Quantization of Simple Parametrized Systems

By

GIULIO RUFFINI
B.A. (University of California, Berkeley) 1988
M.S. (University of California, Davis) 1990

DISSERTATION

Submitted in partial satisfaction of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Physics

in the

GRADUATE DIVISION

of the

UNIVERSITY OF CALIFORNIA

DAVIS

Approved:

Dr. Ling-Lie Chau, UC Davis

Dr. Emil Mottola, Los Alamos National Laboratory

Dr. Albert Schwarz, UC Davis

Committee in Charge

1995

Abstract

I study the canonical formulation and quantization of some simple parametrized systems using Dirac's formalism for constrained systems and the Becchi-Rouet-Stora-Tyutin (BRST) extended phase space method. These systems include the non-relativistic parametrized particle, the relativistic parametrized particle and minisuperspace.

Using Dirac's formalism for constrained systems—including the Dirac bracket—I analyze for each case the construction of the classical reduced phase space and study the dependence on the gauge fixing used. I show that there are two separate features of these systems that may make this construction difficult:

a) Because of the boundary conditions used, the actions are not gauge invariant at the boundaries.

b) The constraints may have a disconnected solution space.

The relativistic particle and minisuperspace have such complicated constraints, while the non-relativistic particle displays only the first feature.

After studying the role of canonical transformations in the reduced phase space, I show that a change of gauge fixing is equivalent to a canonical transformation in the reduced phase space. This result clarifies the problems associated with the first feature above, which until now have clouded the understanding of reparametrization invariant theories.

I then consider the quantization of these systems using several approaches: Dirac's method, Dirac-Fock quantization, and the BRST formalism. I pay special attention to the development of the inner product in the physical space. In the cases of the relativistic particle and minisuperspace I consider first the quantization of one branch of the constraint at the time and then discuss the gravitational and electromagenetic backgrounds in which it is possible to quantize simultaneously both branches and still obtain a unitary quantum theory which respects space-time covariance. I show that the two branches represent the particle (universe) going back and forth in time, and that to preserve unitarity and space-time covariance, second quantization is in general needed. An exception is provided by the flat case with zero electric field.

I motivate and define the inner product in all these cases, and obtain, for example, the Klein-Gordon inner product for the relativistic case. Then I show how to construct phase space path integral representations for amplitudes in these approaches—the Batalin-Fradkin-Vilkovisky (BFV) and the Faddeev path

integrals—from which one can then derive the path integrals in coordinate space—the Faddeev-Popov path integral and the geometric path integral. In particular I establish the connection between the Hilbert space representation and the range of the lapse in the path integrals, which leads to the Feynman propagator in the BRST-Fock case, for example. The role of the Faddeev determinant in the path integrals in providing the interaction between the branches is established.

I also examine the class of paths that contribute in the path integrals and how they affect space-time covariance in the presence of an electromagnetic field. I show that it is consistent to take paths that move forward in time only when there is no electric field, just as one would expect from studying the conditions for the covariant factorization of the Klein-Gordon equation. The key elements in this analysis are the space-like paths and the behavior of the action under the non-trivial element of Z_2, the disconnected part of the reparametrization group.

Contents

Introduction

The quantization of gravity—the most important parametrized system—remains one of the biggest challenges in physics. In the case of gravity, the quantization program is full of problems, not the least of which is that the theory is not renormalizable. However, another important feature of gravity is that it is a constrained system, and it is difficult in general to quantize classical systems with constraints. When the constraints have solution spaces with complex topologies it becomes extremely difficult to produce a unitary quantum theory, specially if one is also trying to implement some symmetries. General relativity is such a constrained system, as I will now explain.

The action for gravity is given by

$$S_H = \int d^4x \sqrt{{}^{(4)}g}\ ({}^{(4)}R - 2\Lambda)$$

In the ADM (Arnowitt-Deser-Misner) formalism [8–10] it is assumed that the topology of space-time is of the form $\mathbb{R} \times \Sigma$, and with the metric expressed by

$$ds^2 = N^2 dt^2 - g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

it becomes, in the canonical formalism (g and R stand for the 3-geometry metric and curvature in the time slices, unless specified otherwise by a (4) supersript)

$$S_H = \int dt \int_\Sigma d^3x \, (\pi^{ij}\dot{g}_{ij} - N^i \mathcal{H}_i - N\mathcal{H})$$

where the constraints are

$$\mathcal{H}_i = -2\nabla_j \pi_i^j, \quad \mathcal{H} = G_{ijkl}\, \pi^{ij}\pi^{kl} - \sqrt{g}(R - 2\Lambda)$$

with

$$G_{ijkl} = \frac{1}{2\sqrt{g}}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl})$$

and where $\pi^{ij}(x)$ is the momentum conjugate to $g_{ij}(x)$,

$$\{g_{ij}(x), \pi^{kl}(x')\} = \delta_i^k \, \delta_j^l \, \delta^3(x - x')$$

The linear constraints generate the space diffeomorphisms, while the constraint *quadratic in the momenta* generates the dynamics—and is the one that sets gravity apart from Yang-Mills. This is a first class system (in the language of Dirac) with a complicated algebra [9]

$$\begin{aligned}
\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} &= \mathcal{H}_j \delta^3(x - x')_{,i} - \mathcal{H}_i \delta^3(x' - x)_{,j} \\
\{\mathcal{H}_i(x), \mathcal{H}(x')\} &= \mathcal{H}(x)\delta^3(x - x')_{,i} \\
\{\mathcal{H}_i(x), \mathcal{H}(x')\} &= g^{ij}(x)\mathcal{H}_j(x)\delta^3(x - x')_{,i} - g^{ij}(x')\mathcal{H}_j(x')\delta^3(x' - x)_{,i}
\end{aligned}$$

Let me compare this system to Yang-Mills (see, for example [2], ex. 19.4), another first class system. The Yang-Mills action is given by

$$S_{YM} = -\frac{1}{4}\int d^4x \, F^a_{\mu\nu} F_a^{\mu\nu}$$

with

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - C_{bc}{}^a A^b_\mu A^c_\nu$$

and where $C_{bc}{}^a$ are the structure constants of a Lie group. Going to the hamiltonian formalism we find the primary constraints $\pi^0_a(x) \approx 0$, and then the secondary ones

$$\Phi_a = \partial_i \pi^i_a - C^b{}_{ac} A^c_i \pi^i_b \approx 0$$

again, one per index—including the space index. They satisfy the first class algebra

$$\{\Phi_a(x), \Phi_b(x')\} = C^c{}_{ab} \Phi_c \, \delta^3(x - x')$$

The hamiltonian is not zero here

$$H = \int d^3x \, \frac{1}{2}(\vec{\pi}^2 + \vec{B}^2), \quad B^{ai} = \frac{1}{2}\epsilon^{ijk} \partial_j A^a_k$$

unlike the gravitational case. The extended hamiltonian reads

$$\mathcal{H}_E = \int d^3x \, \left(H + \left(A^{0a} + \lambda^a_2\right) \Phi_a + \lambda_1 \pi^0_a\right)$$

Most importantly, *the constraints are linear in the momenta, which, as we will see, is related to the fact that Yang-Mills is a true gauge theory—fully invariant.*

Tied to the problems that arise because the constraint $\mathcal{H}$ is quadratic in the momenta are the issues of the definition of the observables and inner product in quantum gravity, as well as the issue of whether it is necessary to develop the the so-called "third quantization" programme. Is there really universe creation and annihilation?

I will not investigate the quantization of gravity here, but will look instead at some simpler systems that share some of its problems (constraints non-linear in the momenta) while being manageable—systems with a finite number of degrees of freedom. These include the parametrized non-relativistic particle, the relativistic particle—in a variety of background fields—and minisuperspace. The knowledge gained from the study of these lesser problems should help us with the more difficult task. At any rate, there is no hope to quantize gravity if the quantization of simple parametrized systems like the relativistic particle cannot be accomplished! And there are some problems, as we will see.

I will first discuss the quantization of the parametrized non-relativistic particle [51], and investigate the problems associated with the use of a gauge dependent action. It is crucial to understand this simple system before investigating the relativistic case. It will be very helpful to see clearly what problems are associated with the reparametrization invariance alone, as opposed to those associated with a constraint whose solution space is split in two—as in the relativistic case.

The parametrized relativistic particle will prove to be a surprisingly interesting system from the point of view of constrained systems, and understanding it will be helpful in a number of ways. The main reason for studying the relativistic particle is that it is a simple parametrized theory, and it provides us with what should be an almost perfunctory testing ground for some old and some more recent machinery in constrained system quantization: Dirac's method [1, 2], BRST theory [2, 19] and several path integral methods. On the other hand, the type of constraint one finds in these systems is neither linear nor topologically simple, so it provides a serious point of departure from simple constrained systems described essentially by a vanishing momentum, $P \approx 0$. Moreover, mathematically at least, this system is very close to simple minisuperspace models, and streching things a bit, to full gravity theory itself.

The similarities of this system to minisuperspace and—on a grander scale—to gravity itself thus provide the motivation for this research, as these toy models may help us in that more difficult programme. These similarities are the following two:

First, this system contains an inherent complication of typical parametrized theories: all such systems are reparametrization invariant theories, and they suffer from the problem that *their actions are not invariant at the boundaries* [14]—which, as we will see, can be understood to be a consequence of the use of boundary conditions on the gauge degrees of freedom. In the hamiltonian formalism this situation is described by constraints that are not linear in the momenta.

It is hard to define an associated reduced phase space to a system that is not fully invariant! This much is just as true of the non-relativistic case, which also comes with a non-linear constraint.

In addition, *the constraint surface in this case is disconnected* (unlike the corresponding situation in the non-relativistic case), and this complicates considerably the quantization procedure. The disconnectedness of the constraint surface is related to particle creation—as we will see—and unitarity will become a relevant issue. I will discuss when and why one needs to develop a field theory (chapter 6).

There are two main approaches to quantization of constrained systems. In one, the gauge degrees of freedom are eliminated classically, before quantization is considered. The formalism for doing this was developed by Dirac [1]. When these degrees of freedom have been eliminated the resulting theory can be quantized in the usual way. Here the important question is: What is the reduced phase space (i.e., what are the physical degrees of freedom)? The main advantadge of this method is that the most difficult part—the reduction—is performed at the classical level. The disadvantadge is that this classical reduction may force us to brake invariances—or perhaps just to lose sight of them. At any rate, this is the point of view that we will begin with: for constrained systems quantization means the quantization of the reduced phase space, i.e., quantization of some well defined degrees of freedom which are found classically and then quantized.

In the other approach (also pioneered by Dirac), one first quantizes—assigning operators to every degree of freedom, both physical and gauge—and then reduces to a physical "subspace" by demanding that the *physical states* satisfy some condition. The advantadge of this method is the preservation of invariances, and the disadvantadge is that the reduction becomes, mathematically, more difficult. One of the biggest problems will be that it is difficult to define an inner product in the physical "subspace". The reason is that in general the physical "subspace" is not a subspace at all.

One may ask if the two approaches lead to the equivalent theories, and for basic systems they do—modulo operator orderings[1], which are always present. This may not be true of the systems we will study here. So here is an important question: Do reduction and quantization commute? If not, which one should one use?

Let me point out here the main ingredients in standard quantum mechanics:

1. the states (resolution of identity)
2. the inner product
3. a Hamiltonian for unitary *time* evolution
4. a probabilistic interpretation

These are what we are really after, all the technical details notwithstanding—and there will be many, so it will be good to keep this scheme in mind. As a result, some of the basic issues associated with the quantization of *constrained* systems that will need to be addressed are the following:

A) What is the physical Hilbert space?

B) What is the physical inner product?

[1]I will ask the reader to always keep in mind that this ambiguity is part of the transition from classical to quantum, and is not a new feature associated with quantization of constrained systems.

C) What describes "time" and "time evolution"?

D) How can we build a path integral from the above?

E) Is the probabilistic interpretation possible?

In the present work I will consider several different approaches to quantization: reduced phase space quantization [2,18], Dirac's original method [2], the Fock space approach [2], and the Becchi-Stora-Rouet-Tyutin (BRST) [2,19].

I will also consider several path integral methods: in phase space we will study the Faddeev [2,18] and the Batalin-Fradkin-Vilkovisky (BFV) [2,19] path integrals, and in configuration space we will look at the Faddeev-Popov and at the geometric path integrals [16].

One basic goal of this work is to compare all these methods. Do they lead to the same quantum theories?

A more ambitious goal is to provide the connection between the Hilbert space and operator formalisms and the path integrals, which has been a serious fault in the conceptual basis for these path integral approaches. One of the original questions for my research, which was formulated by Emil Mottola, was indeed the following: "We have a path integral for quantum gravity: what does it mean? How do you compute it?" Both of these questions—which I still haven't answered—will find their resolution within a well-defined canonical formalism for quantum gravity from which the path integrals will be constructed.

To refresh the reader's memory—and also to point out some ambiguities—I will review in the first chapter how the quantization picture is constructed in the case of the unconstrained non-relativistic particle, including how the path integral in phase space is built from the Hilbert space and the hamiltonian, and how the path integral in configuration space is then obtained from the phase space path integral by the integration of the momenta.

In general, the picture we want to see emerge is[2]

Q. Mechanics $\longrightarrow$ *Path Integral in PS* $\longrightarrow$ *Path Integral in CS*

The above issues are related to a series of technical questions that have appeared repeatedly in the literature, among which are the following:

o) What form of the constraint (or gauge fixing) should one use? Does it matter?

i) What are the inner product and resolution of the identity in the reduced phase space quantization?

ii) What are the inner product and resolution of the identity in the Dirac quatization?

iii) What are the inner product and the resolution of the identity in the BRST state cohomology space?

iv) How does on go from these quantum spaces to their respective path

[2] *PS* stands for Phase Space, and *CS* for Configuration Space.

integrals?

v) What is the range of the "lapse" and the interpretation of the path integrals?

vi) What paths contribute in the path integrals?

vii) Do we have a consistent first quantization scheme for these systems?

viii) What is the role of the disconnected part of the reparametrization group?

Chapter 2 will discuss the classical aspects of parametrized systems: the actions and their invariances—for the free and interacting cases—the constraints that arise, the notions of gauge-fixing and its relation to the reduced phase space, the Dirac bracket, as well as the concept of canonical transformations in the reduced phase space and how they are related to the gauge-fixing. The BRST formalism will also be introduced in this classical context. Chapter 3 will take a formal look at the different quantization schemes I will consider here: reduced phase space, Dirac, BRST and Fock space quantization. The discussion will cease to be formal in chapter 4, where I will discuss in detail the Hilbert spaces involved in the quantization as well as the inner products defined on them. The path integrals will be derived here, and the Fradkin-Vilkovisky theorem proved.

The path integral formalisms—in phase space and in configuration space—will be further developed in chapter 5. By now it should be clear to the reader that this is the proper place for this chapter. Path integrals are very mysterious objects when one doesn't have a state space, an inner product and operators to describe and build them with, and as I already explained one of the goals of my research has been to find as solid a basis as possible for the development and interpretation of the path integrals for constrained systems[3].

I will also study the connection between the class of paths that appear in the path integrals and space-time covariance and unitarity, and how this, in turn, relates to the behavior of the action under the disconnected part of the diffeomorphism group.

Discussion of equivalence of the different methods will be folded in as we go along, and also reviewed in the general discussion at the conclusion. Second quantization—i.e., the idea of finding and quantizing a field lagrangian which as the constraint as an equation of motion—will be discussed in chapter 6.

[Nov 2005 note: see [53] for a follow up paper on this thesis.]

[3] As far as I know, one cannot really do it the other way around.

Chapter 1

The unconstrained particle

As a reference point I include first the derivation of the standard phase-space path integral for the unconstrained non-relativistic particle, starting from the quantum mechanical expression for the propagator. This example is very important, as it illustrates what one would like to have in the other cases, i.e., a well defined quantum mechanical framework from which to construct the path integral expression for the propagator, first in phase space and then in configuration space, according to the picture mentioned in the introduction:

Q. Mechanics $\longrightarrow$ *Path Integral in PS* $\longrightarrow$ *Path Integral in CS*

This is the model we will try to emulate in the discussion of the physical spaces in the parametrized systems.

I will also comment on the issues of unitarity and on the freedom in the definition of the inner product.

1.1 Quantum mechanics of the unconstrained particle

Classically, we have that the standard action for the non-relativistic particle is

$$I = \int_{t_i}^{t_f} L\,dt = \int_{t_i}^{t_f} dt \left(\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right) \tag{1.1}$$

and the equation of motion that results from extremizing it is

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x} \tag{1.2}$$

One can also dicuss the dynamics in the canonical formalism by defining the momentum

$$p = \frac{\partial L}{\partial(dx/dt)} = m\frac{dx}{dt} \tag{1.3}$$

and the hamiltonian $H = p\frac{dx}{dt} - L = \frac{p^2}{2m} + V$, with the equations of motion

$$\frac{dx}{dt} = \{x, H\} = \frac{p}{m} \tag{1.4}$$

$$\frac{dp}{dt} = \{p, H\} = -\frac{\partial V}{\partial x} \tag{1.5}$$

which are equivalent to the one in the lagrangian formalism.

The quantum mechanical framework is defined here by a Hilbert space of states $|\Psi\rangle$. The basic operators are $\mathbf{x}$ and $\mathbf{p}$, with the commutator Heisenberg algebra $[\mathbf{x}, \mathbf{p}] = i$. A basis for the Hilbert space is provided by the eigenstates of either one of these hermitean operators, $\mathbf{x}|x\rangle = x|x\rangle$, or $\mathbf{p}|p\rangle = p|p\rangle$. Completeness of these states is expressed by the following resolutions of the identity:

$$I = \int dx\, |x\rangle\langle x| = \int dp\, |p\rangle\langle p| \tag{1.6}$$

where the integrations are taken over their full ranges. The following projection can be inferred from the above algebra

$$\langle x|p\rangle = \sqrt{\frac{1}{2\pi}} e^{ipx} \tag{1.7}$$

Indeed, consider the action of $\mathbf{p}$ on the state $|p\rangle$ in the x representation:

$$\langle x|\mathbf{p}|p\rangle = p\langle x|p\rangle = -i\frac{\partial}{\partial x}\langle x|p\rangle \tag{1.8}$$

The solution to this equation is the above projection formula.

Notice that here we are using a representation of the algebra in which

$$\langle x|\mathbf{p}|f\rangle = -i\frac{\partial}{\partial x}\langle x|f\rangle$$

The positive definite inner product[1] is defined by

$$\langle\Psi|\Sigma\rangle = (\Psi(x), \Sigma(x)) = \int_{-\infty}^{\infty} dx\ \Psi^*(x)\Sigma(x) \tag{1.9}$$

where, for example, $\Psi(x) = \langle x|\Psi\rangle$, is the state in the coordinate representation. The probabilistic interpretation is tied to the existence of this inner product, which has the key properties of being positive definite—yielding positive norms—and, as we will see, time-independent.

[1]By this I mean that this inner product yields positive norms for the states.

Time evolution is described by Schrödinger's equation[2]

$$i\frac{\partial}{\partial t}|\Psi\rangle = \mathbf{H}|\Psi\rangle \tag{1.10}$$

In this case we take the hamiltonian to be $H = \frac{p^2}{2m} + V(x)$. The solution to this equation is

$$|\Psi\rangle = e^{-i\mathbf{H}(t-t_o)}|\Psi_0\rangle \tag{1.11}$$

for some initial state $|\Psi_0\rangle$. The time independence of the inner product is now easy to see.

The propagator in the coordinate representation is defined to be

$$U(x_f, t_f; x_i, t_i) \equiv \langle x_f|e^{-i\mathbf{H}(t_f-t_i)}|x_i\rangle \tag{1.12}$$

(=U for short). The path integral in phase space is now easily built by inserting the above resolutions of the identity in this definition. First break the time interval into N steps and use the multiplication property of the exponential N times,

$$\langle x_f|e^{-i\mathbf{H}(t_f-t_i)}|x_i\rangle = \langle x_f|e^{-i\mathbf{H}\epsilon}e^{-i\mathbf{H}\epsilon}...e^{-i\mathbf{H}\epsilon}|x_i\rangle \tag{1.13}$$

where $\epsilon = \frac{t_f-t_i}{N}$. Then insert the above resolutions of the identity in this expression,

$$U = \int \langle x_f|e^{-i\mathbf{H}\epsilon}|p_0\rangle\langle p_0|x_1\rangle\langle x_1|e^{-i\mathbf{H}\epsilon}...|p_N\rangle\langle p_N|e^{-i\mathbf{H}\epsilon}|x_i\rangle dp_0 \prod_{i=1}^{N} dx_i dp_i \tag{1.14}$$

Taking the limit $\epsilon \longrightarrow 0$ we have

$$\langle x|e^{-i\mathbf{H}\epsilon}|p\rangle \approx \langle x|(1 - i\mathbf{H}\epsilon)|p\rangle \tag{1.15}$$

[2] I will refer to any equation of the form $i\partial_t\psi = \hat{H}\psi$ for some hamiltonian, as a "Schrödinger equation".

$$= \langle x|(1 - iH\epsilon)|p\rangle \approx \langle x|e^{-iH\epsilon}|p\rangle \tag{1.16}$$

$$= e^{-iH\epsilon}\langle x|p\rangle = \sqrt{\frac{1}{2\pi}}e^{i(px-H\epsilon)} \tag{1.17}$$

where we used the projection $\langle x|p\rangle = \sqrt{\frac{1}{2\pi}}e^{ipx}$ (which follows from the Heisenberg algebra). Here H is *defined* to be

$$H \equiv \langle x|\mathbf{H}|p\rangle = H(x,p) \tag{1.18}$$

Now we can write

$$U = \int_{t_i}^{t_f} e^{i\Sigma_j(p_j\Delta x_j - iH_j\epsilon)}\frac{dp_0}{2\pi}\prod_{i=1}^{N}\frac{dx_i dp_i}{2\pi} \tag{1.19}$$

The resulting expression for the propagator is symbolized by[3]

$$U = \int DxDp\; e^{i\int_{t_i}^{t_f} pdx - Hdt} = \int DxDpe^{i\int_{t_i}^{t_f} dt(p\dot{x}-H)} \tag{1.20}$$

where the measure here means

$$DxDp \equiv \frac{dp_0}{2\pi}\prod_{i=1}^{N}\frac{dx_i dp_i}{2\pi} \tag{1.21}$$

Notice that we could have simply written

$$e^{i\mathbf{H}\Delta t} = \lim_{N\to\infty}\left(1 + i\mathbf{H}\frac{\Delta t}{N}\right)^N \tag{1.22}$$

to insert the resolutions of the identity. The right hand side is in fact the propagator for a general, time-dependent hamiltonian [4, 6] (see equation 1.28 below). *Thus,*

[3] In this chapter only we define $\dot{a} \equiv da/dt$, instead of $da/d\tau$ —which will be the norm thereafter.

the expression above for the propagator in path integral form is in fact general. The path integral provides us with a solution to the Schrödinger equation that satisfies the boundary condition (initial condition) of becoming a delta function at the initial time—provided we normalize the position eigenstates in such a way.

This is then the path integral in phase space. The momentum integrations are easily done to yield Feynman's original path integral—which he obtained from a different point of view,

$$U = \int \mathcal{D}x \; e^{i\int_{t_i}^{t_f} dt(\frac{1}{2}m\dot{x}^2 - V(x))} \tag{1.23}$$

The measure here means

$$\mathcal{D}x = \left(\frac{m}{2\pi i dt}\right)^{\frac{1}{2}} \prod_{i=1}^{N} dx_i \left(\frac{m}{2\pi i dt}\right)^{\frac{1}{2}} \tag{1.24}$$

where dt stands for ϵ (or ϵ_i).

For the free particle ($V = 0$) we obtain

$$U(\Delta x, \Delta t) = \left(\frac{m}{2\pi i \Delta t}\right)^{\frac{1}{2}} e^{i\frac{(\Delta x)^2 m}{2\Delta t}} \tag{1.25}$$

At this point we should make a note on the class of paths that contribute in the coordinate space path integrals. Notice that the paths are described in the form $x = x(t)$, so by definition of the path integral they go forward in time—there is no chance to describe paths going back in time unless we do something strange like changing the sign of the hamiltonian.

Also notice that one could compute the expectation value of the "operator" $\Theta(\Delta t)$, where $\Theta(z)$ is the Heaviside theta function, and then obtain the causal Green's function for the Schrödinger equation—see section 5.1.4 for more on this—

$$\Gamma'_G = \frac{\Theta(\Delta t)}{2\pi} \int dp_0 \ \exp\{i(p_0 \Delta x - \frac{\Delta t p_0^2}{2m})\} = \Theta(\Delta t) U \tag{1.26}$$

which satisfies

$$(i\frac{\partial}{\partial t} - \frac{\mathbf{p}^2}{2})\Gamma'_G = \delta(\Delta t)\delta(\Delta x) \tag{1.27}$$

Let us now look at the case of electromagnetic interaction and at the issue of unitarity. Recall that *unitarity of the propagator results when the hamiltonian is hermitean, whether it is time independent or not* (see for example Shankar's book on quantum mechanics [4]). In general the propagator is given by

$$U(t_f, t_i) = \mathcal{T}[e^{-i\int_{t_i}^{t_f} dt' H(t')}] \equiv \prod_{i=1}^{N} e^{-iH(t_j)\Delta t_j} \tag{1.28}$$

and in the hermitean case it is a product of unitary operators and therefore unitary. It has the properties [4]

$$\begin{gathered} U(t_3, t_2)U(t_2, t_1) = U(t_3, t_1) \\ U^\dagger(t_2, t_1) = U^{-1}(t_2, t_1) = U(t_1, t_2) \end{gathered} \tag{1.29}$$

Any action that we can write in the form $\int dt\, L[\dot{x}(t), x(t), t]$ is therefore going to yield a unitary quantum theory with the usual procedure for constructing a hermitean hamiltonian, *if ordering problems are not encountered.*

Consider now the case of the non-relativistic case with an electromagnetic interaction. The lagrangian for such a case is given by

$$\mathcal{L}_{EM} = \frac{1}{2}mv_i^2 - e\phi + \frac{e}{c}v_i A_i \tag{1.30}$$

(see for example [4]), and the hamiltonian turns out to be

$$\mathcal{H}_{EM} = \frac{(p_i - eA_i/c)^2}{2m} + e\phi \tag{1.31}$$

The point to observe is that this hamiltonian will become, with the usual quantization recipe, a hermitean operator, as can readily be seen by expanding it:

$$\mathcal{H} = \frac{1}{2m}(p_i^2 + \frac{e^2}{c^2}A_i^2 - \frac{e}{c}(p_i A_i + A_i p_i)) + e\phi \tag{1.32}$$

Unitarity is therefore guaranteed with this action.

This completes the review of the non-relativistic unconstrained case. Keep in mind that this is the model we will try to imitate when we study the constrained case, as it is *the* example of well-defined quantum mechanics.

Let us now study a model for the relativistic unconstrained particle. One point of view is to use the hamiltonian

$$h = +\sqrt{p^2 + m^2} \tag{1.33}$$

as we will see. This follows from the action

$$s = -m\int_{t_i}^{t_f} dt\,\sqrt{1 - \dot{x}^2} \tag{1.34}$$

(one could also consider $h' = -h$ which follows from $s' = -s$). Notice that faster-than-light paths are allowed, although they will come in with a real exponential weight[4].

The form

$$s'' = \int_{t_i}^{t_f} dt \left(\frac{1 - \dot{x}^2}{\lambda(t)} + \lambda(t) m^2 \right) \tag{1.35}$$

—where $\lambda(t)$ is also to be varied—is very similar. The equation of motion for $\lambda(t)$ is just

$$\lambda(t) = \pm \frac{1}{2m} \sqrt{1 - \dot{x}^2}. \tag{1.36}$$

Substituting this in the action s'' yields either s or $-s$. Notice, though, that in a path integral λ will not be imaginary unless forced by a rotation of the integration contours.

What kind of quantum mechanics and path integral do we get from this hamiltonian? The path integral is easily computed,

$$U = \int DxDp\, e^{i \int_{t_i}^{t_f} dt (p\dot{x} - h)} = \int DxDp\, e^{i \int_{t_i}^{t_f} dt (p\dot{x} - \sqrt{p^2 + m^2})} = \tag{1.37}$$

$$\frac{1}{2\pi} \int dp\, e^{i(p\Delta x - \Delta t \sqrt{p^2 + m^2})} = \langle x_f | e^{-i\mathbf{h}\Delta t} | x_i \rangle \tag{1.38}$$

Faster than light "propagation" is indeed possible, as hinted by the above lagrangian; see reference [22] for more on this path integral.

[4]This is somewhat deceiving, as there is no such path integral in lagrangian form and with a simple measure [22].

Other relevant actions are ($i = 1, 2, 3;\ \mu = 0, 1, 2, 3;\ x^\mu = t, x$)

$$E = \int_{t_i}^{t_f} dt\ \left(-m\sqrt{1-\dot{x}_i^2} + A_0 - A_i\dot{x}_i^2\right) \tag{1.39}$$

with $h_E = -A_0 + \sqrt{(p_i + A_i)^2 + m^2}$ for the electromagnetic interacting case ($A_\mu = A_\mu(x^\alpha)$), and

$$G = -m\int_{t_i}^{t_f} dt\sqrt{c(x^\mu) - b(x^\mu)(\dot{x} + a(x^\mu))^2} \tag{1.40}$$

with $h_G = \sqrt{c(p^2/b + m^2)}$, for the gravitational background case. These are special cases of the general action for the relativistic particle in a curved space-time and in a background electromagnetic field—with $g_{i0} = g_{0i} = 0$,

$$A_{EG} = \int_{\tau_i}^{\tau_f} d\tau\left(-m\sqrt{g(x^\alpha)_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} - e\frac{dx^\mu}{d\tau}A_\mu(x^\alpha)\right) \tag{1.41}$$

or of the other form

$$A'_{EG} = \int_{\tau_i}^{\tau_f} d\tau\left(\frac{1}{\lambda(\tau)}g(x^\alpha)_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + m\lambda(\tau) - e\frac{dx^\mu}{d\tau}A_\mu(x^\alpha)\right) \tag{1.42}$$

in the gauge $t = \tau$, which altogether eliminates traveling back and forth in time (something as we will see that is related to particle creation and unitarity). The hamiltonian in the first case is just

$$h_{EG} = -eA_0 + \sqrt{g_{00}\{m^2 - (p_i + A_i)g^{ij}(p_j + A_j)\}} \tag{1.43}$$

1.2 Other inner products for the unconstrained particle

In this section I want to point out an ambiguity in the standard construction of a quantum formalism from a simple classical system.

Consider again the simple *unconstrained* classical system described by the coordinate x and the momentum p_x, and a hamiltonian H. How do we quantize it? Introduce the operators $\mathbf{x}$, $\mathbf{p_x}$, with $[\mathbf{x}, \mathbf{p_x}] = \mathbf{i}$, and a Hilbert space of states $|\psi\rangle$, together with a dual $\langle\psi|$. To be more specific we have the representation of the above operators in the form $\sim \mathbf{x}$, $\mathbf{p} \sim -\mathbf{i}\partial_{\mathbf{x}}$ with $(\psi_a, \psi_b) = \int dx\, \psi_a^*(x)\, \psi_b(x)$, where one has, as usual, defined $\psi(x) \equiv \langle x|\psi\rangle$.

Now, we could define

$$\tilde{\psi}(x) \equiv \langle x|\mathbf{A}|\psi\rangle \tag{1.44}$$

as long as the operator $\mathbf{A}$ has an inverse, and keep, for example, the dual as before,

$$\psi^D(x) \equiv \langle\psi|x\rangle \tag{1.45}$$

In terms of the old inner product, using the identity

$$\mathbf{I} = \int dx\, |x\rangle\langle x| \tag{1.46}$$

we obtain the expression

$$\langle\psi_a|\psi_b\rangle = \int dx\, \langle\psi_a|x\rangle\langle x|\mathbf{A^{-1}}\mathbf{A}|\psi_b\rangle = \int dx dx'\, \psi_a^D(x)\langle x|\mathbf{A^{-1}}|x'\rangle\langle x'|\mathbf{A}|\psi_b\rangle \tag{1.47}$$

Now, by definition of the representation we have (modulo ordering)

$$\langle x|\mathbf{A}|x'\rangle \equiv [\hat{A}(x,-i\partial_x)]^{-1}\langle x|x'\rangle \tag{1.48}$$

and $\langle x|x'\rangle = \delta(x-x')$, hence

$$\langle\psi_a|\psi_b\rangle = \int dx\ \psi_a^D(x)\ \left[\hat{A}\left(x,-i\partial_x\right)\right]^{-1}\tilde{\psi}_b(x) \tag{1.49}$$

As for the matrix elements of operators, they can be written as

$$\langle\psi_a|\mathbf{O}|\psi_b\rangle \equiv \int dx\ \psi_a^D(x)\ \hat{O}\ \hat{A}^{-1}\ \tilde{\psi}_b(x) \tag{1.50}$$

Physical quantites are of course left unchanged.

The point is that we can modify the representation of the states if we agree to compensate for the changes when we evaluate physical quantities, which must be left unchanged.

Alternatively, we can also trace the ambiguity to the definition of the observables. Assume we fixed the inner product. Is **O** the operator we want, or is it some "multiple" of it, $\mathbf{O}\,\mathbf{A^{-1}}$? If we have have a classical reference point—or experimental data—we will know what to do.

As an example, consider the free, unconstrained, *relativistic* particle. This system is,as we saw, equivalent to the unconstrained non-relativistic case, except that the hamiltonian is given by the square-root form $h=\sqrt{p_x^2+m^2}$. Now, there are two standard quantization schemes[5]. One involves using the Klein-Gordon inner

[5]This is discussed in the paper by Hartle and Kuchar, [31].

product, the other is the so-called Newton-Wigner quantization, which parallels the non-relativistic case very closely—indeed, the only difference is in the hamiltonian.

Let us look at these two cases. We can start in both situations with our standard quantum states and base kets, $|x\rangle, \langle x|x'\rangle = \delta(x - x')$, (with the same for the p_x kets). In terms of wave-functions defined with these kets, $\psi(x) \equiv \langle x|\psi\rangle$, the inner product is the usual one. However, we could choose to work instead with the kets

$$|\tilde{x}\rangle = (\mathbf{p_x^2} + \mathbf{m^2})^{-\frac{1}{4}}|x\rangle \tag{1.51}$$

Following the reasoning as above we will see that the wave-functions defined by $\tilde{\psi}(x) \equiv \langle \tilde{x}|\psi\rangle$ are designed to work with the Klein-Gordon inner product:

$$\langle \psi_a|\psi_b\rangle = \int dx\ \tilde{\psi}_a^*(x)\ \sqrt{\mathbf{p_x^2} + \mathbf{m^2}}\ \tilde{\psi}_b(x) \tag{1.52}$$

(The reason for calling this the Klein-Gordon inner product is that the action of this operator on the states that satisfy the Klein-Gordon equation is the same as the time derivative operator.)

[Note: The way this is described in the paper I mentioned above, [31], is as follows. First the new momentum states are defined,

$$|\tilde{p_x}\rangle \equiv \left(\sqrt{\mathbf{p_x^2} + \mathbf{m^2}}\right)^{\frac{1}{2}}\ |p_x\rangle \tag{1.53}$$

Now,

$$\mathbf{I} = \int dp_x\ |p_x\rangle\langle p_x| = \int \frac{dp_x}{\sqrt{p_x^2 + m^2}}\ |\tilde{p_x}\rangle\langle\tilde{p_x}| \tag{1.54}$$

Then the kets

$$|x_{NW}\rangle \equiv (2\pi)^{-1/2} \int \frac{dp_x}{\left(2\sqrt{p_x^2 + m^2}\right)^{\frac{1}{2}}} e^{ip_\alpha x^\alpha} |\tilde{p_x}\rangle = e^{itp_t} |x\rangle \tag{1.55}$$

are defined. These are unitarily equivalent to the old ones. The other states,

$$|x^\alpha\rangle \equiv (2\pi)^{-1/2} \int \frac{dp_x}{2\sqrt{p_x^2 + m^2}} e^{itp_t} |\tilde{p}\rangle \tag{1.56}$$

are essentially my $|\tilde{x}\rangle$'s.]

1.3 Comments on unitarity and causality

As it has already been remarked, the question of unitarity in this case boils down to a question of ordering of the operators that make up the hamiltonian. If we can find an ordering that makes the hamiltonian hermitean, then we are set, whether the hamiltonian is time dependent or not. However, in general there are other constraints on the possible orderings one can use. The need for space-time covariance, for example, may conflict with hermicity. The particular form of the hamiltonian we get also follows from the "gauge choice", from the definition of time and the corresponding foliation of space-time. Is there a nice, covariant ordering such that

$$\sqrt{g_{00}\{m^2 - (p_i + A_i)g^{ij}(p_j + A_j)\}} \tag{1.57}$$

is hermitean when made into an operator in the usual way? Well, the specific ordering I already used (by just interpreting the above equation as an operator equation) makes the hamiltonian hermitean, as long as

$$[\hat{g}_{00},\ (\hat{p}_i + \hat{A}_i)\hat{g}^{ij}(\hat{p}_j + \hat{A}_j)] = 0 \tag{1.58}$$

How about space-time covariance? For consistency *we need that the hamiltonian operator transform as the zero component of a vector*. If this demand is met, then we will have a consistent theory. Of course, we will have broken the strongest version of the equivalence principle, because our foliation choice picks a special direction in space-time.

As for *causality*, the square root operator leads to trouble, as may be expected from the appeareance of arbitrary orders of derivatives in the Taylor expansion of such an operator. This type of operator is non-local, as can be seen from writing the corresponding "square-root" Schrödinger equation[6] for the free case in semi-integral from [6]:

$$i\hbar\psi(x,t) = \int d^3x' \; K(x-x')\,\psi(x',t) \tag{1.59}$$

with

$$K(x-x') = \int \frac{d^3p}{(2\pi\hbar)^3}\, e^{ip(x-x')/\hbar}\sqrt{p^2+m^2} \tag{1.60}$$

The kernel is sizable within a Compton wavelength of the particle—and non-local. This leads to a violation of causality, because $\partial_t\psi$ depends on the values of ψ outside of the light cone.

