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Research Article

Mohamed Al-Masaeed, Eqab. M. Rabei, Ahmed Al-Jamel*, and Dumitru Baleanu

Quantization of fractional harmonic oscillator using creation and annihilation operators

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Abstract: In this article, the Hamiltonian for the conformable harmonic oscillator used in the previous study [Chung WS, Zare S, Hassanabadi H, Maghsoodi E. The effect of fractional calculus on the formation of quantum-mechanical operators. Math Method Appl Sci. 2020;43(11):6950–67.] is written in terms of fractional operators that we called α-creation and α-annihilation operators. It is found that these operators have the following influence on the energy states. For a given order α, the α-creation operator promotes the state while the α-annihilation operator demotes the state. The system is then quantized using these creation and annihilation operators and the energy eigenvalues and eigenfunctions are obtained. The eigenfunctions are expressed in terms of the conformable Hermite functions. The results for the traditional quantum harmonic oscillator are found to be recovered by setting $\alpha = 1$.

Keywords: harmonic oscillator, conformable derivative, fractional order creation, annihilation operators

1 Introduction

The fractional derivative extends the classical derivative by allowing the operators of differentiation to take fractional orders, and it has played an important role in physics, mathematics, and engineering sciences [1–10]. The definition of fractional derivative and fractional integral subject to several approaches such as Riemann–Liouville fractional, Caputo, Riesz, Riesz–Caputo, Weyl, Grünwald–Letnikov, Hadamard, and Chen fractional derivatives [1–12]. Recently, a new definition of fractional derivative was presented by Khalil et al. [13] called the conformable derivative. This definition is suggested as a natural extension of the usual derivative in the following senses. Given a function $f \in [0, \infty) \to \mathbb{R}$, the conformable derivative of f with order α is defined by ref. [13]

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \quad (1)$$

for all $t > 0$, $\alpha \in (0,1]$. Here, we utilize $D^\alpha f$ as a shorthand notation for the conformable derivative of f of order α, $T_\alpha(f)(t)$. According to this definition, the conformable derivative of the constant is zero; it satisfies the standard properties of the traditional derivative, i.e., the derivative of the product and the derivative of the quotient of two functions and also satisfies the known chain rule. In addition, one can say that the conformable derivative is simple and similar to the standard derivative. To read more about conformable derivative, its properties, and its applications, we refer you to refs. [14–20]. Because of these properties, in the last few years, the conformable calculus is applied successfully in various fields [13,21,22]. Besides, the conformable Euler–Lagrange equation and Hamiltonian formulation were discussed by Lazo and Torres [23]. In ref. [24], the conformable (2 + 1)-dimensional Ablowitz–KaupNewell–Segur equation was investigated to verify the existence of complex combined dark-bright soliton solutions. For this purpose, the sine-Gordon expansion method is used, which is an effective method. The 2D and 3D surfaces under some suitable values of parameters are also plotted. In ref. [25], new solutions to the fractional (2 + 1)-dimensional Boussinesq dynamical model with the local M-derivative with the aid of the modified exponential function method were obtained. The complex and combined dark-bright characteristic

* **Corresponding author: Ahmed Al-Jamel,** Physics Department, Faculty of Science, Al Al-Bayt University, P.O. Box 130040, Mafraq 25113, Jordan, e-mail: aaljamel@aabu.edu.jo, aaljamel@gmail.com
Mohamed Al-Masaeed: Physics Department, Faculty of Science, Al Al-Bayt University, P.O. Box 130040, Mafraq 25113, Jordan, e-mail: Mohamed94@st.aabu.edu.jo, moh.almssaeed@gmail.com
Eqab. M. Rabei: Physics Department, Faculty of Science, Al Al-Bayt University, P.O. Box 130040, Mafraq 25113, Jordan, e-mail: eqabrabei@gmail.com
Dumitru Baleanu: Department of Mathematics, Cankaya University, Ankara, Turkey; Institute of Space Sciences, Magurele-Bucharest, Romania; Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan, e-mail: dumitru@cankaya.edu.tr

properties of nonlinear Date–Jimbo–Kashiwara–Miwa equation with conformable derivative are extracted in ref. [26].

The quantization of fractional system is of prime importance in physics. Rabei et al. [27] discussed how to find the solution of the Schrodinger equation for some systems that have a fractional behavior in their Lagrangian and obey the WKB approximation. Besides, the canonical quantization of a system with Brownian motion is carried out using fractional calculus by Rabei et al. [8]. However, the quantization of fractional singular Lagrangian systems using WKB approximation is studied by Rabei and Horani [28].

