \in \Gamma : 0 \leq x_2 \leq L/2\}$ resp. $X_2 = \{(x_1, x_2) \in \Gamma : 0 \leq x_1 \leq L/2\}$ and the associated charges $Q_j = Q_{X_j}$.

By charge conservation, $[Q_i, H]$ is supported on ∂X_i. If $L/2 > R$ then ∂X_1 consists of two disjoint ribbons of width $2R$ and centered around the lines $x_2 = 0$ and $x_2 = L/2$. We will denote these ribbons by ∂X_1^- and ∂X_1^+ respectively, and introduce the analogous sets for X_2. Finally, we define

$$\Sigma = \partial X_1^- \cup \partial X_2^-, \qquad \Delta = \partial X_1^- \cap \partial X_2^-.$$

We now introduce for every two flux pairs $\phi^+, \phi^- \in \mathbb{T}^2$ a Hamiltonian $H(\phi^+, \phi^-)$ in the following way. We define first a family of unitaries parametrized by fluxes $\phi \in \mathbb{T}^2$ through

$$U(\phi) := e^{-\mathrm{i}\langle \phi, Q \rangle},$$

where $\langle \phi, Q \rangle = \phi_1 Q_1 + \phi_2 Q_2$. Then we define the family of *twist* Hamiltonians

$$\tilde{H}(\phi) := U^*(\phi) H^{\Sigma} U(\phi) + H_{\Gamma \setminus \Sigma},$$

and finally the family of *twist-antitwist* Hamiltonians (which contain the twist Hamiltonians)

$$H(\phi^+, \phi^-) = U^*(-\phi^-) \tilde{H}(\phi^+ + \phi^-) U(-\phi^-). \tag{1.3}$$

We shall call them all 'flux Hamiltonians'. Note that $H(\phi^+, \phi^-)$ depends on four angles. Note further that $H(\phi) := H(\phi, -\phi)$ is just a unitary transformation of the original Hamiltonian H:

$$H(\phi) = U^*(\phi) H U(\phi). \tag{1.4}$$

In contrast, it is important to stress that the Hamiltonians $\tilde{H}(\phi)$ are *not* related by unitary conjugation. The physical picture for the twist $\tilde{H}(\phi)$ is that a flux pair $\phi = (\phi_1, \phi_2) \in \mathbb{T}^2$ is threaded through the torus along the x_1, x_2 axes. For the twist-antitwist Hamiltonians $H(\phi^+, \phi^-)$, the threaded flux is $\phi^+ + \phi^-$. Indeed a gauge transformation as (1.3) can not change the total flux. Finally, it is good to point out that the fluxes mentioned here are fluxes on top of those possibly contained in H, hence not necessarily the total physical fluxes.

We can now state our main assumption

Assumption 1.1 (Gap for all ϕ)**.** $\tilde{H}(\phi)$ has a non-degenerate ground state whose distance to the rest of the spectrum, i.e. the *gap*, is bounded below by $\gamma(\phi) > 0$, uniformly in L. Moreover, $\inf_{\phi\in\mathbb{T}^2}\gamma(\phi) \geq \gamma$ for some $\gamma > 0$.

This assumption will be in place throughout the entire paper, so we do not repeat it. Making the assumption is standard in the context of the quantum Hall effect. Hastings and Michalakis [1] assume the gap condition only at one point $\phi = \phi_0$ of the flux torus. In Section 1.5 we explain why one could believe this assumption to hold true for any ϕ if it holds for $\phi = \phi_0$.

1.4. **Results.** We denote by $\tilde{P}(\phi)$ the ground state projection of $\tilde{H}(\phi)$ and, for the sake of recognizability, we write

$$\omega_\phi(O) = \omega_{\phi,L}(O) = \mathrm{Tr}(\tilde{P}(\phi)O)$$

for the ground state expectations. The Hall adiabatic curvature is defined by

$$\kappa(\phi) = \mathrm{i}\omega_\phi([\partial_1\tilde{P}(\phi), \partial_2\tilde{P}(\phi)]),$$

where we denoted $\partial_j = \partial/\partial\phi_j$. The main point proven in this note is

Proposition 1.2. *The Hall adiabatic curvature is asymptotically ϕ-independent, in that, for any $N > 0$,*

$$\sup_{\phi,\phi'\in\mathbb{T}^2} |\kappa(\phi) - \kappa(\phi')| \leq C(N)L^{-N},$$

where $C(N)$ is independent of L.