Now, consider the D'Alembertian

$$g^{\mu\nu}\nabla_\mu\nabla_\nu = \frac{1}{\sqrt{g}}\,\partial_\mu[g^{\mu\nu}\sqrt{g}\;\partial_\nu\circ] \tag{1.61}$$

This is a scalar operator. If this operator can be factorized in a hermitean fashion we have an answer to the above question about ordering.

Suppose, for example, with $g_{0i} = 0$ as before, that we have

$$[\partial_0, \hat{g}^{\mu\nu}] = 0 = [\partial_i, \hat{g}^{00}] = 0 \tag{1.62}$$

[6]Again, I will refer to any equation of the form $i\partial_t\psi = \hat{H}\psi$ for some hamiltonian, as a "Schrödinger equation".

Then

$$\sqrt{\hat{g}_{00}\{m^2 - \frac{1}{\sqrt{\hat{g}}}(\hat{p}_i + \hat{A}_i)\hat{g}^{ij}\sqrt{\hat{g}}(\hat{p}_j + \hat{A}_j)\}} \tag{1.63}$$

will be hermitean with respect to the space integration. The issue of space-time covariance is far from simple, though, as we will see in the next section.

Also notice that the mass term should really be substituted by the more general possibility

$$\tilde{m}^2 = m^2 + \xi R \tag{1.64}$$

For small enough ξ there is no classical effect from this term.

All these issues are extremely relevant, because they will help settle the question of whether one can pick a branch and save unitarity and/or space-time covariance.

How are these issues related to the factorization of the Klein-Gordon equation? Can we bring these together under the particle creation point of view? We will discuss these questions in the next section, and come back to them later as well.

1.4 The Schrödinger equation and space-time covariance

In this section I would like to discuss the effect of Lorentz transformations on the Schrödinger equation.

Let us first state how a covariant Schrödinger equation should behave under Lorentz transformations. The wave-function must transform as a relative scalar 3-density of weight 1/2, if we are to construct a probability interpretation. What is imply meant by this is that

$$\int_{-\infty}^{\infty} d^3x \; \psi(x)^* \; \psi(x) = \int_{-\infty}^{\infty} d^3x \; \gamma\psi(\Lambda^{-1}x)^* \; \psi(\Lambda^{-1}x) \tag{1.65}$$

Similarly, if $|\psi(x^\mu)|^2$ is the probability density of finding the particle at x^μ, then it must transform as above so that $\int_{Vol} d^3x \; \psi(x)^* \; \psi(x)$ remains a constant. Thus, under a change of coordinates the wave-function changes by $\Psi(x^\mu) \longrightarrow \gamma^{1/2}\Psi(\Lambda_\nu^{-1\mu}x^\nu)$. This would be the ideal invariant behavior, but *we must check that under this change the wavefunction still satisfies the Schrödinger equation.* In terms of the differential operators involved, by a change of coordinates we can then see that we can get the same equation in the other coordinate system by requiring that the hamiltonian operator transform as the zero component of a 4-vector—again up to a factor of γ. To be specific consider the ground state of a hydrogen atom at rest, $\Psi_{00}(x^\mu)$. For another observer this state will appear as $\gamma^{1/2}\Psi_{00}(\Lambda_\nu^{-1\mu}x^\nu)$. This observer may ask if this state satisfies the Schrödinger equation—which he will write just as the other observer did. The only thing to be careful about is with *background* fields. If there is a background field, the correct covariance statement for an equation $D(\frac{\partial}{\partial x^\mu}, A_\mu(x^\mu))f(x^\mu) = 0$ is that this equation imply that

$$D(\frac{\partial}{\partial x^\mu}, \Lambda A_\mu(\Lambda^{-1}x^\mu))\ f(\Lambda^{-1}x^\mu) = 0$$

in other words, changing variables to $y = \Lambda^{-1}x$

$$D(\Lambda^\mu_\nu \frac{\partial}{\partial y^\nu}, \Lambda^\mu_\nu A_\nu(y^\mu))\ f(y^\mu) = 0 \tag{1.66}$$

Let us see how this is true for the Klein-Gordon equation. Suppose that

$$\left[(\partial_{x^\mu} - A_\mu(x^\alpha))\ \eta^{\mu\nu}\ (\partial_{x^\nu} - A_\nu(x^\alpha)) - m^2\right]\ \phi(x^\alpha) = 0 \tag{1.67}$$

It is easy to see that it then holds that

$$[(\partial_{x^\mu} - \Lambda A_\mu(\Lambda^{-1}x^\alpha))\,\eta^{\mu\nu}\,(\partial_{x^\nu} - \Lambda A_\nu(\Lambda^{-1}x^\alpha)) - m^2]\ \phi(\Lambda^{-1}x^\alpha) = 0 \tag{1.68}$$

To check, simply change the varibles to $y = \Lambda^{-1}x^\alpha$. The equation becomes the earlier one, written in terms of y, a dummy index.

It would be unreasonable to ask that $f(x^\mu)$ satisfy the exact same equation: the background field brakes absolute covariance. Thus he will take the point of view that otherwise the laws of physics should be the same in all inertial frames, and this includes the Schrödinger equation. Thus, he will perform a change in variables in his equation and check if the new equation he gets is true,

$$i\Lambda^\mu_0\,\frac{\partial}{\partial x'^\mu}\Psi(x'^\mu) = \hat{H}(\Lambda^\mu_\nu A^\nu(x'^\nu), \Lambda^\mu_\nu \partial'_\mu)\Psi(x'^\mu) \tag{1.69}$$

For the usual non-relativistic hydrogen atom hamiltonian, the appropiate transformations are those of the Galilean group. Notice that we are not asking that the energy of the state be the same, but that the Schrödinger equation be satisfied. What must happen, in the free case, is that the effect of a boost on a state of definite energy and momentum be transformed into a a state of definite momentum and energy again—the boosted ones. In general, solutions of the equation must be boosted into new solutions (modulo readjustment of the functional dependence of the potentials).

Consider for simplicity the equation ($i = 1, 2, 3$)

$$|p_0| = \sqrt{-p^i p_i + m^2} \tag{1.70}$$

This equation is equivalent to $p_0^2 - p_i^2 = m^2$ of course, so if it is true in one frame of reference it will hold in all of them. Let us check this explicitely by boosting both sides,

$$|\Lambda^\mu_0 p_\mu| = \sqrt{-\Lambda^\mu_i p_\mu \Lambda^i_\nu p^\nu + m^2} =$$

$$\sqrt{-\Lambda^\mu_\alpha p_\mu \Lambda^\alpha_\nu p^\nu + \Lambda^\mu_0 p_\mu \Lambda^0_\nu p^\nu + m^2} = \sqrt{\Lambda^\mu_0 p_\mu \Lambda^0_\nu p^\nu} \tag{1.71}$$

so it holds after a boost, as it should.

Consider next the square-root Schrödinger equation,

$$i\partial_0 \Psi(x^\mu) = \sqrt{\partial^i \partial_i + m^2}\, \Psi(x^\mu) \tag{1.72}$$

and suppose that $\Psi(x^\mu)$ indeed satisfies this equation. I will now show that after a boost this equation is still satisfied. Indeed, after the boost—as discussed before—we have to check that

$$i\Lambda^\mu_0 \frac{\partial}{\partial x'^\mu}\Psi(x'^\mu) = \sqrt{\Lambda^\mu_i \partial'_\mu \Lambda^i_\nu \partial'^\nu + m^2}\,\Psi(x'^\mu) =$$

$$\sqrt{\Lambda^\mu_\alpha \partial'_\mu \Lambda^\alpha_\nu \partial'^\nu - \Lambda^\mu_0 \partial'_\mu \Lambda^0_\nu \partial'^\nu + m^2}\,\Psi(x'^\mu) = \sqrt{-\Lambda^\mu_0 \partial'_\mu \Lambda^0_\nu \partial'^\nu}\,\Psi(x'^\mu) \tag{1.73}$$

which is consistent. Notice that the crucial part of the proof was that

$$i\partial_0 \Psi(x^\mu) = \sqrt{\partial^i \partial_i + m^2}\,\Psi(x^\mu) \longrightarrow [\partial_\mu \partial^\mu + m^2]\,\Psi(x^\mu) = 0 \tag{1.74}$$

which is true because

$$\partial_\mu \partial^\mu + m^2 = \left(\partial_0 - \sqrt{\partial^i \partial_i + m^2}\right)\left(\partial_0 + \sqrt{\partial^i \partial_i + m^2}\right) \tag{1.75}$$

since

$$[\partial_0, \partial_i] = 0 \tag{1.76}$$

Notice that we also needed that

$$[\partial_\mu, \Lambda^\nu_0 \partial_\nu] = 0 \tag{1.77}$$

How about the case of an electromagnetic interaction? One needs to define what the square root means, because a momentum states expansion definition will no longer work. As long as the operator in the square-root is hermitean and positive one can build a basis with its eigenstates, though in general these will be time-dependent. Let us assume that all this is so.

All that changes is that we need to use gauge-covariant derivatives—i.e., minimal coupling—defined as follows[7]

$$D_\mu \equiv \partial_\mu + ieA_\mu \tag{1.78}$$

One can show [11] that this equation—with the minimal coupling—is gauge covariant, meaning that if $\psi(x^\mu)$ is a solution and one changes

$$\psi(x^\mu) \longrightarrow e^{-ie\Lambda(x^\mu)}\,\psi(x^\mu), \quad A_\mu \longrightarrow A_\mu + \partial_\mu\Lambda \tag{1.79}$$

the equation is still valid. How about space-time covariance? This is a much trickier issue (see [12]). Just as before we will need

$$i\Lambda^\mu_0\, D'_0\Psi(x'^\mu) = \sqrt{\Lambda^\mu_i D'_\mu \Lambda^i_\nu D'^\nu + m^2}\;\Psi(x'^\mu) =$$
$$\sqrt{\Lambda^\mu_\alpha D'_\mu \Lambda^\alpha_\nu D'^\nu - \Lambda^\mu_0 D'_\mu \Lambda^0_\nu D'^\nu + m^2}\;\Psi(x'^\mu) = \sqrt{D'^2 + m^2 - \Lambda^\mu_0 D'_\mu \Lambda^0_\nu D'^\nu}\;\Psi(x'^\mu) \tag{1.80}$$

The sufficient condition for this equation to be true is that the electric field is zero in some frame (no particle creation condition),

$$[D_0, D_i] = F_{0i} = E_i \tag{1.81}$$

as I will now show.

Indeed, if this condition is met we know that the Klein-Gordon equation decouples, because

$$D_\mu D^\mu + m^2 = \left(D_0 \pm \sqrt{D^i D_i + m^2}\right)\left(D_0 \mp \sqrt{D^i D_i + m^2}\right) \tag{1.82}$$

[7]Here $A_\mu = (\phi, -\vec{A})$, $\partial_\mu = (\partial_t, \vec{\partial}_x)$, with our choice of metric convention, time-space $\sim (+,-,-,-)$.

since the electric field is zero. Hence we know that for a solution of the Klein-Gordon equation

$$[\left(D_0 \pm \sqrt{D^i D_i + m^2}\right)\left(D_0 \mp \sqrt{D^i D_i + m^2}\right)]_x \; \phi(x) = 0 \tag{1.83}$$

Moreover, because this equation is relativistic we also know that

$$[\left(D'_0 \pm \sqrt{D'^i D'_i + m^2}\right)\left(D'_0 \mp \sqrt{D'^i D'_i + m^2}\right)]_x \; \phi(x) = 0 \tag{1.84}$$

where $D' = \Lambda D$, as discussed earlier. This means that

$$0 = [D^\Lambda_0, \sqrt{D^{\Lambda i} D^\Lambda_i + m^2}]_x \; \phi(x) =$$

$$[D^\Lambda_0, m^2(1 + \frac{1}{2}\frac{D^{\Lambda i} D^\Lambda_i}{m^2} + ...]_x \; \phi(x) = 0 \tag{1.85}$$

The fact that this equation is true for *any boost* Λ now means that

$$0 = [D^\Lambda_0, D^{\Lambda i} D^\Lambda_i]_x \; \phi(x) =$$

$$[D^\Lambda_0, D^{\Lambda \mu} D^\Lambda_\mu]_x \; \phi(x) = [D^\Lambda_0, D^\mu D_\mu]_x \; \phi(x) = 0 \tag{1.86}$$

which is all we need to show that

$$\sqrt{D'^2 + m^2 - \Lambda^\mu_0 D'_\mu \Lambda^0_\nu D'^\nu} \; \Psi(x'^\mu) = i\Lambda^\mu_0 D'_\mu \tag{1.87}$$

Thus, this equation, the square-root Schrodinger equation, is covariant *if there is a frame in which the electric field is zero.*

1.5 Conclusions, summary

In this chapter I have first reviewed the standard quantization of the unconstrained particle, emphasizing the fact that the construction and interpretation of the path integral are straightforward when one knows what the states, the inner product and the hamiltonian are in the theory.

I have discussed the usee of different hamiltonians: the non-relativistic case as well as the square-root relativistic one, with or without interactions. We have seen that unitarity of the resulting theory hinges on whether the hamiltonian is hermitean—time dependence of the hamiltonian is not a problem *per se*.

Then I have discussed the fact that it is possible—after choosing a set of observables and an inner product—to change the inner product in the theory, as long as this change is compensated by a change in the normalization of the states and a modification of the observables, which must be hermitean in the new inner product. Although this is not a new idea, it has been overlooked in the literature as a source of ambiguities in the quantization of constrained systems, a point which I will come back to in the next chapters.

Then I have looked at the issue of whether the Schrödinger equation is space-time covariant, as well as gauge-covariant. For the square-root case the answer is that it is always gauge-covariant—a result of Samarov [11]—and I have showed that relativistic covariance demands that there exist a frame in which the electric field is zero. We will see that this is a recurring theme.

Chapter 2

Classical aspects of parametrized systems

Parametrized systems are constrained systems of a special kind. In this chapter I will review the formalism developed by Dirac and apply it to our systems, and see how to obtain the physical—as opposed to gauge—degrees of freedom and their dynamics.

2.1 The non-relativistic particle: general considerations

The action for the parametrized non-relativistic particle is[1]

$$S = \int_{\tau_i}^{\tau_f} d\tau \; L = \int_{\tau_i}^{\tau_f} d\tau \; \frac{m}{2}\frac{\dot{x}^2}{\dot{t}} \tag{2.1}$$

This form follows from the trick suggested by Dirac [1] in which the time coordinate is added to the degrees of freedom in the Lagrangian: let $x, t = x(\tau), t(\tau)$ so that τ is the new "time" parameter, then

$$S = \int_{\tau_i}^{\tau_f} dt \; L[x, \frac{dx}{dt}, t] = \int_{\tau_i}^{\tau_f} d\tau \; L[x, \frac{dx}{d\tau}\frac{1}{\dot{t}}, t]\frac{dt}{d\tau} \tag{2.2}$$

This action is invariant under reparametrizations that do not affect the boundaries. Indeed, let $\tau \longrightarrow f(\tau)$ with $f(\tau_i) = \tau_i$ and $f(\tau_f) = \tau_f$, and with $df/d\tau > 0$; then the action becomes

$$S \longrightarrow \int_{\tau_i}^{\tau_f} d\tau \; \frac{m}{2}\frac{\dot{x}(f(\tau))^2}{\dot{t}(f(\tau))} = \int_{\tau_i}^{\tau_f} d\tau \frac{df}{d\tau} \; \frac{m}{2}\frac{x'(f)^2}{t'(f)} \tag{2.3}$$

so it is invariant. Notice that if $df/d\tau < 0$ and $f(\tau_i) = \tau_f$, $f(\tau_f) = \tau_i$, then the action is not left unchanged but changes sign, $S \longrightarrow -S$, so for this action the invariance requires $df/d\tau > 0$.

We can think of the full reparametrization group as being the direct product of Z_2 and the reparametrizations connected with the identity. We can say that this

[1] Unless otherwise stated we define $a = da/d\tau$.

action carries a faithful representation of the Z_2 part of the reparametrization group. The action of the Z_2 part of the reparametrization group can be described by two types of reparametrization functions: f_+, which maps τ_i and τ_f into themselves, and f_-, which maps τ_i into τ_f and viceversa. The group multiplication is then described by

$$Z_2 = \begin{cases} f_+ \cdot f_+ & = f_+ \\ f_+ \cdot f_- & = f_- \\ f_- \cdot f_- & = f_+ \end{cases} \tag{2.4}$$

The full diffeomorphism group is given by $\mathcal{G} = Z_2 \otimes \mathcal{F}_+$ where $\mathcal{F}_+$ denotes the part connected to the identity.

The Euler-Lagrange equations of motion that follow from this action are

$$\frac{d}{d\tau}\left(\frac{\dot{x}}{\dot{t}}\right) = 0 = \frac{d}{d\tau}\left(\frac{\dot{x}^2}{\dot{t}^2}\right) \tag{2.5}$$

Let us now go to the hamiltonian formulation in the usual way, by defining the momenta

$$p_x = m\frac{\dot{x}}{\dot{t}} \qquad p_t = -\tfrac{m}{2}\tfrac{\dot{x}^2}{\dot{t}^2} \tag{2.6}$$

We find the constraint[2]

$$\Phi \equiv p_t + \frac{p_x^2}{2m} \approx 0 \tag{2.7}$$

together with the zero hamiltonian $H = p_t\dot{t} + p_x\dot{x} - L \equiv 0$. The equations of motion

[2]Indeed $det\frac{\partial L}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = 0$.

are generated from the extended hamiltonian $H_E = v\Phi$:

$$\begin{aligned} \dot{x} &= \{x, H_E\} = v\frac{p_x}{m} \qquad \dot{p}_x = 0 \\ \dot{t} &= \{t, H_E\} = v \qquad\quad \dot{p}_t = 0 \end{aligned} \tag{2.8}$$

This matches the lagrangian formulation above—in the sense that the dynamics are reproduced—with the identification $v = 1$. The same equations of motion can be obtained from the so-called first order action in the phase space coordinates

$$A = \int_{\tau_i}^{\tau_f} d\tau (p_t \dot{t} + p_x \dot{x} - v\Phi) \tag{2.9}$$

Notice that this action is invariant under the gauge transformations [14]

$$\begin{aligned} &\delta x = \epsilon(\tau)\{x, \Phi\} \quad \delta p_x = \epsilon(\tau)\{p_x, \Phi\} \\ &\delta t = \epsilon(\tau)\{t, \Phi\} \quad \delta p_t = \epsilon(\tau)\{p_t, \Phi\} \\ &\delta v = \dot{\epsilon}(\tau) \end{aligned} \tag{2.10}$$

as long as the gauge parameter vanishes at the boundaries, i.e., $\epsilon(\tau_i) = \epsilon(\tau_f) = 0$. It is not hard to see also that this symmetry is the same as the one in the lagrangian form, with the identification $f(\tau) = \tau + \epsilon(\tau)$. It is very important to be aware that this situation in which there is a restriction in the gauge freedom at the boundaries is very different from the standard concept one has of a gauge theory, where there is no such restriction. Indeed, one usually understands the quantization of a system with symmetries as the quantization of the "true" and "underlying" degrees of freedom one assumes exist. The present situation is from this point of view really troublesome.

As explained in reference [14], this new twist in the concept of invariance is a consequence of the form of the constraint, which is non-linear in the coordinates conjugate to what is fixed at the boundaries, i.e., the momenta. Under the above gauge transformation, the first order action changes, as a boundary term appears:

$$A \longrightarrow A + \epsilon(\tau)\ (p_i \frac{\partial \Phi}{\partial p_i} - \Phi)\Big|_{\tau_i}^{\tau_f} \tag{2.11}$$

which vanishes when the constraint Φ is linear in the momenta—or with some other boundary conditions on the phase space variables. Reference [14] elaborates more on this point, as well as on the idea of modifying the action at the boundaries so that it becomes fully invariant. We will study these ideas in more detail shortly.

As one would expect the above dynamics match those of the unconstrained action—equation (1.1)—when the gauge $t = \tau$ is used. *That one has to use a specific gauge to recover the "physical" coordinates is already an indication of trouble to come*, as we will see when we study the reduced phase space of this constrained system.

The electromagnetic interaction case—which comes from applying Dirac's trick to the unconstrained lagrangian in equation (1.30)—is given by the action

$$\mathcal{S}_{EM} = \int_{\tau_i}^{\tau_f} d\tau\ L_E = \int_{\tau_i}^{\tau_f} d\tau\ \left(\frac{1}{2} m \frac{\dot{x}_i^2}{\dot{t}} - e\phi \dot{t} + \frac{e}{c} \dot{x}_i A_i \right) \tag{2.12}$$

and it has the same reparametrization invariance properties as the free case: $\tau \longrightarrow f(\tau)$ with $f(\tau_i) = \tau_i$ and $f(\tau_f) = \tau_f$, and with $df/d\tau > 0$, leaves the action unchanged. The covariant part, however, is gone. In the hamiltonian formulation we have,

$$p_i = m\frac{\dot{x}_i}{\dot{t}} + \frac{e}{c}A_i \qquad p_t = -m\frac{\dot{x}_i^2}{2\dot{t}^2} - e\phi \tag{2.13}$$

and we find the constraint[3]

$$\Phi_{EM} \equiv p_t + e\phi + \frac{(p_i - \frac{e}{c}A_i)^2}{2m} = p_t + \mathcal{H}_{EM} \approx 0 \tag{2.14}$$

together with the zero hamiltonian $H = p_t\dot{t} + p_i\dot{x}_i - L \equiv 0$. As usual, the equations of motion are generated by the extended hamiltonian $H_E = v\Phi_{EM}$:

$$\begin{aligned} &\dot{x}_i = \{x_i, H_E\} = \{x_i, \mathcal{H}_{EM}\} = v(\frac{p_i}{m} - \frac{e}{mc}A_i) \qquad && \dot{p}_i = v[\frac{e}{mc}A_{j,i}\,(p_j - \frac{e}{c}A_j) - e\phi_{,i}] \\ &\dot{t} = \{t, H_E\} = v && \dot{p}_t = v[\frac{e}{mc}A_{j,0}\,(p_j - \frac{e}{c}A_j) - e\phi_{,0}] \end{aligned} \tag{2.15}$$

This matches the lagrangian formulation with the identification $v = 1$. The first order action is $A = \int_{\tau_i}^{\tau_f} d\tau\ (p_t\dot{t} + p_i\dot{x}_i - v\Phi_{EM})$. Notice that this action is invariant—as before—under the gauge transformations generated by the constraint [14] ($\delta z = \epsilon(\tau)\{z, \Phi_{EM})\}$ and $\delta v = \dot{\epsilon}(\tau)$, where z stands for all q's and p's), with the same condition on the gauge parameter (i.e., that it vanishes at the boundaries: $\epsilon(\tau_i) = \epsilon(\tau_f) = 0$). Again as before, it is not hard to see that this symmetry is the same as the one in the lagrangian form, with the identification $f(\tau) = \tau + \epsilon(\tau)$.

[3] Again, $det\frac{\partial L}{\partial\dot{x}^\mu\partial\dot{x}^\nu} = 0$.

2.2 The relativistic particle: general considerations

The action for the free parametrized relativistic particle is ($c = 1$)

$$S = \int_{\tau_i}^{\tau_f} d\tau \; L = -m \int_{\tau_i}^{\tau_f} d\tau \; \sqrt{\dot{t}(\tau)^2 - \dot{x}(\tau)^2} \tag{2.16}$$

(where from now on $\dot{a} \equiv \frac{da}{d\tau}$). This is just a possible form—basically the proper time—and it is not well defined when the mass is zero. Another form is

$$S' = \int_{\tau_i}^{\tau_f} d\tau \; L' = \int_{\tau_i}^{\tau_f} d\tau \; \left(\frac{\dot{t}(\tau)^2 - \dot{x}(\tau)^2}{\lambda(\tau)} + m\lambda(\tau) \right) \tag{2.17}$$

and it is well defined also in the case $m = 0$. This action is invariant under reparametrizations that do not affect the boundaries. Indeed, let $\tau \longrightarrow f(\tau)$ with $f(\tau_i) = \tau_i$ and $f(\tau_f) = \tau_f$, and with $df/d\tau > 0$; then the action becomes

$$S \longrightarrow -m \int_{\tau_i}^{\tau_f} d\tau \; \sqrt{\dot{t}(f(\tau))^2 - \dot{x}(f(\tau))^2} = -m \int_{f(\tau_i)}^{f(\tau_f)} d\tau \; |\dot{f}(\tau)| \sqrt{t'(f)^2 - x'(f)^2} \tag{2.18}$$

so it is invariant. Notice that if $df/d\tau < 0$ and $f(\tau_i) = \tau_f$, $f(\tau_f) = \tau_i$, then the action is also left unchanged, $S \longrightarrow S$, *so for this action the invariance allows both* $df/d\tau > 0$ *or* $df/d\tau < 0$. We can say that this action carries the trivial representation of the Z_2 part of the reparametrization group. This is a very important point, as we will see later.

The Euler-Lagrange equations of motion that follow from this action are

$$\frac{d}{d\tau}\left(\frac{\dot{t}}{L}\right) = 0 = \frac{d}{d\tau}\left(\frac{\dot{x}}{L}\right) \tag{2.19}$$

Let us now go to the hamiltonian formulation in the usual way; defining the momenta

$$p_x = -m^2\frac{\dot{x}}{L} \qquad p_t = m^2\frac{\dot{t}}{L} \tag{2.20}$$

we find the constraint[4]

$$\Phi \equiv p_t^2 - p_x^2 \approx 0 \tag{2.21}$$

together with the zero hamiltonian $H = p_t\dot{t} + p_x\dot{x} - L \equiv 0$. The equations of motion are generated from the extended hamiltonian $H_E = v\Phi$:

$$\begin{aligned} \dot{x} &= \{x, H_E\} = -2vp_x \qquad \dot{p}_x = 0 \\ \dot{t} &= \{t, H_E\} = 2vp_t \qquad \dot{p}_t = 0 \end{aligned} \tag{2.22}$$

This matches the lagrangian formulation above—in the sense that the dynamics are reproduced—with the identification $v = -L/2m^2$. The same equations of motion can be obtained from the so-called first order action in the phase space coordinates

$$A = \int_{\tau_i}^{\tau_f} d\tau(p_t\dot{t} + p_x\dot{x} - v\Phi) \tag{2.23}$$

Notice that this action is invariant under the gauge transformations [14]

$$\begin{aligned} \delta x &= \epsilon(\tau)\{x, \Phi\} \quad \delta p_x = \epsilon(\tau)\{p_x, \Phi\} \\ \delta t &= \epsilon(\tau)\{t, \Phi\} \quad \delta p_t = \epsilon(\tau)\{p_t, \Phi\} \\ \delta v &= \dot{\epsilon}(\tau) \end{aligned} \tag{2.24}$$

[4]indeed $det\frac{\partial L}{\partial\dot{x}^\mu\partial\dot{x}^\nu} = 0$

as long as the gauge parameter vanishes at the boundaries, i.e. $\epsilon(\tau_i) = \epsilon(\tau_f) = 0$. It is not hard to see also that this symmetry is the same as the orientation-preserving one in the lagrangian form, with the identification $f(\tau) = \tau + \epsilon(\tau)$. Again, it is very important to be aware that this situation in which there is a restriction in the gauge freedom at the boundaries is very different from the standard concept one has of a gauge theory, where there is no such restriction. Indeed, one usually understands the quantization of a system with symmetries as the quantization of the "true" and "underlying" degrees of freedom one assumes exist, and, just as in the non-relativistic case, the present situation is from this point of view really troublesome.

As explained in reference [14], this new twist in the concept of invariance is a consequence of the form of the constraint, which is non-linear in the coordinates conjugate to what is fixed at the boundaries, i.e., the momenta. Under the above gauge transformation, the first order action changes, as a boundary term appears:

$$A \longrightarrow A + \epsilon(\tau)(p_i \frac{\partial \Phi}{\partial p_i} - \Phi)\Big|_{\tau_i}^{\tau_f} \tag{2.25}$$

which vanishes when the constraint Φ is linear in the momenta—or with some other boundary conditions on the phase space variables. Reference [14] elaborates more on this point, as well as on the idea of modifying the action at the boundaries so that it becomes fully invariant. We will study these ideas in more detail shortly—see also reference [51].

As one would expect the above dynamics match those of the unconstrained action—equation (1.34)—when the gauge $t = \tau$ is used. That one has to use a specific gauge to recover the "physical" coordinates is already an indication of trouble to come.

Consider next the more general actions for the interacting particle, equations (1.41) and (1.42). Notice that both actions are invariant under the *connected* part of the reparametrization (or diffeomorphism) group:

$\tau \longrightarrow f(\tau)$ with $f(\tau_i) = \tau_i$ and $f(\tau_f) = \tau_f$, and with $df/d\tau > 0$.

The invariance—or covariance—under the disconnected part of the group is lost when going to the interacting case.

As mentioned, we can think of the full reparametrization group as being the direct product of Z_2 and reparametrizations connected with the identity. It appears that in the interacting case we have lost "half" of the representation.

Let us first look at the action A_{EG} in equation (1.41). Because of the reparametrization invariance we find—using Dirac's formalism as usual—the constraint

$$\Phi_{EG} = (p_\mu - A_\mu) g^{\mu\nu} (p_\nu - A_\nu) - m^2 \approx 0, \tag{2.26}$$

together with a zero hamiltonian, $H_{EG} \equiv 0$.

Using the action A'_{EG} in equation (1.41), the situation is just a bit more complicated. The lagrangian equations of motion are

$$\frac{g_{\mu\nu,\beta}}{\lambda}\dot{x}^\mu\dot{x}^\nu - e\dot{x}^\mu A_{\mu,\beta} - \frac{d}{d\tau}\left(2\frac{g_{\mu\beta}}{\lambda}\dot{x}^\mu - eA_\beta\right) = 0 \tag{2.27}$$

and from varying λ

$$\lambda = \pm\sqrt{\frac{1}{m}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \tag{2.28}$$

In the hamiltonian formalism the situation is the following: first we find the constraint $p_\lambda \approx 0$, since the action is independent of $\dot{\lambda}$. The hamiltonian is not zero, $H'_{EG} \equiv \lambda\Phi_{EG}$. However, one now needs—as Dirac explains [1]— $\dot{\lambda} = 0$, and this implies $\Phi_{EG} \approx 0$, i.e., the constraint above appears here as a secondary constraint.

Again, the constraint contains two branches. It can be rewritten as

$$\Pi_0 = -\Pi_i\tilde{g}^{0i} \pm \sqrt{(\Pi_i\tilde{g}^{0i})^2 - \Pi_i\tilde{g}^{ij}\Pi_j + m^2/g^{00}} \tag{2.29}$$

where $\Pi \equiv p - A$, and $\tilde{a} \equiv a/g^{00}$.

2.3 Minisuperspace: general considerations

The action we will use for our minisuperspace model is

$$S_M = \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau \left(\frac{g_{AB}\dot{Q}^A\dot{Q}^B}{N} + NU(Q)\right) \tag{2.30}$$

which is again equivalent to

$$S_M = -\frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau \sqrt{U(Q)g_{AB}\dot{Q}^A\dot{Q}^B} \tag{2.31}$$

where the signature of g_{AB} is Lorentzian. In this homogeneous cosmological model we have a lapse N, which is essentially the time-time component of the metric, and a parameter Q^0 characterizing the scale of the universe. The other parameters describe spatial anisotropies. We see that, mathematically, we have been discussing these models all along when studying the relativistic particle.

In order to understand the origin of these models it helps to think of the dynamical field as an infinite number of of variables that depend on τ, $g \sim g_{x^i}(\tau)$. It is not hard to imagine that by demanding enough symmetries on the solutions most of these infinite variables will be fixed. In practice one writes the most general form of the metric that satisfies the symmetries imposed and plugs that in into the action—or Einstein's equations.

Since mathematically this system is essentially equivalent to that of the relativistic particle in a curved background and no electromagnetic field, the earlier discussions on symmetries etc. will all apply here. Notice, in particular, that both the orientation preserving and orientation reversing diffeomorphism symmetries are present in this case, since there is no analog of the electromagnetic field to brake them—for this model at least.

In fact, the only difference of importance is in the interpretation of the theory, and it will be essential to remember this when we discuss the quantum aspects of minisuperspace, in particular when we discuss unitarity, space-time covariance and causality.

The equations of motion are given by [49]

$$\frac{d}{d\tau}\left(\frac{2g_{AB}\dot{Q}^B}{N}\right) + N\frac{\partial U}{\partial Q^A} = 0 \tag{2.32}$$

as well as

$$N = \pm\sqrt{\frac{g_{AB}\dot{Q}^A\dot{Q}^B}{U(Q)}} \tag{2.33}$$

The constraint that follows from this form of the action is

$$\Phi_M \equiv P_A P_B g^{AB} - U(Q) \approx 0 \tag{2.34}$$

which looks like that of a relativistic particle in a curved background with a coordinate dependent mass.

2.4 Gauge fixing, the Dirac bracket, and the Reduced Phase Space

Next we consider the idea that at the classical level one should be able to get rid of the gauge, or unphysical degrees of freedom. As Dirac explained 30 years ago [1], the appearance of a first class constraint in the process of defining the momenta is an indication of gauge freedom, or invariance. This can be seen when one considers the dynamics of the system: the phase space trajectories are defined only up to gauge transformations, and the constraints themselves are the generators of the gauge transformation (in the canonical sense). This is because of the inherent ambiguity in the hamiltonian. A possible approach to quantization is to quantize and then constrain, but one would think that it should be possible to eliminate the unphysical gauge degrees of freedom at the classical level. This is the idea behind the reduced phase space approach: there is gauge freedom at the classical level, so why not fix the gauge (or identify gauge-equivalent points) immediately? Gauge-fixing can be done by adding an additional constraint to the system, as we shall see.

Before studying the reduced phase space (RPS) associated with this system, let us consider the simple example in which the phase space is described by (q, p, Q, P) and where we have the linear constraint $\Phi = P \approx 0$. This constraint can be thought to generate a gauge transformation through $Q \longrightarrow Q + \epsilon\{Q, P\}$. This is because the hamiltonian is defined only up to an arbitrary term of the form $v(\tau)P$.

At any rate the point is that the coordinate Q describes a gauge fiber, and the RPS, the physical space, is described by q and p: the physical phase space is recovered by the reduction process defined by:

a) $\Phi \approx 0$, and

b) $Q \sim Q + \epsilon\{Q, P\}$.

This similarity relation means that it is sufficient to consider the gauge invariant functions, which are indeed described by

$$0 = \{C_\Phi, \Phi\} = \{C_\Phi, P\} = \frac{\partial C_\Phi}{\partial Q} \tag{2.35}$$

i.e., they are functions independent of Q.

Consider next the following similar idea: take any function $F(q, p, Q, P)$ on the constrained surface. Then fix $Q = Q(q, p)$. The function $\tilde{F} \equiv F(Q(q, p), q, p)$ lives on a subspace of the full phase space but it can be defined anywhere by a gauge invariant extension. Indeed, if we think of $\tilde{F}$ as living on the whole space, we automatically have

$$\{\tilde{F},\, P\} = \{F(Q(q, p), q, p),\, P\} = \frac{\partial \tilde{F}}{\partial Q} = 0 \tag{2.36}$$

so that $\tilde{F}$ is independent of Q, of course.

Now suppose that the original system has dynamics generated by

$$H_E = h(q, p) + v(\tau)\Phi = h(q, p) + v(\tau)P \tag{2.37}$$

where $v(\tau)$ is an arbitrary function of τ-time. The dynamics, in the language of gauge invariant functions, are described by

$$\dot{C}_\Phi = \frac{\partial C_\Phi}{\partial \tau} + \{C_\Phi, H_E\} = \frac{\partial C_\Phi}{\partial \tau} + \{C_\Phi, h\} \tag{2.38}$$

so that we have dynamics for the physical degrees of freedom only. Suppose we decide to go the "gauge-fixing way" by adding a gauge-fixing constraint to the system

$$\chi = Q - Q(q,p) = 0 \tag{2.39}$$

The first thing to do is to make sure that both this new constraint and the original one are preserved in τ-time, and for that purpose the hamiltonian is "extended" further: $H_{E'} \equiv h(q,p) + v\Phi + w\chi$. Then one demands $\dot{\Phi} = \dot{\chi} = 0$, and this yields

$$\begin{aligned} 0 &= \dot{\chi} = \{\chi, H_{E'}\} = \{\chi, h\} + v \\ 0 &= \dot{P} = w \end{aligned} \tag{2.40}$$

which fixes v and w. Now let $F(q, q, Q, P)$ be an arbitrary function in phase space. The dynamics are then given by

$$\dot{F} = \frac{\partial F}{\partial \tau} + \{F, H_{E'}\} \approx \frac{\partial F}{\partial \tau} + \{F, h\} + v\{F, \Phi\} \tag{2.41}$$

2.4.1 The Dirac bracket

One can also just define $\{\ ,\ \}_*$, the Dirac bracket [1,2]:

$$\{Q, F\}_* = \{P, F\}_* = 0 \tag{2.42}$$

for all F. In general, the Dirac bracket is defined by

$$\{A,B\}_* = \{A,B\} - \{A,\chi_i\}(c_{ij})^{-1}\{\chi_j,B\} \tag{2.43}$$

where the matrix c_{ij} is given by

$$c_{ij} = \{\chi_i,\chi_j\} \tag{2.44}$$

and where χ_i is short for the constraints, $\chi_i = \chi, \Phi$. This is just

$$\{A,B\}_* = \{A,B\} - (\{A,\chi\},\{A,\Phi\})\frac{1}{\{\chi,\Phi\}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \{\chi,B\} \\ \{\Phi,B\} \end{pmatrix} \tag{2.45}$$

The equation of motion above can then be written as

$$\begin{aligned} \dot{F} &= \frac{\partial F}{\partial \tau} + \{F,H_{E'}\} \\ &= \frac{\partial F}{\partial \tau} + \{F,H_{E'}\}_* + \{F,\chi_i\}(c_{ij})^{-1}\{\chi_j,H_{E'}\} \\ &= \frac{\partial F}{\partial \tau} + \{F,H_{E'}\}_* = \frac{\partial F}{\partial \tau} + \{F,h\}_* \end{aligned} \tag{2.46}$$

since $\{\chi_j, H_{E'}\} = 0$ for this case of τ-time independent gauge-fixing (see equation (2.41)). Moreover

$$\{F,H_{E'}\} = \{F,H_{E'}\}_* = \{F,H_E\}_* = \{F,h\}_* \tag{2.47}$$

which follows from $H_{E'} - H_E = a\Phi + b\chi,\ \ H_E - h = v\Phi$, and $\{\chi_j, F\}_* = 0$ for any[5] function F.