Recently, the deformation of the ordinary quantum mechanics based on the idea of fractional calculus is considered by Chung et al. [29]. They adopted the conformable fractional calculus, which depends on the basic limit definition of the derivative. The authors proposed the α-position operator and α-momentum operator and they constructed the α-Hamiltonian operator and fractional Schrodinger equation. Also by considering the fractional calculus, they have formulated the conformable quantum mechanics and they have discussed some physical examples such as the harmonic oscillator problem.

The harmonic oscillator problem is of great importance in quantum mechanics. The treatment of this problem using the algebraic method based on the creation and annihilation operators rather than solving the Schrodinger equation is well known in quantum mechanics. It plays a central role in modeling various physical phenomena as well as its importance in the canonical field quantization. It is then a natural step to extend the algebraic method within the frame of conformable quantum mechanics. The main purpose of this article is to treat the conformable harmonic oscillator using an algebraic method with newly defined operators that we call the α-creation and α-annihilation operators. Their names are justified by noting that the α-Hamiltonian of the system is factored in terms of these operators and that they have the effect of promotion and demotion of the α-states. It should be mentioned that this treatment is presumably needed to lay out the transition into any possible conformable quantum field theory.

This article is organized as follows. In Section 2, we present a brief review of the formulation of conformable quantum mechanics. In Section 3, we present and discuss the quantization of fractional harmonic oscillator using the α-creation and α-annihilation operators. In Section 4, we present our summary and conclusions of this work.

2 The conformable quantum mechanics

Recently, Chung et al. [29] proposed a formulation of the ordinary quantum mechanics in fractional form using the conformable derivative. Here, we present the main definitions and relations needed for our work. According to Chung et al. [29], the fractional Schrodinger equation takes the form

$$\begin{aligned} H_\alpha(\hat{x}_\alpha, \hat{p}_\alpha)\psi(x,t) &= \left(\frac{\hat{p}_\alpha^2}{2m^\alpha} + V_\alpha(\hat{x}_\alpha)\right)\psi(x,t) \\ &= E^\alpha\psi(x,t), \end{aligned} \tag{2}$$

with the position and momentum operators $\hat{x}_\alpha$, $\hat{p}_\alpha$ defined as

$$\hat{x}_\alpha = x, \quad \hat{p}_\alpha = -i\hbar_\alpha^\alpha D_x^\alpha \tag{3}$$

where $\hbar_\alpha^\alpha = \frac{h}{(2\pi)^{\frac{1}{\alpha}}}$. The inner product is defined as ref. [29]

$$\langle f|g\rangle = \int_{-\infty}^{\infty} g^*(x)f(x)|x|^{\alpha-1}\mathrm{d}x, \tag{4}$$

and the expectation value of an observable A for a system in the state $\psi(x,t)$ is

$$\begin{aligned} \langle A\rangle &= \langle\psi(x,t)|A|\psi(x,t)\rangle \\ &= \int_{-\infty}^{\infty}\psi^*(x,t)A\psi(x,t)|x|^{\alpha-1}\mathrm{d}x. \end{aligned} \tag{5}$$

One can refer to ref. [29] for more illustrations.

3 Quantization of conformable harmonic oscillator

The Hamiltonian for the conformable harmonic oscillator is given as ref. [29]:

$$H_\alpha = \frac{\hat{p}_\alpha^2}{2m^\alpha} + \frac{1}{2}m^\alpha\omega^{2\alpha}x^{2\alpha}, \tag{6}$$

and the Schrodinger equation for this system takes the form

$$\left(\frac{\hat{p}_\alpha^2}{2m^\alpha} + \frac{1}{2}m^\alpha\omega^{2\alpha}x^{2\alpha}\right)\psi(x,t) = E^\alpha\psi(x,t). \tag{7}$$

By defining an arbitrary potential depending on $X_\alpha(x)$ as ref. [29]

$$X_\alpha(x) = \begin{cases} \dfrac{(x)^\alpha}{\alpha}, & \text{if } x > 0 \\ \dfrac{-(-x)^\alpha}{\alpha}, & \text{if } x < 0, \end{cases} \tag{8}$$

one may re-express equations (6) and (7) in the forms

$$H_\alpha = \frac{\hat{p}_\alpha^2}{2m^\alpha} + \frac{\alpha^2}{2} m^\alpha \omega^{2\alpha} X_\alpha^2(x) \tag{9}$$

and

$$\left(\frac{\hat{p}_\alpha^2}{2m^\alpha} + \frac{\alpha^2}{2} m^\alpha \omega^{2\alpha} X_\alpha^2(x)\right)\psi(x,t) = E^\alpha \psi(x,t), \tag{10}$$

respectively.