Since the integral of curvature is an integer multiple of 2π, see e.g. [2], this immediately implies

Theorem 1.3. *For any $\phi \in \mathbb{T}^2$ and any $N > 0$*

$$\inf_{n\in\mathbb{Z}} |\kappa(\phi) - 2\pi n| \leq C(N)L^{-N}.$$

Moreover, the minimizer n_0 is independent of ϕ.

It is common lore that $\kappa(0)$ is the Hall conductance of the original model described by H, see [12]. The arguments used up to now do not give any information on how $\kappa(\phi)$ depends on L, and indeed the integer n_0 may a priori depend on L. To clarify this, it natural to assume that the state ω_0 has a thermodynamic limit:

Theorem 1.4. *Assume that for any operator $O \in \mathcal{B}(\mathcal{H}_X)$ with X finite, the limit $\lim_{L\to\infty}\omega_{0,L}(O)$ exists. Then, the thermodynamic limit of the Hall curvature exists and it is quantized:*

$$\lim_{L\to\infty} \kappa(0) \in 2\pi\mathbb{Z}.$$

Recent works [13, 14, 15] provided a proof that the Hall conductance equals the Hall curvature (also) in interacting systems.

1.5. **The rationale for Assumption 1.1.** Consider, for a function $\alpha : \Gamma \to \mathbb{R}$, the unitary (gauge transformation) $U_\alpha = \prod_{x\in\Gamma} e^{-\mathrm{i}\alpha(x)Q_x}$ and choose, for given $\phi = (\phi_1, \phi_2)$

$$\alpha_\phi(x_1, x_2) = -(1 - x_1/L)\phi_1 - (1 - x_2/L)\phi_2.$$

Then we check that

$$U_{\alpha_\phi} \tilde{H}(\phi) U^*_{\alpha_\phi} = H + W(\phi) \tag{1.5}$$

where $W = W(\phi)$ is of the form $W = \sum_{X\subset\Gamma} W(X)$ with $W(X) \in \mathcal{B}(\mathcal{H}_X)$ and such that

i. $W(X) = 0$ whenever $\mathrm{diam}(X) > R$,

ii. $\sup_{X\subset\Gamma} \|W(X)\| \leq \epsilon$.

In fact, we have here that $\epsilon = C/L$ for some L-independent constant C. Although this can be checked by a direct calculation, it is best understood as follows. First of all, local charge conservation (1.2) implies that the effect of a U_α on a local interaction term, say $\Phi(X)$, depends only on the *change* of $\alpha(x)$ over X. In the proposed α_ϕ, this is of order L^{-1} everywhere but across the site $L - 1$ and 0. There however, this abrupt jump of size $-\phi_1$ is precisely compensated by the twist induced by $U(\phi)$ in $\tilde{H}(\phi)$. Put differently, a twist-antitwist can be removed by a gauge transformation using a vector potential that is a single-valued function on the torus. A twist cannot be removed globally as it corresponds to a multivalued vector potential, but as such its effect can still be made locally small everywhere.

The stability of the spectral gap for a Hamiltonian can be formulated as follows. A Hamiltonian H with a non-degenerate ground state has a stable spectral gap, if for any W satisfying conditions i, ii, with ϵ sufficiently small but L-independent, $H + W$ has a non-degenerate ground state with a gap, uniformly in L. At the time of writing, stability of the spectral gap has been proven in the case H is *frustration-free* [16, 17] or the second quantization of free fermions [10, 11]. The latter case being, arguably, the most relevant for quantum Hall effect. Yet, *if* $H = H(0)$ has a stable gap in the precise sense above, then, by (1.5), Assumption 1.1 holds true as well, i.e. all $\tilde{H}(\phi)$ are gapped uniformly in ϕ.