[5]Indeed, the Dirac bracket is designed so that one can set the constraints to zero *before* its computation.

Let us look now at the case of a more general constraint and also at more general gauge-fixings, including the possibility of τ-dependent ones. The goal is to obtain the reduced phase space and describe its dynamics. We can start by looking at the extended hamiltonian $H_{E'} = h(q,p) + v\Phi(Q,P) + w\,\chi(Q,P,\tau)$. Again, we require that $\dot{\Phi} = \dot{\chi} = 0$. These equations imply $w = 0$ and

$$v = \frac{1}{\{\Phi,\chi\}}\frac{\partial\chi}{\partial\tau} \tag{2.48}$$

and the dynamics are again described by equation (2.41). What happens in the Dirac bracket formalism? Equation (2.46) is still correct, but equation (2.46) is not since $\{\chi_j, H_{E'}\} = 0$ doesn't hold anymore. In fact $\dot{\chi} = 0$ now means that $\{\chi, H_{E'}\} = -\partial\chi/\partial\tau$. One can check after some simple algebra that the equation of motion is now

$$\dot{F} = \frac{\partial F}{\partial\tau} + \{F,h\}_* + \frac{\partial\chi}{\partial\tau}\frac{\{F,\Phi\}}{\{\Phi,\chi\}} \tag{2.49}$$

which looks different than (2.46). In fact, *for entirely general constraints and gauge-fixing functions*, it is easy to see that the equation of motion for an arbitrary function in phase space is

$$\dot{F} = \frac{\partial F}{\partial\tau} + \{F,h\}_* + \frac{\partial\chi}{\partial\tau}\frac{\{F,\Phi\}}{\{\Phi,\chi\}} + \frac{\partial\Phi}{\partial\tau}\frac{\{F,\chi\}}{\{\chi,\Phi\}} \tag{2.50}$$

which is equivalent to the extended hamiltonian description with

$$H_{E'} = h(q,p) + v(\tau)\Phi(q,p,Q,P,\tau) + w(\tau)\chi(q,p,Q,P,\tau),$$

where $v,\ w$ are fixed as usual by $\dot{\Phi} = \dot{\chi} = 0$:

$$\frac{\partial \Phi}{\partial \tau} + \{\Phi, H_{E'}\} = 0 = \frac{\partial \chi}{\partial \tau} + \{\chi, H_{E'}\} \tag{2.51}$$

Notice that in general these equations don't have solutions—in which case equation (2.50) is undefined, of course. Since this possibility is in fact going to appear when we consider the constant of the motion coordinate system in section 2.4.4, let us study it in more detail with a simple toy case which illustrates nicely how and when the dynamics of a constrained system lead to the concept of reduced phase space (see Dirac [1]).

2.4.2 RPS vs. RPS^*

Recall that the idea of reduced phase space comes from the fact that some coordinates in the phase space of a constrained system have arbitrary dynamics. Consider the simple case in which we have a phase space described by the coordinates Q, P, and q^i, p_i, with a first class hamitonian $h(q^i, p_i)$, and a constraint $\Phi = P^2/2 \approx 0$. Now, this form of the constraint may appear to be unusual, and perhaps one would expect the system to be equivalent to one in which the constraint is $P \approx 0$: the constraint surfaces are definitely the same! But before jumping to conclusions let us look at the dynamics.

In the original case they would be described by the extended hamiltonian $H_E = h + v(\tau)P^2/2$, and the equation of motion for Q would be $\dot{Q} = v(\tau)P = 0$

in the constraint surface! The constraint and the constrained dynamics then reduce the phase space to the space described by $Sp \sim [Q, P{=}0, q^i, p_i]$ with dynamics

$$\dot{A}(q^i, p_i, Q, \tau) = \frac{\partial A}{\partial \tau} + \{A, h(q^i, p_i)\} \tag{2.52}$$

Let us see how the above methodology fares in this case. Introduce an additional constraint, $\chi = Q = 0$, and define as usual

$$H'_E = h + v(\tau)\Phi + w(\tau)\chi.$$

Demanding $\dot{\chi} = \dot{\Phi} = 0$ then yields $vP = 0 = -wP$. As for the Dirac bracket, *it cannot be defined, since the matrix* $\{\chi_i, \chi_j\}$ *has no inverse.* This is not due to a bad choice of gauge-fixing, as any such function will run into this problem as long as it is not singular itself (e.g. $\chi = Q/P$). Here we see the direct connection between the existence of a reduced phase space and the existence of a well-defined Dirac bracket.

In the second case we would have $\dot{Q} = v(\tau)$. This is the usual situation, where we see clearly that the coordinates Q, P are pure gauge, and that the system *reduces* to the system $q^i, p_i, h(q^i, p_i)$. Indeed, the Dirac bracket allows one to set $Q = P = 0$ immediately,

$$\{Q, F(Q, P, q^i, p_i)\}_* = 0 = \{P, F(Q, P, q^i, p_i)\}_*$$

This system may seem artificial, and one may wonder if it would ever arise from a lagrangian formulation, say. Both the relativistic particle and minisuperspace, for instance, have constraints of the form $P^2 - a^2 = 0$. Even though this system is not as degenerate as the one above, with this constraint *it is not possible to reduce by the identification procedure alone the above phase space to the "physical" coordinates* q^i, p^i. One also needs to stay in the constrained surface—and this is different from the usual situation of linear constraints. Indeed, the dynamics for the "gauge" coordinate Q are still given by $\dot{Q} = v(\tau)P$ —which is totally arbitrary in the constrained surface (unless $a = 0$), but is not if $P = 0$. What does this mean?

One aspect of this situation can be understood in the following terms. Consider the simpler problem in which we want to minimize a function $f(x)$ in the space spanned by the coordinates (x, y) subject to the constraint $x - x_0 = 0$. The solution is clearly given by the set $\{x_0, y, \text{ for all } y\}$. This is an example of a very simple "gauge" system. Using the lagrange multiplier method (see for example Lanczos' book on classical mechanics [7]), this problem is solved by minimizing the function $F(x) = f(x) + \lambda g(x)$, where $g(x)$ is some "version" of the constraint, and also by demanding $g(x) = 0$. This means that we have to solve the equations

$$\begin{aligned} &\frac{\partial f(x)}{\partial x} + \lambda \frac{\partial g(x)}{\partial x} = 0 \\ &\frac{\partial F(x)}{\partial y} = 0 \text{ (contains no information)} \\ &g(x) = 0 \end{aligned} \tag{2.53}$$

Using $g = x - x_0$ the solution is immediate. However, we can also check that using $g = (x - x_0)^2$ leads to trouble, as then we find no solution! The bottom line is that—as any calculus student knows–the function $g(x)$ has to be chosen so that

$$\left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right)\Bigg|_{g=0} \neq \vec{0} \tag{2.54}$$

Similarly, the choice $\Phi = P^2$ is not good, as *with it the full solution space is not found*, just as we saw before. The condition on the constraints for finding the full solution space to the extremization problem is

$$\frac{\delta\Phi}{\delta z(\tau)}\Bigg|_{\Phi=0} \neq 0 \text{ for all } \tau \tag{2.55}$$

where $z(\tau)$ stands for all q's and p's. The constraint $P^2 = 0$ fails this test, and indeed we find less solutions when we use it—$\dot{Q} = 0$ instead of $\dot{Q} = Pv(\tau)$ =arbitrary.

Let us now discuss in more detail the concept of reduced phase space—or RPS, for short. One point of view is that, ideally, we can divide the full phase space into its pure gauge and its gauge invariant—or physical—degrees of freedom. Conventionally we call the gauge invariant subspace the reduced phase space, RPS. However, we could also *fix the gauge part* and work with the resulting space. One easy way to think about it is to go to the "gauge-invariant" coordinate system (see section 2.4.4), fix the gauge coordinates there with some gauge-fixing function, and then transform back to the original coordinates. The resulting space—the full phase space after gauge fixing—we call RPS^*. Mathematically, this space is the direct product of the two spaces

$$RPS^* = \{\text{gauge invariant}\} \otimes \{\text{pure gauge—fixed}\} \tag{2.56}$$

and it depends clearly on the gauge-fixing used. Usually this distinction is not made, the reason being that one usually works with actions that are fully gauge invariant and therefore indifferent to gauge-fixing choices. However, for the case of actions that are not fully gauge invariant it is important to distinguish these two concepts. The reduced phase space proper is described by the gauge invariant coordinates only, and it is different from RPS^*, although isomorphic to it.

To denote the constraint and the gauge fixing used we will write RPS_Φ and $RPS^*_{\Phi,\chi}$. Let $Q, P, q, p \equiv q^a, p_a$ be the coordinates for the full space, in which Q and P stand as usual for the pure gauge part—with $P \approx 0$—and q, p for the invariant part. Also, let the first class extended hamiltonian be given by $H_E = h(q,p) + v(\tau)\Phi$. The space RPS_Φ is described by q, p only, and $RPS^*_{\Phi,\chi}$ by the full set—with Q, P fixed by χ and Φ respectively. We will consider two simple gauge- fixings,

a) $\chi_1 = Q - f_1(\tau)$ and

b) $\chi_2 = Q - f_2(\tau)$

for some arbitrary functions f_1 and f_2 of the parameter τ. Their respective spaces will be described by

a) q_1, p_1, Q_1, P_1, and $H_1 = h(q,p) + \dot{f}_1 P_1$

b) q_2, p_2, Q_2, P_2, and $H_2 = h(q,p) + \dot{f}_2 P_2$

(but we expect the true physical coordinates q, p's to be unchanged.)

Let us now consider the following proposition:

Proposition 1 *The spaces* $RPS^*_{\Phi,\chi}$ *are all isomorphic to each other and to* RPS_Φ*. In fact ,* RPS^*_{Φ,χ_1} *and* RPS^*_{Φ,χ_2} *are related by a canonical transformation. The generator of the canonical transformation depends on* χ_1 *and* χ_2 *: it is given by*

$$\mathcal{G}^{\Phi}_{\chi_1\rightarrow\chi_2} = q_1p_2 + Q_1P_2 + (f_2 - f_1)P_2 = \mathcal{G}(q_1, p_2, Q_1, P_2) \tag{2.57}$$

The corresponding canonical transformation is

$$p_1 = \frac{\partial\mathcal{G}}{\partial q_1} = p_2 \qquad\qquad q_2 = \frac{\partial\mathcal{G}}{\partial p_2} = q_1$$

$$P_1 = \frac{\partial\mathcal{G}}{\partial Q_1} = P_2 \qquad\qquad Q_2 = \frac{\partial\mathcal{G}}{\partial P_2} = Q_1 + f_2 - f_1 \tag{2.58}$$

$$H_2 = H_1 + \frac{\partial\mathcal{G}}{\partial\tau} = H_1 + (\dot{f}_2 - \dot{f}_1)P_2$$

Symbolically we can write this result as $Can_{\mathcal{G}}[RPS^*_{\chi_1}] = RPS^*_{\chi_2}$.

This proposition is trivial in the situation described by the decoupled coordinates, but it holds in *all* coordinate systems—see the next section—as long as the constraint can be made into a momentum. Let us review a little bit of the general canonical transformation theory (see for example Goldstein). Suppose that two canonical coordinate systems

$$(q, p, h(q, p)) \text{ and } (\tilde{q}, \tilde{p}, \tilde{h}(\tilde{q}, \tilde{p}))$$

yield the same dynamics. This will occur if, in fact

$$\int_{\tau_i}^{\tau_f} (pq - h(q,p))d\tau = \int_{\tau_i}^{\tau_f} (\tilde{p}\tilde{q} - \tilde{h}(\tilde{q},\tilde{p}))d\tau + \int_{\tau_i}^{\tau_f} dW$$

(careful with the boundary conditions for extremization!) i.e.,

$$pq - h(p,q) = \tilde{q}\tilde{p} - \tilde{h}(\tilde{q},\tilde{p}) + \frac{dW}{d\tau}$$

The above transformation equations come from these identities.

2.4.3 RPS analysis for the non-relativistic particle

For the non-relativistic particle, we have a phase space described by the coordinates t, x, p_x and p_t, and a constraint $\Phi = p_t + p_x^2/2m \approx 0$. A possible approach is to immediately do a canonical transformation such that Φ becomes a momentum, and to proceed as above. Indeed, for simple constraints this is always possible, and is equivalent to what follows (see section 2.4.4).

The gauge invariant functions are those that satisfy[6] $\{C_\Phi(q^\alpha, p_\alpha), \Phi\} = 0$, i.e.,

$$0 = (\frac{\partial}{\partial t} + \frac{p_x}{m}\frac{\partial}{\partial x})C_\Phi(q^\alpha, p_\alpha) = \frac{\partial C_\Phi(q^\alpha, p_\alpha)}{\partial t} + \{C_\Phi(q^\alpha, p_\alpha), \frac{p_x^2}{2m}\} \qquad (2.59)$$

which is solved by

$$C_\Phi = C_\Phi(x - (t - t_0)\frac{p_x}{m},\, p_x, p_t) \qquad (2.60)$$

These are the functions, then, of the constants of the motion.

Let us now consider fixing the gauge with a τ-dependent gauge, $\chi_\alpha = t - f(\tau)$, where $\alpha \equiv \dot{f}(\tau)$. As before, we need $\dot{\chi}_\alpha = 0 = \partial\chi_\alpha/\partial\tau + \{\chi_\alpha, H_{E'}\}$ which implies

[6]Greek indices denote the full space-time range.

$v = \alpha$, so we have

$$H_{E'} = \alpha\Phi = \alpha(p_t + \frac{p_x^2}{2m}) \tag{2.61}$$

and the equations of motion in this gauge are

$$\begin{aligned} &\dot{t} = \alpha \quad \dot{x} = \{x, \alpha\frac{p_x^2}{2m}\} \\ &\dot{p}_t = 0 \quad \dot{p}_x = 0 \end{aligned} \tag{2.62}$$

so the system is reduced to the coordinates x, p_x, with a hamiltonian

$$H_\alpha = \alpha\frac{p_x^2}{2m} \tag{2.63}$$

The Dirac bracket computation in this gauge is easily done, since $\{\chi_\alpha, \Phi\} = 1$: let $\chi_j = \chi_\alpha, \Phi$ as before. Then

$$\{A, B\}_* = \{A, B\} - (\{A, \chi_\alpha\}, \{A, \Phi\}) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \{\chi_\alpha, B\} \\ \{\Phi, B\} \end{pmatrix} \tag{2.64}$$

and this yields

$$\{x, p_x\}_* = 1,\ \{t, p_t\}_* = 0,\ \{x, p_t\}_* = -\frac{p_x}{m}$$

with all others zero.

Recall that we run into a subtlety with the Dirac bracket description of the dynamics. In this gauge the correct dynamics are described by the equation

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial \tau} + \{F, \alpha\Phi\} \tag{2.65}$$

which follows from equation (2.50) and, of course, matches the above ones ([15], and see ex. 4.8 in reference [2]).

In the gauge $\alpha = 1$ the connection to the unconstrained case of section 1 is clear, but let us look at the interpretation of the more general gauge-fixings. First notice that equation (2.65) has a very simple interpretation: for functions of x, p_x, say, one can simply factor out the term $\dot{f}$ to read

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, \frac{p_x^2}{2m}\}$$

Let us show, anyhow, that this reduced phase space is the same as the phase space for the unconstrained non-relativistic particle in section 1, in some unusual τ-dependent coordinates. Recall that the idea here is that when one picks a gauge, a coordinate system for the reduced phase space is implicitly chosen—or an active transformation occurs, depending on your point of view—and that with τ-dependent choices the dynamics will look different. Let us go back to the unconstrained system of chapter 1: there, t was the time, not τ. However, as we know from our later point of view t and τ are one and the same in the gauge that leads to this system. So keep in mind in what follows that $t = \tau$. What canonical transformation corresponds to

$$H = \frac{p_x^2}{2m} \longrightarrow H_\alpha = \alpha \frac{p_x^2}{2m} \ ?$$

Consider

$$q_\alpha = x + (\alpha - 1)\frac{p_x}{m}(t - t_0), \quad p_\alpha = p_x \tag{2.66}$$

where clearly $\{q_\alpha, p_\alpha\} = 1$. Moreover

$$\frac{dq_\alpha}{dt} = \frac{dx}{dt} + (\alpha - 1)\frac{p_x}{2m} = \{q_\alpha, H_\alpha\} = \alpha \frac{p_x}{m}$$

since $dp_x/dt = 0$ and $dx/dt = p_x/m$.

Let us study this picture some more. Let $\epsilon = \alpha - 1$, then

$$q_\alpha = x + \epsilon \frac{p_x}{m}(t - t_0) = x + \epsilon\{x, p_x^2 \frac{(t-t_0)}{2m}\}$$

The generator of this infinitesimal canonical transformation is

$$G = p_x^2 \frac{(t-t_0)}{2m} = p_\alpha^2 \frac{(t-t_0)}{2m} \tag{2.67}$$

The full canonical transformation is indeed given by

$$W = xp_\alpha + \epsilon G = xp_\alpha + \epsilon p_\alpha^2 \frac{(t-t_0)}{2m} \tag{2.68}$$

then it follows that

$$\begin{aligned} p_x &= \frac{\partial W}{\partial x} = p_\alpha & \qquad q_\alpha &= \frac{\partial W}{\partial p_\alpha} = x + \epsilon \frac{\partial G}{\partial p_\alpha} \\ H_\alpha &= H + \frac{\partial W}{\partial t} = \alpha H \end{aligned} \tag{2.69}$$

which is what we had before.

In fact, the choice $\epsilon = -1$, i.e., $\alpha = 0$ solves

$$H(q, p{=}\partial W/\partial q) + \frac{\partial W}{\partial t} = 0 \tag{2.70}$$

This is the so-called Principal function equation, the equation that defines the generating function for the canonical transformation that makes the hamiltonian zero (here we have actually described the method for any hamiltonian independent of the q's, $H = H(p)$).

Let us now look at the interacting case. With the gauge-fixing choice $\chi_\alpha = t - f(\tau)$, where $\alpha \equiv \dot{f}(\tau)$— as before. It is easy to see that the equations now read

$$\begin{aligned} \dot{x}_i &= \dot{f}\ \{x_i, \mathcal{H}_{EM}\} \qquad & \dot{p}_i &= \dot{f}\ \{p_i, \mathcal{H}_{EM}\} = -\dot{f}\frac{\partial \mathcal{H}_{EM}}{\partial x_i} \\ \dot{t} &= \dot{f} & \dot{p}_t &= \dot{f}\ \{p_t, \mathcal{H}_{EM}\} = -\dot{f}\frac{\partial \mathcal{H}_{EM}}{\partial t} \end{aligned} \tag{2.71}$$

which is not terribly surprising. The Dirac bracket computation is easily done, since again $\{\chi_\alpha, \Phi\} = 1$, and it yields

$$\{x_i, p_j\}_* = \delta_{ij}, \ \ \{t, p_t\}_* = 0, \ \ \{x_i, p_t\}_* = -\{x_i, \mathcal{H}_{EM}\}_*$$

with all others zero. The dynamics for functions of x_i, p_j are once more given by

$$\dot{F} = \frac{\partial F}{\partial \tau} + \dot{f}\ \{F, \mathcal{H}_{EM}\}$$

Thus, the behavior with respect to changes of gauge-fixing—and their effects on the RPS^*—is just as in the free case. The interacting case does not bring in any new conceptual problems, although the dynamics are more complicated.

****

Let us review what the use of a parameter has done and comment on the peculiarities of this system. First notice that the original unconstrained case is recovered through the use of a very special gauge: $t = \tau$. This can easily be seen in the parametrized second order action. In effect, *the original system is reached through a gauge choice, not through the gauge invariant content of the theory.* In fact, *the gauge invariant content of the parametrized case is described by the constants of the motion*—as we will see—*which are true reparametrization invariants!* The "time evolution" effect is provided by the use of τ-dependent gauges, that is, by the use of a different gauge at each τ-time—and of gauge dependent coordinates, of course. Different τ-time-varying gauge choices produce different systems, which as we have seen, are related by canonical transformations.

How is this discussion particular to parametrized systems, i.e., what is the effect in general of τ-time dependent gauge-fixings? The answer is that the same effect in the dynamics will occur in any situation where τ-dependent gauge fixings are used. Consider again the simple system Q, P, q, p with the constraint $P = 0$ and first class hamiltonian $h(q,p)$. Let us use the τ-dependent gauge fixing $\chi = Q - f(\tau)$. It is easy to see that the extended hamiltonian is $H_{E'} = h(q,p) + \dot{f}P$, as follows from the usual requirement that $\dot{\chi} = 0$. Notice that none of this affects the dynamics in the RPS, the space described by the coordinates q, p and hamiltonian $h(q,p)$. However, if we now did a canonical transformation into some new canonical pairs $\mathcal{Q}^a, \mathcal{P}_a$ that describe the full phase space after gauge fixing—or RPS^*—we will have some complicated dynamics.

Parametrized systems are special, then, in that we are forced to use such gauge fixings, because the boundary conditions demand it. Indeed, the lack of gauge freedom at the boundaries means that one needs to use gauges that match the boundary requirements. This, as we saw, is a consequence of the non-linearity of the constraint. We will see this point in a different and clearer light at the end of the next section.

2.4.4 Constant of the motion coordinate system: an example

We will study the non-relativistic free case. In this coordinate system we take the constraint Φ to be a new coordinate, say a momentum, $P \equiv \Phi$. Now we need its conjugate coordinate Q, which we can define by

$$\{Q, P\} = 1 = \frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x}\frac{p_x}{m}$$

The other coordinates q, p must commute with Q, P—and we saw that the solution to $\{C_\Phi, P\} = 0$ is given by any function of p_x, p_t or $x - p_x t/m$—and have the right commutation relation among themselves.

A possible solution is given by

$$\begin{aligned} Q &= \frac{1}{2}(t + m\frac{x}{p_x}) \quad & P &= p_t + \frac{p_x^2}{2m} \quad & &\textit{gauge degrees of freedom} \\ q &= \frac{1}{2}(t - \frac{x}{p_x}) \quad & p &= p_t - \frac{p_x^2}{2m} \quad & &\textit{physical degrees of freedom} \end{aligned} \tag{2.72}$$

There is, however a problem with this solution. In fact, in the surface $\Phi = 0$, we have $p_t = -p_x^2/2m \leq 0$, so in that surface $p = -p_x^2/m \leq 0$. The physical momentum coordinate doesn't take values over the full range! It is easy to see that the map

$$(t, x, p_t, p_x) \longrightarrow (Q, P, q, p)$$

is not one-to-one. Indeed the two points (t, x, p_t, p_x), and $(t, -x, p_t, -p_x)$ are mapped to the same (Q, P, q, p). That there are problems with this transformation is not surprising, as it is already undefined at $p_x = 0$. If we fix, say, $p_x > 0$, then the map will be one-to-one, of course.

This bad choice of coordinates is an example of the problem we discussed in section 2.4. Our coordinate choice made the constraint into a momentum, but for that it picked up the "x part" of the constraint, which suffers from the problem illustrated above. Either we change the constraint to $p_x = +\sqrt{-2mp_t}$ —say—or have to deal with the bad square form. It is simply not possible to make a constraint like $P^2/2 \approx 0$ into a momentum. It doesn't cover the full range to begin with.

This analysis will be of relevance again when we deal with the relativistic case and its quadratic constraint [52].

A proper solution to the problem of finding a set of canonical coordinates in which the constraint is a momentum is given by

$$\begin{array}{lll} Q = t - t_0 & P = p_t + \dfrac{p_x^2}{2m} & \textit{gauge degrees of freedom} \\ q = p_x(t - t_0) - mx & p = -\dfrac{p_x}{m} & \textit{physical degrees of freedom} \end{array} \tag{2.73}$$

The system seems to be telling us that to make life simple it is t that has to be considered pure gauge. The first order action in this coordinates is

$$A_{Q,P,q,p} = \int_{\tau_i}^{\tau_f} d\tau \; (P\dot{Q} + p\dot{q} - w(\tau)P) \tag{2.74}$$

Now, that is simple! Where did all the invariance-at-the-boundaries problems go?

Well, we need to know what the original boundary conditions look like in this coordinate system. Recall that t and x where fixed at the boundaries. This corresponds to fixing Q and $-q/m - pQ$ at the boundaries. The above action doesn't have an extremum with such boundary conditions; a surface term needs to be added, as explained in reference [14]. Indeed, consider the variation of the action:

$$\begin{aligned}\delta A_{Q,P,q,p} &= \delta \int_{\tau_i}^{\tau_f} (P\dot{Q} + p\dot{q} - w(\tau)P)d\tau \\ &= \int_{\tau_i}^{\tau_f} (\delta P(\dot{Q} - w) + \frac{d(P\delta Q)}{d\tau} - \dot{P}\delta Q - \delta w P + \delta p\dot{q} + \frac{d(p\delta q)}{d\tau} - \dot{p}\delta q)d\tau \\ &= \int_{\tau_i}^{\tau_f} (\delta P(\dot{Q} - w) - \dot{P}\delta Q - \delta w P + \delta p\dot{q} - \dot{p}\delta q)d\tau + (P\delta Q + p\delta q)|_{\tau_i}^{\tau_f}\end{aligned} \tag{2.75}$$

With the boundary conditions on x, t, the surviving surface term is

$$(p\delta q)|_{\tau_i}^{\tau_f} = -\, p_x(t - t_0)\frac{\delta p_x}{m}\bigg|_{\tau_i}^{\tau_f} \tag{2.76}$$

To get rid of it we need to add the surface term $B|_{\tau_i}^{\tau_f}$ to the action, with

$$(p\delta q + \delta B)|_{\tau_i}^{\tau_f} = 0 \tag{2.77}$$

A solution is

$$B = (t - t_0)\frac{p_x^2}{2m} = \frac{m}{2}Qp^2 \tag{2.78}$$

Unsurprisingly, the generator of canonical transformations of the previous section reappears. This is where the lack of gauge invariance at the boundaries comes from, as this term depends on Q. We can rephrase the subtleties of this system as follows:

There is nothing special about the action—or the constraint, as it is simple enough, $P \approx 0$. However, *our insistence on peculiar boundary conditions makes the addition of a gauge dependent boundary term necesssary for the existence of an extremum. Therefore gauge invariance at the boundaries is lost.*

This illustrates the fact that it is not the form of the constraint only that matters, but also what is fixed at the boundaries: the recipe for no trouble is that *the constraint should be linear in the coordinates conjugate to whatever is fixed at the boundaries.*

2.4.5 RPS analysis for the relativistic particle

For the non-relativistic particle, we have a phase space described by the coordinates t, x, p_x and p_t, and a constraint $\Phi = p_t^2 - p_x^2 - m^2 \approx 0$. A possible approach is to try to immediately perform a canonical transformation such that Φ becomes a momentum, and to proceed as above. We will look at this approach in section 2.4.4.

One approach is to say that the gauge invariant functions are those that satisfy $\{C_\Phi(q^\alpha, p_\alpha), \Phi\} \equiv 0$, i.e.,

$$0 \equiv (2p_t \frac{\partial}{\partial t} - 2p_x \frac{\partial}{\partial x} - m^2) C_\Phi(q^\alpha, p_\alpha) \tag{2.79}$$

which is solved by

$$C_\Phi = g(tp_x + xp_t, p_x, p_t) e^{tp_x + xp_t - m^2 \frac{x}{2p_x}} \tag{2.80}$$

A more symmetric way to write this is

$$C_\Phi = g(tp_x + xp_t, p_x, p_t)e^{tp_x + xp_t + \frac{m^2}{p_x p_t}(tp_x - xp_t)} \tag{2.81}$$

For the general interacting case it is harder to write the solution, of course.

One really only needs to ask for a weak equality, i.e., that the bracket be zero on the constraint surface,

$$\{C_\Phi(q^\alpha, p_\alpha), \Phi\} \approx 0 \tag{2.82}$$

In this case the constraint surface can be understood to mean either the full constraint suraface—both branches included—or only one branch. For the interacting case we have $\{C_\Phi, \Phi_{EG}\} = \{C_\Phi, (\Pi^0 - B_+)g_{00}(\Pi^0 - B_-)\} =$

$$\{C_\Phi, (\Pi^0 - B_+)\}g_{00}(\Pi^0 - B_-) + \{C_\Phi, (\Pi^0 - B_-)\}g_{00}(\Pi^0 - B_+) = 0 \tag{2.83}$$

where

$$B_\pm \equiv -\Pi^i \tilde{g}_{0i} \pm \sqrt{(\Pi^i \tilde{g}_{0i})^2 - \Pi^i \tilde{g}_{ij}\Pi^j + m^2/g_{00}}$$

We see that if only one branch is taken the equation is just

$$\{C_\Phi, (\Pi^0 - B_+)\}g_{00}(\Pi^0 - B_-) = 0$$

a simpler equation than the ones above! This corresponds to "choosing a branch". On the other hand, demanding weak equality on the *full* surface yields

$$\{C_\Phi, (\Pi^0 - B_+)\} \approx \{C_\Phi, (\Pi^0 - B_-)\} \approx 0 \tag{2.84}$$

Let us now consider fixing the gauge with a τ-dependent gauge, $\chi_\alpha = t - f(\tau)$, where $\alpha \equiv \dot{f}(\tau)$. As before, we need $\dot{\chi}_\alpha = 0 = \partial\chi_\alpha/\partial\tau + \{\chi_\alpha, H_{E'}\}$ which implies $v = \alpha/2p_t$, so we have

$$H_{E'} = \frac{\alpha}{2p_t}\Phi = \frac{\alpha}{2p_t}(p_t^2 - p_x^2 - m^2) \tag{2.85}$$

and the equations of motion in this gauge are

$$\begin{aligned} &\dot{t} = \alpha \quad \dot{x} = -\frac{\dot{f}}{p_t}p_x \\ &\dot{p}_t = 0 \quad \dot{p}_x = 0 \end{aligned} \tag{2.86}$$

Using

$$p_t = \pm\sqrt{p_x^2 + m^2} \tag{2.87}$$

we have

$$\dot{x} = \pm\{x, \alpha\sqrt{p_x^2 + m^2}\} = \pm\dot{f}\{x, \sqrt{p_x^2 + m^2}\} \tag{2.88}$$

so the system is reduced to the coordinates x, p_x, with a hamiltonian

$$H_\alpha = \pm\alpha\sqrt{p_x^2 + m^2} \tag{2.89}$$

Notice that the hamiltonian comes with two signs: thus propagation back and forth in time are both described in this system

The Dirac bracket computation in this gauge is easily done, since $\{\chi_\alpha, \Phi\} = 2p_t$: let $\chi_j = \chi_\alpha, \Phi$ as before. Then $\{A, B\}_* =$

$$\{A, B\} - (\{A, \chi_\alpha\}, \{A, \Phi\}) \begin{pmatrix} 0 & -\{\chi_\alpha, \Phi\}^{-1} \\ \{\chi_\alpha, \Phi\}^{-1} & 0 \end{pmatrix} \begin{pmatrix} \{\chi_\alpha, B\} \\ \{\Phi, B\} \end{pmatrix} \tag{2.90}$$

and this yields

$$\{x, p_x\}_* = 1,\ \{t, p_t\}_* = 0,\ \{x, p_t\}_* = \pm\{x, \sqrt{p_x^2 + m^2}\}$$

with all others zero.

Recall that we run into a subtlety with the Dirac bracket description of the dynamics. In this gauge, for functions of x, p_x, the correct dynamics are described by equation 2.65

$$\frac{dF}{d\tau} = \frac{\partial F}{\partial \tau} + \{F, H_\alpha\}$$

which follows from equation (2.50) ([15], and see ex. 4.8 in reference [2]).

In the gauge $\alpha = 1$ the connection to the unconstrained case of section 1 (equation (1.34)) with the hamiltonian h is clear, but let us look at the interpretation of the more general gauge-fixings. First notice that equation (2.65) has a very simple interpretation: One can simply factor out the term $\dot{f}$ to read

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, \pm\sqrt{p_x^2 + m^2}\}$$

Let us show, anyhow, that this reduced phase space is the same as the phase space for the unconstrained non-relativistic particle in section 1, in some unusual τ-dependent coordinates. Recall that the idea here is that when one picks a gauge, a coordinate system for the reduced phase space is implicitly chosen—or an active transformation occurs, depending on your point of view—and that with τ-dependent choices the dynamics will look different. Let us go back to the unconstrained system of section 1:

there, t was the time, not τ. However, as we know from our later point of view t and τ are one and the same in the gauge that leads to this system. So keep in mind in what follows that $t = \tau$. What canonical transformation corresponds to

$$H = \sqrt{p_x^2 + m^2} \longrightarrow H_\alpha = \alpha\sqrt{p_x^2 + m^2}\ ?$$

The full canonical transformation is indeed given by

$$W = xp_\alpha + \epsilon G = xp_\alpha + (\alpha - 1)\sqrt{p_x^2 + m^2}(t - t_0) \tag{2.91}$$

then it follows that

$$\begin{aligned} p_x &= \frac{\partial W}{\partial x} = p_\alpha \qquad\qquad q_\alpha = \frac{\partial W}{\partial p_\alpha} = x + (\alpha - 1)\frac{\partial G}{\partial p_\alpha} \\ H_\alpha &= H + \frac{\partial W}{\partial t} = \alpha H \end{aligned} \tag{2.92}$$

The choice $\alpha = 0$ solves

$$H(q, p{=}\partial W/\partial q) + \frac{\partial W}{\partial t} = 0 \tag{2.93}$$

Again, this is the so called Principal function equation, the equation that defines the generating function for the canonical transformation that makes the hamiltonian zero (here we have actually described the method for any hamiltonian independent of the q's, $H = H(p)$).

How about the interacting case? With the gauge-fixing choice $\chi_\alpha = t - f(\tau)$, where $\alpha \equiv \dot{f}(\tau)$—as before—we have $v = \dot{f}/\{t, \Phi_{EG}\}$. We can next compute the bracket

$$\{t, \Phi_{EG}\} = 2g^{00}(p_0 - A_0) + 2g^{0i}(p_i - A_i) = 2g^{\mu 0}\Pi_\mu \tag{2.94}$$

The extended hamiltonian is given then by

$$H_{E'} = \frac{\dot{f}}{2\Pi_\mu g^{\mu 0}}\Phi_{EG} = \frac{\dot{f}}{2\Pi_\mu g^{\mu 0}}(\Pi_0 - B_+)g^{00}(\Pi_0 - B_-) \tag{2.95}$$

where $+$ and $-$ stand for the branches:

$$B_\pm \equiv -\Pi_i \tilde{g}^{0i} \pm \sqrt{(\Pi_i \tilde{g}^{0i})^2 - \Pi_i \tilde{g}^{ij}\Pi_j + m^2/g^{00}}$$

and $\tilde{a} = a/g^{00}$.

Let us now look at the general equation of motion. The idea is to look at the branch decomposition of the constraint above. One has to choose, ultimately, a branch in which to "be", for example when the initial conditions are chosen. It is then easy to work out the general equation of motion,

$$\dot{A} = \frac{\partial A}{\partial \tau} + \dot{f}\{A, \Pi_0 - B_\pm\}\bigg|_{\Pi_0 = B_\pm} \tag{2.96}$$

> *The conclusion is that at the end one of the branches gets chosen, and with the above gauge fixing in t the system in x_i, p_{x_i} behaves as that corresponding to a reduced system with a hamiltonian $h = -A_0 - B_\pm$, just as in the non-relativistic case.*

The Dirac bracket computation can now be done; it yields

$$\{x_i, p_j\}_* = \delta_{ij}, \ \ \{t, p_t\}_* = 0, \ \ \{f(x_i, p_i), p_t\}_* \approx \{f(x_i, p_i), A_0 + B_\pm\}$$

with all others zero, where a branch has been chosen. Thus, the behavior with respect to changes of gauge-fixing—and their effects on the RPS^*—is just as in the free case. The interacting case does not bring in any new conceptual problems, although the dynamics are more complicated. The only confusing aspect may be that it is Π_0 that comes with two signs. Using the Legendre transform it is easy to see, however, that this combination is indeed $\dot{t}$—and this is what comes with two signs. Thus, the interpretation of the presence of two branches as a representation of back and forth motion in time remains valid.

2.4.6 RPS analysis for minisuperspace

This analysis in this section leads to the same situation as in the relativistic case, since mathematically the relativistic particle is just as minisuperspace. As mentioned above the constraint in this system is

$$\Phi_M \equiv P_A P_B g^{AB} - U(Q) \approx 0 \tag{2.97}$$

for a gauge fixing function of the form $\chi = Q^0 - f(\tau)$ —where we tentatively assign the "time-keeping" role to the scale of the universe, it is useful to rewrite the constraint as

$$\Phi_M \equiv (P_0 - \mathcal{B}_+) g^{00} (P_0 - \mathcal{B}_-) \approx 0 \tag{2.98}$$

where

$$\mathcal{B}_\pm \equiv \pm\sqrt{(P^I \tilde{g}_{0I})^2 - P^i \tilde{g}_{IJ} P^J + \tilde{U}} \tag{2.99}$$

With the bracket

$$\{\chi, \Phi_M\} = 2 g_{A0} P^A \tag{2.100}$$

the Dirac Bracket computation yields

$$\{Q^I, P_J\}_* = \delta_{IJ}, \ \ \{Q^0, P_0\}_* = 0, \ \ \{f(Q^I, P_J), P_0\}_* \approx \{f(Q^I, P_J), \mathcal{B}_\pm\}$$

with all others zero, where a branch has been chosen, and the dynamics are once more given—for functions on the reduced coordinates—by

$$\dot{F} = \frac{\partial F}{\partial \tau} + \dot{f}\ \{F, \mathcal{B}_\pm\}$$

Notice that once more if one tries to force the hamiltonian philosophy on a system of this type we find that we need two hamiltonians—with the above gauge fixing. This unusual situation will be examined further when we study the quantization of these systems.

Finally, notice that we have the following interpretation for the appearance of the two branches:

Classicaly we have that the effective hamiltonian in the reduced phase space comes with two signs—one for each branch; this produces a forward and backward propagation in "time", the conjugate variable to the hamiltonian.