3.1 α-creation operator $\hat{a}_\alpha^\dagger$ and α-annihilation operator $\hat{a}_\alpha$

We will develop here a fractional algebraic method for solving equation (10). It involves the definition of two operators, namely, the fractional creation operator of order α ($\hat{a}_\alpha^\dagger$) and fractional annihilation operator of order α ($\hat{a}_\alpha$). We start by rewriting the Hamiltonian equation (9) as

$$H_\alpha = \frac{1}{2m^\alpha}(\hat{p}_\alpha^2 + \alpha^2 m^{2\alpha} \omega^{2\alpha} \hat{X}_\alpha^2). \tag{11}$$

Because $\hat{p}_\alpha$ and $\hat{X}_\alpha$ do not commute, the Hamiltonian could be factored as

$$\begin{aligned} H_\alpha = \frac{1}{2m^\alpha}\{&(i\hat{p}_\alpha + \alpha m^\alpha \omega^\alpha X_\alpha)(-i\hat{p}_\alpha + \alpha m^\alpha \omega^\alpha X_\alpha) \\ &- i\alpha m^\alpha \omega^\alpha [\hat{p}_\alpha, X_\alpha]\}. \end{aligned} \tag{12}$$

The commutator $[\hat{p}_\alpha, X_\alpha]$ can be found as follows.

$$\begin{aligned} [\hat{p}_\alpha, X_\alpha]\psi &= \hat{p}_\alpha X_\alpha \psi - X_\alpha \hat{p}_\alpha \psi \\ &= -i\hbar_\alpha^\alpha [\psi D_x^\alpha X_\alpha + X_\alpha D_x^\alpha \psi - X_\alpha D_x^\alpha \psi] \\ &= -i\hbar_\alpha^\alpha \left[\psi D_x^\alpha \frac{x^\alpha}{\alpha}\right], \end{aligned} \tag{13}$$

and thus

$$[\hat{p}_\alpha, X_\alpha] = -i\hbar_\alpha^\alpha. \tag{14}$$

Substituting this result in equation (12) we obtain

$$\begin{aligned} H_\alpha = \alpha\hbar_\alpha^\alpha \omega^\alpha \Bigg[&\frac{(i\hat{p}_\alpha + \alpha m^\alpha \omega^\alpha X_\alpha)}{\sqrt{2m^\alpha \alpha \hbar_\alpha^\alpha \omega^\alpha}} \\ &\times \frac{(-i\hat{p}_\alpha + \alpha m^\alpha \omega^\alpha X_\alpha)}{\sqrt{2m^\alpha \alpha \hbar_\alpha^\alpha \omega^\alpha}} - \frac{1}{2}\Bigg]. \end{aligned} \tag{15}$$

According to this result, we define the α-creation operator $\hat{a}_\alpha^\dagger$ and the α-annihilation operator $\hat{a}_\alpha$ as

$$\hat{a}_\alpha^\dagger = \frac{(\alpha m^\alpha \omega^\alpha X_\alpha - i\hat{p}_\alpha)}{\sqrt{2m^\alpha \alpha \hbar_\alpha^\alpha \omega^\alpha}}, \tag{16}$$

$$\hat{a}_\alpha = \frac{(\alpha m^\alpha \omega^\alpha X_\alpha + i\hat{p}_\alpha)}{\sqrt{2m^\alpha \alpha \hbar_\alpha^\alpha \omega^\alpha}}, \tag{17}$$

respectively. The commutation relation of $\hat{a}_\alpha$ with $\hat{a}_\alpha^\dagger$ is now straightforwardly calculated using the fundamental relation equation (14) to yield

$$[\hat{a}_\alpha, \hat{a}_\alpha^\dagger] = 1. \tag{18}$$

The fractional Hamiltonian is then expressed in terms of fractional creation and annihilation operators as

$$H_\alpha = \alpha\hbar_\alpha^\alpha \omega^\alpha \left(\hat{a}_\alpha \hat{a}_\alpha^\dagger - \frac{1}{2}\right), \tag{19}$$

or equivalently as

$$H_\alpha = \alpha\hbar_\alpha^\alpha \omega^\alpha \left(\hat{a}_\alpha^\dagger \hat{a}_\alpha + \frac{1}{2}\right). \tag{20}$$

Substituting $\alpha = 1$, one may recover the usual Hamiltonian of harmonic oscillator.