On the other hand, counterexamples of Hamiltonians with an unstable gap were constructed [17] or proposed specifically for our setting [18]. Unlike our result, [1] also covers those cases because the gap assumption there is only made for $\phi = 0$. Therefore, the authors of [1] need a vastly more ingenious proof than we do. However, the observation (1.5) and the fact that stability holds true for free fermions make us believe that Assumption 1.1 for all ϕ is reasonable from the physical point of view.

2. Preliminaries

We recall some standard results on locality of the dynamics of quantum lattice systems that will be crucial for our proofs.

2.1. **Lieb-Robinson bounds and consequences.** As the Hamiltonians $\tilde{H}(\phi)$ are sums of local terms, they all satisfy a Lieb-Robinson bound [19, 20]:

Lemma 2.1. *For any $a > 0$, there exists constants $v, C > 0$ such that for any $O_X \in \mathcal{B}(\mathcal{H}_X), O_Y \in \mathcal{B}(\mathcal{H}_Y)$,*

$$\|[\tau_t^{\tilde{H}(\phi)}(O_X), O_Y]\| \leq C\|O_X\|\|O_Y\| \min(|X|, |Y|)\, e^{-a(\mathrm{dist}(X,Y)-v|t|)}$$

for all $t \in \mathbb{R}$.

Note that in general, the constant a depends on the rate of decay of the interaction. In the present case of a finite range interaction, the bound holds for any positive a.

The same Lieb-Robinson bound extends immediately to all Hamiltonians $H(\phi^-, \phi^+)$ by unitary equivalence through $U(\phi)$, and using the fact that conjugation with any $U(\phi)$ preserves the support of any operator since the charges are single-site operators, see e.g. [20].

In what follows, we only consider the family of 'flux Hamiltonians' introduced above allowing us to bypass some additional setting up and simplify some estimates. In particular, we fix $a = 1$ and the constants (v, C) in the Lieb-Robinson bound which are valid for all flux Hamiltonians.

Here are two direct consequences of the Lieb-Robinson bound.

Lemma 2.2. *Let H, H' be two flux Hamiltonians. Then for any $O_X \in \mathcal{B}(\mathcal{H}_X)$,*

$$\left\| \tau_t^H(O_X) - \tau_t^{H'}(O_X) \right\| \leq C \|O_X\| L |X| \mathrm{e}^{-(\mathrm{dist}(\partial X_1 \cup \partial X_2, X) - v|t|)}.$$

Proof. Starting from

$$\tau_{-t}^H \tau_t^{H'}(O_X) - O_X = -\mathrm{i} \int_0^t \tau_{-s}^H \left([H - H', \tau_s^{H'}(O_X)] \right) \mathrm{d}s,$$

the bound follows by the Lieb-Robinson bound for $\tau^{H'}$, unitarity of the evolution τ^H, and the fact that $\mathrm{supp}(H - H') \subset \partial X_1 \cup \partial X_2$ whose volume is proportional to L. □

The second consequence of the Lieb-Robinson bound is then

Lemma 2.3. *Let H be a flux Hamiltonian. Then for any $O_X \in \mathcal{B}(\mathcal{H}_X)$,*

$$\left\| \tau_t^H(O_X) - \mathrm{tr}_{\Gamma \setminus X^r} \tau_t^H(O_X) \right\| \leq C\, |X| \mathrm{e}^{-(r - v|t|)},$$

where $X^r = \{x \in \Gamma : \mathrm{dist}(x, X) \leq r\}$.

For a proof, we refer e.g. to [21]. Note that the norm of the difference is well-defined by the identification (1.1).

2.2. **Quasi-adiabatic evolution.** For the following result, we refer to [22], whose results apply in this context by our gap Assumption 1.1.