2.5 BRST extended phase space

Let us now review the BRST treatment of a system with a constraint Φ [2, 19]. We will consider some of the above simple systems as examples of straightforward applications of this formalism.The first objects that are introduced into the extended phase space are λ and its conjugate momentum π , $\{\lambda, \pi\} = 1$. Since λ is arbitrary, its momentum is constrained, $\pi \approx 0$. We thus have two constraints. To each constraint, the rules say we must associate a conjugate pair of ghosts, η_0, ρ_0 and η_1, ρ_1 with $\{\eta_0, \rho_0\} = 1$ and $\{\eta_1, \rho_1\} = 1$, other (super)brackets being zero. The total extended phase space is thus described by $t, p_t, x, p_x, \lambda, \pi, \eta_0, \rho_0, \eta_1, \rho_1$. The next thing to do is to define the BRST generator, which generates the gauge transformations in extended phase space:

$$\Omega = \eta_0 \Phi + \eta_1 \pi \tag{2.101}$$

We also need a gauge-fixing function, $\mathcal{O}$. The dynamics are then generated by the hamiltonian $\mathcal{H} = h + \{\mathcal{O}, \Omega\}$ $(= \{\mathcal{O}, \Omega\}$, since the original first class hamiltonian h is zero.) The first order action is then

$$S = \int_{\tau_i}^{\tau_f} (\dot{q}^\alpha p_\alpha - \mathcal{H}) d\tau = \int_{\tau_i}^{\tau_f} (\dot{t} p_t + \dot{x} p_x + \dot{\lambda}\pi + \dot{\eta}_0 \rho_0 + \dot{\eta}_1 \rho_1 - \{\mathcal{O}, \Omega\}) d\tau \tag{2.102}$$

We will use the "non-canonical" gauge fixing $\mathcal{O}_{NC} = \rho_1 f(\lambda) + \rho_0 \lambda$ and also the "canonical" one $\mathcal{O}_C = \rho_1 \chi + \rho_0 \lambda$, where χ is a function of the original phase space variables only—the Dirac bracket gauge-fixing function: $\chi = \chi(t, x, p_x, p_t, \tau)$. The terminology will become clearer as we go on.

Consider first the non-canonical gauge for the non-relativistic particle. The hamiltonian is $\mathcal{H}_{NC} = \{\mathcal{O}_N C, \Omega\} =$

$$\{\rho_1 f(\lambda) + \rho_0 \lambda, \eta_0 \Phi + \eta_1 \pi\} = \rho_1 \eta_1 f'(\lambda) + \pi f(\lambda) + \lambda \Phi + \rho_0 \eta_1 \tag{2.103}$$

Let us assume that the proper boundary conditions are imposed so that all surface terms that arise under the variation of the action vanish. Then the canonical equations of motion are

$$\begin{aligned}
\dot{x} &= \{x, \mathcal{H}_N C\} = \frac{\partial \mathcal{H}_{NC}}{\partial p_x} = \frac{\lambda p_x}{m} & \dot{p_x} &= 0 \\
\dot{t} &= \lambda & \dot{p_t} &= 0 \\
\dot{\lambda} &= f(\lambda) & \dot{\pi} &= -\rho_1 \eta_1 f''(\lambda) - \pi f'(\lambda) - \Phi \\
\dot{\eta_0} &= \eta_1 & \dot{\rho_0} &= 0 \\
\dot{\eta_1} &= 0 & \dot{\rho_1} &= 0
\end{aligned} \tag{2.104}$$

This can be summarized by saying that the coordinates t, x, p_t, p_x and λ, π have dynamics independent from the ghosts, and that it is clear that $dx/dt = p_x/m =$ *constant*—just as in the unconstrained case, $\dot{\Phi} = \dot{\pi} = 0$ for the right initial conditions—and that we have the gauge fixing $d^2t/d\tau^2 = f(\lambda)$. So despite all the new formalism, the dynamical bottom line is always the same.

For the "canonical" gauge we have $\mathcal{H}_C = \{\mathcal{O}_C, \Omega\} =$

$$\{\rho_1 \chi + \rho_0 \lambda, \eta_0 \Phi + \eta_1 \pi\} = \rho_1 \eta_0 \{\chi, \Phi\} + \pi \chi + \lambda \Phi + \rho_0 \eta_1 \tag{2.105}$$

and the equations of motion are

$$\begin{aligned}
\dot{x} &= \{x, \mathcal{H}_C\} = \frac{\partial \mathcal{H}_C}{\partial p_x} = \rho_1\eta_0 \frac{\partial\{\chi,\Phi\}}{\partial p_x} + \pi\frac{\partial \chi}{\partial p_x} + \frac{\lambda p_x}{m} & \dot{p_x} &= -\rho_1\eta_0\frac{\partial\{\chi,\Phi\}}{\partial x} - \pi\frac{\partial\chi}{\partial x} \\
\dot{t} &= \rho_1\eta_0\frac{\partial\{\chi,\Phi\}}{\partial p_t} + \pi\frac{\partial\chi}{\partial p_t} + \lambda & \dot{p_t} &= -\rho_1\eta_0\frac{\partial\{\chi,\Phi\}}{\partial t} - \pi\frac{\partial\chi}{\partial t} \\
\dot{\lambda} &= \chi & \dot{\pi} &= -\Phi \\
\dot{\eta_0} &= \eta_1 & \dot{\rho_0} &= -\rho_1\{\chi,\Phi\} \\
\dot{\eta_1} &= \eta_0\{\chi,\Phi\} & \dot{\rho_1} &= -\rho_0
\end{aligned} \tag{2.106}$$

For illustration purposes let us use the gauge $\chi = t - \alpha\tau - t_0$, so that $\{\chi,\Phi\} = 1$. Then the equations of motion become

$$\begin{aligned}
\dot{x} &= \lambda\frac{p_x}{m} & \dot{p_x} &= 0 \\
\dot{t} &= \lambda & \dot{p_t} &= -\pi \\
\dot{\lambda} &= \chi & \dot{\pi} &= -\Phi \\
\dot{\eta_0} &= \eta_1 & \dot{\rho_0} &= -\rho_1 \\
\dot{\eta_1} &= \eta_0 & \dot{\rho_1} &= -\rho_0
\end{aligned} \tag{2.107}$$

Again, we can summarize this by writing $dx/dt = p_x/m =$ constant as before, $d^2t/d\tau^2 = \chi$ is the gauge-fixing , and $d\Phi/d\tau = -\pi = 0$ if the initial conditions satisfy $\Phi = 0 = \pi$, since $d\pi/d\tau = -\Phi$. So, again, the dynamics stay the same.

Consider next the relativistic case with a non-canonical gauge. Again, the hamiltonian is

$$\mathcal{H}_{NC} = \{\mathcal{O}_{NC},\Omega\} = \{\rho_1 f(\lambda) + \rho_0\lambda, \eta_0\Phi + \eta_1\pi\} = \rho_1\eta_1 f'(\lambda) + \pi f(\lambda) + \lambda\Phi + \rho_0\eta_1 \tag{2.108}$$

Let us assume that the proper boundary conditions are imposed so that all surface terms that arise under the variation of the action vanish. Then the canonical equations of motion for the free case are

$$\begin{aligned}
&\dot{x} = \{x, \mathcal{H}_{NC}\} = -\lambda 2p_x && \dot{p_x} = 0 \\
&\dot{t} = \lambda 2p_t && \dot{p_t} = 0 \\
&\dot{\lambda} = f(\lambda) && \dot{\pi} = -\rho_1\eta_1 f''(\lambda) - \pi f'(\lambda) - \Phi \\
&\dot{\eta_0} = \eta_1 && \dot{\rho_0} = 0 \\
&\dot{\eta_1} = 0 && \dot{\rho_1} = 0
\end{aligned} \tag{2.109}$$

This can be summarized by saying that the coordinates t, x, p_t, p_x and λ, π have dynamics independent from the ghosts, and that it is clear that $dx/dt = -p_x/p_t =$ *constant*—as in the unconstrained case again, $\dot{\Phi} = \dot{\pi} = 0$ for the right initial conditions—and that we have the gauge fixing $d^2t/d\tau^2 = f(\lambda)$. So despite all the new formalism, the dynamical bottom line is always the same.

For the "canonical" gauge we have $\mathcal{H}_C = \{\mathcal{O}_C, \Omega\} =$

$$\{\rho_1\chi + \rho_0\lambda, \eta_0\Phi + \eta_1\pi\} = \rho_1\eta_0\{\chi, \Phi\} + \pi\chi + \lambda\Phi + \rho_0\eta_1 \tag{2.110}$$

and the equations of motion are

$$
\begin{aligned}
&\dot{x} = \{x, \mathcal{H}_C\} = \rho_1\eta_0\frac{\partial\{\chi,\Phi\}}{\partial p_x} + \pi\frac{\partial\chi}{\partial p_x} + -2\lambda p_x && \dot{p_x} = -\rho_1\eta_0\frac{\partial\{\chi,\Phi\}}{\partial x} - \pi\frac{\partial\chi}{\partial x} \\
&\dot{t} = \rho_1\eta_0\frac{\partial\{\chi,\Phi\}}{\partial p_t} + \pi\frac{\partial\chi}{\partial p_t} + 2\lambda p_t && \dot{p_t} = -\rho_1\eta_0\frac{\partial\{\chi,\Phi\}}{\partial t} - \pi\frac{\partial\chi}{\partial t} \\
&\dot{\lambda} = \chi && \dot{\pi} = -\Phi \\
&\dot{\eta_0} = \eta_1 && \dot{\rho_0} = -\rho_1\{\chi,\Phi\} \\
&\dot{\eta_1} = \eta_0\{\chi,\Phi\} && \dot{\rho_1} = -\rho_0
\end{aligned}
\tag{2.111}
$$

For illustration purposes let us use the gauge $\chi = t - \alpha\tau - t_0$, so that $\{\chi, \Phi\} = 1$. Then the equations of motion become

$$
\begin{aligned}
&\dot{x} = \lambda\frac{p_x}{m} && \dot{p_x} = 0 \\
&\dot{t} = \lambda && \dot{p_t} = -\pi \\
&\dot{\lambda} = \chi && \dot{\pi} = -\Phi \\
&\dot{\eta_0} = \eta_1 && \dot{\rho_0} = -\rho_1 \\
&\dot{\eta_1} = \eta_0 && \dot{\rho_1} = -\rho_0
\end{aligned}
\tag{2.112}
$$

Again, we can summarize this by writing $dx/dt = -p_x/p_t$ = constant as before, $d^2t/d\tau^2 = \chi$ is the gauge-fixing , and $d\Phi/d\tau = -\pi = 0$ if the initial conditions satisfy $\Phi = 0 = \pi$, since $d\pi/d\tau = -\Phi$. So, again, the dynamics stay the same.

Finally, let us remind the reader that in previous work we showed that a constraint rescaling in the BRST action is equivalent—via a canonical tranformation—to a rescaling of the gauge fixing function. We will elaborate when we discuss the BFV path integral (see section 5.1.4).

Notice that in this formalism there are no constraints—as is usually said, "the local gauge invariance is made rigid". The gauge-fixing has been implemented in the dynamics, *and* the dynamics of BRST-invariant functions do not depend on the gauge-fixing. This is what sets BRST apart from Dirac. We can use a gauge-fixed action in the Dirac case as well, but we have to use the Dirac bracket—solve the constraints—or alternatively, find functions that commute with both the constraints and the gauge-fixing. Here the ghosts take care of this, i.e., of the reduced simplectic geometry (we will see this explicitely when we study the path integrals.) And we don't even have to reduce. Gauge-fixing does not interfere with BRST invariance.

2.6 Conclusions, summary

In this chapter I have begun by reviewing the classical aspects of the actions for the non-relativistic and relativistic particles as well as minisuperspace, and studied their invariances at the lagrangian level. We have seen that the actions are invariant only up to boundary terms [14].

Some of the actions, we have seen, carry a representation of the full reparametrization group, including its disconnected part (Z_2). The disconnected part is broken by a background electromagnetic field, however. Continuing with the review, Dirac's formalism has been applied to our parametrized systems, and we have seen that they are constrained systems of a special kind: because the actions are invariant only up to boundary terms, the constraints are not linear in the momenta, and, moreover, in the more interesting cases of the relativistic particle and minisuperspace, the constraint surfaces are disconnected. These are both well-known facts, and they are the source of serious conceptual and technical problems in the quantization process of parametrized systems.

I have then argued that to be consistent at the classical level we have to pick a branch in the cases where the constraint surface splits—the relativistic case as well as minisuperspace. The two branches correspond to two hamiltonians in the reduced phase space that generate either forward or backward time displacements.

I have then made use of the reparametrization invariance to construct—with the Dirac bracket—the reduced phase spaces, and showed that this construction depends on the gauge-fixing employed—a disturbing new feature of these "gauge" systems, and a direct consequence of the actions' lack of invariance at the boundaries. However,I have showed that these reduced phase spaces, RPS^*_χ, are related by time-dependent canonical transformations.

Finally, I have reviewed the BRST phase phace construction, emphasizing that it possesses the advantadge of incorporating both the constraint and the gauge-fixing in the action—the ghosts take care of the "reduced phase space" simplectic properties.

Chapter 3

Canonical Quantization

In this chapter I will review the different quantization schemes that we will apply to our systems. I will pay special attention to the approach in which the reduced phase space is quantized, that is, to the *constrain then quantize* method, since it is the most intuitive and immediate. I will also give a formal introduction to the *quantize then constrain* approaches, but the more difficult questions of the precise definition of the Hilbert spaces involved and the fundamental questions of the definition and properties of the inner product and unitarity will be delayed until the next chapter. The application of all these ideas to the construction of path integrals will be considered in the chapter after that.

For the relativistic case two approaches will be taken. In the first one, one branch of the constraint will be chosen. The solution space of the general constraint for the relativistic particle is given by equation (2.29),

$$\Pi_0 = -\Pi_i \tilde{g}^{0i} \pm \sqrt{(\Pi_i \tilde{g}^{0i})^2 - \Pi_i \tilde{g}^{ij} \Pi_j + m^2/g^{00}} \tag{3.1}$$

where $\Pi \equiv p - A$, and $\tilde{a} \equiv a/g^{00}$. The situation for minisuperspace is very similar since the constraint is essentially the same as the above one without the electromagnetic field.

In the second approach we will look at the possibility of quantizing both the branches at the same time.

3.1 Constrain, then quantize: RPS quantum mechanics

Reduced phase space quantization is the simplest and most transparent quantization approach. If one knows what the true degrees of freedom in the system are it should be simple to quantize them. There are, in the present situation some problems which make the systems we have been studying interesting: one of them is the issue of lack of invariance at the boundaries. In the previous chapter I explained that different gauge fixings correspond to a different choice of coordinates to describe the reduced phase space. These correspond to different coordinate choices in the unconstrained case, and as Dirac explains canonical transformations correspond to unitary transformations in quantum mechanics. The only complication may arise from the time dependence of the transformation.

Other problems in the qunatization process involve the branching of the constraint surface and unitarity.

Let us first look at the non-relativistic case. There are no branches and the RPS hamiltonian is always unitary. This example will teach us about the relation that exists between *unitary transformations, "pictures", canonical transformations and different gauge fixings.*

In the RPS approach we have—after the gauge fixing

$$\chi_\alpha = t - f(\tau)$$

some coordinates/operators $\mathbf{q}_\alpha, \mathbf{p}_\alpha$ and a hamiltonian $\mathbf{H}_\alpha$, acting on a Hilbert space of states $|\psi\rangle_\alpha$. Time evolution is then given by the Schrödinger equation

$$i\frac{\partial}{\partial\tau}|\psi\rangle_\alpha = \mathbf{H}_\alpha|\psi\rangle_\alpha \tag{3.2}$$

Taking the point of view that different gauge fixings correspond to canonical transformations of the unconstrained system in chapter 1 generated by

$$W_\alpha = q_1 p_\alpha + (\alpha - 1)\frac{p_\alpha^2(t-t_0)}{2m} \equiv q_1 p_\alpha + G_\alpha \tag{3.3}$$

and remembering—see for example [3]—that this corresponds to a unitary transformation of the form

$$\mathbf{O} \longrightarrow \mathbf{O}_\alpha = \mathbf{U}_\alpha \mathbf{O} \mathbf{U}_\alpha^{-1} \approx \mathbf{O} - \mathbf{i}[\mathbf{G}_\alpha, \mathbf{O}] \tag{3.4}$$

with

$$\mathbf{U}_\alpha = \mathbf{e}^{\mathbf{i}\mathbf{G}_\alpha} \tag{3.5}$$

we see that these correspond to ($\alpha = 1$ is the reference system, e.g., $x_1 = x$)

$$\mathbf{x} \longrightarrow \mathbf{x}_\alpha = \mathbf{U}_\alpha \mathbf{x} \mathbf{U}_\alpha^{-1} = \mathbf{U}_\alpha(\mathbf{U}_\alpha^{-1}\mathbf{x} - [\mathbf{U}_\alpha^{-1}, \mathbf{x}]) = \mathbf{x} + (\alpha - 1)(t - t_0)\frac{\mathbf{p}_x}{m} \tag{3.6}$$

just as in the classical Poisson bracket formulation. This transformation also affects the eigenstates of operators. The new basis states for the operator $\mathbf{B}$, for instance, change as

$$|b\rangle_1 \longrightarrow |b\rangle_1 = \mathbf{U}_\alpha |b\rangle_\alpha. \tag{3.7}$$

For example, the case $\alpha = 0$, which we saw corresponds to $H = 0$ classically, leads to

$$x_0 = x - (t - t_0)p/m, p_0 = p$$

Then the system is described by the eigenstates of x_0,

$$\mathbf{x}_0|x\rangle_0 = \mathbf{x}|x\rangle \tag{3.8}$$

which we can check are

$$|x\rangle_0 = e^{-i(t-t_0)\frac{\mathbf{p}_x^2}{2m}}|x\rangle_1 = \mathbf{U}_0|x\rangle_1 \tag{3.9}$$

since

$$\mathbf{x}_0|x\rangle_0 = \mathbf{U_0 x U_0^{-1}}|x\rangle_0 = \mathbf{U_0 x}|x\rangle_1 = x|x\rangle_0 \tag{3.10}$$

Notice also that

$$\langle p|x\rangle_\alpha = e^{-ip_x x - i(t-t_0)\frac{p_x^2}{2m}} \tag{3.11}$$

In the old coordinates ($\alpha = 1, G = 0$), the new basis states are

$$\langle x|q\rangle_0 = \left(\frac{m}{2\pi i(t-t_0)}\right)^{1/2} e^{\frac{i(x-q)^2 m}{2(t-t_0)}} \tag{3.12}$$

These states form a complete set and they are also orthonormal,

$$\begin{aligned} I &= \int dq\ |q\rangle_0\ {}_0\langle q| \\ \langle x|x'\rangle &= \int dq\ \langle x|q\rangle_0\ {}_0\langle q|x'\rangle = \delta(x - x') \end{aligned} \tag{3.13}$$

and ${}_0\langle q|q'\rangle_0 = \delta(q - q')$

As for the wave-functions we can take two different points of view—continuing with the case $\alpha = 0$:

i) they are unaffected. $|\psi\rangle_0 = |\psi\rangle$, i.e.,

$$i\frac{\partial}{\partial t}|\psi\rangle_0 = \mathbf{H}|\psi\rangle_0.$$

Then

$$\frac{d}{dt}\,{}_0\langle q|\psi\rangle = \frac{d}{dt}\langle q|\mathbf{U_0^{-1}}e^{-i(t-t_0)\frac{p_x^2}{2m}}|\psi_{in}\rangle = \frac{d}{dt}\langle q|\psi_{in}\rangle = 0 \tag{3.14}$$

This is " all we have done is a change of coordinates" point of view.

ii) The states do change, $|\psi\rangle_0 = \mathbf{U_0^{-1}}|\psi\rangle$. Then $i\frac{d}{dt}|\psi\rangle_0 = 0$, but

$$i\frac{d}{dt}({}_0\langle q|\psi\rangle_0) = i\frac{d}{dt}\langle q|\psi\rangle = \frac{\mathbf{p}_x^2}{2m}\langle q|\psi\rangle \tag{3.15}$$

which connects nicely with the RPS description above (write τ for t)

Regardless of the point of view, it is easy to see now that this formalism corresponds to the Heisenberg picture of quantum mechanics. The second point of view makes this more obvious perhaps, as the states are frozen.

The punchline is that the quantization of the reduced phase space yields the quantum mechanics of the unconstrained case, although possibly in different coordinates—or pictures.

Let us now study the relativistic case. For the relativistic case the situation can be similar to the non-relativistic case above. The reduced phase space structure is complicated because of the branches, it is disconnected; in consequence one avenue is to quantize one (or both) of the branches separately. This would seem the lowest efford extension of the ideas in the simpler case above. Indeed we can proceed in this manner for the free case, as well as for some other situations which we will study below. Let us look at the free case first.

As before, in the RPS approach we have (after the gauge fixing $\chi_\alpha = t - f(\tau)$) some coordinates/operators $\mathbf{q}_\alpha, \mathbf{p}_\alpha$ and a hamiltonian $\mathbf{H}_\alpha$, acting on a Hilbert space of states $|\psi\rangle_\alpha$ whith time evolution given by the Schrödinger equation

$$i\frac{\partial}{\partial\tau}|\psi\rangle_\alpha = \mathbf{H}_\alpha|\psi\rangle_\alpha \tag{3.16}$$

Taking the point of view that different gauge fixings correspond to canonical transformations of the unconstrained system in chapter 1 generated by

$$W_\alpha = q_1 p_\alpha + (\alpha - 1)\sqrt{p_x^2 + m^2}\;(t - t_0) \equiv q_1 p_\alpha + G_\alpha$$

and remembering—see for example [3]—that this corresponds to a unitary transformation of the form

$$\mathbf{O} \longrightarrow \mathbf{O}_\alpha = \mathbf{U}_\alpha \mathbf{O} \mathbf{U}_\alpha^{-1} \approx \mathbf{O} - \mathbf{i}[\mathbf{G}_\alpha, \mathbf{O}] \tag{3.17}$$

with

$$\mathbf{U}_\alpha = \mathbf{e}^{\mathbf{i}\mathbf{G}_\alpha} \tag{3.18}$$

we see that these correspond to ($\alpha = 1$ is the reference system, e.g., $x_1 = x$)

$$\mathbf{x} \longrightarrow \mathbf{x}_\alpha = \mathbf{U}_\alpha \mathbf{x} \mathbf{U}_\alpha^{-1} = \mathbf{U}_\alpha(\mathbf{U}_\alpha^{-1}\mathbf{x} - [\mathbf{U}_\alpha^{-1}, \mathbf{x}]) = \mathbf{x} + (\alpha - 1)(t - t_0)\frac{\mathbf{p}_x}{\sqrt{p_x^2 + m^2}} \quad (3.19)$$

and as before the new basis states for the operator $\mathbf{B}$

$$|b\rangle_1 \longrightarrow |b\rangle_1 = \mathbf{U}_\alpha |b\rangle_\alpha. \quad (3.20)$$

For example, the case $\alpha = 0$, which we saw corresponds to $H = 0$ classically, leads to $x_0 = x - (t - t_0)p_x/\sqrt{p_x^2 + m^2}, p_0 = p$. Then the system is described by the eigenstates of x_0,

$$\mathbf{x}_0 |x\rangle_0 = \mathbf{x}|x\rangle \quad (3.21)$$

which we can check are

$$|x\rangle_0 = e^{-i(t-t_0)\sqrt{\mathbf{p}_x^2 + m^2}} |x\rangle_1 = \mathbf{U}_0 |x\rangle_1 \quad (3.22)$$

Notice also that

$$\langle p|x\rangle_\alpha = e^{i(p_x - (t-t_0)\sqrt{p_x^2 + m^2})} \quad (3.23)$$

In the old coordinates ($\alpha = 1, G = 0$), the new base states are

$$\langle x|q\rangle_0 = \int dp_x e^{i(p_x(x-q) - (t-t_o)\sqrt{p_x^2 + m^2})} \quad (3.24)$$

These states form a complete set and they are also orthonormal, just as before, and as for the wave-functions, we can, as before, take two different points of view—continuing with the case $\alpha = 0$:

i) they are unaffected. $|\psi\rangle_0 = |\psi\rangle$, i.e.,

$$i\frac{\partial}{\partial t}|\psi\rangle_0 = \mathbf{H}|\psi\rangle_0.$$

Then

$$\frac{d}{dt}{}_0\langle q|\psi\rangle = \frac{d}{dt}\langle q|\mathbf{U_0^{-1}}e^{-i(t-t_0)\sqrt{\mathbf{p}_x^2+m^2}}|\psi_{in}\rangle = \frac{d}{dt}\langle q|\psi_{in}\rangle = 0 \tag{3.25}$$

the " all we have done is a change of coordinates" point of view, or

ii) The states do change, $|\psi\rangle_0 = \mathbf{U_0^{-1}}|\psi\rangle$. Then $i\frac{d}{dt}|\psi\rangle_0 = 0$, but

$$i\frac{d}{dt}({}_0\langle q|\psi\rangle_0) = i\frac{d}{dt}\langle q|\psi\rangle = \sqrt{\mathbf{p}_x^2 + m^2}\langle q|\psi\rangle \tag{3.26}$$

which connects nicely with the RPS description above (write τ for t)

Now, there are some problems with the use of a square root hamiltonian, like causality. The free case—as well as some others we will discuss later—can be first-quantized fairly easily. Unitarity is not a problem.

Let us look at the general case. The constraint is essentially

$$\Pi_0 = -\Pi_i\tilde{g}^{0i} \pm \sqrt{(\Pi_i\tilde{g}^{0i})^2 - \Pi_i\tilde{g}^{ij}\Pi_j + m^2/g^{00}} \tag{3.27}$$

We could try to choose a branch. However, now the problem could be unitarity. We saw that the hamiltonian in the reduced phase space is

$$h = -A_0 - B_\pm = \mp\sqrt{(\Pi_i\tilde{g}^{0i})^2 - \Pi_i\tilde{g}^{ij}\Pi_j + m^2/g^{00}} \tag{3.28}$$

Now this hamiltonian will become an operator—with some suitable ordering. Under what circumstances there exists an ordering such that the hamiltonian operator is hermitean? We can ignore the first term, A_0, as it is already hermitean. It is in fact sufficient to find an ordering such that

$$(\Pi_i \tilde{g}^{0i})^2 - \Pi_i \tilde{g}^{ij} \Pi_j + m^2/g^{00} \tag{3.29}$$

is hermitean, and hopefully positive definite, since the square root of a positive definite hermitean operator is hermitean. For example, consider the case of a flat background. It is easy to see then that the above hamiltonian is hermitean, no matter what the gauge potential happens to be. The problem, however, is that we are not assured at all that the resulting theory will be covariant. As we discussed earlier, in the unconstrained situation—which is where we are after reduction, it may not be possible to find a covariant and hermitean ordering. It is possible, however, if there is a frame in which the electric field is zero.

In the case of a curved background there are many questions. What about our foliation choice? Our choice of time coordinate? Do these affect the resulting physics? Well, our discussion before indicates that if we change the gauge the resulting theories will be related by a canonical transformation. But, of course, the starting point occurs when we choose a branch. It is unclear, in the general case, what will result if we solve the constraint equation in a different way—i.e., by choosing the other branch, or by solving for p_x instead, say. For the flat background case we have only Lorentz transformations to worry about.

3.1.1 Quantization of both branches

Let us look next at the full constraint—at both the branches. We can start by constructing one quantum theory with each branch. We can then take the direct sum of the theories by keeping the inner product and hamiltonian and the rest of the operators "diagonal". For example, we represent the states by arranging them into two-vectors, with one entry for each branch. The inner product prevents that the two sectors talk to each other, and time evolution doesn't allow a transition either.

This is the trivial construction. It is consistent as long as there are no covariance issues (like demanding that the theories thus constructed by different observers be the same, which will fail to be true in complicated metric cases, or if there is an electric field).

But we can also try to allow for interaction. We would like to keep our one-branch energy eigenstates in our new theory. However, we run into an immediate problem. If the resulting "universal" hamiltonian is to be hermitean in this new "universal" inner product, then it follows that eigenstates of the hamiltonian corresponding to different eigenvalues will have to be orthogonal. Thus, for hermicity of the hamiltonian and unitarity of the theory we need

$$(\Psi_E, \hat{H}\Psi_{E'}) = E'(\Psi_E, \Psi_{E'}) = E(\Psi_E, \Psi_{E'}) = 0$$

if $E' \neq E$. This implies that time evolution cannot take you from one sector to the other, and the theory is unitary within each sector already unless the sectors overlap—unless they have states with the same eigenvalues. For the free case there is no such overlap. There is no overlap either if the zero component of the electromagnetic potential is zero.

As a first example, consider the free case. The two sectors, as explained, are made to decouple to preserve unitarity. Moreover, the theory is Lorentz invariant as long as we define the inner product right, because a Lorentz transformation cannot change the sign of the energy (or, paths going forward in time in one frame also go forward in other inertial frames.) All inertial observers will agree on the construction of this theory.

Consider next the particle in a flat background but with an electromagnetic field. We can try to do the trivial construction, but in general we run into the problem that within each sector the theory is not space-time covariant, as we discussed earlier. We have to abandon the picture in which the old "square-root" hamiltoinian eigenstates are going to be eigenstates of the new diagonal 2x2 hamiltonian. We have to invent a new hamiltonian. This is the route to the two component formalism (see [6]).

Can we attach any special significance to the case where there exists a frame in which

$$[\partial_0 - A_0, \partial_i - A_i] = 0 = F_{0i} = E_i \tag{3.30}$$

i.e., the situation in which there is no particle creation in the corresponding field theory? Notice that Lorentz invariant statements (and particle number is a Lorentz invariant concept) in a field theory will show particle creation corrections only throgh the use of Lorentz invariant quantities regarding the electromagnetic field. The quantities

$$\vec{E} \cdot \vec{B}, \;\; E^2 - B^2 \tag{3.31}$$

are Lorentz invariants, and if there is a frame with zero electric field one of these invariants immediately vanishes, while the other is forever negative.

For one thing, solutions to the square-root Schrödinger equation

$$i\partial_0 \psi = -A_0 \mp \sqrt{m^2 + (-i\partial_j - A_j)^2}\, \psi \tag{3.32}$$

are also solutions to the Klein-Gordon equation

$$[(-i\partial_0 - A_0)^2 - (-i\partial_j - A_j)^2]\phi = 0 \tag{3.33}$$

As a trivial example, we know that if we have the free case solution, and we perform a gauge transformation on the electromagnetic potential,

$$A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda \tag{3.34}$$

the wavefunction changes in a simple way,

$$\psi \longrightarrow e^{i\Lambda}\psi \tag{3.35}$$

Consider the Klein-Gordon inner product,

$$(\psi_a, \psi_b) = -i \int \psi_a^* \, \frac{1}{2} \left(\overrightarrow{D}_0 - \overleftarrow{D}_0 \right) \psi_b \, d^3x = \int \psi_a^* \, \frac{1}{2} \left(\Pi_0 + \Pi_0^\dagger \right) \psi_b \, d^3x \tag{3.36}$$

This is a very nice, covariant inner product. Notice that if

$$[\partial_0 - A_0, \partial_i - A_i] = 0 \tag{3.37}$$

then the solutions to the Klein-Gordon equation match those of our Schrödinger equation. Moreover, we can use the above inner product, since it decouples the two sectors:

$$(\psi_E, \psi_{E'}) = \int \psi_E^* \, \frac{1}{2} \left(\Pi_0 + \Pi_0^\dagger \right) \psi_{E'} \, d^3x = \tag{3.38}$$

$$\int \psi_E^* \, \frac{1}{2} \left(\hat{H} + \hat{H}^\dagger \right) \psi_{E'} \, d^3x = \int \psi_E^* \, \frac{1}{2} \left(E' + E \right) \psi_{E'} \, d^3x \sim \left(E' + E \right) \delta_{E'E}$$

So the Klein-Gordon inner product provides us with the decoupling relativistic inner product we were looking for.

Essentially, the vanishing of the above commutator enables us to speak of "energy eigenstates", to describe the solution space of the Klein-Gordon equation in terms of the eigenfunctions of the hamiltonian operator. Schematically, we solve the equation

$$D_0 \phi = H(D_i) \phi \tag{3.39}$$

in terms of the solution to

$$H(D_i)\phi = E\phi \tag{3.40}$$

i.e., solve then

$$D_0\phi = E\phi \tag{3.41}$$

We can apply this criterion to the more general case where there is a gravitational background. What is required for the decoupling to occur? Here we also have a conserved Klein-Gordon inner product. Recall that the Klein-Gordon equation is given by

$$\left(D_\mu D^\mu - m^2 + \xi R\right)\phi = 0 \tag{3.42}$$

which we can "foliate" as (assume that $g^{00} = 1,\ g^{i0} = 0$)

$$\left(D_0 D_0 + g^{ij} D_i D_j - m^2 + \xi R\right)\phi = 0 \tag{3.43}$$

The covariant derivative here stands for the fully covariant one—both gravitational and electromagnetic. The theory then decouples if we have

$$[D_0,\ g^{ij} D_i D_j + \xi R] = 0 \tag{3.44}$$

This will yield a decoupled situation, no particle creation in some sense, unitarity within the one particle sector, etc. The Klein-Gordon inner product,

$$(\psi_a, \psi_b) = -i \int \psi_a^* \,\frac{1}{2}\left(\overrightarrow{D}_0 - \overleftarrow{D}_0\right)\psi_b\, d^3\Sigma \tag{3.45}$$

provides us with a nice inner product in the one-branch approach. A solution is provided by a zero electromagnetic field and by a static metric (torsion free[1]

[1] $[\nabla_a, \nabla_b] = 0$

situation), or the existence of a time-like killing vector field.

The Klein-Gordon equation can then be rewritten in the form

$$\left(iD_0 - \sqrt{g^{ij}D_iD_j - m^2 + \xi R}\right)\left(iD_0 + \sqrt{g^{ij}D_iD_j - m^2 + \xi R}\right)\phi = 0 \tag{3.46}$$

How about minisuperspace? Recall that the constraint there was

$$\Phi_M \equiv P_A P_B g^{AB} - U(Q) \approx 0 \tag{3.47}$$

and that for a gauge fixing function of the form $\chi = Q^0 - f(\tau)$ —where we tentatively assign the "time-keeping" role to the scale of the universe, it is useful to rewrite the constraint as

$$\Phi_M \equiv (P_0 - \mathcal{B}_+)g^{00}(P_0 - \mathcal{B}_-) \approx 0 \tag{3.48}$$

where

$$\mathcal{B}_\pm \equiv \pm\sqrt{(P^I\tilde{g}_{0I})^2 - P^i\tilde{g}_{IJ}P^J + \tilde{U}} \tag{3.49}$$

We can proceed much as above.

3.2 Quantize, then constrain

3.2.1 Dirac's formalism

In this formalism [1] we do not assume the existence of a classical reduced phase space. We do begin with first class constraints and a first class hamiltonian, that is, with constraints that commute with the hamiltonian as well as among themselves. This ensures—as we will see—that the physical states can be defined consistently: once defined "physical" at a certain time they will stay physical under time evolution. Physical states are defined by the conditions

$$\hat{\Phi}_\alpha \psi = 0 \tag{3.50}$$

Notice that having first class constraints and hamiltonian ensures that the operator algebra is well represented,

$$[\hat{\Phi}_\alpha, \hat{\Phi}_\beta]\psi = 0$$

(the commutator better be a linear combination of the constraints!) and that time evolution respect physicality,

$$\hat{\Phi}_\alpha e^{-i\hat{H}T}\psi = 0$$

In this approach we start by ignoring the fact that there are constraints, and we quantize all the degrees of freedom, both physical and gauge. Then we select a subspace[2] of the full Hilbert space: the kernel of the constraint operator.

[2] Well, not really a subspace, as I will explain.

One interesting—and simple—way to look at this quantization scheme is to use the gauge invariant coordinates of section 2.4.4. In that case the states in the coordinate representation are initially of the form[3] $\psi(Q,q)$, but after imposing the "physicality" condition

$$\mathbf{\Phi}\ \psi(Q,q) = \mathbf{P}\ \psi(Q,q) = 0, \tag{3.51}$$

we are left with the states $\psi(q)$. Here I have implicitely used the representation

$$\mathbf{P} \sim -i\frac{\partial}{\partial Q}$$

I could have used

$$\mathbf{P} \sim -i\frac{\partial}{\partial Q} - f(Q)$$

instead, and get the states

$$\psi(q)e^{i\int_{Q_0}^{Q} f(Q')dQ'} \tag{3.52}$$

The first problem we run into is that the states that satisfy the constraint are not in the original Hilbert space. Indeed those are given by $\psi \neq \psi(Q)$, which have infinite norm in the original Hilbert space,

$$\int_{-\infty}^{\infty} dQdq\ \psi(q)^*\, \psi(q) = \infty$$

and gauge fixing is needed (although one can also start from a compact coordinate space—by imposing periodicity conditions, say.) The inner product in the "reduced" Hilbert space can then be defined by

$$(\phi,\psi) \equiv \int_{-\infty}^{\infty} dQdq\, \delta(Q)\, \phi(q)^*\psi(q) \tag{3.53}$$

[3]I remind the reader that Q stands for the gauge coordinate, q for the physical one, etc.

or by

$$(\phi, \psi) \equiv \frac{1}{V_Q} \int_{-\infty}^{\infty} dQdq\, \phi(q)^* \psi(q) \tag{3.54}$$

where V_Q is the Q-volume, or more generally by [2]

$$(\varphi, \psi) = \int dQdq\, \varphi^*(q)\, \delta(\hat{\chi}) |\{\hat{\chi}, \hat{P}\}|\, \psi(q) = \int dq\, \varphi^*(q)\, \psi(q) \tag{3.55}$$

after using the gauge $\chi = Q - f(q, p, P)$.

This is clearly equivalent to the reduced phase space quantization—in the gauge $\alpha = 1$. The gauge invariant operators—i.e., those that commute with the constraint—are by definition, those that commute with $\mathbf{P}$, and after eliminating Q dependence from the states, they are reduced to operators in the physical coordinates only. So it is easy to see that this approach is the same as the reduced phase space method *for simple constraints and topologies.* For the general situation the equivalence of the methods is not clear, although we will try to shed some light with the cases at hand.

We will discuss these issues at length in the next chapter.