3.2 The eigenfunction and eigenvalue

First of all let us operate by the fractional Hamiltonian on $\hat{a}_\alpha^\dagger \psi$, then we have

$$H_\alpha(\hat{a}_\alpha^\dagger \psi) = \alpha\hbar_\alpha^\alpha \omega^\alpha \left(\hat{a}_\alpha^\dagger \hat{a}_\alpha + \frac{1}{2}\right)(\hat{a}_\alpha^\dagger \psi). \tag{21}$$

Using equation (18), we obtain

$$\begin{aligned} H_\alpha(\hat{a}_\alpha^\dagger \psi) &= \alpha\hbar_\alpha^\alpha \omega^\alpha \left(\hat{a}_\alpha^\dagger(\hat{a}_\alpha^\dagger \hat{a}_\alpha + 1) + \hat{a}_\alpha^\dagger \frac{1}{2}\right)(\psi) \\ &= \hat{a}_\alpha^\dagger \left(\alpha\hbar_\alpha^\alpha \omega^\alpha \left(\hat{a}_\alpha^\dagger \hat{a}_\alpha + \frac{1}{2}\right)\psi + \alpha\hbar_\alpha^\alpha \omega^\alpha \psi\right) \\ &= \hat{a}_\alpha^\dagger (H_\alpha + \alpha\hbar_\alpha^\alpha \omega^\alpha)\psi \\ &= (E^\alpha + \alpha\hbar_\alpha^\alpha \omega^\alpha)(\hat{a}_\alpha^\dagger \psi). \end{aligned}$$

Similarly, one can show that

$$H_\alpha(\hat{a}_\alpha \psi) = \alpha\hbar_\alpha^\alpha \omega^\alpha \left(\hat{a}_\alpha \hat{a}_\alpha^\dagger - \frac{1}{2}\right)(\hat{a}_\alpha \psi), \tag{22}$$

and that

$$H_\alpha(\hat{a}_\alpha \psi) = (E^\alpha - \alpha\hbar_\alpha^\alpha \omega^\alpha)(\hat{a}_\alpha \psi). \tag{23}$$

It follows that $(\hat{a}_\alpha^\dagger \psi)$ is an eigenfunction of the Hamiltonian H_α with an eigenenergy E^α increased by $\alpha\hbar_\alpha^\alpha \omega^\alpha$ and $\hat{a}_\alpha \psi$ is an eigenfunction of the Hamiltonian H_α with an eigenenergy E^α decreased by $\alpha\hbar_\alpha^\alpha \omega^\alpha$. This justifies the names given above to $\hat{a}_\alpha^\dagger$ and $\hat{a}_\alpha$.

To calculate wave function for ground state ψ_0, we stipulate that

$$\hat{a}_\alpha \psi_0 = 0. \tag{24}$$

Then, by making use of equation (17), we have

$$\frac{(\alpha m^\alpha\omega^\alpha X_\alpha + i\hat{p}_\alpha)}{\sqrt{2m^\alpha\alpha\hbar_\alpha^\alpha\omega^\alpha}}\hat{a}_\alpha\psi_0 = (\alpha m^\alpha\omega^\alpha X_\alpha + \hbar_\alpha^\alpha D_x^\alpha)\psi_0 = 0.$$

It is then straightforward to obtain the ground state eigenfunction as

$$\psi_0 = \left(\frac{\alpha m^\alpha\omega^\alpha}{\pi\hbar_\alpha^\alpha}\right)^{\frac{1}{4}}\exp\left(-\frac{\alpha m^\alpha\omega^\alpha x^{2\alpha}}{\hbar_\alpha^\alpha 2\alpha^2}\right) \tag{25}$$

$$= \left(\frac{\alpha m^\alpha\omega^\alpha}{\pi\hbar_\alpha^\alpha}\right)^{\frac{1}{4}}\exp\left(-\frac{\alpha m^\alpha\omega^\alpha X_\alpha^2}{2\hbar_\alpha^\alpha}\right), \tag{26}$$

where we make use of equation (4) to normalize ψ_0. Making use of equations (2), (20), and (24) one may calculate the ground state energy E_0^α. The result is

$$E_0^\alpha = \frac{1}{2}\alpha\hbar_\alpha^\alpha\omega^\alpha. \tag{27}$$

In the same manner, we calculate the energy for the first excited state (E_1^α) from