Lemma 2.4. *Let $s \mapsto (\phi_-(s), \phi_+(s)) \in \mathbb{T}^2 \times \mathbb{T}^2$ for $s \in [0, 1]$ be a differentiable curve of fluxes, and denote by $H(s) = H(\phi_-(s), \phi_+(s))$ the corresponding family of Hamiltonians. Then there is a family of unitaries $V(s)$ such that the ground state projection $P(s)$ of the Hamiltonian $H(s)$ is given by*

$$P(s) = V(s) P(0) V^*(s).$$

These unitaries are the unique solution of

$$-\mathrm{i}\partial_s V(s) = K(s) V(s), \qquad U(0) = \mathbb{1},$$

where the generator $K(s)$ can be written as

$$K(s) = \int \mathrm{d}t\, W(t) e^{\mathrm{i}tH(s)} \partial_s H(s) e^{-\mathrm{i}tH(s)}. \tag{2.1}$$

Here, $W \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ is a specific function [22] *such that*

$$|W(t)| = \mathcal{O}(t^{-\infty}), \qquad \left| \int_t^\infty \mathrm{d}t'\, W(t') \right| = \mathcal{O}(t^{-\infty}), \tag{2.2}$$

as $t \to \infty$.

Here and below, we use the notation $\mathcal{O}(t^{-\infty})$ for a function that decays to zero faster then any rational function.

3. Proofs

The main point is to prove Propostion 1.2. Using the quasi-adiabatic generators $\tilde{K}_j$ associated to changes of $\tilde{P}$ in directions ϕ_j, namely

$$\partial_j \tilde{P}(\phi) = \mathrm{i}[\tilde{K}_j(\phi), \tilde{P}(\phi)], \qquad \tilde{K}_j(\phi) = \int_{\mathbb{R}} W(t) e^{\mathrm{i}t\tilde{H}(\phi)} \left(\partial_j \tilde{H}(\phi)\right) e^{-\mathrm{i}t\tilde{H}(\phi)} \mathrm{d}t$$

and the cyclicity of the trace (the Hilbert space $\mathcal{H}_\Gamma$ is finite dimensional), we have

$$\kappa(\phi) = \mathrm{i}\mathrm{Tr}(\tilde{P}(\phi)[\tilde{K}_1(\phi), \tilde{K}_2(\phi)]). \tag{3.1}$$

We are going to show that this is asymptotically constant by comparing the expression inside the trace with such expression for P, K associated to the Hamiltonian H. Although for technical reasons, the proof below is phrased slightly differently, the heart of the argument can be presented in the following brief way. By (1.4), the family $H(\phi)$ is isospectral and hence

$$P(\phi) = e^{\mathrm{i}\langle \phi, Q\rangle} P e^{-\mathrm{i}\langle \phi, Q\rangle}. \tag{3.2}$$

Furthermore, $[Q_i, Q_j] = 0$ so that

$$K_j(\phi) = e^{\mathrm{i}\langle \phi, Q\rangle} K_j e^{-\mathrm{i}\langle \phi, Q\rangle} \tag{3.3}$$

as well, and hence

$$P(\phi)[K_1(\phi), K_2(\phi)] = e^{\mathrm{i}\langle \phi, Q\rangle} P[K_1, K_2] e^{-\mathrm{i}\langle \phi, Q\rangle}. \tag{3.4}$$

As illustrated in Figure 1, $K_j(\phi)$ are supported in a neighbourhood of both ribbons of ∂X_j while $\tilde{K}_j(\phi)$ are supported in a neighbourhood of ∂X_j^- only. Hence the commutator $[K_1(\phi), K_2(\phi)]$ is supported in the neighbourhood of four corners, while $[\tilde{K}_1(\phi), \tilde{K}_2(\phi)]$ is supported in a neighbourhood of the single corner Δ — we shall take an $L/4$-fattening of Δ — where it is approximately equal to $[K_1(\phi), K_2(\phi)]$. Hence,

$$\begin{aligned}
\kappa(\phi) = \mathrm{i}\mathrm{Tr}(\tilde{P}(\phi)[\tilde{K}_1(\phi), \tilde{K}_2(\phi)]) &= \mathrm{i}\mathrm{Tr}\left(\tilde{P}(\phi)\mathrm{tr}_{\Gamma\setminus\Delta^{L/4}}[\tilde{K}_1(\phi), \tilde{K}_2(\phi)]\right) + \mathcal{O}(L^{-\infty}) \\
&= \mathrm{i}\mathrm{Tr}\left(\tilde{P}(\phi)\mathrm{tr}_{\Gamma\setminus\Delta^{L/4}}[K_1(\phi), K_2(\phi)]\right) + \mathcal{O}(L^{-\infty}) \\
&= \mathrm{i}\mathrm{Tr}(P(\phi)\mathrm{tr}_{\Gamma\setminus\Delta^{L/4}}[K_1(\phi), K_2(\phi)]) + \mathcal{O}(L^{-\infty}),
\end{aligned}$$