Observables are defined by operators that commute weakly with the constraints, i.e., first class operators. This ensures that we can work with them within the physical space. For example, let $\hat{A}$ be a first class operator. Then, by defifinition

$$[\hat{\Phi}_\alpha, \hat{A}] = \hat{C}^\beta \hat{\Phi}_\beta \tag{3.56}$$

Then we can ask, if $|\psi\rangle$ is a physical state,

$$\hat{\Phi}_\alpha |\psi\rangle = 0$$

is $\hat{A}|\psi\rangle$ physical? Well,

$$\hat{\Phi}_\alpha \hat{A}|\psi\rangle = [\hat{\Phi}_\alpha, \hat{A}]|\psi\rangle = \hat{C}^\beta \hat{\Phi}_\beta |\psi\rangle = 0 \tag{3.57}$$

indeed.

Notice also that if $\hat{A}$ is an observable, then $\hat{A} + \hat{C}^\alpha \hat{\Phi}_\alpha$ is also observable and *physically indistinguishable* from $\hat{A}$. These operators are related by a gauge transformation. One can gauge fix by demanding that the observables also commute with an extra set of constraints—which are chosen not to commute with the original constraints, in the sense that the matrix formed by their Poisson bracket is non- singular. This is very similar to the Dirac bracket approach in the classical development of the previous chapter.

3.2.2 Fock space quantization

Let us introduce the Fock quantization method. To begin with, we will need an even number of constraints—a recurring theme. Let me state why immediately: we will, in essence, pair the constraints and assign to each combination opposite sign norm states so that their effect in the theory cancels after we select the physical space. Thus, gauge–degrees of freedom effectively disappear from the theory. The key new ingredient here is the appearance of states with negative norms.

We assume for now that the constraints can be canonically transformed to momenta[4],

$$P_1 \approx 0 \approx P_2$$

A very important assumption is that we will represent these as hermitean operators—as we will see. Let us now define the operators

$$\hat{a} = \hat{P}_1 + i\hat{P}_2,\ \ \hat{a}^\dagger = \hat{P}_1 - i\hat{P}_2 \tag{3.58}$$

and

$$\hat{b} = -\frac{i}{2}(\hat{Q}^1 + i\hat{Q}^2),\ \ \hat{b}^\dagger = \frac{i}{2}(\hat{Q}^1 - i\hat{Q}^2) \tag{3.59}$$

As we see, we need an even number of constraints.

The commutation relation that follow from this definition are

$$[\hat{a}, \hat{b}^\dagger] = [\hat{b}, \hat{a}^\dagger] = 1$$

and the rest zero.

Notice that it is implied by the notation here that both $\hat{P}_1$ and $\hat{P}_2$ are hermitean. For example, $\hat{a} + \hat{a}^\dagger$ is hermitean, and is equal to $2P_1$. This fact is crucial for the development of the formalism, and is a subtle assumption—it selects an indefinite inner product when we define the vacuum.

The states on this space are defined by the following construction:

a) it is assummed that there is a "vacuum" state, $|0\rangle$, satifying the conditions

$$\hat{a}|0\rangle = \hat{b}|0\rangle = 0$$

[4]This is always true locally, but, in general, not globally.

a') This state is also assumed to have unit norm, $\langle 0|0\rangle = 1$.

b) The rest of these states are defined by acting on the "vacuum" above with the creation operators.

Before we work out some of the consequences of this prescription, let us look at what it is doing in terms of the $\hat{P}_i$ operators. What is the vacuum? We need a state that satisfies

$$(\hat{P}_1 + i\hat{P}_2)|0\rangle = -\frac{i}{2}(\hat{Q}^1 + i\hat{Q}^2)|0\rangle = 0$$

In a standard Hilbert space there is no such state! Although these operators commute, they are not hermitean with the usual inner product, so we do not expect to construct a basis with their eigenstates. This points the way to a cure.

Consider the states $(\hat{a}^\dagger + \hat{b}^\dagger)|0\rangle$, $(\hat{a}^\dagger - \hat{b}^\dagger)|0\rangle$, and $\hat{a}^\dagger|0\rangle$. They have positive, negative, and zero norm respectively. Our Hilbert space is not a positive definite inner product space.

Now comes the constraint. The constraints above are equivalent classically to demanding that

$$a \approx 0 \approx a^\dagger$$

However, we cannot demand this condition from the states, since there is no state in our construction that will satisfy $\hat{a}^\dagger|\psi\rangle = 0$! By the way, this is another clue that our definition of the Hilbert space—our representation—is not the usual one (there is no such vacuum in the Hilbert space we learned in kindergarden.... There is a solution to the above equations that define the vacuum, but the solution has infinite norm.)

We can only demand that the physical states satisfy

$$\hat{a}|\psi_F\rangle = 0$$

Now, what of the other "half" of the constraint? The physical states do not satisfy it! However, *notice that expectation value of the constraints in between physical states in always zero.* So in this very important sense we are safe. Moreover, we will now see that effectively—as claimed at the beginning of this section, the Fock space is reduced to a space isomorphic to the Dirac states—the space of states that satisfy the constraint in Dirac quantization:

> *A physical state in the Fock space is either the vacuum or a linear combination of the vacuum and a physical state that has zero inner product with all the physical states.*

Indeed, the other physical states are given by the null states

$$f(\hat{a}^\dagger)|0\rangle \tag{3.60}$$

since $[\hat{a}, \hat{a}^\dagger] = 0$, which decouple from the physical states, since $\hat{a}^\dagger \equiv (\hat{a})^\dagger$.

Let us describe the observables in this representation. It is clear that the observables of the Dirac formalism will not work here, unless they are observable in the strong sense. For example, if $\hat{A}$ is again an observable in the Dirac sense, we know that (let $\hat{\Phi}_\alpha = \hat{P}_1,\ \hat{P}_2$)

$$[\hat{\Phi}_\alpha, \hat{A}] = \hat{O}^1_\alpha(\hat{a} + \hat{a}^\dagger) + \hat{O}^2_\alpha(\hat{a} - \hat{a}^\dagger)$$

Now, if $|\psi\rangle$ is a physical state in the Fock sense,

$$\hat{a}|\psi\rangle = 0$$

is $\hat{A}|\psi\rangle$ physical? Well,

$$\hat{a}\hat{A}|\psi\rangle = [\hat{a}, \hat{A}]|\psi\rangle = [\hat{P}_1 + i\hat{P}_2, \hat{A}]|\psi\rangle =$$

$$\hat{K}^1(\hat{a} + \hat{a}^\dagger)|\psi\rangle + \hat{K}^2(\hat{a} - \hat{a}^\dagger)|\psi\rangle = (\hat{K}^1 - \hat{K}^2)\hat{a}^\dagger|\psi\rangle$$

So, in general $\hat{A}|\psi\rangle$ is not physical.

Only classical observables that are strongly observable are acceptable in general in the Fock scheme.

This is nor entirely surpising, though. In this approach there is no need for gauge-fixing, which means that in order to set up a map from the Fock space to the earlier Dirac one, gauge-fixing will be needed in the second. In the Dirac approach, one can always make a weak observable strong by adding to it a linear combination of the constraints. Such observables are physically equivalent gauge cousins, and after fixing the gauge only one will remain. Thus at the end of the day we do have an isomorphism between the two approaches. After we are done with the constraining we end up with the same reduction as in the Dirac case. However, in the intermediate steps things will look very different—as we will discuss later—and things can get to be very tricky with theories that are not completely gauge-invariant, like the ones we will discuss here. We will continue this discussion in the next chapter.

3.2.3 BRST quantization

In BRST [2] the states we have are originally in the extended space, so in this respect the philosophy is as in the Dirac approach. One may start asking if one shouldn't reduce classically even in this approach— and then quantize. As remarked, however, there are no constraints in the BRST approach—"the invariance has been made rigid"—so it makes sense to quantize as usual when there are no constraints.

Recall the following from the classical development of a system with a constraint Φ [2,19]. The first objects that are introduced into the extended phase space are the multiplier λ and its conjugate momentum π, i.e., $\{\lambda, \pi\} = 1$. Since λ is arbitrary, its momentum is constrained, $\pi \approx 0$. We thus have two constraints. To each constraint, the rules say we must associate a conjugate pair of ghosts, η_0, ρ_0 and η_1, ρ_1 with $\{\eta_0, \rho_0\} = 1$ and $\{\eta_1, \rho_1\} = 1$—with the other (super)brackets being zero. The total extended phase space is thus described by $t, p_t, x, p_x, \lambda, \pi, \eta_0, \rho_0, \eta_1, \rho_1$. The next thing to do is to define the BRST generator, which generates the gauge transformations in extended phase space:

$$\Omega = \eta_0 \Phi + \eta_1 \pi \tag{3.61}$$

We also need a gauge-fixing function, $\mathcal{O}$. The dynamics are then generated by the hamiltonian $\mathcal{H} = h + \{\mathcal{O}, \Omega\}$ ($= \{\mathcal{O}, \Omega\}$, since for our systems the original first class hamiltonian h is zero.)

The BRST generator has the crucial property that

$$\{\Omega, \Omega\} = 0$$

Now, the above is translated into the quantum recipe in the usual way. Observables become operators, and the (super)Poisson bracket structure is translated into the (super)commutator language in the usual way,

$$\{A, B\} \longrightarrow i\hbar[\hat{A}, \hat{B}]$$

In this case we have both commutators and anticommutators. For example, we will need

$$[\hat{\Omega}, \hat{\Omega}] = \hat{\Omega}^2 = 0 \tag{3.62}$$

In the particle case this means that we start with the states $|\Psi\rangle$ with the basis $|t, x, \lambda, \eta_0, \eta_1\rangle$, say. In the "coordinate" representation we have

$$\langle t, x, \lambda, \eta_0, \eta_1|\Psi\rangle \equiv \Psi = \psi + \psi_0\eta_0 + \psi_1\eta_1 + \psi_{01}\eta_0\eta_1 \tag{3.63}$$

where the ψ's are functions of x, t, λ. The inner product in this original extended space is given by

$$(\Sigma, \Psi) \equiv \int dtdxd\lambda d\eta_0 d\eta_1 \; \Sigma^*(z^A)\, \Psi(z^A) \tag{3.64}$$

Now, in order to get to the BRST physical space we need to do two things, and in both the central object is the BRST generator $\mathbf{\Omega}$ and its properties[5]:

[5] The first one is a subtle assumption about the representation of this algebra. See below.

a) $\hat{\Omega}^\dagger = \hat{\Omega}$,

b) $\hat{\Omega}^2 = 0$,

which it inherits from the classical description: Ω is real, and $\{\Omega, \Omega\} = 0$ [2, 19]. The BRST physical space is defined by:

i) the BRST physical condition

$$\hat{\Omega}|\Psi\rangle_{Ph} \equiv 0 \tag{3.65}$$

where recall that the BRST generator is $\Omega = \eta_0 \Phi + \eta_1 \pi$

ii) we need the BRST cohomology, i.e we need to identify

$$|\Psi\rangle_{Ph} \sim |\Psi\rangle_{Ph} + \hat{\Omega}|\Delta\rangle \tag{3.66}$$

since the factor $\hat{\Omega}|\Delta\rangle$ is physical ($\hat{\Omega}^2 = 0$), but has zero inner product with any physical state[6] ($\hat{\Omega}^\dagger = \hat{\Omega}$, $\hat{\Omega}|\Psi\rangle_{Ph} = 0$).

Consider, for example, a gauge theory with constraints $G_a \approx 0$ and algebra

$$[G_a, G_b] = C^c_{ab} G_c \tag{3.67}$$

with the C^c_{ab} constants. This is what reference [2] calls "Constraints that close according to a group", since the above are the structure constants and the Poisson algebra reproduces the Lie algebra of a group. The Jacobi identity of the Poisson bracket implies that for consistency the constants satify the Jacobi identity—as they

[6] see last footnote...

do for a Lie Group,

$$C^c_{ab}C^e_{cd} + C^c_{bd}C^e_{ca} + C^c_{da}C^e_{cb} = 0 \tag{3.68}$$

In such a case the BRST generator is defined by

$$\Omega = \eta^a G_a - \frac{1}{2}\eta^b\eta^c C^a_{cb}\mathcal{P}_a \tag{3.69}$$

Now,

$$[\Omega, \Omega] = 0 \tag{3.70}$$

is equivalent to the algebra above. Indeed[7]

$$[\Omega, \Omega] = \eta_a\eta_b([G_a, G_b] - C^c_{ab}G_c) + \frac{1}{4}[\eta^b\eta^c C^a_{cb}\mathcal{P}_a, \eta^{b'}\eta^{c'}C^{a'}_{c'b'}\mathcal{P}_{a'}] \tag{3.71}$$

Now,

$$\frac{1}{4}[\eta^b\eta^c C^a_{cb}\mathcal{P}_a, \eta^{b'}\eta^{c'}C^{a'}_{c'b'}\mathcal{P}_{a'}] = C^d_{ab}C^e_{cd}\mathcal{P}_e\eta_a\eta_b\eta_c = 0 \tag{3.72}$$

is the Jacobi identity.

What are the equations for physicality? In this simple case there are no ordering ambiguities in the writing of the operator

$$\hat{\Omega} = \hat{\eta}^a\hat{G}_a - \frac{1}{2}\hat{\eta}^b\hat{\eta}^c C^a_{cb}\hat{\mathcal{P}}_a \tag{3.73}$$

thanks in part to the total antisymmetry of the structure constants. Let us look at the solutions to the equation

$$\hat{\Omega}|\Psi_{BRST}\rangle = 0 \tag{3.74}$$

[7]Following the convention in reference [2], $[\eta^a, \mathcal{P}_b] = -\delta^a_b$.

For this purpose, let us use the semi-abstract, semi-coordinate representation (for the ghosts) expression for the states that we used before. This is indeed the one that results from projecting

$$\langle \eta^0, \eta^1, \eta^2 | \left(\sum |\psi(z), \lambda\rangle\rangle \otimes |f(\eta^0, \eta^1, \eta^2)\rangle \right) \tag{3.75}$$

where

$$\hat{\eta}^a |\eta^0, \eta^1, \eta^2\rangle = \eta^a |\eta^0, \eta^1, \eta^2\rangle \tag{3.76}$$

Now, doing a Taylor expansion

$$\langle \eta^0, \eta^1, \eta^2 | \sum |\psi(z), \lambda\rangle\rangle \otimes |f(\eta^0, \eta^1, \eta^2)\rangle = |\psi\rangle + |\psi_a\rangle \eta^a + |\psi_{ab}\rangle \eta^a \eta^b + |\psi_{abc}\rangle \eta^a \eta^b \eta^c \tag{3.77}$$

$\equiv |\Psi\rangle$ and where, w.l.g., we assume that the wavefunctions are totally antisymmetric in their indices. Then

$$\hat{\Omega}|\Psi\rangle = \left(\hat{\eta}^a \hat{G}_a - \frac{1}{2} \hat{\eta}^b \hat{\eta}^c C^a_{cb} \hat{\mathcal{P}}_a \right) \left(|\psi\rangle + |\psi_a\rangle \eta^a + |\psi_{ab}\rangle \eta^a \eta^b + |\psi_{abc}\rangle \eta^a \eta^b \eta^c \right) =$$
$$\eta^a \hat{G}_a |\psi\rangle + \eta^a \eta^b \left(\hat{G}_a |\psi_b\rangle + \frac{i}{2} C^c_{ba} |\psi_c\rangle \right) + \eta^a \eta^b \eta^c \left(\hat{G}_a |\psi_{bc}\rangle + i C^d_{cb} |\psi_{da}\rangle \right) \tag{3.78}$$

The general solution is then given by

$$\hat{G}_a |\psi\rangle = \hat{G}_a |\psi_b\rangle - \hat{G}_b |\psi_a\rangle + i C^c_{ba} |\psi_c\rangle = \sum_{antiSymm} \left(\hat{G}_a |\psi_{bc}\rangle + i C^d_{cb} |\psi_{da}\rangle \right) = 0 \tag{3.79}$$

or more simply by

$$\epsilon^{abc} \hat{G}_a |\psi\rangle = \epsilon^{abc} \left(\hat{G}_a |\psi_b\rangle - \hat{G}_b |\psi_a\rangle + i C^c_{ba} |\psi_c\rangle \right) = \epsilon^{abc} \left(\hat{G}_a |\psi_{bc}\rangle + i C^d_{cb} |\psi_{da}\rangle \right) = 0 \tag{3.80}$$

with $|\psi_{abc}\rangle$ totally unrestricted.

To check the above, multiply the above equation by η_a, η_b and use

$$\eta_{i_1} \cdots \eta_{i_N} T^{i_1 \cdots i_N} = \eta_1 \cdots \eta_N \epsilon^{i_1 \cdots i_N} T^{i_1 \cdots i_N} \tag{3.81}$$

when $i = 1, ..., N$.

Evidently, the above also means that $|\psi\rangle$ satisfies the constraints (i.e., it is a Dirac state). These two sectors represent the extremes. The "middle" sectors fall somewhat in between. An example is given by the state

$$|\eta_0 = \eta_1 = G_2 = 0\rangle \sim \eta_0 \eta_1 \varphi(z, \lambda)_{G_2=0}$$

It is clear that this state is annihilated just by looking at the BRST generator—and using the antisymmetry of the structure constants.

In the abelian case, it is usually stated that the states we end up with—i.e., *the BRST cohomology*—are

$$\Psi = \psi + \psi_0 \eta_0 + \psi_1 \eta_1 + \psi_{01} \eta_0 \eta_1$$

with

$$\hat{\Phi}\psi = \hat{\pi}\psi = \hat{\Phi}\psi_0 = \hat{\pi}\psi_0 = \hat{\Phi}\psi_1 = \hat{\pi}\psi_1 = \hat{\Phi}\psi_{01} = \hat{\pi}\psi_{01} = 0, \tag{3.82}$$

in other words, with the wave functions independent of Q, λ (in the λ representation, say), where Q is the hypothetical coordinate conjugated to the constraint.... This, as we will see in the next chapter, is incorrect.

The hamiltonian is given by $\hat{\mathcal{H}} = \{\hat{\mathcal{O}}, \hat{\Omega}\}$, for some gauge fixing operator $\hat{\mathcal{O}}$, and we could expect that it has no effect on physical states due to the two conditions above. We will see that this is also false.

If we use the above "cohomology" states, the inner product on the physical space and in the coordinate representation can be defined by

$$(\Sigma, \Psi) \equiv \int dt dx d\pi d\eta_0 d\eta_1 \eta_0 \eta_1 \, \Sigma^*(z^A) \, \delta(\hat{\chi}) |\{\hat{\chi}, \hat{\Phi}\}| \, \Psi(z^A) \tag{3.83}$$

Gauge fixing is needed, although here the situation is different than the one in the Dirac case, since there the divergence of the unregularized inner product was genuine, whereas here there isn't a true divergence (recall that the ghosts carry "negative degrees of freedom"), but a $\delta(0) \times \infty$ situation. We will motivate this definition later. We will see, though, that choosing the correct states in the cohomology will solve the regularization problem.

Although it is unclear yet how to connect naturally the BFV path integral with the BRST cohomology canonical description, it is easy to write the path integral when one works in the full BRST space, using resolutions of the identity like

$$\begin{aligned} I &= \int dt dx d\lambda d\eta_0 d\eta_1 |t, x, \lambda, \eta_0, \eta_1\rangle \langle t, x, \lambda, \eta_0, \eta_1| \\ &= \int dp_t dp_x d\pi d\rho_0 d\rho_1 |p_t, p_x, \pi, \rho_0, \rho_1\rangle \langle p_t, p_x, \pi, \rho_0, \rho_1| \end{aligned} \tag{3.84}$$

projections like

$$\langle t, x, \lambda, \eta_0, \eta_1 | p_t, p_x, \pi, \rho_0, \rho_1 \rangle = e^{i(tp_t + xp_x + \lambda\pi + \eta_0\rho_0 + \eta_1\rho_1)}, \tag{3.85}$$

and a hamiltonian like $\hat{\mathcal{H}} = \{\hat{\mathcal{O}}, \hat{\Omega}\}$. This is formally obvious: recall that the action is

$$S = \int_{\tau_i}^{\tau_f} (\dot{t}p_t + \dot{x}p_x + \dot{\lambda}\pi + \dot{\eta}_0\rho_0 + \dot{\eta}_1\rho_1 - \{\mathcal{O}, \Omega\})d\tau \tag{3.86}$$

We will study this issues in better and finer detail in the next chapter.

3.2.4 BRST-Fock quantization

In this approach the basic BRST formalism is not changed (unlike in the transition from Dirac to Dirac-Fock.) The physical condition is still $\hat{\Omega}|\psi\rangle = 0$. However, the representation of this Hilbert space is different than the one we used before.

Again we define the operators

$$\hat{a} = \hat{P}_1 + i\hat{P}_2, \;\; \hat{a}^\dagger = \hat{P}_1 - i\hat{P}_2 \tag{3.87}$$

and

$$\hat{b} = -\frac{i}{2}(\hat{Q}^1 + i\hat{Q}^2), \;\; \hat{b}^\dagger = \frac{i}{2}(\hat{Q}^1 - i\hat{Q}^2) \tag{3.88}$$

where as before we assume that we have an even number of constraints—which is true with the multipliers.

Now we add the ghosts to this picture:

$$\hat{c} = \hat{\eta}_1 + i\hat{\eta}_2, \;\; \hat{c}^\dagger = \hat{\eta}_1 - i\hat{\eta}_2 \tag{3.89}$$

and

$$\hat{\bar{c}} = \frac{i}{2}(\hat{\rho}^1 + i\hat{\rho}^2), \;\; \hat{\bar{c}}^\dagger = \frac{i}{2}(\hat{\rho}^1 - i\hat{\rho}^2) \tag{3.90}$$

The BRST generator now reads

$$\hat{\Omega} = \hat{c}^\dagger \hat{a} + \hat{a}^\dagger \hat{c} \tag{3.91}$$

The Hilbert space is constructed as usual, starting from a vacuum that is annihilated by all the destruction operators. Then the rest of the states are created with the creation operators, just as before, except that we have the ghost part too now. These degrees of freedom are fermionic (the creation operators squared are zero, as follows from the brackets.) Thus, the BRST condition is equivalent (in this Fock representation) to asking that the states be annihilated by the annihilation operators of the ghosts and of the non-ghost parts.

As discussed in reference [2], in this representation any physical state is either the vacuum or $\hat{\Omega}$ of something, i.e., BRS exact. This is similar to what we found in Dirac-Fock, i.e., that a physical state there was either the vacuum or null. Note, however that all the classical observables here are good within the physical space.

Finally, there are no inner product regularization problems here, no hermicity problems, and the operator cohomology duality theorems are also true of the states—since the only nontrivial state cohomology class is at the zero ghost number [2].

Moreover, the invariance of the physical states amplitude under changes in gauge-fixing $\hat{K}$ is direct,

$$\langle \Psi | e^{[\hat{\Omega}, \hat{K}]} | \Psi' \rangle = \langle \Psi | \Psi' \rangle \tag{3.92}$$

because there are no hermicity questions.

We will see that this representation leads to the Feynman propagator in the relativistic case. This is essentially because in the Fock representation the multiplier is imaginary and half-ranged (see section 5.1.3).

3.3 Conclusions, summary

This has been mainly a review chapter in which I have introduced the different quantization schemes that I will apply to the systems under consideration. I have also described the problems associated with each quantization approach, thus setting the stage for the developments in the next chapter. The main point to come back to is the definitions of the state spaces and their inner products, which are not clearly spelled out by the different prescriptions.

I have, however, completely carried out the reduced phase space quantization, the "constrain, then quantize" approach, for the one-branch situations—which I have argued are the only legitimate situations for the reduced phase space approach. I have showed that since the construction of the classical reduced phase spaces can only be completed up to time-dependent canonical transformations, we will get, upon quantization, quantum theories in different pictures.

I have also introduced the idea that a path integral can be immediately developed in the BRST quantization approach in the full extended phase space, since it is, in essence, an approach in which the invariance is made "rigid"—there are no constraints. I will come back to this path integral in the next chapters—the BFV path integral.

Chapter 4

The physical inner product

In this chapter I study the problem of the development of an inner product in the physical space. This applies to the "quantize, then constrain" approaches I have introduced in the previous chapter (the problems in the "constrain then quantize" approach were already discussed, where we saw that there is a relationship between different gauge choices and different representations).

The ambiguity in the definition of the inner product in the unconstrained case was also pointed out in the first chapter. This ambiguity is present anytime we have to define a new inner product.

I explained, in the previous chapters, what the problems are in quantization of parametrized systems, namely, the definition of the state spaces, the inner products and the subsequent introduction of path integrals. I will explain how to solve these problems.

4.1 Introduction

In this section we will look at the question of the development of the inner product, starting with some general comments on the problems which have clouded this issue. Until this point we have not been able to give a satisfactory, well-motivated definition for the inner product in the Dirac or BRST quantization approaches. We will do so now. We will discuss the problems that arise because of the arbitrariness that exists in defining the constraint, and because of the fact that the states that satisfy the constraint are in general not normalizable.

Consider the Dirac quantization scheme, with a given constraint Φ. Recall that in the Dirac approach we start with a full Hilbert space—that is, we ignore the fact that there is a constraint and we make the transition from classical to quantum as usual. The classical constraint, which is assumed to be a first class constraint, is made into an operator and then imposed on the states, that is, a subspace of the Hilbert space is selected by demanding that the states in it be zero eigenvectors of the constraint,

$$\mathbf{\Phi}|\psi\rangle = 0 \tag{4.1}$$

Notice, however, that the definition of the constraint is not unique. Two constraints $\Phi \approx 0$, and $\Phi' \approx 0$ are equally valid starting points for the quantization if they have equivalent solution spaces. There is no way to choose one over the other in general, so we should understand what happens if we change the form of the constraint. Do we get different quantum theories? Let us start by observing that if ψ satisfies $\mathbf{\Phi}\psi = 0$, it follows that the state $\mathbf{v}\psi$ does as well, provided that $[\mathbf{v}, \mathbf{\Phi}] = 0$. In fact, the constraints $\Phi \approx 0$ and $v\Phi \approx 0$ are equivalent in the quantum sense as long as they commute as operators and as long as $\mathbf{v}$ has no zero modes.

How do we define an inner product on this subspace? Well, if we had an inner product in the original, big, Hilbert space it should follow that the states selected by the constraint can use it. After all, the constraint selects states in the original Hilbert space, doesn't it? Well, it should, but in general it doesn't. The simplest example is the locally general one, $\Phi = P$. The problem is that

momentum states are not normalizable when the coordinate space is infinite

To see what kind of problems this will lead us to, consider for example the quantity

$$\langle P\!=\!0|[\mathbf{Q}, \mathbf{P}]|P\!=\!0\rangle \tag{4.2}$$

What is it? We will reach a contradiction whether we assume that the coordinate space is infinite, finite—there are no $P = 0$ states in such a case—or periodic. This is a problem, because the states $|P = 0\rangle$ are the starting point in the Dirac quantization approach! If we must have them, then either $\mathbf{P}$ cannot be a hermitean operator (infinite range case), or $\mathbf{Q}$ does not exist (periodic boundary conditions). *It is dangerous to assume in general that there exists a definition of inner product in the full space that we can use with the physical states.* This observation applies also to the BRST quantization approach. It is because of this that we basically have to "redefine"—in general—the inner product in the physical space, the only exception being the periodic boundary condition case and the Fock approach.

Let us look at this problem a bit closer. A Hilbert space is described by a set of linear operators and a vector space of states, together with a good[1] definition of inner product. Consider then a Hilbert space with the operators $\mathbf{Q}, \mathbf{P}$ such that

i) $\mathbf{P} = \mathbf{P}^\dagger$, $\mathbf{Q} = \mathbf{Q}^\dagger$

ii) $[\mathbf{Q}, \mathbf{P}] = i$

and let us discuss possible state spaces and inner products.

It is not to hard to see that this Hilbert space cannot contain any eigenvectors of either one of the above operators. Indeed, suppose for example that $|P_a\rangle$ is an eigenvector of $\mathbf{P}$. Then (i is false),

$$\langle P_a|[\mathbf{Q}, \mathbf{P}]|P_a\rangle = i\langle P_a|P_a\rangle \tag{4.3}$$

but also (then ii is false),

$$\langle P_a|[\mathbf{Q}, \mathbf{P}]|P_a\rangle = (P_a - P_a)\langle P_a|\mathbf{Q}|P_a\rangle = 0 \tag{4.4}$$

The point is that if one includes such states in the Hilbert space the operators are not hermitean. In the infinite range case the inner products diverge.

As a different situation, consider for example the states representing a particle in a box. There are then $\mathbf{Q}$ eigenstates—although they are not normalized to one. There are no $\mathbf{P}$ eigenstates in the coordinate representation. This is easy to see

[1] i.e., finite!

since in the coordinate representation this is a first order differential operator and we have too many boundary conditions. In fact, **P** takes states out of the Hilbert space.

We can also consider the periodic boundary condition case. Then there exist momentum eigenstates, and they are normalizable. However, the operator **Q** is not well-defined: it takes states outside the Hilbert space.

In the situation with no boundary conditions and an infinite coordinate space we do have momentum eigenstates. These are not normalizable, though, and more to the point, the **P** operator is not really hermitean with respect to them[2]. Yet another point is that when we quantize the constraint *we will ask for normalizable momentum eigenstates.* It is usually ok to work with momentum states that are not normalizable and really not in the Hilbert space, because they are just mathematical tools. In the quantization of constrained systems they are not treated as tools—they are the building blocks of the physical Hilbert space!

We will have to be careful then that we don't assume that the constraint is a hermitean operator in such an inner product space.

> *The main thing to remember is that if one insists on having states of well-defined momentum—say—as well as a finite inner product, the corresponding momentum operator will not be hermitean. This is not necessarily a problem if we keep it in mind and are careful with the algebra. In*

[2]Unless you are willing to say that $\delta'(0) \cdot 0 = \infty$!

the Dirac formalism, for example, if the constraint is $P \approx 0$*, the states satisfying the constraint make the operator non-hermitean. Similarly, in BRST,* $\mathbf{\Omega}$ *will not be hermitean in the physical sector. However, in both cases the* operators have the hermitian property when used between "conjugate states", like $\langle Q_a|\mathbf{P}|P_a\rangle$.

Since the gauge coordinates are not going to be to essential in the resulting theory—once we take care of them—there is a way to deal with this, although the interpretation of the method was, until now, unclear. I will motivate here the definition of the Dirac inner product, as well as point out and clarify some important aspects that have created confusion in the literature.

One of these aspects is an ambiguity that we can trace to the definition of the constraint:

$$(\psi_a, \psi_b) \equiv \int \psi_a^* \, |\{\boldsymbol{\chi}, \mathbf{\Phi}\}| \delta(\boldsymbol{\chi}) \, \psi_b \;\; dV \tag{4.5}$$

where the gauge-fixing condition is[3] $\chi \approx 0$, and $dV = dqdQ$. If the constraint is multiplied by a function the inner product will indeed be different, and one would reason that the same inner product and quantum theory should result from constraints that are equivalent.

I will also point out that we can further rewrite this inner product as

$$(\psi_a, \psi_b) \equiv \int dV dw dc d\bar{c} \; \psi_a^* \; e^{i(w\boldsymbol{\chi} + c|\{\boldsymbol{\chi}, \mathbf{\Phi}\}|\bar{c})} \; \psi_b \tag{4.6}$$

[3]As usual, we will have q represent the physical degrees of freedom and Q the gauge ones.

This leads to the BRST quantization approach.

As for the BRST inner product we would expect that it may reduce to the Dirac inner product given above. Recall that the BRST states for a system with one constraint as above are described by four sectors: one no-ghost sector, two one ghost sectors and finally one double ghost sector. Recall also the duality theorems; essentially, we find that the sectors are all isomorphic in pairs. However, we will find that they are not all isomorphic to the Dirac states.

The inner product can then be formally defined in the following way (see [23] and also [2] p. 325, and ex. 14.23):

$$(\Psi_a, \Psi_b) \equiv \int d\rho_1 d\eta_0 d\pi dV \; \Psi_a^* \, e^{i\{K,\Omega\}} \, \Psi_b \tag{4.7}$$

How are the states defined? Physical? Zero ghost? Isn't the exponent just one when evaluated between physical states—states annihilated by $\mathbf{\Omega}$? Can we motivate this definition? How does it compare to the Dirac inner product? Can we tie this to the path integrals? What about the gauge fixing function K, is it arbitrary? We will address all these questions.

Yet another approach we will discuss is Dirac-Fock quantization. In this approach the constraints are not imposed on the states, and there is no normalization problem. All the states have a well defined norm from the beginning to the end. Only, for some this norm will be negative. We will compare this quantization apprach to the other ones, and establish their equivalence for well-defined gauge systems. For parametrized systems something peculiar happens, of course.

At this stage we have that both the Dirac approach and the BRST approach suffer from the same problem. *It is not clear at all how to implement the constraints by starting with a bigger quantum space, and obtain from a quantum reduction a well defined quantum theory.* The basic reason for this is that one usually works with an infinite-range coordinate space; in such a situation the states selected by the constraint are in general not normalizable, and, as we will see, it is hard to implement the physical condition through a projecting operator—but not impossible.

We will look at the finite coordinate space situation, and also at the unbounded one. Both cases will be shown to be suitable for quantization.

As for the path integrals, if one hopes to interpret the BFV path integral 'a la Dirac'—by looking at the state cohomology, inner product, etc.—it seems reasonable to expect that one should also understand the Faddeev path integral in such terms, from the Dirac state space, and this has not been done yet: *It is unclear at this point if the Dirac quantization scheme (and the corresponding quantization scheme in the BRST formalism) can be related to the Faddeev (BFV) path integral, or if this path integral is really related only to the quantized reduced phase space (this relation has already been established).* We will clarify these connections, and show that the Faddeev path integral can be constructed within the Dirac approach.

4.2 Dirac quantization and quantum gauge transformations

The key idea in the Dirac approach to constrained systems is that *a first class constraint introduces arbitrariness in the dynamics*, something that is reflected in the arbitary term in the hamiltonian, $H_E = h + v\Phi$. Now, in the quantum world—in the Schrödinger picture—this means that will have an extra term in the Schrödinger equation; the general solution will be

$$|\Psi\rangle = e^{-it\mathbf{H}_E}\,|\Psi_0\rangle = e^{-it\mathbf{h}}\,e^{-it\mathbf{v}\mathbf{\Phi}}\,|\Psi_0\rangle \tag{4.8}$$

where we have used the fact that the constraint is first class and commutes with the original hamiltonian $\mathbf{h}$. One has to be careful about the $\mathbf{v}$ function/operator as well.

From a different perspective, a momentum operator $\mathbf{P}$ can be understood to produce a gauge transformation in the sense that

$$e^{-ia\mathbf{P}}\,|Q\rangle = |Q + a\rangle \tag{4.9}$$

since

$$\mathbf{Q}\,e^{-ia\mathbf{P}}\,|Q\rangle = \Big([\mathbf{Q}, e^{-ia\mathbf{P}}] + e^{-ia\mathbf{P}}Q\Big)\,|Q\rangle = (a + Q)\,e^{-ia\mathbf{P}}\,|Q\rangle \tag{4.10}$$

This also implies that in the coordinate representation

$$e^{-ia\mathbf{P}}\,\psi(Q) = \psi(Q - a) \tag{4.11}$$

In a sense, what is being said here is that we have

$$|\psi\rangle \sim |\psi\rangle + \mathbf{P}|Any\rangle \tag{4.12}$$

since the state $\mathbf{P}|Any\rangle$ should decouple from any physical state: let us try[4]

$$\langle\Phi\!=\!0|\boldsymbol{\Phi}\,|Any\rangle = \langle\Phi\!=\!0|\boldsymbol{\Phi}\,|\Phi\!=\!0\rangle\langle\Phi\!=\!0|Any\rangle = 0\cdot\infty\cdot\langle\Phi = 0|Any\rangle \tag{4.13}$$

The answer is no in general—we run into hermiticity problems. *However, if one uses* $|Any\rangle = |\chi\!=\!0\rangle$ *where* $[\chi, \Phi] \neq 0$ *then the decoupling occurs.*

The quantum gauge transformation behavior matches the classical one. The expectation value of the position operator behaves as it should classically—whether we let the above transformation operator act on the states or on the operators. The $\mathbf{Q}$ operator correponds to the classical Q—the gauge degree of freedom.

Notice, though, that all we needed for the previous statement was the Heisenberg algebra. So this idea provides us with an interpretation for the gauge transformation ideas in the quantum version of a theory with constraints. Let us try to obtain the gauge invariant states and an inner product for them.

Gauge invariant states will be defined by

$$\mathbf{P}\,|\Psi\rangle = 0 \tag{4.14}$$

The reason for the use of this definition is that these states will clearly be gauge-invariant:

$$e^{-ia\mathbf{P}}\,|\Psi\rangle = |\Psi\rangle \tag{4.15}$$

[4] here $\mathbf{P} = \boldsymbol{\Phi}$.

Whether this condition is too strong or not will be discussed later; for now let us try to work with it. Now, there is only one class of states that meet the above criterion:

$$|\Psi\rangle = |P\!=\!0\rangle \otimes |phys\rangle \tag{4.16}$$

where the label "*phys*" refers to the other, *physical* degrees of freedom.

Where do we need to talk about "gauge fixing"? Well, the old inner product definition may run into trouble:

$$\langle \Psi_a|\Psi_b\rangle = \langle P\!=\!0|P\!=\!0\rangle\langle phys_a|phys_b\rangle \tag{4.17}$$

As mentioned, the states above have infinite norm in the unbounded coordinate case: $\langle P\!=\!0|P\!=\!0\rangle = \delta(0)$. *We need to fix this normalization problem, as well as provide for a resolution of the identity in this Hilbert subspace.*

The resolution of the identity would naively be

$$\mathbf{I}_{\mathbf{\Phi}} = \sum |P\!=\!0\rangle \otimes |phys\rangle\langle phys| \otimes \langle P\!=\!0| \tag{4.18}$$

This would be a resolution of the identity on the physical subspace; on the full space it would be a projector into the physical subspace. Indeed, we would have $(\mathbf{I}_{\mathbf{\Phi}})^2 = \mathbf{I}_{\mathbf{\Phi}}$ if it weren't for the infinite norm problem. In the next section we will study the situation with periodic boundary conditions. Then we will look at the infinite coordinate space, and see the connection between the Dirac quantization approach and the Faddeev path integral in a direct way.