$$H_\alpha(\psi_1) = E_1^\alpha(\psi_1).$$

By making use of equation (21), we have

$$\begin{aligned} H_\alpha(\hat{a}_\alpha^\dagger\psi_0) &= (E_0^\alpha + \alpha\hbar_\alpha^\alpha\omega^\alpha)(\hat{a}_\alpha^\dagger\psi_0) \\ &= \left(\frac{1}{2}\alpha\hbar_\alpha^\alpha\omega^\alpha + \alpha\hbar_\alpha^\alpha\omega^\alpha\right)(\hat{a}_\alpha^\dagger\psi_0) \\ &= \left(\frac{3}{2}\alpha\hbar_\alpha^\alpha\omega^\alpha\right)(\hat{a}_\alpha^\dagger\psi_0), \end{aligned}$$

from which we obtain the energy for first excited state as

$$E_1^\alpha = \frac{3}{2}\alpha\hbar_\alpha^\alpha\omega^\alpha. \tag{28}$$

Repeating similar steps for the n-excited state, one finds that the energy eigenvalues are

$$E_n^\alpha = \alpha\hbar_\alpha^\alpha\omega^\alpha\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots \tag{29}$$

The n-excited state (ψ_n) is determined from

$$\psi_n = A_n(\hat{a}_\alpha^\dagger)^n\psi_0. \tag{30}$$

The constant A_n is calculated from the normalization condition as

$$\begin{aligned} \langle\psi_n|\psi_n\rangle &= |A_n|^2\int_{-\infty}^{\infty}((\hat{a}_\alpha^\dagger)^n\psi_0)^*((\hat{a}_\alpha^\dagger)^n\psi_0)x^{\alpha-1}\mathrm{d}x \\ &= |A_n|^2\int_{-\infty}^{\infty}(\hat{a}_\alpha^\dagger(\hat{a}_\alpha^\dagger)^{n-1}\psi_0)^*((\hat{a}_\alpha^\dagger)^n\psi_0)x^{\alpha-1}\mathrm{d}x \\ &= |A_n|^2\int_{-\infty}^{\infty}((\hat{a}_\alpha^\dagger)^{n-1}\psi_0)^*(\hat{a}_\alpha(\hat{a}_\alpha^\dagger)^n\psi_0)x^{\alpha-1}\mathrm{d}x. \end{aligned} \tag{31}$$

Using (see Appendix A)

$$[\hat{a}_\alpha, (\hat{a}_\alpha^\dagger)^n] = n(\hat{a}_\alpha^\dagger)^{n-1}, \tag{32}$$

we then have

$$\begin{aligned} \langle\psi_n|\psi_n\rangle = |A_n|^2 &\int_{-\infty}^{\infty}((\hat{a}_\alpha^\dagger)^{n-1}\psi_0)^*((\hat{a}_\alpha^\dagger)^n\hat{a}_\alpha\psi_0 \\ &+ n(\hat{a}_\alpha^\dagger)^{n-1}\psi_0)x^{\alpha-1}\mathrm{d}x \\ = |A_n|^2 &\int_{-\infty}^{\infty}((\hat{a}_\alpha^\dagger)^{n-1}\psi_0)^*(0 + n(\hat{a}_\alpha^\dagger)^{n-1}\psi_0)x^{\alpha-1}\mathrm{d}x \\ = n\,|A_n|^2 &\int_{-\infty}^{\infty}((\hat{a}_\alpha^\dagger)^{n-1}\psi_0)^*((\hat{a}_\alpha^\dagger)^{n-1}\psi_0)x^{\alpha-1}\mathrm{d}x \\ = n\,|A_n|^2 &\int_{-\infty}^{\infty}|(\hat{a}_\alpha^\dagger)^{n-1}\psi_0|^2x^{\alpha-1}\mathrm{d}x. \end{aligned} \tag{33}$$

Making use of equation (30) and interchanging n by $n-1$, we have

$$\psi_{n-1} = A_{n-1}(\hat{a}_\alpha^\dagger)^{n-1}\psi_0 \to \frac{\psi_{n-1}}{A_{n-1}} = (\hat{a}_\alpha^\dagger)^{n-1}\psi_0$$

$$\begin{aligned} \langle\psi_n|\psi_n\rangle &= n\,|A_n|^2\int_{-\infty}^{\infty}\left|\frac{\psi_{n-1}}{A_{n-1}}\right|^2x^{\alpha-1}\mathrm{d}x \\ &= n\left|\frac{A_n}{A_{n-1}}\right|^2\int_{-\infty}^{\infty}|\psi_{n-1}|^2x^{\alpha-1}\mathrm{d}x, \end{aligned}$$

where $X_\alpha = \frac{x^\alpha}{\alpha} \to \mathrm{d}X_\alpha = x^{\alpha-1}\mathrm{d}x$