where in the last equation, we used the fact that in the neighbourhood of Δ, the ground states $P(\phi)$ and $\tilde{P}(\phi)$ are approximately equal since $H - \tilde{H}$ is supported away from Δ, and the fact that local perturbations perturb locally, see [22, 23]. But by (3.4), the fact that the local charge is on-site and cyclicity again, this is independent of ϕ, which concludes the argument. It is interesting to note that the corner Δ has an echo in the analysis of the non-interacting situation, see [24, 25, 26].

3.1. **The case of the fractional quantum Hall effect.** The description of the simple mechanism of the proof above allows us to explain how the results of this paper can be extended to cover fractional conductance, as also explained in Section 9 of the original [1]. Let us modify Assumption 1.1 by allowing that there is a q-dimensional spectral subspace of $\tilde{H}(\phi)$, the range of a spectral projector $\tilde{P}(\phi)$. It is not important that the Hamiltonian $\tilde{H}(\phi)$ is degenerate on this space, but we still call the range of $\tilde{P}(\phi)$ the ground state space and we require an L-independent gap to other parts of the spectrum. By construction, q is ϕ-independent and we also assume it to be L-independent, for L large enough. Additionally, we require a *topological order* condition, see (3.5) below.

The argument has two parts. First of all, let $\omega_\phi = q^{-1}\tilde{P}(\phi)$ be the chaotic ground state, i.e. the incoherent superposition of all ground states. The argument above runs unchanged but for a factor q^{-1} that is carried through from the definition (3.1). Hence $\phi \mapsto \mathrm{Tr}(\tilde{P}(\phi)[\tilde{K}_1(\phi), \tilde{K}_2(\phi)])$ remains approximately constant and integrates to an integer, proving that the expression $\omega([\tilde{K}_1, \tilde{K}_2]))$ (we suppress the ϕ-dependence) is of the form p/q for $p \in \mathbb{N}$.

Secondly, let $\tilde{\omega} = \tilde{\omega}_\phi$ be any (pure) ground state, i.e. a positive normalized functional that is supported on $\tilde{P}(\phi)$ and let us assume the topological order condition: for any local observable O with support independent of L, we have

$$\tilde{\omega}(O) = \omega(O) + \mathcal{O}(L^{-\alpha}) \tag{3.5}$$

for some $\alpha > 0$. Then, since $[\tilde{K}_1, \tilde{K}_2]$ can be approximated by an observable located in the corner $\Delta^{L/4}$, the topological order condition implies that

$$\tilde{\omega}([\tilde{K}_1, \tilde{K}_2]) = \omega[\tilde{K}_1, \tilde{K}_2]) + \mathcal{O}(L^{-\alpha})$$

This proves fractional quantization for any ground state. Although the argument is compelling, one should keep in mind that there is to date no proven example of an interacting Hamiltonian exhibiting such fractional quantization with $p/q \notin \mathbb{Z}$.

3.2. **The actual proof.** In the following lemma, we compare $[\tilde{K}_1(\phi), \tilde{K}_2(\phi)]$ with $G^\Delta(\phi)$ (defined below), which is an adequate replacement of $\mathrm{tr}_{\Gamma\setminus\Omega}[K_1(\phi), K_2(\phi)]$. We denote by τ_t^ϕ and $\tilde{\tau}_t^\phi$ the time-evolutions generated by $H(\phi)$ and $\tilde{H}(\phi)$. Let

$$G^\Delta(\phi) := \int \mathrm{d}t\, W(t) \int \mathrm{d}t'\, W(t') \left[\tau_t^\phi(\partial_1 H_{\Delta^{L/8}}(\phi)), \tau_{t'}^\phi(\partial_2 H_{\Delta^{L/8}}(\phi))\right]$$

The following lemma establishes that $G^\Delta(\phi)$ is localized in a neighbourhood of Δ and that it is a good approximation of $[\tilde{K}_1(\phi), \tilde{K}_2(\phi)]$.