4.2.1 Periodic boundary conditions

How can we solve the infinite norm problem? It originates with the infinite volume in coordinate space. *One possible solution is to use a finite volume coordinate space.* If we use a finite coordinate space—by imposing periodic boundary conditions, say—then the norm of the momentum states will be finite. The momentum operator will now have a discrete spectum, $P_n = n\frac{2\pi}{L}$, but otherwise this representation will have the same properties that the infinite volume case has. In the coordinate representation, for example, we have the operators $\mathbf{Q}, \mathbf{P}$ with the commutator $[\mathbf{Q}, \mathbf{P}] = \mathbf{i}$ represented as $\mathbf{P} = -i\partial/\partial Q$ and[5] $\mathbf{Q} = Q$. The $\mathbf{P}$ e-states are given by $\phi_n = e^{-iP_nQ}/\sqrt{L}$, where $P_n = 2\pi n/L$ as before, and the $\mathbf{Q}$ e-states by $\delta_L(Q - Q_0)$, where the delta function is also periodic with period L. Just as in the infinite volume case it is true that

$$e^{-ia\mathbf{P}}|Q\rangle = |Q + a\rangle \tag{4.19}$$

and also that

$$e^{ik\mathbf{Q}}|P_n\rangle = |P_n + k\rangle \tag{4.20}$$

although notice that k is immediately restricted to be of the form $k = 2\pi n/L$, since all the operators in the theory come from classical functions defined in the periodic coordinate space—that is, periodic functions! At any rate, the gauge transformation ideas still apply in this context. Notice that the state $P_n = 0$ is still in the theory.

[5] Strictly speaking the coordinate degree of freedom should be described by $\exp(\pm i2\pi n\mathbf{Q}/L)$.

The resolution of the identity in such a space is given by

$$I = \sum |P_n\rangle\langle P_n| = \int_0^L dQ|Q\rangle\langle Q| \tag{4.21}$$

How do the earlier definitions of inner product match with this philosophy? We see that here there is no need for gauge fixing! However, we can artificially introduce a "gauge-fixing normalization effect"' by altering the normalization of the gauge invariant states, say by multiplying them by a constant factor, or by a function. This is ok, the representations are isomorphic, although not unitarily so (recall that there are different ways to normalize states even in the unconstrained case).

Notice also that if the constraint has multiple solutions the solution states—the physical states—can always be chosen to be orthogonal in the original bigger space if the constraint is hermitean. For example, say $\Phi = (P-a)(P-b)$. Then the solution states will be orthogonal in the original bigger space if the original inner product in the full space is such that the operator $\mathbf{P}$ is hermitean. This is the case for the relativistic particle in some situations, as in the free case. In a more complicated case this will not apply, of course. In the present discussion, orthogonality is preserved under quantum reduction.

How do we describe the dynamics? As explained before, the general solution to the Schrödinger equation is given by

$$|\Psi\rangle = e^{-it\mathbf{H}_E}\,|\Psi_0\rangle = e^{-it\mathbf{h}}\,e^{-it\mathbf{vP}}\,|\Psi_0\rangle \tag{4.22}$$

Consider the transition amplitude

$$\langle \Psi_f | \Psi_i \rangle = \langle \Psi_f | e^{-it\mathbf{h}} e^{-it\mathbf{v}\mathbf{P}} | \Psi_i \rangle \tag{4.23}$$

If the initial or final states satisfy the constraint the amplitude will not depend on $\mathbf{v}$. If not, just insert the projector $\mathbf{I}_{\mathbf{\Phi}}$. Notice that this definition of physical/non-physical states is an orthonormal definition if the constraint is hermitean—or equivalent to a hermitean one. The physical and non-physical sectors decouple, and this decoupling is respected by the—first class—hamiltonian.

Let us now consider the following question:

Can we recover the Faddeev path integral?

We can certainly use

$$\mathbf{I}_{\mathbf{\Phi}} \equiv |P{=}0\rangle\langle P{=}0| = \sum \delta_{0P_n} |P_n\rangle\langle P_n| \tag{4.24}$$

and insert this in the propagation amplitude. Consider then the physical amplitude

$$\langle Phys| e^{-it\mathbf{H}_E} \, |Phys\rangle \equiv U_{Phys}(q, q'; t) \tag{4.25}$$

where q represents the physical degrees of freedom, so that $|Phys\rangle \sim |P = 0, q\rangle$. Then

$$\langle P = 0, q_f | e^{-it\mathbf{H}_E} \, |P = 0, q_i\rangle = \tag{4.26}$$

$$\langle q_f | e^{-it\mathbf{H}_E} \, |q_i\rangle \tag{4.27}$$

This doesn't look like a good way to get to the Faddeev path integral at all!

Consider instead the amplitude

$$\langle\psi|e^{-it\mathbf{H}_E}\,\delta(\mathbf{P})\,|\psi\rangle \equiv U_{Phys} \tag{4.28}$$

We have projected the transition to the physical states. However, this is not a gauge-invariant object, because the coefficients of the projection, $\langle P\!=\!0|\psi\rangle$, are not. This is the boundary effect that we will discuss later—such is the situation in the particle case. Now, we can use the projector above, but this will not be a very illuminating experience. Notice that, for a constraint of the form $\mathbf{\Phi} = \mathbf{P} - a = 0$,

$$(|P\!=\!a\rangle\langle P\!=\!a|Q\!=\!Q_0\rangle\langle Q\!=\!Q_0|)\cdot|P\!=\!a\rangle\langle P\!=\!a| = \tag{4.29}$$

$$|P\!=\!a\rangle\langle P\!=\!a| \tag{4.30}$$

and

$$(|P\!=\!a\rangle\langle P\!=\!a|Q\!=\!Q_0\rangle\langle Q\!=\!Q_0|)^2 = \tag{4.31}$$

$$(|P\!=\!a\rangle\langle P\!=\!a|Q\!=\!Q_0\rangle\langle Q\!=\!Q_0|) \equiv \mathbf{K} \tag{4.32}$$

Using these two properties, $\mathbf{K}\mathbf{I}_{\mathbf{\Phi}} = \mathbf{I}_{\mathbf{\Phi}}$, and $\mathbf{K}^2 = \mathbf{K}$, we are ready to build a path integral, using the resolution of the identity

$$\hat{I} = \int dQdq\,|Q,q\rangle\langle Q,q| = \sum_{P_n}|P_n\rangle\langle P_n| \tag{4.33}$$

The amplitude can be written as

$$\langle\psi|e^{-it\mathbf{H}_E}\,\delta(\mathbf{P})\,|\psi\rangle \equiv \tag{4.34}$$

$$\int DQ \prod \{\sum_{P_n}\} \int DqDp\, \delta(P_0 - a) \left(\prod_{i=1}^{N} \delta(P_i - a)\delta(Q_i - f(\tau_i))\right) e^{i\int [d\tau P\dot{Q} + p\dot{q} - h]} \quad (4.35)$$

Example: Dirac quantization of the particle with periodic boundary conditions

The constraint is $\Phi = p_t + p_x^2/2m = 0$. Assume that the t coordinate space is finite, with periodic boundary conditions. Now, this operator $\mathbf{\Phi}$ is conjugate to t. We will represent it by $-i\partial_t - \partial_x^2$. The solution space to the constraint in this coordinate representation is given by the states

$$\psi(x,t) = e^{itp_t + ixp_x} \frac{1}{\sqrt{TL}} \quad (4.36)$$

where $p_t = 2\pi n'/T$, and thus we need

$$p_t = -\frac{p_x^2}{2m} = -\frac{(2\pi n L)^2}{2m} = -2\frac{\pi^2}{L^2 m} n^2 \equiv 2\frac{\pi}{T} n' \quad (4.37)$$

We see that it is hard to have both space and time bounded. Indeed, we are then looking for the solutions to the above differential equation in a torus. I guess we could put restrictions on the allow mass, but still, we would not have all the p_t momenta.

We could go to the constant of the motion coordinate system, and put periodicity there.

So, anyhow, let the x coordinate be unbounded. Then we see: the solution space to the constraint is in one-to-one correspondence to the x momentum states, i.e, in one-to-one correspondence to the x coordinate Hilbert space. And these Dirac states are correctly normalized. We can use the inner product

$$(\psi_a, \psi_b) = \int_0^T dt \int dx\ \psi_a^*\ \psi_b = \int dx\ \psi_a^*\ \psi_b \tag{4.38}$$

This is because there are no normalization problems to begin with.

4.2.2 Unbounded gauge coordinate space

In this section we will produce the connection between the Dirac inner product and the Faddeev path integral. This reasoning will also take us to the BRST inner product and the rest of the BRST quantization approach, including the BFV path integral.

The key to the Fadeev path integral is in the Dirac inner product. We must obtain a resolution of the identity in the physical space, and for that we will use the Dirac inner product. Let us start with the quasi-projector

$$\mathbf{Y} \equiv |P\!=\!0\rangle\langle P\!=\!0| \tag{4.39}$$

This operator will take any state into the physical space (quite clearly). We can look at it in the coordinate representation, using the resolution of identity:

$$\mathbf{Y} = \int dQdQ'\ |Q\rangle\langle Q|P\!=\!0\rangle\langle P\!=\!0|Q'\rangle\langle Q'| = \frac{1}{2\pi}\int dQdQ'\ |Q\rangle\langle Q'| \tag{4.40}$$

where the normalization $\langle Q|P\!=\!0\rangle = 1/\sqrt{2\pi}$ has been used. Notice that the state $\mathbf{Y}|\psi\rangle$ is clearly left unchanged by a gauge transformation.

Now, I have been calling the above operator a projector, and indeed it would be a projector in the case where the coordinate Q is defined with boundary conditions, but in the infinite case $\mathbf{Y}$ is not a projector. By defintion a projector $\mathbf{K}$ satisfies the property $\mathbf{KK} = \mathbf{K}$, and instead we have $\mathbf{YY} = \mathbf{Y}\delta(0)$. Here is the source of all our headaches.

Consider then the projector

$$\mathbf{K_\Phi} \equiv |P\!=\!0\rangle\langle P\!=\!0|\delta(\mathbf{Q}-\mathbf{a}) = \left(\int dQ\,|Q\rangle\right)\langle Q=a| \tag{4.41}$$

which we can also rewrite as

$$\mathbf{K_\Phi} = \delta(\mathbf{P})\delta(\mathbf{Q}-\mathbf{a}) = |P\!=\!0\rangle\langle Q=a| \tag{4.42}$$

It satisfies

$$\mathbf{K_\Phi}\mathbf{K_\Phi} = \mathbf{K_\Phi} \tag{4.43}$$

However, this operator is not hermitean, since $\mathbf{Q}$ and $\mathbf{P}$ don't commute. Notice that the basic trick is to realize that

$$\langle Q|P\!=\!0\rangle \sim 1 \tag{4.44}$$

(ignoring normalization factors) and

$$\langle P\!=\!0|\delta(\mathbf{Q}-\mathbf{a})|P\!=\!0\rangle \sim 1 \tag{4.45}$$

For this reason we can think of $\delta(\mathbf{Q}-\mathbf{a})$ as a regularizing term—which is not needed in the case of periodic boundary conditions.

The operator $\mathbf{K_\Phi}$ *has the properties that*

a) It leaves physical states (kets) unchanged

a') $\delta(\mathbf{Q}-\mathbf{a})\mathbf{K_\Phi} = \delta(\mathbf{Q}-\mathbf{a})$

b) and $\mathbf{K_\Phi}\mathbf{K_\Phi} = \mathbf{K_\Phi}$

Marnelius' idea [23] is along similar lines. The constraint imposes us to work with the states $|P\!=\!0\rangle$. With these we construct the space. But to define an inner product we use their duals, $\langle Q_a|$. For physical quantities, however, we want to define an inner product *within the physical space alone*—the $|P\rangle$'s. Solution: find an operator that maps the first into the second

$$|P\rangle \longleftrightarrow \langle Q| \tag{4.46}$$

This is achieved by

$$|P\rangle \longleftrightarrow \delta(\mathbf{Q}-a)|P\rangle \tag{4.47}$$

for example, since

$$\langle P\!=\!0|\delta(\mathbf{Q}-a)|P\!=\!0\rangle = 1 \tag{4.48}$$

Notice, though, that there are many more choices for the operator that will do this!

This duality map role is also taken by the BRST exponential operator—as we will see— $\exp[\mathbf{\Omega},\mathbf{K}]$: we can understand it as a "duality map" operator. In the case of discrete spaces it reduces to the unity—trivial duality map.

Consider the *physical* states $|\psi_a\rangle$ and $|\psi_b\rangle$, and let us try to build the *physical* amplitude "$\langle\psi_a|\mathbf{O}|\psi_b\rangle$", where the operator $\mathbf{O}$ is physical—i.e., it commutes with the constraint $\mathbf{\Phi} = \mathbf{P}$, i.e., it is independent of $\mathbf{Q}$. As mentioned, this amplitude needs regularization:

$$\langle\psi_a|\mathbf{O}\delta(\mathbf{Q}-\mathbf{a})|\psi_b\rangle \tag{4.49}$$

because physical states are $\mathbf{P}$ eigenstates.

Let me summarize: *We define the inner product between physical states to be*

$$(\psi_a|\psi_b) \equiv \langle\psi_a|\delta(\mathbf{Q}-\mathbf{a})|\psi_b\rangle \tag{4.50}$$

Notice that *physical operators will be hermitean in the regularized inner product if they were hermitean to begin with in the bigger space.* This is because we can write such an operator in the form

$$\mathbf{O} = \mathbf{O}(\mathbf{P},\mathbf{q},\mathbf{p}) = \mathbf{O}'(\mathbf{q},\mathbf{p}) + \mathbf{O}''(\mathbf{P},\mathbf{q},\mathbf{p})\mathbf{P} \tag{4.51}$$

with $\mathbf{O}''$ regular in $\mathbf{P}$. The first term is hermitean, and the second one is as well—it drops out.

We can now proceed to construct the path integral. The resolutions of the identity we need to use are the usual ones[6],

$$\mathbf{I} = \int dQdq\, |Q,q\rangle\langle Q,q| = \int dPdp\, |P,p\rangle\langle P,p| \tag{4.52}$$

[6]As usual, the coordinates Q, P refer to the gauge degrees of freedom, and q, p to the physical ones.

which we insert in the amplitude

$$\mathcal{A} = \langle\psi_a|e^{-i\tau(\mathbf{h}+v\mathbf{P})}\delta(\mathbf{Q}-\mathbf{a})|\psi_b\rangle = \langle\psi_a|e^{-i\tau(\mathbf{h}+v\mathbf{P})}\delta(\mathbf{Q}-\mathbf{a})\mathbf{K}_{\mathbf{\Phi}}\mathbf{K}_{\mathbf{\Phi}}...\mathbf{K}_{\mathbf{\Phi}}|\psi_b\rangle \quad (4.53)$$

to obtain

$$\mathcal{A} = \int dQdqdPdp\,\delta(Q-Q(\tau))\delta(P)e^{-i\int d\tau(P\dot{Q}+p\dot{q}-h(q,p))} \quad (4.54)$$

Before getting into more details, notice also the following trick: if we have a conjugate pair like Q, P it follows that

$$\langle P\!=\!0|e^{i\mathbf{QA}}|P\!=\!0\rangle = \delta(\mathbf{A}) \quad (4.55)$$

where the operator $\mathbf{A}$ is understood to commute with $\mathbf{Q}, \mathbf{P}$. We will use this trick to produce the delta functions in the theory. The phase space keeps getting bigger. Consider also the fact that for two ghosts, η_1, η_2 and their conjugate momenta ρ_1, ρ_2 we have

$$\langle\rho_1\!=\!\rho_2\!=\!0|e^{i\boldsymbol{\eta_1}\mathbf{O}\boldsymbol{\eta_2}}|\rho_1\!=\!\rho_2\!=\!0\rangle = det(\mathbf{O}) \quad (4.56)$$

(all these can be seen using the coordinate resolutions of the identity, as we will explain in a second), so finally we can write that the physical quantities are to be obtained by using the amplitudes

$$\langle\psi_a, \pi\!=\!\rho_1\!=\!\rho_2\!=\!0|\mathbf{A}e^{i\boldsymbol{\lambda\chi}+\boldsymbol{\eta_1}|\{\boldsymbol{\chi},\mathbf{\Phi}\}|\boldsymbol{\eta_2}}|\psi_b, \pi\!=\!\rho_1\!=\!\rho_2\!=\!0\rangle \quad (4.57)$$

which is the equivalent to our earlier expression with the Dirac inner product in ghost form. To see this we insert

$$\mathbf{I} = \int dqdQd\lambda d\eta_1 d\eta_2\ \ |q, Q, \lambda, \eta_1, \eta_2\rangle\,\langle q, Q, \lambda, \eta_1, \eta_2| \quad (4.58)$$

and use the projections

$$\langle q, Q, \lambda, \eta_1, \eta_2 | p, P, \pi, \rho_1, \rho_2 \rangle = e^{i(qp+QP+\lambda\pi+\eta_1\rho_1+\eta_2\rho_2)} \tag{4.59}$$

It is assumed here that the terms in the exponent commute as operators, otherwise write

$$\langle \psi_a, \pi = \rho_1 = \rho_2 = 0 | \mathbf{A} e^{i\boldsymbol{\lambda\chi}} e^{i\boldsymbol{\eta_1}|\{\boldsymbol{\chi}, \boldsymbol{\Phi}\}|\boldsymbol{\eta_2}} | \psi_b, \pi = \rho_1 = \rho_2 = 0 \rangle \tag{4.60}$$

instead.

This is essentially the BRST inner product and the point of departure for the BFV path integral; we will discuss this more throughly in the next section.

Example[7] 1: Constraints of the form $G_i = p_i - \partial V / \partial q_i = 0$

We consider the constraints

$$G_i = p_i - \partial V / \partial q_i = 0 \tag{4.61}$$

for some subset of the indices i, and where V depends on the q_i only. *a)* The gauge transformation generated by the constraints is given by

$$A \to A + \delta_\epsilon A = A + \epsilon^i [A, p_i - \partial V / \partial q_i] \tag{4.62}$$

where the generator of the canonical transformation is just

$$\mathcal{G} = \epsilon^i G_i, \quad A \to A + [A, \mathcal{G}] \tag{4.63}$$

[7] This is essentially exercise 13.5 in Henneaux & Teitelboim's book [2].

This is complicated in general. However, we have

$$q_j \to q_j + \delta_\epsilon q_j = q_j + \epsilon^i [q_j, p_i - \partial V/\partial q_i] = q_j + \epsilon^i \delta_{ij} \tag{4.64}$$

b) The Dirac state condition is given by

$$\left(-i \frac{\partial}{\partial q_i} - \frac{\partial V}{\partial q_i} \right) \psi(q) = 0 \tag{4.65}$$

which implies that

$$\psi(q_j + \delta_\epsilon q_j) = e^{i\delta_\epsilon q_k \, \hat{\mathbf{p}}_\mathbf{k}} \psi(q_j) = e^{i\delta_\epsilon q_k \partial V/\partial q_k} \psi(q_j) \tag{4.66}$$

so we have "classical" invariance up to a phase.

c) Let us study this Hilbert space, including the inner product. We will compare the result with the reduced phase space approach.

The Dirac inner product for these states is given by

$$(\varphi_a,\, \varphi_b) = \int dV \varphi_a^* \left(\prod \delta(q_i) \right) \varphi_b \tag{4.67}$$

say. We have that the q_i degree of freedom if absent...just as if we had started by quantizing a theory with the constraints $G_i = 0$ as above, and $\chi_i = q_i = 0$. The solution to the Dirac conditions equation is given by

$$\psi(q) \sim e^{iV(q)} \tilde{\psi}(q \neq q_i) \tag{4.68}$$

With this additional piece of information it is clear that the inner product will not depend on the gauge fixing χ.

What about the expectation value of *observables*? Observables need to be hermitean and have expectation values that are not gauge dependent. Thus they need to commute with both the constraints and the gauge-fixing. The solution to

$$[\hat{A}, \hat{\Phi}] \approx 0 \tag{4.69}$$

is given by

$$\hat{A} = e{-}i\hat{V}\hat{O}e{-}i\hat{V} + \hat{\alpha}\hat{\Phi} \tag{4.70}$$

The correct interpretation is that, again, when we change gauge-fixing we change representations. A unitary transformation is involved: for a given gauge-fixing we have the states

$$\psi(q)_\chi \sim e^{iV(q)}\Big|_{\chi=0} \tilde{\psi}(q \neq q_i) \tag{4.71}$$

for another

$$\psi(q)_{\chi'} \sim e^{iV(q)}\Big|_{\chi'=0} \tilde{\psi}(q \neq q_i) \tag{4.72}$$

The difference is a unitary (canonical) transformation. In the same way, for the purposes of this isomorphism, the operators must transform. Of course, the question to go back to is: what is my reference point, or, what is the inner product, or, how do I choose my observables?

To summarize, we have the following rules

a) use *observables* in the usual sense ($[\mathbf{A}, \mathbf{G}] = \mathbf{0}$), for they map the physical space on itself, and they lead to gauge-fixing invariant expectation values, and

b) pick the right $\hat{A} \sim \hat{A} + \lambda\hat{G}$ in the equivalence class to ensure hermiticity with respect to the chosen gauge-fixing.

Gauge-fixing picks an element in the equivalence class of observables, and this is tied to the hermicity properties of the observables.

Example 2: the non-relativistic particle and similar cases

As discussed, in the situation where we do not impose boundary conditions the solution to the Dirac constraint is not normalizable. The solution is given by $\psi = e^{itp_t+ixp_x}$, with the condition $p_t + p_x^2/2m = 0$, but there are no further conditions. And the inner product/norm for these states yields immediately a $\delta(0)$ from the dt integration. This is the reason why some "gauge fixing" is introduced—to take care of the "gauge infinite volume". The inner product for the physical states can be defined by[8]

$$(\psi_a, \psi_b) = \int dxdt\, \psi_a^* \, \delta(\mathbf{K}(x,t,\tau))\, |\{\mathbf{K}, \mathbf{\Phi}\}|\, \psi_b \tag{4.73}$$

This we can rewrite as

$$(\psi_a, \psi_b) = \frac{1}{2\pi} \int dxdtd\lambda dcd\bar{c}\, \psi_a^* \, e^{i\lambda\mathbf{K}+ic|\{\mathbf{K},\mathbf{\Phi}|\}\bar{c}}\, \psi_b \tag{4.74}$$

As long as we choose $\mathbf{K}$ so that $K = 0$ can be rewritten as $t = t(x, \tau)$ we are ok. However, notice that the final form of the inner product will depend on the form of the constraint—not on the gauge-fixing. This ambiguity is related to the

[8]Here there is no need to reorder the inner product operators, since they will turn out to be hermitean immediately.

above discussion on the possible different representations. Indeed, this inner product reduces to the inner product of the unconstrained case we discussed above—with the same "ambiguity".

The constraint is

$$\Phi = p_t + \frac{p_x^2}{2m} \tag{4.75}$$

so, according to Dirac [1], the states are defined by starting with a full Hilbert space,

$$|x, t>, \ |\psi>, \ < x, t|\psi >= \psi(x, t, \tau), \tag{4.76}$$

and then by imposing the condition

$$(\mathbf{p}_t + \frac{\mathbf{p}_x^2}{2m})|\psi >= 0 \tag{4.77}$$

In this Hilbert space the operators are $\mathbf{x}, \mathbf{t}, \mathbf{p}_t, \mathbf{p}_x$ with the usual commutation relations $[\mathbf{x}, \mathbf{p}_x] = i = [\mathbf{t}, \mathbf{p}_t]$ with the others zero. The hamiltonian in this system is zero, so the Schrödinger equation of motion is just

$$\frac{\partial}{\partial \tau}|\psi >= 0. \tag{4.78}$$

The states are frozen with respect to the τ parameter, and are thus gauge invariant. In this case this means that the physical states are solutions to the Schrödinger equation in t-time: just use the definition of

$$\mathbf{p}_t = -ih\frac{\partial}{\partial t} \tag{4.79}$$

in the position representation. These states are in one-to-one correspondence with the states of the unconstrained non-relativistic particle after we impose some gauge condition—like $\chi = \chi_\alpha = 0$. Indeed, the inner product is defined to be

$$(\varphi, \psi) = \int dxdt\, \varphi^* \, \overbrace{\delta(\chi)|\{\chi, \Phi\}|}\, \psi \qquad (4.80)$$

and with the above gauge it reduces to the inner product of the unconstrained case. Notice that the inner product is independent of " time", i.e., independent of gauge fixing. Also, from the point of view of this inner product, gauge fixing has to be chosen so that the operator $|\{\boldsymbol{\chi}, \boldsymbol{\Phi}\}|$ has no zero modes, e.g, $\chi = t$ is ok, but $\chi = x$ is not. This description can be seen to correspond to the Schrödinger picture.

Let me be more explicit. Consider the constraint

$$\hat{\Phi} = \hat{p}_t + \hat{A}(\hat{x}, \hat{p}_x) \qquad (4.81)$$

where $\hat{A}$ is a hermitean operator (in the full space sense). The states that solve the Dirac physical condition, $\hat{\Phi}\,|\psi^D\rangle = 0$ are given by[9]

$$|\psi^D\rangle = e^{-i\hat{t}\,\hat{A}}\;|\varphi(x)\rangle \otimes |p_t\!=\!0\rangle \qquad (4.82)$$

To see this, write a general state in the form $|\psi\rangle = \exp(-i\hat{t}\,\hat{A})\,|\eta\rangle$ and use

$$[\hat{p}_t + \hat{A}, e^{-i\hat{t}\,\hat{A}}] = e^{-i\hat{t}\,\hat{A}}\hat{A} \qquad (4.83)$$

which means

$$\hat{p}_t + \hat{A}(\hat{x}, \hat{p}_x)\; e^{-i\hat{t}\,\hat{A}} = e^{-i\hat{t}\,\hat{A}}\hat{p}_t \qquad (4.84)$$

[9]These can be *rewritten* as $|\psi^D\rangle = \exp(-i\hat{t}\,(\hat{A} - k_t))|\varphi(x)\rangle \otimes |p_t\!=\!k_t\rangle$

so we need the zero eigenstate of $\hat{p}_t$—because of the way we chose to write the solution. Again, life is easy when $[\hat{p}_t, \hat{A}] = 0$. In the coordinate representation, with $\psi(t,x) \equiv \langle t, x|\psi\rangle$, $\hat{p}_t \sim -i\partial_t$, etc., the Dirac states are given by

$$\psi^D(t,x) = e^{-i\hat{A}\,t}\,\varphi(x) \tag{4.85}$$

Then the Dirac inner product is given by $(\chi = t - f(\tau))$

$$(\psi_a^D, \psi_b^D) \equiv \langle\psi_a^D|\,\overbrace{\delta(\chi)|\{\chi,\Phi\}|}\,|\psi_b^D\rangle =$$

$$\int dtdx\,\left(\psi_a^D\right)^*\,\delta(t - f(\tau))\,\psi_b^D = \int dx\,\left(\psi_a^D\right)^*\,\psi_b^D =$$

$$\int dx\,e^{+i\hat{A}\,t}\,\left(\varphi_a(x)\right)^*\,e^{-i\hat{A}\,t}\,\varphi_b(x) = \int dx\,\left(\varphi_a(x)\right)^*\,\varphi_b(x) \tag{4.86}$$

because of the hermiticity of $\hat{A}$. This is the usual inner product of the unconstrained case.

Example 3: the Klein-Gordon inner product—one branch only.

We begin with the phase-space described by the coordinates t, x, p_x, p_t and the constraint (which we write relativistically)

$$\Phi = p_t^2 - p_x^2 - m^2 \approx 2\sqrt{p_x^2 + m^2}\left(p_t - \sqrt{p_x^2 + m^2}\right) \tag{4.87}$$

The last (weak) equality follows when we restrict ourselves to the positive p_t branch—which we do in this simple example. That is, we will pick a branch in the decomposition

$$\delta(\Phi) = \delta\left(2\sqrt{p_x^2 + m^2}\left(p_t - \sqrt{p_x^2 + m^2}\right)\right) + \delta\left(2\sqrt{p_x^2 + m^2}\left(p_t + \sqrt{p_x^2 + m^2}\right)\right) \tag{4.88}$$

which is equivalent to working with the constraint expressed as

$$\Phi' = \left(p_t - \sqrt{p_x^2 + m^2}\right) \sqrt{p_x^2 + m^2} \tag{4.89}$$

We now go to the quantum theory, where the variables above become operators in the usual way. For example, in the coordinate representation the variable p_t becomes the operator $-i\partial_t$.

The physical states are then defined beginning with the full Hilbert space—the one corresponding to the above phase-space—and then imposing the physical condition

$$\mathbf{\Phi}|\psi\rangle = 0 \tag{4.90}$$

just as before. The states are as in the previous example, since the multiplicative operator does not have any zero modes. They are thus given by

$$|\psi^D\rangle = e^{-i\hat{t}\,\hat{A}}\ |\varphi(x)\rangle \otimes |p_t\!=\!0\rangle \tag{4.91}$$

with $\hat{A} = \sqrt{\hat{p}_x^2 + m^2}$, or in the coordinate representation as

$$\psi(t,x) = e^{-it\sqrt{\hat{p}_x^2+m^2}}\ \varphi(x) \tag{4.92}$$

With this choice we will find that the inner product is given by

$$(\psi_a, \psi_b)_{\Phi'} = \int d^4x\ \psi_a^*\ \sqrt{\hat{p}_x^2 + m^2}\ \psi_b\ \delta(t - f(\tau)) \tag{4.93}$$

For the states that satisfy the constraint we have

$$\sqrt{\hat{p}_x^2 + m^2}\ \psi = i\frac{\partial}{\partial t}\psi \tag{4.94}$$

so this form of the inner product is equivalent to the Klein-Gordon inner product,

$$(\psi_a, \psi_b) = \int d^3x \; \psi_a^* \, \frac{1}{2} \left(i \overrightarrow{\partial}_t - i \overleftarrow{\partial}_t \right) \, \psi_b \tag{4.95}$$

In general we can say that the effect of rescaling the constraint on the inner product has been that with $\Phi' = \Gamma(q,p)\Phi$ we have that the new inner product reads

$$(\psi_a, \psi_b)_{\Phi'} = (\psi_a, \hat{\Gamma}\, \psi_b)_{\Phi} \tag{4.96}$$

Let us now anticipate the BRST inner product. The above states are now embedded in the extended space: they are given by

$$|\Psi\rangle \equiv |\psi^D\rangle \otimes |\lambda{=}\rho_1{=}\rho_2{=}0\rangle \tag{4.97}$$

and the inner product is given by

$$\langle \Psi_a | \, e^{i\hat{\pi}\hat{\chi}} \, e^{i\hat{\eta}_1 \overbrace{|\{\chi, \Phi\}|} \hat{\eta}_2} \, |\Psi_b\rangle = \langle \psi_a | \; \overbrace{\delta(\hat{\chi}) \, |\{\chi, \Phi\}|} \; |\psi_b\rangle \tag{4.98}$$

which we further rewrite as

$$= \int dt dx \; \psi_a^*(t,x) \; \overbrace{\delta(\hat{\chi}) \, |\{\chi, \Phi\}|} \; \psi_b(t,x) = \int dx \; \psi_a^*(t,x) \, \sqrt{\hat{p}_x^2 + m^2} \, \psi_b(t,x) \tag{4.99}$$

or finally, again as

$$= \int dx \; \psi_a^*(t,x) \, \frac{1}{2} \left(i \overrightarrow{\partial}_t - i \overleftarrow{\partial}_t \right) \, \psi_b(t,x) \tag{4.100}$$

which is the BRST inner product (notice that in the last equality we have used the fact that the states satisfy the constraint). The gauge $\chi = t - f(\tau)$ leads to this result very directly, but any gauge that can be rewritten in the form $t = t(p_t, p_x, x)$ will give the same answer. The only restriction, as usual, is that $\{\chi, \Phi\} \neq 0$.

Also notice that the result does depend on the way we write the constraint—which is by far not unique. But this arbitrariness can be eliminated by demanding that some observables be hermitean—which is what Ashtekar does [48]. But how do you pick your observables?

The thing to keep in mind is that our choice in the way in which we write the constraint will affect the normalization. Notice, however, that even in the unconstrained case one is always free to change the states by

$$|\psi\rangle \longrightarrow |\psi'\rangle \equiv \boldsymbol{\alpha}^{-1}|\psi\rangle \tag{4.101}$$

as long as the inner product and expectation values are modified accordingly,

$$\langle\psi|\mathbf{A}|\psi\rangle \longrightarrow \langle\psi'|\boldsymbol{\alpha}^{\dagger}\mathbf{A}\boldsymbol{\alpha}|\psi'\rangle \tag{4.102}$$

In fact the solution spaces to the constraint

$$\boldsymbol{\Phi}|\psi\rangle = 0 \tag{4.103}$$

and

$$\boldsymbol{\Phi}\mathbf{v}|\psi\rangle = 0 \tag{4.104}$$

are isomorphic as long as the operator $\mathbf{v}$ has no zero modes—if $|\psi_0\rangle$ is a solution to the first then $\mathbf{v}^{-1}|\psi_0\rangle$ solves the second—and viceversa. This equivalence/ambiguity appears in the definition of the constraint and then in the above definitions of the inner product—it appears in the determinant.

One can fix this ambiguity by demanding that certain operators in the theory be hermitean [48]—although this procedure just transfers the ambiguity from the inner product to the definition of observables.

> *The punchline is that we are free to normalize as we wish. There is extensive freedom on the way we represent the Heisenberg algebra—something that applies to constrained as well as unconstrained systems. The physical quantities in theory must remain the same, of course.*

4.2.3 Dirac inner product for the two branches case: ordering problems and unitarity

What problems do we encounter when the relativistic constraint's full solution space is used? Let us review the description of the inner product and the observables and their hermiticity properties.

But first we need *a good inner product:*

a) it satisfies $((\psi_a|\psi_b))^* = (\psi_b|\psi_a)$

b) it is invariant under changes of gauge-fixing

c) it is conserved under time evolution—the hamiltonian is hermitean, and

d) $(\psi|\psi) \geq 0, = 0$ only if $|\psi) = 0$.

The last requirement can be postponed until we decide to equate norms with probabilities.

Consider now the definition of observables.

i) Observables are chosen by asking that

$$[\hat{A}_D, \hat{\Phi}] \approx 0 \tag{4.105}$$

but we will have to be careful with the weak equality.

ii) Also, $\hat{A}_D \sim \hat{A}_D + \hat{\lambda}\hat{\Phi}$, since the effect on physical states is the same, and they are both observables.

iii) We can use this freedom to ask that

$$[\hat{A}_D, \hat{\chi}] \approx 0 \tag{4.106}$$

for some $\hat{\chi}$ with $[\hat{\chi}, \hat{\Phi}]\Big|_{\Phi=0} \neq 0$. This is "gauge-fixing".

iv) Now we need hermiticity of the observables with respect to the inner product on physical states given by

$$(\psi_a^D, \psi_b^D) = \int dV \, \left(\psi_a^D\right)^* \, \widehat{\prod_a \delta(\chi_a) \mathrm{sdet}\{\chi_a, G_b\}} \, \psi_b^D \tag{4.107}$$

(where, again, the big hat just means that the whole will become an operator—and that we haven't commited to a specific ordering yet). But first, we need to demand that the operator $\widehat{\prod_a \delta(\chi_a) \mathrm{sdet}\{\chi_a, G_b\}}$ be hermitean and positive definite. Let us see why. A good inner product needs to satisfy

$$[(\psi_a, \psi_b)]^* = (\psi_b, \psi_a) \tag{4.108}$$

as we wrote above. This ensures that states have real norms,

$$\parallel \psi \parallel^2 \equiv (\psi, \psi) = [(\psi, \psi)]^* \in \mathbb{R} \tag{4.109}$$

This, for example, tells us that the inner product for the free relativistic case can be defined to be the Klein-Gordon inner product,

$$(\psi_a^D, \psi_b^D) = \int dV \, \left(\psi_a^D\right)^* \, \frac{1}{2} \left(\delta(t - f(\tau))\, \hat{p}_t + \hat{p}_t^\dagger \, \delta(t - f(\tau))\right) \, \psi_b^D \tag{4.110}$$

Why is this the Klein-Gordon inner product? Consider the operator

$$\hat{O} = \delta(t - f(\tau)) \left(-i \, \frac{\overrightarrow{d}}{dt}\right) \circ \tag{4.111}$$

It is easy to see explicitely that its hermitean conjugate is given by

$$\hat{O}^\dagger = \left(-i \, \frac{\overrightarrow{d}}{dt}\right) [\delta(t - f(\tau))\circ] = \left(i \, \frac{\overleftarrow{d}}{dt}\right) [\delta(t - f(\tau))\circ] \tag{4.112}$$

i.e., $\hat{p}_t \, \delta(t - f(\tau))$, because the delta function takes care of any boundary condition problems at $t = \pm\infty$. Then

$$\hat{O} + \hat{O}^\dagger = \delta(t - f(\tau)) \left(-i \, \frac{\overrightarrow{d}}{dt}\right) \circ + \left(i \, \frac{\overleftarrow{d}}{dt}\right) [\delta(t - f(\tau))\circ] \tag{4.113}$$

After integrating the delta function, the inner product becomes the Klein-Gordon one,

$$(\psi_a^D, \psi_b^D) = \int dx \, \left(\psi_a^D(t,x)\right)^* \, \frac{1}{2} \left(-i \, \frac{\overrightarrow{d}}{dt} + i \, \frac{\overleftarrow{d}}{dt}\right) \, \psi_b^D(t,x) \Big|_{t=f(\tau)} \tag{4.114}$$

which is real, as promised (recall that it is the charge of the Klein-Gordon field).