$$\begin{aligned} \langle\psi_n|\psi_n\rangle &= n\left|\frac{A_n}{A_{n-1}}\right|^2\int_{-\infty}^{\infty}|\psi_{n-1}|^2\mathrm{d}X_\alpha \\ &= n\left|\frac{A_n}{A_{n-1}}\right|^2\langle\psi_{n-1}|\psi_{n-1}\rangle = 1 \\ &\to A_n = \frac{A_{n-1}}{\sqrt{n}}, \end{aligned}$$

where the inner product for ψ_{n-1} is equal $\langle\psi_{n-1}|\psi_{n-1}\rangle = 1$. Then we obtain

$$\begin{aligned} \langle\psi_n|\psi_n\rangle = n\left|\frac{A_n}{A_{n-1}}\right|^2 = 1 \to A_n = \frac{A_{n-1}}{\sqrt{n}} \\ A_n = \frac{A_{n-1}}{\sqrt{n}}, \end{aligned} \tag{34}$$

where $A_0 = 1$. The eigenfunction for nth-excited state is then

$$\psi_n = \frac{1}{\sqrt{n!}}(\hat{a}_\alpha^\dagger)^n\psi_0. \tag{35}$$

3.3 Eigenfunctions in terms of Hermite polynomial

The eigenfunctions for the excited state equation (35) can be expressed in terms of the Hermite polynomials as follows. First, we rewrite equation (16) as

$$\begin{aligned}\hat{a}_\alpha^\dagger &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{\alpha m^\alpha \omega^\alpha}{\hbar_\alpha^\alpha}}X_\alpha - \sqrt{\frac{\hbar_\alpha^\alpha}{\alpha m^\alpha \omega^\alpha}}D_x^\alpha\right]\\ &= \frac{1}{\sqrt{2}}\left[\sqrt{\frac{\alpha m^\alpha \omega^\alpha}{\hbar_\alpha^\alpha}}X_\alpha - \sqrt{\frac{\hbar_\alpha^\alpha}{\alpha m^\alpha \omega^\alpha}}x^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\right].\end{aligned}$$

Using $X_\alpha = \frac{x^\alpha}{\alpha}$, then $\frac{\mathrm{d}}{\mathrm{d}X_\alpha} = x^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}x}$, and

$$\hat{a}_\alpha^\dagger = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{\alpha m^\alpha \omega^\alpha}{\hbar_\alpha^\alpha}}X_\alpha - \sqrt{\frac{\hbar_\alpha^\alpha}{\alpha m^\alpha \omega^\alpha}}\frac{\mathrm{d}}{\mathrm{d}X_\alpha}\right].$$

We define $Y = \sqrt{\frac{\alpha m^\alpha \omega^\alpha}{\hbar_\alpha^\alpha}}X_\alpha$, then $\mathrm{d}Y = \sqrt{\frac{\alpha m^\alpha \omega^\alpha}{\hbar_\alpha^\alpha}}\mathrm{d}X_\alpha$ and $\frac{\mathrm{d}}{\mathrm{d}Y} = \sqrt{\frac{\hbar_\alpha^\alpha}{\alpha m^\alpha \omega^\alpha}}\frac{\mathrm{d}}{\mathrm{d}X_\alpha}$. In terms of Y, we have

$$\hat{a}_\alpha^\dagger = \frac{1}{\sqrt{2}}\left(Y - \frac{\mathrm{d}}{\mathrm{d}Y}\right). \tag{36}$$

Substituting this equation in (35) we obtain

$$\psi_n = \frac{1}{\sqrt{2^n n!}}\left(Y - \frac{\mathrm{d}}{\mathrm{d}Y}\right)^n \psi_0. \tag{37}$$

Making use of equation (26) in (37), and following ref. [30], one may show

$$\psi_n = \left(\frac{\alpha m^\alpha \omega^\alpha}{\pi\hbar_\alpha^\alpha}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(Y)\exp\left(-\frac{Y^2}{2}\right). \tag{38}$$

The eigenfunction for n-excited state in terms of Hermite polynomials is then given as

$$\begin{aligned}\psi_n &= \left(\frac{\alpha m^\alpha \omega^\alpha}{\pi\hbar_\alpha^\alpha}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n\left(\sqrt{\frac{\alpha m^\alpha \omega^\alpha}{\hbar_\alpha^\alpha}}X_\alpha\right)\\ &\quad\times \exp\left(-\frac{\alpha m^\alpha \omega^\alpha X_\alpha^2}{2\hbar_\alpha^\alpha}\right).\end{aligned} \tag{39}$$