Lemma 3.1. *We have*

$$\left\| [\tilde{K}_1(\phi), \tilde{K}_2(\phi)] - \mathrm{tr}_{\Gamma\setminus\Delta^{L/4}}\left(G^\Delta(\phi)\right) \right\| = \mathcal{O}(L^{-\infty})$$

and

$$\mathrm{tr}_{\Gamma\setminus\Delta^{L/4}}\left(G^\Delta(\phi)\right) = U^*(\phi)\, \mathrm{tr}_{\Gamma\setminus\Delta^{L/4}}\left(G^\Delta(0)\right)\, U(\phi).$$

Proof. To prove the first estimate, we pick up a $\phi \in \mathbb{T}^2$ and drop the ϕ dependence in this proof for notational clarity. First of all, we note that the operator G^Δ is concentrated around the set Δ. Indeed, the commutator $[\partial_1 H_{\Delta^{L/8}}, \partial_2 H_{\Delta^{L/8}}]$ is strictly supported on $\Delta^{L/8}$, the time evolution τ_t can be controlled using the Lieb-Robinson bound for short times, and the good decay properties of W take care of long times, see also [22]. To make this precise, we show that

$$\|G^\Delta - \mathrm{tr}_{\Gamma\setminus\Delta^{L/4}}(G^\Delta)\| = \mathcal{O}(L^{-\infty}). \tag{3.6}$$

Using the good decay properties (2.2) of W we can restrict the integrals to $[-T, T]$ with $T = L/(32v)$, making an error of order $\mathcal{O}(L^{-\infty})$:

$$\|G^\Delta - \mathrm{tr}_{\Gamma\setminus\Delta^{L/4}}(G^\Delta)\| \leq \left\| \int_{-T}^{T} \mathrm{d}t\, W(t) \int_{-T}^{T} \mathrm{d}t'\, W(t') f(t,t') \right\| + \mathcal{O}(L^{-\infty}),$$

where the integrand is given by

$$f(t,t') = \left(\mathbb{1} - \mathrm{tr}_{\Gamma\setminus\Delta^{L/4}}\right) [\tau_t(\partial_1 H_{\Delta^{L/8}}), \tau_{t'}(\partial_2 H_{\Delta^{L/8}})].$$

To estimate $\|f(t,t')\|$ we use Lemma 2.3 with $X = \Delta^{L/8}$, $r = L/8$, $|t| \leq L/32v$ and the fact that $\|\partial_j H_{\Delta^{L/8}}\| \leq CL$ to bound the integral by $\mathcal{O}(L^{-\infty})$. The claim then follows.

The next step is to show that $[\tilde{K}_1, \tilde{K}_2]$ is close in norm to G^Δ. Note first of all that, like in the argument above, we can restrict the integrals to $[-T,T]$ making an error of order $\mathcal{O}(L^{-\infty})$. Therefore, it is sufficient to bound the norm of

$$\int_{-T}^{T} \mathrm{d}t\, W(t) \int_{-T}^{T} \mathrm{d}t'\, W(t') \left([\tilde{\tau}_t(\partial_1 \tilde{H}), \tilde{\tau}_{t'}(\partial_2 \tilde{H})] - [\tau_t(\partial_1 H_{\Delta^{L/8}}), \tau_{t'}(\partial_2 H_{\Delta^{L/8}})] \right). \tag{3.7}$$

Since $\mathrm{dist}(\Delta^{L/8}, \mathrm{supp}(H - \tilde{H})) = 3L/8 - 2R$, it follows from Lemma 2.2 that

$$\|\tau_t(\partial_j H_{\Delta^{L/8}}) - \tilde{\tau}_t(\partial_j H_{\Delta^{L/8}})\| = \mathcal{O}(L^{-\infty})$$

for any $t \in [-T,T]$. Thus we can replace the evolutions τ_t by $\tilde{\tau}_t$, making an error that is again a $\mathcal{O}(L^{-\infty})$. We can further replace $\partial_j H_{\Delta^{L/8}}$ by $\partial_1 \tilde{H}_{\Delta^{L/8}}$ without any error since by construction $H_{\Delta^{L/8}} = \tilde{H}_{\Delta^{L/8}}$ (as functions of ϕ). Altogether, we estimate the integrand of (3.7) by