Now we need to ensure that the definition does not depend on the choice of gauge-fixing. Consider the gauge-fixing $\chi = t - \tau_1$. The inner product with this gauge-fixing reads

$$(\psi_a^D, \psi_b^D)_1 = \int dV \, \left(\psi_a^D\right)^* \, \left(\delta(\hat{\chi})[\hat{\chi}, \hat{\Phi}] + [\hat{\chi}, \hat{\Phi}]^\dagger \delta(\hat{\chi})\right) \, \psi_b^D \tag{4.115}$$

It is not hard to see that it we change the gauge-fixing to $\chi = t - \tau_2$ the inner product will remain the same as long as the old $\hat{p}_t^\dagger$ is the hermitean conjugate of $\hat{p}_t$ in the new inner product as well. Indeed,

$$(\psi_a^D, \psi_b^D)_2 = \int dV \, \left(\psi_a^D\right)^* \, \left(\delta(t-\tau_2)[\hat{\chi}, \hat{\Phi}] + [\hat{\chi}, \hat{\Phi}]^\dagger \delta(t-\tau_2)\right) \, \psi_b^D = \tag{4.116}$$

$$\int dV \, e^{-i\hat{p}_t^\dagger \, \Delta\tau} \left(\psi_a^D\right)^* \, \left(\delta(t-\tau_1)[\hat{\chi}, \hat{\Phi}] + [\hat{\chi}, \hat{\Phi}]^\dagger \delta(t-\tau_1)\right) \, e^{i\hat{p}_t \, \Delta\tau} \psi_b^D \tag{4.117}$$

Thus, we just need

$$\left(\delta(\chi)[\hat{\chi}, \hat{\Phi}] + [\hat{\chi}, \hat{\Phi}]^\dagger \delta(\chi)\right) \hat{p}_t = \hat{p}_t^\dagger \left(\delta(\chi)[\hat{\chi}, \hat{\Phi}] + [\hat{\chi}, \hat{\Phi}]^\dagger \delta(\chi)\right) \tag{4.118}$$

in the physical subspace.

Observe that this is always true: our inner product definition matches the Klein-Gordon definition—which we know is conserved. Let us recall how this Klein-Gordon inner product is produced for the interacting case. We start from a classical action which has the Klein-Gordon equation as its extremizing equation of motion,

$$\int dV \sqrt{g} \mathcal{L} = \int dV \sqrt{g} \left(\nabla_a \Phi \nabla^b \Phi^* + m^2 \Phi \Phi^*\right) \tag{4.119}$$

and, from the global (or local if ∇ stands for the fully covariant derivative) $U(1)$ symmetry,

$$\Phi \longrightarrow e^{i\alpha}\Phi \tag{4.120}$$

(and complex conjugate) of the action we infer the existence of conserved Noether current,

$$\partial_\mu J^\mu = 0 \tag{4.121}$$

$$J^\mu = \Phi\nabla^\mu\Phi^* - \Phi^*\nabla^\mu\Phi \tag{4.122}$$

We then use this conserved current to define a conserved inner product—by integrating the first over some surface.

Now, for the case in which we can factor the Klein-Gordon equation and the corresponding solution space, this means that for gauge-invariance the sectors must decouple, because this is is the only way for the operator $\hat{p}_t^\dagger$ to be the hermitean conjugate of $\hat{p}_t$ in the new inner product. Thus, unitarity implies decoupling. Indeed, gauge-invariance of the inner product is equivalent to unitarity, (or conservation of the inner product), since time evolution is a gauge-transformation for parametrized theories. This inner product is not going to yield positive norms, though. However, we can stick to one branch if the Klein-Gordon equation factorizes.

We know that our inner product decouples nicely if the Klein-Gordon equation does. This is then a criterium for unitarity in the one particle sector. We already *saw that this will occur if*

$$[D_0, D_i] = 0 \tag{4.123}$$

Now, as mentioned, for the purposes of a probabilistic interpretation we would also like to have an inner product that gives states a positive norm. When the inner product decouples the course to take is clear: just stick to one branch. If the ordering that we choose—say, because of minimal coupling and/or space-time covariance—and the interactions don't allow decoupling then we are in trouble in this respect. We will not have space-time covariance and/or minimal coupling and unitarity in the one particle sector. We will come back to this point later.

Let us go back to the observables. They need to be hermitean in this new inner product. This will happen provided they were hermitean with respect to the old inner product (in the physical subspace is sufficient), and provided they commute with $\overbrace{\prod_a \delta(\chi_a)\text{sdet}\{\chi_a, G_b\}}$. Conditions *i)* and *iii)* cover this last point, together with the Jacobi identity.

We can look at the problem from the point of view of our old projector $\mathbf{K_\Phi}$.

Let us start by playing with the simpler constraint

$$\Phi = P^2 - a^2 \approx 0 \tag{4.124}$$

where a is positive. Then

$$\delta(P^2 - a^2) = \frac{\delta(P - a)}{a} + \frac{\delta(P + a)}{a} \tag{4.125}$$

How do we define the projector here? Following our previous discussion, the inner product between physical states is defined by

$$(\psi_a, \psi_b) \equiv \langle\psi_a|\frac{1}{2}\left(\delta(\mathbf{Q}-\mathbf{Q_0})\mathbf{P}+\mathbf{P}^\dagger\delta(\mathbf{Q}-\mathbf{Q_0})\right)|\psi_b\rangle \tag{4.126}$$

This is the inner product that we had before (Klein-Gordon). Notice that the two sectors decouple. Now we need the projector (which leaves the physical states alone.) A first guess is given by

$$\mathbf{K_\Phi} \equiv \delta(\mathbf{P^2}-\mathbf{a^2})\frac{1}{2}\left(\delta(\mathbf{Q}-\mathbf{Q_0})\mathbf{P}+\mathbf{P}^\dagger\delta(\mathbf{Q}-\mathbf{Q_0})\right) \tag{4.127}$$

This operator, however, is not the identity on the physical states[10],

$$\mathbf{K_\Phi}|P\!=\!\pm a\rangle = \pm|P\!=\!\pm a\rangle \tag{4.128}$$

We can form a projetor by simply taking the absolute value of this operator—see the next chapter:

$$\mathbf{K'_\Phi} = |\mathbf{K_\Phi}| = \delta(\mathbf{\Phi})\mathbf{\frac{1}{2}}|\delta(\mathbf{Q}-\mathbf{Q_0})\mathbf{P}+\mathbf{P}^\dagger\delta(\mathbf{Q}-\mathbf{Q_0})| \tag{4.129}$$

Consider now

$$\begin{aligned}\mathbf{K_\Phi} = \frac{1}{2}\delta(\mathbf{P}^2-a^2)\,|[\mathbf{Q}-\mathbf{Q_0},\mathbf{P}^2-a^2]|\,\delta(\mathbf{Q}-\mathbf{Q_0}) = \\ \frac{1}{2}\left(\delta(\mathbf{P}-a)+\delta(\mathbf{P}+a)\right)\delta(\mathbf{Q}-\mathbf{Q_0})\end{aligned} \tag{4.130}$$

where now we have inserted an absolute value. Is this a projector? Only if we insert $\Theta(\mathbf{P})$, which is equivalent to picking a branch.

[10]It does satisfy $\mathbf{K_\Phi K_\Phi K_\Phi} = \mathbf{K_\Phi}$.

The reason I mention these cases is because they will help clarify later some results in the Faddeev path integral. We will come back to the search for a projector in the next section.

Let us now study a bit more the interacting relativistic particle. What happens if the determinant itself doesn't commute with the gauge-fixing? Well, as we saw this is as if we were computing the expectation value of an operator that is not hermitean: *our inner product doesn't even have to be real, it will be a bad inner product. Moreover, the hermiticity properties of the rest of the operators are then in question.*

Now, for hermicity we need commutation of the observables—which already commute with the constraint—with the operator

$$\overbrace{\prod_a \delta(\chi_a)\text{sdet}\{\chi_a, G_b\}} \tag{4.131}$$

In the particle case this corresponds to asking commutation with $\delta(\hat{t} - f(\tau))\hat{p}_t$—and $\hat{\Phi}$ of course. Easily done—use operators in the x coordinates. For interacting case it may prove harder to find such "hermitean" operators.

As we discussed (and there is already the Klein-Gordon solution to this problem), however, we can solve part of our problems if we come up with a hermitean ordering like

$$i\,\overbrace{\prod_a \delta(\chi_a)\text{sdet}\{\chi_a, G_b\}} = \delta(\hat{\chi}_a) sdet[\hat{\chi}_a, \hat{G}_b]/2 + sdet[\hat{\chi}_a, \hat{G}_b]^\dagger \delta(\hat{\chi}_a^\dagger)/2 \tag{4.132}$$

For the free case, this decouples the worlds—it yields minus one-half $\delta(\hat{t} - f(\tau))i\partial_t +$

$(i\partial_t)^\dagger \delta(\hat{t} - f(\tau))$.

For the interacting case it yields

$$2\hat{g}^{\mu 0}(\hat{p}_\mu - \hat{A}_\mu)\delta(\hat{t} - f(\tau)) + \delta(\hat{t} - f(\tau))\{2\hat{g}^{\mu 0}(\hat{p}_\mu - \hat{A}_\mu)\}^\dagger \tag{4.133}$$

which must be Klein-Gordon in curved space, if we started with a covariant ordering of the constraint, of course.

Then we can try to find observables that commute with such an object.

Some concepts I discussed can be confusing. One of them is unitarity. Unitarity means that the hamiltonian is hermitean, that the inner product is conserved under time evolution (or invariant under gauge transformations.) This we can always produce, and its definition is independent of whether there is particle creation. That has to do with the decoupling of the two sectors, because the conserved inner product that we have is not positive definite: if we want unitarity within one sector we will get it if the sectors decouple, because then we will have a conserved inner product within a sector—an inner product with a definite sign.

4.2.4 From Dirac quantization to the Faddeev path integral

Now that we have developed a projector language and a regularized inner product we can write an expression for the path integral.

For the simplest case, the regularized physical amplitude is just given by

$$\langle P=0, \varphi_a(q)|\delta(\mathbf{Q} - Q')\, |P=0, \varphi_b(q)\rangle =$$

$$\langle P=0, \varphi_a(q)|Q=Q', \varphi_b(q)\rangle = \langle\varphi_a(q)|\varphi_b(q)\rangle \tag{4.134}$$

where I am using the usual old q, Q etc. notation for the gauge and physical degrees of freedom. Notice that we could write, just as well,

$$\langle Q=Q', \varphi_a(q)|\delta(\mathbf{P})\ |Q=Q', \varphi_b(q)\rangle =$$

$$\langle P=0, \varphi_a(q)|Q=Q', \varphi_b(q)\rangle = \langle\varphi_a(q)|\varphi_b(q)\rangle \tag{4.135}$$

Once we have this it is immediate to write the corresponding path integral,

$$\langle Q=Q', \varphi_a(q)|\mathbf{K_\Phi K_\Phi ... K_\Phi}\, \delta(\mathbf{P})\ |Q=Q', \varphi(q)_b\rangle \tag{4.136}$$

The key is to have $\mathbf{K_\Phi K_\Phi} = \mathbf{K_\Phi}$ and $\mathbf{K_\Phi}\delta(\mathbf{P}) = \delta(\mathbf{P})$, or $\delta(\mathbf{Q})\mathbf{K_\Phi} = \delta(\mathbf{Q})$ etc.

This operator, $\mathbf{K_\Phi}$, is then the composition law operator.

It is crucial for full gauge invariance that we choose the gauge-fixing states/terms to satisfy

$$\langle\chi=0|\Phi=0\rangle = constant \tag{4.137}$$

If this condition is not met we loose gauge-invariance. In the path integral this will appear in the form of an action that is not invariant at the boundaries.

For the particle we can write

$$U \equiv \langle t_f, x_f|\delta(\mathbf{\Phi})\ |t_i x_i\rangle \tag{4.138}$$

It is easy to see that this is the usual propagator in the non-relativistic case. We can write a path integral by repeated insertion of

$$\hat{K}_\Phi = \delta(\mathbf{\Phi})\delta(\mathbf{t} - f(\tau)) \tag{4.139}$$

This procedure yields the Faddeev path integral. For the relativistic case we also need $\langle t_f, x_f|\delta(\mathbf{\Phi})|t_i x_i\rangle$. Remember that we tried

$$\mathbf{K_\Phi} \equiv \delta(\mathbf{P^2}-\mathbf{a^2})\frac{1}{2}\left(\delta(\mathbf{Q}-Q_0)\mathbf{P}+\mathbf{P}^\dagger\delta(\mathbf{Q}-Q_0)\right) \tag{4.140}$$

This operator, however, is neither a projector nor the identity on the physical states,

$$\mathbf{K_\Phi}|P{=}\pm a\rangle = \pm|P{=}\pm a\rangle \tag{4.141}$$

This can be fixed in a number of ways. We can change the amplitude, or the operator. For example, we can work with the projector[11] (it is a projector)

$$\mathbf{K'_\Phi} = |\mathbf{K_\Phi}| = sign(\mathbf{P})\ \mathbf{K_\Phi} \tag{4.142}$$

and with any amplitude of the form

$$U \equiv \langle t_f, x_f|(\alpha\,\delta(\mathbf{P}-\mathbf{a})+\beta\,\delta(\mathbf{P}+\mathbf{a}))|t_i x_i\rangle \tag{4.143}$$

This will work because this projector leaves the operator $\alpha\,\delta(\mathbf{P}-\mathbf{a})+\beta\,\delta(\mathbf{P}+\mathbf{a})$ unchanged. However, to build a composition law we need to work with the amplitude

$$U \equiv \langle t_f, x_f|\ sign(\mathbf{P})\delta(\mathbf{P^2}-\mathbf{a^2})\ |t_i, x_i\rangle \tag{4.144}$$

so that this amplitude will be the one appearing in the composition law, after insertions of the projector. The composition law here is the usual Klein-Gordon one, generated by the second part of the projector:

$$\frac{1}{2}\left(\delta(\mathbf{Q}-Q_0)\mathbf{P}+\mathbf{P}^\dagger\delta(\mathbf{Q}-Q_0)\right)$$

[11]To see the second equality, think of the projector as a diagonal matrix in the p basis with only two non-zero eigenvalues.

Notice that these projectors contain two important ingredients: the physical states (represented by some delta function), and the inner product.

We have a choice on how to look at the projector: which part represents the state and which the composition law. We can instead take the "signed" part into the composition law, and rewrite

$$\mathbf{K}'_{\mathbf{\Phi}} = |\mathbf{K}_{\mathbf{\Phi}}| = \delta(\mathbf{\Phi})\, \frac{\mathbf{1}}{\mathbf{2}}\, |\delta(\mathbf{Q} - Q_0)P + P^{\dagger}\delta(Q - Q_0)|$$

which works nicely with the amplitude

$$\langle x_\mu|\, \delta(\mathbf{\Phi})\, |y_\mu\rangle$$

The composition law here is the Klein-Gordon one with an absolute value.

This description also holds—quite clearly—if there is "decoupling", i.e., if the above constant a is made into an operator that nonetheless commutes with $\mathbf{P}$. This describes, then, the relativistic particle with no electric field, for example. What about the interacting case? It corresponds, in our simple description here, to considering a constraint of the form

$$\mathbf{\Phi} = \mathbf{P^2} - f(\mathbf{Q})^2 \tag{4.145}$$

Does our projector description still hold? To answer this question we need to understand the quantities

$$\langle \chi_i|\, \left(\delta(\mathbf{Q} - Q_0)\mathbf{P} + \mathbf{P}^{\dagger}\delta(\mathbf{Q} - Q_0)\right)\, |\chi_j\rangle \tag{4.146}$$

where $|\chi_i\rangle$ are the two solutions to the constraint equation. I don't think it will be easy to construct a projector theory if there is no decoupling. Thus, we will also miss an understanding of the Faddeev path integral if there is no decoupling.

We can also consider the "Feynman" amplitude. It is given by

$$\langle t_f, x_f | \frac{1}{\hat{\Phi} - i\epsilon} | t_i x_i \rangle \tag{4.147}$$

In what sense is this an amplitude in the physical subspace?

We will be able to interpret this amplitude in the next section. Let us investigate it a bit more here, though.

Consider again the simple constraint case. Then it is easy to see that

$$\langle Q | \frac{1}{\hat{P} - a - i\epsilon} | Q' \rangle = \Theta(Q - Q') \langle Q | \delta(\hat{P} - a) | Q' \rangle \tag{4.148}$$

(times $2\pi i$...) Similarly,

$$\langle Q | \frac{1}{\hat{P}^2 - a^2 - i\epsilon} | Q' \rangle = \frac{1}{a} \Theta(Q - Q') \langle Q | \delta(\hat{P} - a) | Q' \rangle - \frac{1}{a} \Theta(Q' - Q) \langle Q | \delta(\hat{P} + a) | Q' \rangle \tag{4.149}$$

To see this just use

$$\frac{1}{P^2 - a^2 - i\epsilon} = \frac{1}{a} \left(\frac{1}{P - a - i\epsilon} - \frac{1}{P + a + i\epsilon} \right) \tag{4.150}$$

Here we can use the projector

$$\frac{1}{\mathbf{\Phi} - i\epsilon} \frac{1}{2} \Big(\delta(\mathbf{Q} - \xi) \mathbf{P} + \mathbf{P}^\dagger \delta(\mathbf{Q} - \xi) \Big) \tag{4.151}$$

This also works, as this operator is a projector and it leaves the above amplitude unchanged.

With this operators and amplitudes we can now build path integrals. These amplitudes we have discussed are the various Green function for the Klein-Gordon equation, provided we use the proper constraint.

The following have been discussed repeatedly in the literarature [26, 43, 44]: The **Hadamard** Green function,

$$\Delta_1(x-y) = \frac{1}{(2\pi)^3}\int d^4k\ \delta(k^2-m^2)\ e^{ik(x-y)} = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\lambda \int d^4k\ e^{ik(x-y)+i\lambda(k^2-m^2)} \tag{4.152}$$

which is a solution to the Klein-Gordon equation. In term of our projectors it can be written as

$$\Delta_1(x-y) \sim \langle x_\mu, \pi\!=\!0|\ e^{i\hat{\lambda}\hat{\Phi}}|y_\mu, \pi\!=\!0\rangle \sim \langle x_\mu|\ \delta(\hat{\Phi})|y_\mu\rangle \tag{4.153}$$

as discussed before. As we saw it satisfies a composition law generated by

$$\frac{1}{2}\left|\delta(\mathbf{Q}-\mathbf{Q_0})\mathbf{P} + \mathbf{P}^\dagger\delta(\mathbf{Q}-Q_0)\right| \tag{4.154}$$

instead of the Klein-Gordon one,

$$A\cdot B \equiv -i\int d\sigma^\mu\ A(x,z)\ \overleftrightarrow{\partial}_\mu\ B(z,x) = \int d\sigma^\mu\ A(x,z)[\hat{P}_\mu + \hat{P}^\dagger_\mu]B(z,x) \tag{4.155}$$

The **causal** Green function—also as solution of the Klein-Gordon equation,

$$i\Delta(x-y) = \frac{1}{(2\pi)^3}\int d^4k\ sign(k_0)\delta(k^2-m^2)\ e^{ik(x-y)} \sim \langle x_\mu|\ sign(\mathbf{P})\delta(\mathbf{P^2}-\mathbf{a^2})\ |y_\mu\rangle \tag{4.156}$$

We also saw earlier that this "signed" amplitude does satisfy the Klein-Gordon composition law.

The **Feynman** amplitude,

$$i\Delta_F(x-y) = \frac{1}{(2\pi)^4}\int d^4k\; \frac{e^{-ik(x-y)}}{k^2-m^2+i\epsilon} = \frac{-i}{(2\pi)^4}\int_0^\infty d\lambda \int d^4k\; e^{-ik(x-y)-\lambda(k^2-m^2+i\epsilon)} \tag{4.157}$$

$$\sim \langle x_\mu|\; \frac{1}{\hat{\Phi}+i\epsilon}\; |y_\mu\rangle$$

which, again, satisfies the Klein-Gordon compostion law,

The **Wightman** functions

$$G^{\pm}(x,y) = \frac{1}{(2\pi)^3}\int d^4k\; \theta(\pm k_0)\; \delta(k^2-m^2)\; e^{ik(x-y)} \tag{4.158}$$

which obey the Klein-Gordon composition laws.

Finally we have the **Newton-Wigner propagators**, which are not manifestly covariant objects

$$G^{\pm}_{NW}(x,y) = \frac{1}{(2\pi)^3}\int d^4k\; \theta(\pm k_0)\; k_0\delta(k^2-m^2)\; e^{ik(x-y)} \tag{4.159}$$

and which satisfy the non-relativistic compostion law instead of the relativistic one.

We have deduced in a simple way the composition laws of these amplitudes in the physical space as well as their path integral representations in the extended space:

$$A = \int dqdpdQdP\; \delta(\Phi_0)\prod K_{i\Phi} e^{i\int d\tau(p\dot{q}+P\dot{Q}-h(q,p))} \tag{4.160}$$

Notice also that we can write

$$\langle t_f, x_f|\delta(\hat{\Phi})|t_i x_i\rangle = \langle t_f, x_f, \pi\!=\!0|e^{i\hat{\lambda}\hat{\Phi}}|t_i x_i, \pi\!=\!0\rangle \tag{4.161}$$

by adding this new degree of freedom.

Again, it is easy to write the path integrals in all these cases. *We obtain, for the "decoupling" cases the Faddeev path integrals discussed in the next chapter, as well as the BFV path integral—as we will show.*

How about the composition laws for these amplitudes? If we have the amplitudes above and the correponding nice projectors we have, schematically,

$$\langle t_f, x_f|\delta(\hat{\Phi})|t_i x_i\rangle = \langle t_f, x_f|\mathbf{K_\Phi}\delta(\hat{\Phi})|t_i x_i\rangle = \tag{4.162}$$

$$\int d^4x \langle t_f, x_f|\mathbf{K_\Phi}|t, x\rangle\langle t, x|\delta(\hat{\Phi})|t_i x_i\rangle = \int d^4x \langle t_f, x_f|\delta(\hat{\Phi})|t, x\rangle \mathbf{O}\langle t, x|\delta(\hat{\Phi})|t_i x_i\rangle \tag{4.163}$$

for some differential operator $\mathbf{O}$. It is not very hard to find these composition laws. For example, for the "signed" case the compostion law operator is simply $-i\partial_t$, and the Feynman amplitude satisfies a composition law of the "Klein-Gordon" form.

Let me remark once again, that my construction applies only to the decoupling case. If there are interesting interactions my projector formalism may noy be applicable, especially in the on-shell situation. It may be possible to extend these ideas to the case of the causal projectors.

4.3 The Fock space inner product.

The state space is defined in terms of the raising and lowering operators, just as for the harmonic oscillator case. There is a crucial difference: the commutation relations differ by a sign, and they induce an indefinite inner product (the definition of inner product is inherent in the algebra, since one discusses the commutation relations of the *hermitean* conjugates of operators.) Recall that the algebra is given by the definitions

$$\hat{a} = \hat{P}_1 + i\hat{P}_2, \;\; \hat{a}^\dagger = \hat{P}_1 - i\hat{P}_2 \tag{4.164}$$

and

$$\hat{b} = -\frac{i}{2}(\hat{Q}^1 + i\hat{Q}^2), \;\; \hat{b}^\dagger = \frac{i}{2}(\hat{Q}^1 - i\hat{Q}^2) \tag{4.165}$$

As explained in reference [2], one needs an even number of constraints. This is similar to what we usually do in BRST when we add a constrained multiplier degree of freedom for each of the constraints.

The commutation relations that follow from this definition are

$$[\hat{a}, \hat{b}^\dagger] = [\hat{b}, \hat{a}^\dagger] = 1 \tag{4.166}$$

and the rest zero.

It is implied by the notation here that both $\hat{P}_1$ and $\hat{P}_2$ are hermitean. For example, $\hat{a} + \hat{a}^\dagger$ is hermitean, and is equal to $2\hat{P}_1$. This fact is crucial for the development of the formalism, and is a subtle assumption—it selects an indefinite inner product when we define the vacuum.

We discussed in the previous chapter that the states on this space are defined by, a) it is assummed that there is a "vacuum" state, $|0\rangle$, satifying the conditions $\hat{a}|0\rangle = \hat{b}|0\rangle = 0$. This state is also assumed to have unit norm, $\langle 0|0\rangle = 1$, and b), the rest of these states are defined by acting on the "vacuum" above with the creation operators.

Recall also that this vacuum was a puzzling object: we need a state that satisfies

$$(\hat{P}_1 + i\hat{P}_2)|0\rangle = -\frac{i}{2}(\hat{Q}^1 + i\hat{Q}^2)|0\rangle = 0 \tag{4.167}$$

Can we understand what is going on in more pedestrian terms? Following Marnelius' work [23] let us map this representation into coordinate space. For that let us map $\hat{Q}^2 \to i\hat{Q}^2$, $\hat{P}_2 \to -i\hat{P}_2$. This is a canonical transformation. However, you may say that in this representation these operators are not hermitean—but they will be provided we fix the inner product. The inner product has to become indefinite! Consider the mixed representation in which we write the states in the form $\varphi = \varphi(Q^1, P_2)$. The vacuum is defined by

$$(\hat{P}_1 + \hat{P}_2)\varphi_0(Q^1, P_2) = (-i\frac{\partial}{\partial Q^1} + P_2)\varphi_0(Q^1, P_2) = 0 \tag{4.168}$$

This is solved by

$$\varphi_0(Q^1, P_2) = e^{-iP_2Q^1} \tag{4.169}$$

This is already the vacuum: it automatically satisfies

$$(\hat{Q}^1 - \hat{Q}^2)\varphi_0(Q^1, P_2) = (Q^1 - i\frac{\partial}{\partial P_2})\varphi_0(Q^1, P_2) = 0 \tag{4.170}$$

What do we use for inner product? Let us go back to the description in terms of the creation and annihilation operators, and then proceed. In fact, I will now give a description of the Fock space quantization and inner product in terms of:

a) The creation and annihilation operators

b) The holomorphic representation—continuum, and

c) q and p space

As I already discussed, we can represent any state by the holomorphic function of the creation operators that acting on the vacuum produces the desired state. Consider for example the eigenstates of the destruction operators (not normalized yet),

$$|\psi_{\alpha\beta}\rangle = e^{\alpha\hat{b}^\dagger+\beta\hat{a}^\dagger}\;|0\rangle \sim |\hat{a}\!=\!\alpha,\;\hat{b}\!=\!\beta\rangle \tag{4.171}$$

The computation of the inner product of these states is straightforward:

$$\langle\psi_{\alpha\beta}|\psi_{\alpha'\beta'}\rangle = \langle 0|e^{\alpha^*\hat{b}+\beta^*\hat{a}}\;e^{\alpha'\hat{b}^\dagger+\beta'\hat{a}^\dagger}\;|0\rangle = \tag{4.172}$$

$$\langle 0|e^{\alpha^*\hat{b}}e^{\beta'\hat{a}^\dagger}\;e^{\beta^*\hat{a}}e^{\alpha'\hat{b}^\dagger}\;|0\rangle = \tag{4.173}$$

$$\langle 0|\left([e^{\alpha^*\hat{b}},e^{\beta'\hat{a}^\dagger}]+1\right)\left([e^{\beta^*\hat{a}},e^{\alpha'\hat{b}^\dagger}]+1\right)|0\rangle \tag{4.174}$$

hence we need to compute

$$\langle 0|[e^{\alpha^*\hat{b}},e^{\beta'\hat{a}^\dagger}]|0\rangle = \langle 0|\sum_{n=0}^{\infty}\frac{\alpha^{*n}}{n!}[\hat{b}^n,e^{\beta'\hat{a}^\dagger}]|0\rangle = \tag{4.175}$$

$$\langle 0|\sum_{n=1}^{\infty}\frac{(\alpha^*\beta')^n}{n!}|0\rangle = e^{\alpha^*\beta'}-1 \tag{4.176}$$

since

$$\langle 0|[\hat{b}^n, f(\hat{a}^\dagger)]|0\rangle = \langle 0|\frac{d^n}{d\hat{a}^{\dagger n}} f(\hat{a}^\dagger)|0\rangle = \frac{d^n}{dx^n} f(x)|_{x=0} \tag{4.177}$$

The result is then

$$\langle \psi_{\alpha\beta}|\psi_{\alpha'\beta'}\rangle = e^{\alpha^*\beta' + \beta^*\alpha'} \tag{4.178}$$

Notice that the spectrum is given by the whole complex plane! This is not surprising, since we are dealing with the spectrum of *normal* operators. Other useful facts are

$$\langle 0|[\hat{a}^n, \hat{b}^{\dagger n}]\,|0\rangle = n! \tag{4.179}$$

and

$$\begin{aligned} &\| \hat{a}^{\dagger n}\hat{b}^{\dagger n} \| \equiv \langle 0|\hat{a}^n \hat{b}^n \hat{b}^{\dagger n} \hat{a}^{\dagger n}\,|0\rangle = \\ &\langle 0|[\hat{a}^n, \hat{b}^{\dagger n}][\hat{b}^n, \hat{a}^{\dagger n}]|0\rangle = (n!)^2 \end{aligned} \tag{4.180}$$

The more general statement is that

$$\langle 0|\hat{a}^{n_1}\hat{b}^{n_2}\hat{a}^{\dagger n_3}\hat{b}^{\dagger n_4}|0\rangle = (n_1)!(n_2)!\delta_{n_1 n_4}\delta_{n_2 n_3} \tag{4.181}$$

Now, for

$$|\psi\rangle \sim \sum c_{nm}\hat{a}^{\dagger n}\hat{b}^{\dagger m} = \psi(\hat{a}^\dagger, \hat{b}^\dagger) \tag{4.182}$$

we have

$$\| \psi \| = \langle 0| \sum c^*_{n_1 n_2}\hat{a}^{n_1}\hat{b}^{n_2} c_{n_3 n_4}\hat{a}^{\dagger n_3}\hat{b}^{\dagger n_4}|0\rangle = \tag{4.183}$$

$$\sum c^*_{mn}c_{nm}n!m! = \sum \frac{1}{n!m!}\,[\partial^n_{a^\dagger}\partial^m_{b^\dagger}\psi(\hat{a}^\dagger, \hat{b}^\dagger)]\,[\partial^m_{a^\dagger}\partial^n_{b^\dagger}\psi(\hat{a}^\dagger, \hat{b}^\dagger)]^*\Big|_{a^\dagger = b^\dagger = 0} \tag{4.184}$$

Consider next the eigenvalue equation

$$\hat{Q}_2\psi_\alpha = (\hat{b} + \hat{b}^\dagger)\psi_\alpha = \frac{\partial\psi_\alpha}{\partial a^\dagger} + b^\dagger\psi_\alpha \equiv \alpha\psi_\alpha \tag{4.185}$$

It is solved by

$$\psi_\alpha = A\, e^{-a^\dagger(b^\dagger-\alpha)} \tag{4.186}$$

where A is an arbitrary function of $b^\dagger$. Without loss of generality we choose A such that the resulting state is an eigenfunction of $\hat{P}_1 = (\hat{a} + \hat{a}^\dagger)/2$. The resulting eigenvalue equation is easily solved by

$$\psi_{Q^2 P_1} = N e^{-(a^\dagger - P_1)(b^\dagger - Q^2)} \tag{4.187}$$

These are the candidates. But we have to check that they are normalizable. It is hard to compute the inner product using the formula above. It will be interesting to try, though. We can compute

$$(\frac{d}{da^\dagger})^n \psi_{Q^2 P_1} = (-(b^\dagger - Q_2))^n \psi_{Q^2 P_1} \tag{4.188}$$

easily enough. Next, one can check[12] that

$$(\frac{d}{db^\dagger})^m (\frac{d}{da^\dagger})^n \psi_{Q^2 P_1} = \psi_{Q^2 P_1} (\frac{d}{db^\dagger} - (a^\dagger - P_1))^m (-(b^\dagger - Q_2))^n \tag{4.189}$$

which at zero becomes

$$(\frac{d}{db^\dagger})^m (\frac{d}{da^\dagger})^n \psi_{Q^2 P_1}\Big|_{a^\dagger = b^\dagger = 0} = N[(\frac{-d}{db^\dagger} + P_1)^m (-(b^\dagger - Q_2))^n]\Big|_{b^\dagger=0} = \tag{4.190}$$

$$N(\frac{d}{dQ_2} + P_1)^m (Q_2)^n = N e^{-P_1 Q_2} (\frac{d}{dQ_2})^m [e^{P_1 Q_2} Q_2^n] \tag{4.191}$$

since it can be easily checked that

$$(\frac{d}{dx} + P)^m [f(x)] = e^{-xP} (\frac{d}{dx})^m [e^{xP} f(x)] \tag{4.192}$$

[12] Because $d_x^n[g(x)e^{ax}] = e^{ax}(d_x + a)^n g(x)$.

(see the last footnote). So, finally, we have

$$(\frac{d}{db^\dagger})^m(\frac{d}{da^\dagger})^n\psi_{Q^2P_1}\Big|_{a^\dagger=b^\dagger=0} = Ne^{-P_1Q_2}(\frac{d}{dQ_2})^m(\frac{d}{dP_1})^n[e^{P_1Q_2}] \tag{4.193}$$

The inner product is hence given by

$$\parallel \psi \parallel = \sum c^*_{mn}c_{nm}n!m! = \sum \frac{1}{n!m!}\,[\partial^n_{a^\dagger}\partial^m_{b^\dagger}\psi(\hat{a}^\dagger,\hat{b}^\dagger)]\,[\partial^m_{a^\dagger}\partial^n_{b^\dagger}\psi(\hat{a}^\dagger,\hat{b}^\dagger)]^*\Big|_{a^\dagger=b^\dagger=0} = \tag{4.194}$$

$$\sum\frac{1}{n!m!}\left(Ne^{-P_1Q_2}(\frac{d}{dQ_2})^n(\frac{d}{dP_1})^m[e^{P_1Q_2}]\right)^* Ne^{-P_1Q_2}(\frac{d}{dQ_2})^m(\frac{d}{dP_1})^n[e^{P_1Q_2}] \tag{4.195}$$

and more generally, $(\psi_{Q_2P_1},\psi_{Q_2'P_1'})=$

$$\sum\frac{1}{n!m!}\left(Ne^{-P_1Q_2}(\frac{d}{dQ_2})^n(\frac{d}{dP_1})^m[e^{P_1Q_2}]\right)^* Ne^{-P_1'Q_2'}(\frac{d}{dQ_2'})^m(\frac{d}{dP_1'})^n[e^{P_1'Q_2'}] \tag{4.196}$$

We will now try a different approach. Let us ask the following question: *can we come up with an inner product in the a, b etc. space that matches the above inner product?* We also want the algebra to be respected, which for us now means that the hermiticity properties of the operators must be preserved. The commutator algebra is already respected by the representation in which the destruction operators become simple derivatives and the creation operators act by multiplication. *The answer is yes.* This inner product is given by

$$(\psi_k(a^*,b^*),\psi_l(a^*,b^*)) \equiv \int\frac{da db da^* db^*}{(2\pi i)^2}e^{-aa^*-bb^*}\psi^*_k(a^*,b^*)\psi_l(b^*,a^*) \tag{4.197}$$

where here I use the notation in which the operator $\psi_k(a^\dagger,b^\dagger)$ becomes in this representation the state $\psi_k(a^*,b^*)$. Notice the swap: the state $\psi_l(a^*,b^*)$ becomes $\psi_l(b^*,a^*)$

in the integral. To calculate this integration it is convenient to go to the variables $a = r_a e^{i\theta_a}$, $a^* = r_a e^{-i\theta_a}$ and $b = r_b e^{i\theta_b}$, $b^* = r_b e^{-i\theta_b}$. Then it is easy to see that

$$\frac{da db da^* db^*}{(2\pi i)^2} = \frac{1}{\pi^2} r_a dr_a d\theta_a r_b dr_b d\theta_b \tag{4.198}$$

This expression has an easy interpretation in terms of the eigenstates of the destruction operators

$$|\psi_{\alpha\beta}\rangle = e^{\alpha\hat{b}^\dagger + \beta\hat{a}^\dagger}|0\rangle \equiv |\hat{a}{=}\alpha,\ \hat{b}{=}\beta\rangle \tag{4.199}$$

if one uses them in a decomposition of unity as follows,

$$\hat{I} = \int \frac{d\alpha d\alpha^* d\beta d\beta^*}{(2\pi i)^2} |\alpha\rangle_{\hat{a}} \otimes |\beta\rangle_{\hat{b}}\ {}_{\hat{a}}\langle\beta^*| \otimes\ {}_{\hat{b}}\langle\alpha^*|\, e^{-\alpha\alpha^* - \beta\beta^*} \tag{4.200}$$

For one, one can check that the inner product between these states is the same as was calculated before. What really matters, though, is that we have a true representation of the above Hilbert space. We can see this in two ways. It is enough to check that the vacuum has unit norm and that the algebra and hermiticity properties are preserved. Or we can simply check that

$$\langle 0|\hat{a}^{n_1}\hat{b}^{n_2}\hat{a}^{\dagger n_3}\hat{b}^{\dagger n_4}|0\rangle = (n_1)!(n_2)!\delta_{n_1 n_4}\delta_{n_2 n_3} = (a^{*n_1}b^{*n_2}, a^{*n_3}b^{*n_4}) \tag{4.201}$$

as defined above. This is true, indeed, as follows from the following model calculation: $(a^{*n}, b^{*m}) \sim$

$$\int \frac{da da^*}{2\pi i} a^n (a^*)^m e^{-aa^*} = \frac{1}{\pi}\int_0^\infty r_a dr_a \int_0^{2\pi} d\theta_a r_a^{n+m} e^{i\theta_a(n-m)} e^{-r_a^2} = \delta_{nm} n! \tag{4.202}$$

(see Faddeev and Slavnov's book, in references [18].) As a check, we can compute the earlier inner product, $(\psi_{Q_2 P_1}, \psi_{Q_2' P_1'}) =$

$$\sum \frac{1}{n!m!} \left(N e^{-P_1 Q_2} (\frac{d}{dQ_2})^n (\frac{d}{dP_1})^m [e^{P_1 Q_2}] \right)^* N e^{-P_1' Q_2'} (\frac{d}{dQ_2'})^m (\frac{d}{dP_1'})^n [e^{P_1' Q_2'}] = \tag{4.203}$$

$$\int \frac{dadbda^*db^*}{(2\pi i)^2} e^{-aa^*-bb^*} N^* e^{-(a-P_1^*)(b-Q^{*2})} N e^{-(b^*-P_1)(a^*-Q^2)} = \tag{4.204}$$

$$\frac{|N|^2}{\pi^2} \int r_a dr_a d\theta_a r_b dr_b d\theta_b e^{-r_a^2 - r_b^2 - 2r_a r_b cos(\theta_a+\theta_b) + r_a(e^{i\theta_a} y^* + e^{-i\theta_a} x) + r_b(e^{i\theta_b} x^* + e^{-i\theta_b} y) - 2Re[xy]} \tag{4.205}$$

a difficult integral. Let us try instead yet another approach: *we can try to map the above situation into the original coordinate system.* Recall that the algebra above was given by the definitions

$$\hat{a} = \hat{P}_1 + i\hat{P}_2, \;\; \hat{a}^\dagger = \hat{P}_1 - i\hat{P}_2 \tag{4.206}$$

and

$$\hat{b} = -\frac{i}{2}(\hat{Q}^1 + i\hat{Q}^2), \;\; \hat{b}^\dagger = \frac{i}{2}(\hat{Q}^1 - i\hat{Q}^2). \tag{4.207}$$

Consider the mixed representation in which we write the states in the form $\varphi = \varphi(Q^1, P_2)$. The vacuum is defined by

$$(\hat{P}_1 + i\hat{P}_2)\varphi_0(Q^1, P_2) = (i\frac{\partial}{\partial Q^1} + iP_2)\varphi_0(Q^1, P_2) = 0 \tag{4.208}$$

This is solved by

$$\varphi_0(Q^1, P_2) = e^{-P_2 Q^1} \tag{4.209}$$

Can we come up with an inner product here that respects the algebra, the hermiticity properties of the operators and that gives unit norm to the vacuum? The answer is yes once more. The inner product is given by

$$(\psi_a, \psi_b) \equiv \int_{-i\varepsilon}^{i\infty} dQ^1 \int_{-\infty-i\epsilon}^{\infty-i\epsilon} dP_2 [\psi_a(Q_1^*, P_2^*)]^* \psi_b(Q_1, P_2) \tag{4.210}$$

First notice that the vacuum is normalizable to unity:

$$\begin{gathered} \parallel \psi_0 \parallel = \int_{-i\varepsilon}^{i\infty} dQ^1 \int_{-\infty-i\epsilon}^{\infty-i\epsilon} dP_2 e^{-2Q_1 P_2} = \int_{-\varepsilon}^{\infty} idq \int_{-\infty}^{\infty} dp e^{-2iq(p-i\epsilon)} = \\ \int_{-\varepsilon}^{\infty} idq \delta(2q) e^{-2q\epsilon} = i\pi \end{gathered} \tag{4.211}$$

or

$$\begin{gathered} = i \int_{-\infty}^{\infty} dp \frac{1}{-2(ip+\epsilon)} \, e^{-2iqp-2q\epsilon} \Big|_{-\varepsilon}^{\infty} = \\ 2\pi i \int_{-\infty}^{\infty} dp \frac{1}{2(ip+\epsilon)} e^{2i\varepsilon p + 2\varepsilon\epsilon} = i\pi \end{gathered} \tag{4.212}$$

It is very important to check that the operators are hermitean. This could be troublesome, for example, for the operator

$$\hat{Q}_2 \sim i\partial_{P_2} \tag{4.213}$$

Indeed, for the case of the vacuum we need, for hermiticity, that the boundary term

$$\int_{-i\varepsilon}^{i\infty} dQ^1 \int_{-\infty-i\epsilon}^{\infty-i\epsilon} dP_2 i\partial_{P_2} e^{-2Q_1 P_2} \tag{4.214}$$

vanish, and it does,

$$= -\int_{-\varepsilon}^{\infty} dq \int_{-\infty}^{\infty} dp \partial_p e^{-2iq(p-i\epsilon)} =$$
$$-\int_{-\infty}^{\infty} dp \partial_p \frac{1}{-2(ip+\epsilon)} \; e^{-2iqp-2q\epsilon}\Big|_{-\varepsilon}^{\infty} = \tag{4.215}$$
$$-\int_{-\infty}^{\infty} dp \partial_p \frac{1}{2(ip+\epsilon)} e^{2i\varepsilon p+2\varepsilon\epsilon} = 0$$

It is easily checked that $\hat{P}_1$ and $\hat{Q}_1$, $\hat{P}_2$ are also hermitean in this inner product, as they should. So we have yet another representation.