3.4 Eigenfunctions in terms of conformable Hermite polynomial

The eigenfunctions for the excited state equation (35) can be expressed in terms of the Hermite polynomials as follows. First, we rewrite equation (16)

$$\hat{a}_\alpha^\dagger = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{\alpha m^\alpha \omega^\alpha}{\hbar_\alpha^\alpha}}\frac{x^\alpha}{\alpha} - \sqrt{\frac{\hbar_\alpha^\alpha}{\alpha m^\alpha \omega^\alpha}}D_x^\alpha\right].$$

We define $y^\alpha = \sqrt{\frac{\alpha m^\alpha \omega^\alpha}{\hbar_\alpha^\alpha}}\frac{x^\alpha}{\alpha}$, then $\alpha y^{\alpha-1}\mathrm{d}y = \sqrt{\frac{\alpha m^\alpha \omega^\alpha}{\hbar_\alpha^\alpha}}x^{\alpha-1}\mathrm{d}x$, and $\frac{y^{1-\alpha}}{\alpha}\frac{\mathrm{d}}{\mathrm{d}y} = \sqrt{\frac{\hbar_\alpha^\alpha}{\alpha m^\alpha \omega^\alpha}}x^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}x}$, so we get $\frac{1}{\alpha}D_y^\alpha = D_x^\alpha$. In terms of Y, we have

$$\hat{a}_\alpha^\dagger = \frac{1}{\sqrt{2}}\left(y^\alpha - \frac{1}{\alpha}D_y^\alpha\right). \tag{40}$$

Substituting this equation in (35) we obtain

$$\psi_n = \frac{1}{\sqrt{2^n n!}}\left(y^\alpha - \frac{1}{\alpha}D_y^\alpha\right)^n \psi_0. \tag{41}$$

From equation (26) $\psi_0 = \left(\frac{\alpha m^\alpha \omega^\alpha}{\pi\hbar_\alpha^\alpha}\right)^{\frac{1}{4}}\exp\left(-\frac{y^{2\alpha}}{2}\right)$, we have

$$\psi_n = \frac{1}{\sqrt{2^n n!}}\left(\frac{\alpha m^\alpha \omega^\alpha}{\pi\hbar_\alpha^\alpha}\right)^{\frac{1}{4}}\left(y^\alpha - \frac{1}{\alpha}D_y^\alpha\right)^n \exp\left(-\frac{y^{2\alpha}}{2}\right). \tag{42}$$

Making use of conformable Hermite polynomial [31]:

$$H_n^\alpha(y) = \frac{(-1)^n}{\alpha^n}\exp(y^{2\alpha})D_y^{n\alpha}\exp(-y^{2\alpha}). \tag{43}$$

Thus, using the relation (see Appendix B)

$$\left(y^\alpha - \frac{1}{\alpha}D_y^\alpha\right)^n \exp\left(-\frac{y^{2\alpha}}{2}\right) = \exp\left(-\frac{y^{2\alpha}}{2}\right)H_n^\alpha(x), \tag{44}$$

we adopt $D^{n\alpha}$ to denote the conformable derivative n-times.

The eigenfunction for n-excited state in terms of conformable Hermite polynomials is then given as

$$\psi_n = \frac{1}{\sqrt{2^n n!}}\left(\frac{\alpha m^\alpha \omega^\alpha}{\pi\hbar_\alpha^\alpha}\right)^{\frac{1}{4}}\exp\left(-\frac{y^{2\alpha}}{2}\right)H_n^\alpha(y). \tag{45}$$

4 Summary and conclusions

In this article, an algebraic method, using α-creation operator $\hat{a}_\alpha^\dagger$ and α-annihilation operator $\hat{a}_\alpha$, is established for the conformable harmonic oscillator. The Hamiltonian for the systems is written in terms of these operators. It is found that for a given order α, the α-creation operator has the effect of promoting the present state of the system while the α-annihilation operator demotes the state. The system is quantized in terms of $\hat{a}_\alpha^\dagger$ and $\hat{a}_\alpha$ and the energy eigenvalues and eigenfunctions are obtained. The eigenfunctions are expressed in terms of the conformable Hermite functions. The results for the traditional quantum harmonic oscillator are found to be recovered by setting $\alpha = 1$. The formulation of the harmonic oscillator using $\hat{a}_\alpha^\dagger$ and $\hat{a}_\alpha$ may be useful in the formulation of conformable field quantization.