$$\begin{aligned}
&\|W\|_\infty^2 \|[\tilde{\tau}_t(\partial_1 \tilde{H}), \tilde{\tau}_{t'}(\partial_2 \tilde{H})] - [\tilde{\tau}_t(\partial_1 \tilde{H}_{\Delta^{L/8}}), \tilde{\tau}_{t'}(\partial_2 \tilde{H}_{\Delta^{L/8}})]\| \\
&\quad \le \|W\|_\infty^2 \left(\|[\tilde{\tau}_{t-t'}(\partial_1 \tilde{H}), \partial_2 \tilde{H} - \partial_2 \tilde{H}_{\Delta^{L/8}}]\| + \|[\tilde{\tau}_{t-t'}(\partial_1 \tilde{H}_{\Delta^{L/8}} - \partial_1 \tilde{H}), \partial_2 \tilde{H}_{\Delta^{L/8}}]\| \right) \\
&\quad = \mathcal{O}(L^{-\infty}),
\end{aligned}$$

for any $t, t' \in [-T,T]$. To obtain the last estimate we used the Lieb-Robinson bound, noting that the supports of $\partial_1 \tilde{H}$ and $\partial_2 \tilde{H} - \partial_2 \tilde{H}_{\Delta^{L/8}}$ are separated by a distance $L/8$ while $v|t - t'| \le L/16$. Hence,

$$\left\| [\tilde{K}_1, \tilde{K}_2] - G^\Delta \right\| = \mathcal{O}(L^{-\infty}),$$

which, together with (3.6), concludes the proof of the first claim.

To get the covariance we note that $G^\Delta(\phi) = U^*(\phi) G^\Delta(0) U(\phi)$ which follows directly from $H(\phi) = U^*(\phi) H U(\phi)$. Then, the covariance of $\mathrm{tr}_{\Gamma \setminus \Delta^{L/4}} G^\Delta(\phi)$ follows upon noting that $U(\phi)$ is a product over single site unitaries. □

Proof of Proposition 1.2. By (3.1) and Lemma 3.1,

$$\kappa(\phi) = \mathrm{Tr}\left(\tilde{P}(\phi)\, \mathrm{tr}_{\Gamma \setminus \Delta^{L/4}} \left(G^\Delta(\phi) \right) \right) + \mathcal{O}(L^{-\infty}).$$

Recalling the notation of (1.3) and Assumption 1.1, the Hamiltonians $H(\phi) = H(\phi, -\phi)$ and $\tilde{H}(\phi) = H(\phi, 0)$ can be smoothly deformed into each other through $[0,1] \ni s \mapsto H(\phi, (s-1)\phi)$ without closing the spectral gap above the ground state energy. Let $V(s)$ be the quasi-adiabatic unitaries corresponding to this homotopy, as provided by Lemma 2.4. We note first that the derivative

$$\partial_s H(\phi, (s-1)\phi) = -\mathrm{i} U^*((1-s)\phi) [\langle \phi, Q \rangle, H_{\Gamma \setminus \Sigma}] U((1-s)\phi)$$

is supported on $(\partial X_1^+ \cup \partial X_2^+) \setminus \Sigma$. For any observable O, we can write

$$V^*(1) O V(1) = O + \mathrm{i} \int_0^1 V^*(s) [K(s), O] V(s) \mathrm{d}s.$$

Let us now assume that O is supported in $\Delta^{L/4}$. Then by the expression (2.1) of $D(s)$, the fact the distance of the support of $\partial_s H(\phi, (s-1)\phi))$ to $\Delta^{L/4}$ is $3L/4 - 2R$, the decay of $W(t)$ and the Lieb-Robinson bound, we deduce that

$$\|[K(s), O]\| = \mathcal{O}(L^{-\infty}).$$

Therefore, applying this with $O = \mathrm{tr}_{\Gamma \setminus \Delta^{L/4}} G^\Delta(\phi)$ and using $\tilde{P}(\phi) = V(1) P(\phi) V^*(1)$ and cyclicity of the trace, we get

$$\mathrm{Tr}\left(\tilde{P}(\phi)\, \mathrm{tr}_{\Gamma \setminus \Delta^{L/4}} \left(G^\Delta(\phi) \right) \right) = \mathrm{Tr}\left(P(\phi)\, \mathrm{tr}_{\Gamma \setminus \Delta^{L/4}} \left(G^\Delta(\phi) \right) \right) + \mathcal{O}(L^{-\infty}).$$

Now, the covariance of $P(\phi)$ and of $\mathrm{tr}_{\Gamma\setminus\Delta^{L/4}}\left(G^{\Delta}(\phi)\right)$ provided in (3.2) and Lemma 3.1 respectively, show that the trace on the right is in fact independent of ϕ by cyclicity, settling the claim. □

Proof of Theorem 1.4. Recall from (3.1) that

$$\kappa_L = \mathrm{i}\omega_{0,L}([\tilde{K}_1, \tilde{K}_2]). \tag{3.8}$$

where both the $\tilde{K}_j$'s and the state $\omega_{0,L}$ depend on L (we have made the dependence explicit in the notation). The claim will follow from the fact that $[\tilde{K}_1, \tilde{K}_2]$ can be approximated uniformly in L by a local observable Υ^ℓ supported in Δ^ℓ. The error decays rapidly in ℓ, and the expectation value of Υ^ℓ converges by assumption.

Indeed, (2.1) and the arguments repeatedly used in this article yield that $\tilde{K}_j$ is a sum of local terms in the form $\tilde{K}_j = \sum_{X\subset\Gamma}\Psi^\Gamma(X)$, where $\|\Psi^\Gamma(X)\| = \mathcal{O}(\mathrm{diam}(X)^{-\infty})$ uniformly in L, and that $\Psi^\Gamma(X)$ converges in norm as $L\to\infty$ for any fixed X, see [22]. Hence, for any $\ell\in\mathbb{N}$, the local observable $\mathrm{tr}_{\Gamma\setminus\Delta^\ell}[\tilde{K}_1, \tilde{K}_2]$ converges to a $\Upsilon^\ell \in \mathcal{B}(\mathcal{H}_{\Delta^\ell})$ as $L\to\infty$. Since moreover,

$$(\mathbb{1} - \mathrm{tr}_{\Gamma\setminus\Delta^\ell})([\tilde{K}_1, \tilde{K}_2]) = \mathcal{O}(\ell^{-\infty}),$$

uniformly in L, we have

$$\begin{aligned}[\tilde{K}_1, \tilde{K}_2] - \Upsilon^\ell &= (\mathbb{1} - \mathrm{tr}_{\Gamma\setminus\Delta^\ell})([\tilde{K}_1, \tilde{K}_2]) + \left(\mathrm{tr}_{\Gamma\setminus\Delta^\ell}[\tilde{K}_1, \tilde{K}_2] - \Upsilon^\ell\right)\\ &= \mathcal{O}(\ell^{-\infty}) + \mathcal{O}(L^{-\infty}).\end{aligned}$$

Hence,

$$\lim_{L\to\infty}\left|\kappa_L - \mathrm{i}\omega_{0,L}(\Upsilon^\ell)\right| = \mathcal{O}(\ell^{-\infty}).$$

By assumption $\omega_{0,L}(\Upsilon^\ell)$ converges, concluding the proof. □

4. Acknowledgements

We would like to thank Y. Avron for his careful reading of the first version of this manuscript, and for his many comments which helped improve this article.

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Mathematisches Institut der Universität München, 80333 München, Germany
Current address: Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada
E-mail address: `sbach@math.ubc.ca`

Instituut Theoretische Fysica, KU Leuven, 3001 Leuven, Belgium
E-mail address: `alexander.bols@kuleuven.be`

Instituut Theoretische Fysica, KU Leuven, 3001 Leuven, Belgium
E-mail address: `wojciech.deroeck@kuleuven.be`

Instituut Theoretische Fysica, KU Leuven, 3001 Leuven, Belgium
Current address: Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123, USA
E-mail address: `fraas@vt.edu`