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Appendix A

$$[\hat{a}_\alpha, (\hat{a}_\alpha^\dagger)^n] = n(\hat{a}_\alpha^\dagger)^{n-1}. \tag{46}$$

Proof. Using number operator

$$\hat{N} = \hat{a}_\alpha \hat{a}_\alpha^\dagger. \tag{47}$$

We calculate this commutation relation

$$\begin{aligned} N(\hat{a}_\alpha^\dagger)^n &= N\hat{a}_\alpha^\dagger(\hat{a}_\alpha^\dagger)^{n-1} \\ &= (\hat{a}_\alpha^\dagger + \hat{a}_\alpha^\dagger N)(\hat{a}_\alpha^\dagger)^{n-1} \\ &= (2(\hat{a}_\alpha^\dagger)^2 + (\hat{a}_\alpha^\dagger)^2 N)(\hat{a}_\alpha^\dagger)^{n-2} \\ &= \cdots = (n(\hat{a}_\alpha^\dagger)^n + (\hat{a}_\alpha^\dagger)^n N). \end{aligned}$$

So, we have

$$[N, (\hat{a}_\alpha^\dagger)^n] = n(\hat{a}_\alpha^\dagger)^n, \tag{48}$$

and substituting equation (47) in this commutation relation we have

$$\begin{aligned} [\hat{a}_\alpha^\dagger \hat{a}_\alpha, (\hat{a}_\alpha^\dagger)^n] &= \hat{a}_\alpha^\dagger[\hat{a}_\alpha, (\hat{a}_\alpha^\dagger)^n] + [\hat{a}_\alpha^\dagger, (\hat{a}_\alpha^\dagger)^n]\hat{a}_\alpha \\ &= \hat{a}_\alpha^\dagger[\hat{a}_\alpha, (\hat{a}_\alpha^\dagger)^n] + 0 = n(\hat{a}_\alpha^\dagger)^n. \end{aligned}$$

Thus, we obtain

$$\hat{a}_\alpha^\dagger[\hat{a}_\alpha, (\hat{a}_\alpha^\dagger)^n] = n(\hat{a}_\alpha^\dagger)^n \to [\hat{a}_\alpha, (\hat{a}_\alpha^\dagger)^n] = n(\hat{a}_\alpha^\dagger)^{n-1}$$

and $\hat{a}_\alpha(\hat{a}_\alpha^\dagger)^n = (\hat{a}_\alpha^\dagger)^n \hat{a}_\alpha + n(\hat{a}_\alpha^\dagger)^{n-1}$. □

Appendix B

Proof.

$$\exp\left(-\frac{y^{2\alpha}}{2}\right)\left(y^\alpha - \frac{1}{\alpha}D_y^\alpha\right)\exp\left(\frac{y^{2\alpha}}{2}\right)f(y) = -\frac{1}{\alpha}D_y^\alpha f(y), \tag{49}$$

for n-times we get

$$\begin{aligned} &\left(y^\alpha - \frac{1}{\alpha}D_y^\alpha\right)^n \exp\left(\frac{y^{2\alpha}}{2}\right)f(y) \\ &= \frac{(-1)^n}{(\alpha)^n}\exp\left(\frac{y^{2\alpha}}{2}\right)D_y^\alpha f(y), \end{aligned} \tag{50}$$

let $f(y) = \exp\left(-\frac{y^{2\alpha}}{2}\right)$, we obtain

$$\left(y^\alpha - \frac{1}{\alpha}D_y^\alpha\right)^n = \frac{(-1)^n}{\alpha^n}\exp\left(\frac{y^{2\alpha}}{2}\right)D_y^{n\alpha}\exp\left(-\frac{y^{2\alpha}}{2}\right), \tag{51}$$

multiply this equation from right side by $\exp\left(-\frac{y^{2\alpha}}{2}\right)$, thus, we have

$$\begin{aligned} &\left(y^\alpha - \frac{1}{\alpha}D_y^\alpha\right)^n \exp\left(-\frac{y^{2\alpha}}{2}\right) \\ &= \exp\left(-\frac{y^{2\alpha}}{2}\right)\left[\frac{(-1)^n}{\alpha^n}\exp(y^{2\alpha})D_y^{n\alpha}\exp(-y^{2\alpha})\right]. \end{aligned}$$

So, using conformable Hermite polynomial in equation (43), we obtain

$$\left(y^\alpha - \frac{1}{\alpha}D_y^\alpha\right)^n \exp\left(-\frac{y^{2\alpha}}{2}\right) = \exp\left(-\frac{y^{2\alpha}}{2}\right)H_n^\alpha(x). \tag{52}$$

□