How unique is this inner product definition? Can we deform the paths? The actual path definition should reflect the spectra of the operators. If we can deform the paths, then we should be able to also use different spectra. Is this so? For one thing, in the above definition one can exchange the paths for Q^1 and P_2. This will still work, as the vacuum is symmetric in these variables.

Let us return to the issue of the spectra of the operators. Can we produce a working resolution of the identity? Yes, it is given by

$$\hat{I} = \int_{-i\varepsilon}^{i\infty} dQ^1 \int_{-\infty-i\epsilon}^{\infty-i\epsilon} dP_2 |Q^{1*} P_2^*\rangle\langle Q^1 P_2| \tag{4.216}$$

since

$$\langle\psi|Q^1 P_2\rangle = \Big(\langle Q^{1*} P_{2*}|\psi\rangle\Big)^* = (\psi(Q^{1*}, P_2^*))^* \;\; \langle Q^1 P_2|\psi\rangle = \psi(Q^1 P_2) \tag{4.217}$$

Let us investigate some more this algebra. First, let us exchange Q_1 and P_1, taking care that the algebra is preserved. This is just a matter of covenience. Let us write

$$\hat{A} = (\hat{Q}^1 - i\hat{P}_2)/\sqrt{2}, \ \ \hat{A}^\dagger = (\hat{Q}^1 + i\hat{P}_2)/\sqrt{2} \tag{4.218}$$

and

$$\hat{B} = (i\hat{P}_1 - \hat{Q}^2)/\sqrt{2}, \ \ \hat{B}^\dagger = (-i\hat{P}_1 - \hat{Q}^2)/\sqrt{2}. \tag{4.219}$$

The algebra that follows is as before,

$$[\hat{A}, \hat{B}^\dagger] = [\hat{B}, \hat{A}^\dagger]^\dagger = 1 \tag{4.220}$$

To understand the properties of this algebra let us define now

$$\hat{\mathcal{Q}}_- = (\hat{A} - \hat{B})/\sqrt{2} \tag{4.221}$$

and

$$\hat{\mathcal{Q}}_+ = (\hat{A} + \hat{B})/\sqrt{2} \tag{4.222}$$

These have the nice properties

$$[\hat{\mathcal{Q}}_-, \hat{\mathcal{Q}}_-^\dagger] = -1 \tag{4.223}$$

and

$$[\hat{\mathcal{Q}}_+, \hat{\mathcal{Q}}_+^\dagger] = +1 \tag{4.224}$$

Now, the fact that one oscillator comes defined with a minus sign doesn't carry much meaning by itself (imagine exchanging the notation for creation and annihilation operators in the usual oscillator case). What really matters, is that the vacuum as defined above corresponds here to requiring

$$\hat{\mathcal{Q}}_+ |0\rangle = 0 \tag{4.225}$$

and

$$\hat{\mathcal{Q}}_-|0\rangle = 0 \tag{4.226}$$

In the standard case the second equation corresponds to asking that the creation operator have a zero eigenstate. This state did not exist in your quantum mechanics class because although the differential equation has a solution in the coordinate representation[13], *it is not normalizable with the standard inner product.* What this means for us is that we are going to invent a new inner product—not a positive definite one, as is easily seen by "creating" with the operator $\hat{\mathcal{Q}}_-^\dagger$. At any rate, we can further define

$$\hat{z}_+ = \frac{\hat{Q}^1 - \hat{Q}^2}{\sqrt{2}}\,, \quad \hat{P}_{z_+} = \frac{\hat{P}_1 - \hat{P}_2}{\sqrt{2}} \tag{4.227}$$

and

$$\hat{z}_- = \frac{\hat{Q}^1 + \hat{Q}^2}{\sqrt{2}}\,, \quad \hat{P}_{z_-} = \frac{\hat{P}_1 + \hat{P}_2}{\sqrt{2}} \tag{4.228}$$

Notice that the pairs $\hat{z}_+\,,\ \hat{P}_{z_+}$ are canonically conjugated—and the "+" and "−" sides decouple. Then we have

$$\hat{\mathcal{Q}}_+ = \frac{1}{\sqrt{2}}(\hat{z}_+ + i\hat{P}_{z_+}) \quad \hat{\mathcal{Q}}_- = \frac{1}{\sqrt{2}}(\hat{z}_- - i\hat{P}_{z_-}) \tag{4.229}$$

Everything decoupled! It is now easy to write the coordinate expression for the vacuum and the inner products:

—The "+" side: $[\hat{\mathcal{Q}}_+, \hat{\mathcal{Q}}_+^\dagger] = +1$

[13] $\varphi(x) \sim exp(x^2/2)$

We have $\hat{\mathcal{Q}}_+ = (\hat{z}_+ + i\hat{P}_{z_+})/\sqrt{2}$, $\hat{\mathcal{Q}}_+^\dagger = (\hat{z}_+ - i\hat{P}_{z_+})/\sqrt{2}$- -the standard Hilbert space description.

The vacuum is given by

$$\varphi_{0_+}(z_+) = \langle z_+|0_+\rangle = \pi^{-\frac{1}{4}} e^{-(z_+)^2/2} \tag{4.230}$$

where the inner product is given by

$$\Big(\varphi_+(z_+),\ \varphi'_+(z_+)\Big) = \int_{-\infty}^{\infty} dz_+\, \varphi^*_+(z_+)\ \ \varphi'_+(z_+) \tag{4.231}$$

—The "−" side: $[\hat{\mathcal{Q}}_-, \hat{\mathcal{Q}}_-^\dagger] = -1$

We have $\hat{\mathcal{Q}}_- = (\hat{z}_- - i\hat{P}_{z_-})/\sqrt{2}$, $\hat{\mathcal{Q}}_-^\dagger = (\hat{z}_- + i\hat{P}_{z_-})/\sqrt{2}$—*not* the standard Hilbert space description.

The vacuum is given by

$$\varphi_{0_-}(z_-) = \langle z_-|0_-\rangle = \pi^{-\frac{1}{4}} e^{+(z_-)^2/2} \tag{4.232}$$

and is normalized to one. The inner product is now given by

$$\Big(\varphi_-(z_-),\ \varphi'_-(z_-)\Big) = \int_{-i\infty}^{i\infty} dz_-\, \varphi^*_-(z^*_-)\ \ \varphi'_-(z_-) \tag{4.233}$$

The full vacuum is given by

$$\psi_0 = \pi^{-\frac{1}{2}} e^{\frac{-(z_+)^2+(z_-)^2}{2}} = \pi^{-\frac{1}{2}} e^{2Q^1Q^2} \tag{4.234}$$

and the full inner product by

$$(\psi,\ \psi') = \int_{-\infty}^{\infty} dz_+ \int_{-i\infty}^{i\infty} dz_-\ \ \psi(z^*_+, z^*_-)^*\ \ \psi'(z_+, z_-) = \tag{4.235}$$

$$\int dQ^1 \int dQ^2 \;\; \psi(z_+^*(Q^1,Q^2), z_-^*(Q^1,Q^2))^* \;\; \psi'(z_+(Q^1,Q^2), z_-(Q^1,Q^2)) \tag{4.236}$$

where $Q^{1*} = Q^2$.

The next step is to impose the condition

$$\hat{a}|\psi\rangle = 0 \tag{4.237}$$

this is the condition that defines physical states. This yields the vacuum described above plus null states, as we discuss next.

The next factor in this formalism is that we have the so-called null states. Any state of the form $\hat{a}^\dagger|any\rangle$ decouples from any physical state. This situation is remarkably similar to the one that we have in the BRST formalism. They are both characterized by a definition of cohomology: the states are defined by the kernel of some operator. In the BRST case the kernel is nilpotent, so the cohomology is proper,

$$H(\hat{\Omega}) \equiv \frac{Ker\ \hat{\Omega}}{Im\ \hat{\Omega}} \tag{4.238}$$

For the Fock space case we don't have a nilpotent operator. The "cohomology" is defined by

$$H(Fock) \equiv \frac{Ker\ \hat{a}}{Im\ \hat{a}^\dagger} \tag{4.239}$$

The similarities between these two formalisms also include the fact that in both cases the number of constraints is even.

4.3.1 The particle in Fock space

Consider first the case of a constraint of the form[14] $\mathbf{\Phi} = \mathbf{p_t} + \mathbf{A}(\mathbf{x}, \mathbf{p_x}) \approx \mathbf{0}$. This case covers the non-relativistic particle as well as the relativistic case when we choose a branch (i.e., the square root hamiltonian situation). We will examine the full relativistic constraint in a moment.

The physical states are defined by

$$(\mathbf{a} + \mathbf{A})|\Psi\rangle_{Ph} = 0, \tag{4.240}$$

where $\mathbf{a} = \mathbf{p_t} + i\boldsymbol{\pi}$ and—for the non-relativistic case we have

$$\mathbf{A} = \frac{\mathbf{p_x^2}}{2m} \tag{4.241}$$

and for the relativistic one

$$\mathbf{A} = \sqrt{\mathbf{p}_x^2 + m^2} \tag{4.242}$$

The following reasoning can be carried out also when there is an electromagnetic background with no electric field. In such a case we would define $\mathbf{a} = \mathbf{\Pi_t} + i\boldsymbol{\pi}$, which preserves the commutator algebra—we keep the original $\mathbf{b}$ definitions—and $\mathbf{A}$ to be the square root hamiltonian. Indeed, it is vital that

$$[\mathbf{a}, \mathbf{A}] = \mathbf{0} \tag{4.243}$$

which holds when there is no electric field. We also have $[\mathbf{b}, \mathbf{A}] = \mathbf{0}$.

[14] See exercise 13.14 in Teitelboim's book [2]

This equation for the physical states is appropiate because, defining

$$\mathbf{M} = \mathbf{a} + \mathbf{A} \tag{4.244}$$

we have that

$$\mathbf{\Phi} = \mathbf{M} + \mathbf{M}^\dagger \tag{4.245}$$

Hence

$$_{Ph}\langle\Psi|\mathbf{\Phi}\,|\Psi\rangle_{Ph} = 0 \tag{4.246}$$

As we will see, another inportant property of our definition (which holds when there is no electric field) is that $\mathbf{M}$ is *normal*, i.e.,

$$[\mathbf{M}, \mathbf{M}^\dagger] = \mathbf{0} \tag{4.247}$$

The general solution to the physicality condition is given by (recall $[\mathbf{b}, \mathbf{A}] = 0$)

$$|\Psi\rangle_{Ph} = f(\mathbf{a}^\dagger)\, e^{-\mathbf{A}\mathbf{b}^\dagger}\;|0\rangle|f\rangle \tag{4.248}$$

with $f(\mathbf{a}^\dagger)$ totally arbitrary. However, we can rewrite this solution in the following more useful way

$$|\Psi\rangle_{Ph} = e^{-\mathbf{A}\mathbf{b}^\dagger}\;|0\rangle|f'\rangle + \sum_{n>0,\,f} \mathbf{M}^{\dagger\mathbf{n}}\, e^{-\mathbf{A}\mathbf{b}^\dagger}\;|0\rangle|f\rangle \tag{4.249}$$

simply because we can write an arbitrary function of x in terms of a power series around any point, $f(x) = f(x_0) + \sum_{n>1} c_n\,(x - x_0)^n$—and $[\mathbf{a}^\dagger, \mathbf{A}] = \mathbf{0}$. Now, *any state of the form* $\mathbf{M}^{\dagger\mathbf{n}}\,|phys\rangle$ *is clearly null, but it is also physical because* $\mathbf{M}$ *is normal (no electric field).*

Up to null states the physical states are thus given by

$$|\Psi\rangle_{Ph} = e^{-\mathbf{A}\mathbf{b}^\dagger}|0\rangle|f\rangle \tag{4.250}$$

which for the non-relativistic case, for example, are explicitely $\exp(-\frac{i}{2}(\hat{t} - i\hat{\lambda})\frac{\hat{p}_x^2}{2m})\ |0\rangle|f\rangle$. The null states are, as explained, are given by

$$(\mathbf{a}^\dagger + \mathbf{A})^k e^{-\mathbf{A}\mathbf{b}^\dagger}|0\rangle|g\rangle, \tag{4.251}$$

where k is an integer, $k \neq 0$.

As for the time evolution operator—and the dynamics—it is given by

$$\begin{aligned} &U =_{Ph} \langle x'|e^{i\mathbf{p_t}\Delta\tau}\ |x\rangle_{Ph}, \\ &|x\rangle_{Ph} = e^{-\mathbf{A}\mathbf{b}^\dagger}\ |0\rangle|x\rangle \end{aligned} \tag{4.252}$$

so

$$U =_{Ph} \langle x'|e^{-i\mathbf{A}\Delta\tau}\ |x\rangle_{Ph} \tag{4.253}$$

This, again, follows from

a) $\mathbf{p_t} + \mathbf{A} = \frac{1}{2}(\mathbf{a} + \mathbf{A} + (\mathbf{a} + \mathbf{A})^\dagger)$

b) $_{Ph}\langle\Psi'|(\mathbf{p_t} + \mathbf{A})\ |\Psi\rangle_{Ph} = 0 =_{Ph} \langle\Psi'|\mathbf{\Phi}\ |\Psi\rangle_{Ph}$

c) $_{Ph}\langle\Psi'|\mathbf{O_x}\ |\Psi\rangle_{Ph} = \langle f'|\mathbf{O_x}\ |f\rangle$.

Also, recall

$$e^{i\lambda\mathbf{\Phi}}\ |\Psi\rangle_{Ph} = |\Psi\rangle_{Ph} + |null\rangle \tag{4.254}$$

which is almost obvious.

To give another explicit result, consider the free relativistic case with a branch choice—it is not clear yet how to proceed if one doesn't do this first. As before, the physical states are defined by

$$(\mathbf{a}+\mathbf{A})|\Psi\rangle_{Ph}=0, \tag{4.255}$$

where

$$\mathbf{A}=\sqrt{\mathbf{p}_x^2+m^2},\ \ \mathbf{a}=\mathbf{p_t}+i\boldsymbol{\pi}, \tag{4.256}$$

etc., and up to null states they are given by

$$|\Psi\rangle_{Ph}=e^{-\mathbf{Ab}^\dagger}\ |0\rangle|f\rangle=e^{-\frac{i}{2}(\hat{t}-i\hat{\lambda})\sqrt{\hat{p}_x^2+m^2}}\ |0\rangle|f\rangle \tag{4.257}$$

The null states are

$$(\mathbf{a}^\dagger+\mathbf{A})^k e^{-\mathbf{Ab}^\dagger}\ |0\rangle|g\rangle, \tag{4.258}$$

where k is an integer, $k\neq 0$.

As pointed out, we can also consider the case of an electromagnetic field with no electric components.

These state spaces are thus isomorphic to the space of functions of the coordinate x—the old physical coordinates. This includes the resulting inner products.

We can also define the propagation amplitude to be

$$\langle t_f,x_f,\pi\!=\!0|e^{i\hat{\lambda}\hat{\Phi}}|t_ix_i,\pi\!=\!0\rangle=\langle t_f,x_f|\frac{1}{\hat{\Phi}+i\epsilon}|t_ix_i\rangle \tag{4.259}$$

This amplitude is the causal amplitude—leads to the Feynman propagator for the full relativistic case, for example.

This result follows from the discussion earlier on how to represent the Fock space in the coordinate basis.

Let us now consider the full constraint. As usual we will consider first an easier case,

$$\hat{\Phi} = \hat{P}_1^2 - \hat{A}^2 \tag{4.260}$$

where we assume that $\hat{A}$ is a hermitean operator that commutes with $\hat{a}, \hat{b}$—zero electric field, as before. The strategy will be as before: find a normal operator $\hat{M}$, $[\hat{M}, \hat{M}^\dagger] = 0$, and write the constraint as a sum of this operator and its hermitean conjugate,

$$\hat{\Phi} = \hat{M} + \hat{M}^\dagger \tag{4.261}$$

This will ensure two important things:

i) ${}_{Ph}\langle\Psi|\hat{\Phi}\,|\Psi\rangle_{Ph} = 0$

ii) The states $\hat{M}^{\dagger n}\,|phys\rangle = 0$, $n > 0$, are null *and* physical.

Let us write the constraint in terms of the new variables,

$$\hat{\Phi} = \hat{P}_1^2 - \hat{A}^2 = (\frac{\hat{a} + \hat{a}^\dagger}{2})^2 - \hat{A}^2 = \frac{1}{4}\left(\hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}\hat{a}^\dagger\right) - \hat{A}^2 \tag{4.262}$$

The natural definition is thus

$$\hat{M} = \frac{1}{4}\left(\hat{a}^2 + \hat{a}\hat{a}^\dagger\right) - \frac{1}{2}\hat{A}^2 \tag{4.263}$$

which satisfies the above requirements with our assumptions about $\hat{A}$.

To find the physical states we now need to solve the differential equation

$$[\left(\frac{\partial}{\partial b^*}\right)^2 + a^* \frac{\partial}{\partial b^*} - 2\hat{A}^2]\ \psi(a^*, b^*) = 0 \tag{4.264}$$

which is done easily enough

$$\psi(a^*, b^*) = g(a^*)\ \exp\left(\frac{-b^* a^*}{2} \pm \frac{b^*}{2}\sqrt{a^{*2} + 8\hat{A}^2}\right) \tag{4.265}$$

Next we need to discuss the null states. From the properties of $\hat{M}$ we know that the states

$$\hat{M}^{\dagger n}\ \exp\left(\frac{-b^* a^*}{2} \pm \frac{b^*}{2}\sqrt{a^{*2} + 8\hat{A}^2}\right) |0\rangle |f\rangle \tag{4.266}$$

are physical and null. Do these exhaust all the freedom from the function g in the previous equation? If so the physical space reduces to the usual two branches.

Notice that the operator $\hat{M}^\dagger$ has a zero mode,

$$\varphi = \exp\left(-b^*(a^* - 2A)\right) \tag{4.267}$$

Let us compute the effect of $\hat{M}^\dagger$ on a physical state—we know that we will get a physical state! Now $\hat{M}^\dagger |phys\rangle \sim$

$$(a^{*2} + a^* \frac{\partial}{\partial b} - 8\hat{A}^2)\ \exp\left(\frac{-b^* a^*}{2} \pm \frac{b^*}{2}\sqrt{a^{*2} + 8\hat{A}^2}\right) =$$

$$(\frac{a^{*2}}{2} - 8\hat{A}^2 \pm \frac{a^*}{2}\sqrt{a^{*2} + 8\hat{A}^2})\ \exp\left(\frac{-b^* a^*}{2} \pm \frac{b^*}{2}\sqrt{a^{*2} + 8\hat{A}^2}\right) \tag{4.268}$$

a physical state, as promised. Can we now find a function such that

$$F(\hat{M}^\dagger)\,|phys\rangle = \hat{a}^\dagger\ |phys\rangle\ ? \tag{4.269}$$

If so we are set. It is not hard to see that this question is equivalent to asking that the function

$$f_{\pm}(a^*) = \frac{a^{*2}}{2} - 8\hat{A}^2 \pm \frac{a^*}{2}\sqrt{a^{*2} + 8\hat{A}^2} \tag{4.270}$$

have an inverse. And indeed, it does,

$$a^* = \mp \frac{f_{\pm} + 8\hat{A}^2}{\sqrt{f_{\pm} + 2\hat{A} + 8\hat{A}^2}} \tag{4.271}$$

The function $f(a^*)$ is one-to-one (although not onto). Thus, the physical space is one in which we have a set of states for each branch, $|0_+\rangle \otimes |others\rangle + |0_-\rangle \otimes |others\rangle$.

What happens in the fully interacting case? Our decomposition of the constraint, $\hat{\Phi} = \hat{M} + \hat{M}^\dagger$ is still valid, but with the above choice, $\hat{M}$ is not normal. What this means is that the null states in the theory, $\hat{M}^{\dagger n}\,|phys\rangle$ are no longer physical. So we may end up with many more non-null physical states than we bargained for. This issue is an important one—what is the correct description of this space in general? I do not know the answer to this question yet.

4.4 BRST inner product and construction of the path integral

Let us now look at the BRST quantization approach—we already hinted above at how the inner product may look in this formalism. The discussion will become extremely formal at some points, but we will try to draw very "unformal" conclusions at the end. The trouble will always hinge around

a) The assumption of hermiticity of $\mathbf{\Omega}$

b) Operator ordering questions

We will address all these issues.

The physical space is defined by the condition that

$$\mathbf{\Omega}|\Psi\rangle = 0 \tag{4.272}$$

where recall that

$$\Omega = \eta_0 \Phi + \eta_1 \pi \tag{4.273}$$

Now, there are many ways to write the solution to this equation. A key property of the BRST generator is that $\mathbf{\Omega}^2 = 0$, so any state of the form $\mathbf{\Omega}|\Psi\rangle$ is physical. However, such a state has also zero inner product with any other physical state—since the BRST generator is by assumption hermitean[15], $\mathbf{\Omega} = \mathbf{\Omega}^\dagger$. So these states

[15] But remember that it is not hermitean with respect to the physical states.... This is indeed serious trouble for the cohomology idea. The rule is to always compute the commutators first—or first operate and then compute the inner product.

can be factored out—formally. It has been argued (see Henneaux's report in [19]) that a complete set of physical states is given by

$$|\Psi\rangle = |\psi\rangle + |\psi_0\rangle\eta_0 + |\psi_1\rangle\eta_1 + |\psi_{01}\rangle\eta_0\eta_1 \tag{4.274}$$

where all the kets satisfy $\mathbf{\Phi}, \boldsymbol{\pi} \, |\psi\rangle = 0$. We will examine this result. Also, notice that this is by no means *the* way to describe the physical space.

Consider for example the states

$$|\Psi_\chi\rangle = |\psi_{\chi=0}, \pi=\eta_0=\rho_1=0\rangle \tag{4.275}$$

These also satisfy the BRST condition—they are also physical.

These are the states that we will use in the BFV path integral—i.e these determine the boundary conditions we will use. There are many other choices for them, for example

$$|\Psi_\Phi\rangle = |\psi_{\Phi=0}, \lambda=\rho_0=\eta_1=0\rangle \tag{4.276}$$

What was the inner product in the full enlarged space? The original BRST definition of the coordinate spaces is infinitely ranged, and as such it runs into regularization problems. This inner product, when used, for example, in the zero ghost sector of the physical states described above, is just given by

$$(\psi_a, \psi_b) = \int dQdqd\eta_0 d\eta_1 d\lambda \, \psi_a^* \, \psi_b = \int dQd\eta_0 d\eta_1 d\lambda dp \int dq \, \psi_a^* \, \psi_b \tag{4.277}$$

since physical states do not depend on Q, λ. Now, the first part of the integration is where the regularization problems appear (we see, however, that these will go away if the coordinate spaces—including the ghosts'—are finite. In such a case we can define $\int d\eta\eta = 1$, $\int d\eta = 1/L$, so that the ghost normalizations cancel those of the gauge...or something like that!)

As was previously remarked, the BFV path integral can be interpreted as coming from a larger *quantum* space—with no constraints. The fundamental reason is that the multipliers—which "impose the constraints"—are dynamical and can be interpreted in the path integral as legitimate degrees of freedom. Indeed we showed how to obtain the physical amplitude from a full quantum space...the only place where physicality enters is in the boundary conditions. This is the only place where one feels uncomfortable (told me Claudio Teitelboim himself!), as at first sight the choice of these boundary conditions seems *ad hoc*. However, as we will now see, these boundary conditions are easily understood to arise from the required BRST invariance of the "end" states in the amplitude. The BRST boundary conditions do have an interpretation: they can be understood in the context of gauge-invariant states. States that implement such boundary conditions are indeed annihilated by the BRST generator.

Let us look a bit closer into the case where the constraint is $\Phi = P \approx 0$. The path integral is arguably where BRST in phase space started, so let us begin with that too. As mentioned, we can immediately write this path integral if we start from the extended quantum space, which is spanned by the states[16] $|x, \lambda, c, \bar{c}, q\rangle$ where the

[16]Again we remind ourselves that $\eta_0 \equiv c$, $\rho_0 \equiv \bar{\mathcal{P}}$, $\eta_1 \equiv -i\mathcal{P}$, and $\rho_1 \equiv i\bar{c}$.

q degree of freedom denotes the physical, non-gauge sector. Now the hamiltonian is given by

$$\mathcal{H} = h + \{\mathcal{O}, \Omega\} \tag{4.278}$$

where Ω is the BRST generator,

$$\Omega = \eta_0 P + \eta_1 \pi \tag{4.279}$$

formed by combining the two constraints with the ghosts, and where $\mathcal{O}$ is the "gauge-fixing" term.

Now, the propagation amplitude is given by

$$\langle \Psi_a | e^{-i\tau\hat{\mathcal{H}}} | \Psi_b \rangle = \int d\mu(Q) \; \Psi_a^*(Q) \; e^{-i\tau\hat{\mathcal{H}}_Q} \; \Psi_b(Q) \tag{4.280}$$

where the states are physical (boundary conditions).

Now, the BFV path integral provides us with a clue to the possible constructions of the extended Hilbert space. The first step is to look at the boundary conditions that we used in the path integral: $c, \bar{c}$ and π are to vanish at the boundaries. So this defines the physical states. Now, we can obtain the propagation amplitude by using as the propagation operator

$$\hat{\mathcal{U}} = e^{-i\Delta\tau\hat{\mathcal{H}}} \tag{4.281}$$

where the hamiltonian is the extended (super)hamiltonian: $\hat{\mathcal{H}} \equiv \hat{h} + \{\hat{O}, \hat{\Omega}\}$. In the particle cases, for example, $h = 0$. Then we can obtain the correct propagation amplitude from the expression

$$U(t_i, x_i, t_f, x_f) \equiv \langle t_f, x_f, c{=}\bar{c}{=}\pi{=}0|\, \hat{\mathcal{U}}\, |t_i, x_i, c{=}\bar{c}{=}\pi{=}0\rangle \tag{4.282}$$

Notice that our notation anticipates that this amplitude will not depend on τ, which will be the case. From this expression it is easy to obtain the BFV path integral by repeated insertion of the extended space resolutions of the identity

$$\mathbf{I} = \int dt dx d\pi dc d\bar{c}\, |t, x, \pi, c, \bar{c}\rangle\langle t, x, \pi, c, \bar{c}| = \tag{4.283}$$

$$\int dp_t dp_x d\lambda d\bar{\mathcal{P}} d\mathcal{P} |p_t, p_x, \lambda, \bar{\mathcal{P}}, \mathcal{P}\rangle\langle p_t, p_x, \lambda, \bar{\mathcal{P}}, \mathcal{P}|$$

and the projections

$$\langle t, x, \pi, c, \bar{c}|p_t, p_x, \lambda, \bar{\mathcal{P}}, \mathcal{P}\rangle = e^{i\left(tp_t + xp_x + \pi\lambda + c\bar{\mathcal{P}} + \bar{c}\mathcal{P}\right)} \tag{4.284}$$

just as in the unconstrained case.

Let us consider the following two types of gauge-fixing terms:

a) $\mathcal{O}_{NC} = \rho_1 f(\lambda) + \rho_0\lambda$ which yields

$$\{\mathcal{O}_{NC}, \Omega\} = \rho_1\eta_1 f'(\lambda) + \pi f(\lambda) + \lambda\Phi + \rho_0\eta_1 \tag{4.285}$$

b) $\mathcal{O}_C = \rho_1\chi + \rho_0\lambda$ with

$$\{\mathcal{O}_C, \Omega\} = \rho_1\eta_0\{\chi, \Phi\} + \pi\chi + \lambda\Phi + \rho_0\eta_1 \tag{4.286}$$

Notice that the expectation value of these operators on physical states—states annihilated by the BRST generator—would appear to be zero, so that formally

$$\langle\Psi_a|e^{[\hat{\mathcal{O}},\hat{\Omega}]}\,|\Psi_b\rangle = \langle\Psi_a|\Psi_b\rangle \tag{4.287}$$

up to hermiticity issues. So this is where the invariance of the amplitude under changes in gauge-fixing is expected to come from. However, it is crucial to have hermiticity of the BRST operator—and this in general we don't have (recall the discussion at the beginning regarding the Dirac states.) This also implies that the inner product for the case of zero physical hamiltonian is all there is to propagation! (recall, however, that parametrized systems are more problematic because of the action's lack of invariance at the boundaries). Let us see how this goes. For example, take the BRST invariant states $|\Psi_\Phi\rangle$ we defined above. Then[17]

$$\langle\Psi_\Phi|e^{[\hat{\mathcal{O}}_C,\hat{\Omega}]}\,|\Psi'_\Phi\rangle = \tag{4.288}$$

$$\langle\psi_{\Phi=0},\lambda=\rho_0=\eta_1=0|e^{i\left(\hat{\rho}_1\hat{\eta}_0\{\hat{\chi},\hat{\Phi}\}+\hat{\pi}\hat{\chi}+\hat{\lambda}\hat{\Phi}+\hat{\rho}_0\hat{\eta}_1\right)}|\psi'_{\Phi=0},\lambda=\rho_0=\eta_1=0\rangle$$

which, *up to ordering*, is

$$= \langle\psi_{\Phi=0}|\{\hat{\chi},\hat{\Phi}\}\delta(\hat{\chi})\,|\psi'_{\Phi=0}\rangle \tag{4.289}$$

i.e., the Dirac inner product. The mentioned ordering questions are tricky, and we will examine them further below—the Fradkin-Vilkovisky theorem will justify our guess.

At any rate, it is now easy to see now how to proceed to form the BFV path integral: we write

$$e^{[\hat{\mathcal{O}}_C,\hat{\Omega}]} = \lim_{N\to\infty}\left(1+\frac{1}{N}[\hat{\mathcal{O}}_C,\hat{\Omega}]\right)^N \tag{4.290}$$

[17] $i\{\ ,\ \}\sim[\ ,\ ]$

and we insert the resolutions of the identity, etc.

We can also consider the states $|\Psi_\chi\rangle$. These will connect easily with the amplitude we considered before in the Dirac quantization approach. Notice that up to boundary issues it is just as good to work with the constraint $Q \approx 0$. The theory at the end will be the same as using $P \approx 0$. The BRST formalism does not distinguish between these two cases. These states are more to the point, since they match the path integrals we will evaluate in the particle case. It is easy to see, again up to ordering difficulties, that in this case the amplitude becomes

$$\langle\Psi_\chi| e^{[\hat{\mathcal{O}}_C, \hat{\Omega}]} \, |\Psi'_\chi\rangle = \langle\psi_{\chi=0}|\delta(\hat{\Phi}) \, |\psi_{\chi'=0}\rangle \tag{4.291}$$

From this expression it is easy to reach the BFV path integral that we discussed before.

Finally, we can also consider the non-canonical gauge. For example, if we use the case $f(\lambda) = 0$ it is immediate that

$$\langle\Psi_\chi| e^{[\hat{\mathcal{O}}_{NC}, \hat{\Omega}]} \, |\Psi'_\chi\rangle = \langle\psi_{\chi=0}|\delta(\hat{\Phi}) \, |\psi_{\chi'=0}\rangle$$

The amplitude $\langle\Psi_\Phi| e^{[\hat{\mathcal{O}}_{NC}, \hat{\Omega}]} \, |\Psi'_\Phi\rangle$, on the other hand, appears to be undefined $(0 \cdot \infty)$.

Let us now discuss less formally the above issues: the state cohomology, the inner product, ordering problems, and finally, the path integral.

4.4.1 Analysis of Cohomology

Before we begin, is good to keep in mind that we have three worlds to compare:

i) Dirac; $\hat{\Phi}|\psi_D\rangle = 0$

ii) Fock; $\hat{a}|\psi_F\rangle = 0$

iii) BRST; $\hat{\Omega}|\Psi_{BRST}\rangle = (\hat{\eta}_0\hat{\Phi} + \hat{\eta}_1\hat{\pi})|\Psi_{BRST}\rangle = 0$

These are all different methods and the spaces under consideration are also different.

Now, solutions to *iii)* include

$$|\psi_D\rangle \otimes |\pi = 0\rangle \otimes |G\rangle, \quad |\psi\rangle \otimes |\pi = 0\rangle \otimes |\eta_0 = 0\rangle, \quad |\psi = 0\rangle \otimes |f(\pi)\rangle \otimes |\eta_1 = 0\rangle \tag{4.292}$$

and

$$|\psi\rangle \otimes |f(\pi)\rangle \otimes |\eta_0 = \eta_1 = 0\rangle \tag{4.293}$$

or any combination of these. Now, some of these may not be allowed. Or, they may differ by a state of the form

$$\hat{\Omega}|\Lambda\rangle \tag{4.294}$$

which—*if* $\hat{\Omega}$ is hermitean—is null. Consider the state—in the "coordinate" representation

$$\Psi_{\Omega=0} = \psi_{\Phi=\pi=0} + \psi^1_{\Phi=0}\eta_1 + \psi^0_{\pi=0}\eta_0 + \psi^{01}\eta_0\eta_1 \tag{4.295}$$

The standard trick is to argue that since[18] $\hat{\Omega}^2 = 0$ we must identify

$$\Psi \sim \Psi + \hat{\Omega}\Lambda \tag{4.296}$$

The reasoning is that the states $\hat{\Omega}\Lambda$ are physical (true) and that they decouple from other physical states. This, however, is not true in general. This is because in the

[18]This is trivially true in this simple case where there is no place for anomalies to appear in the transition $\{\Omega, \Omega\}_{PB} = 0 \longrightarrow [\mathbf{\Omega}, \mathbf{\Omega}] = 0$

physical space we don't have hermiticity of $\hat{\Omega}$ (with the exception of the bounded situation, which we ignore for now). That is, the above argument is based on the assumption that

$$\langle\Lambda|\hat{\Omega}^{\dagger}|\Psi_{\Omega=0}\rangle = 0 \tag{4.297}$$

This puts restrictions on what $|\Lambda\rangle$ can be.

The trouble spot can be traced to the fact that in general it isn't true that

$$\langle P=0|\mathbf{P}|Any\rangle = 0 \tag{4.298}$$

when for example $|Any\rangle = |P=0\rangle$—though this true in the bounded case.

For this reason it is incorrect to assume, for example, that one can always express a state in the form

$$\Psi_{\Omega=0} = \psi_{\Phi=0=\pi} + \psi^{1}_{\Phi=0=\pi}\eta_1 + \psi^{0}_{\Phi=0=\pi}\eta_0 + \psi^{01}_{\Phi=0=\pi}\eta_0\eta_1 \tag{4.299}$$

by a proper choice of $|\Lambda\rangle$. *Such asumption is not consistent with the existence of an inner product and/or hermicity of the BRST generator in an unbounded space.* Such a state, $|\Lambda\rangle$, does not exist that yields such a representation and such that $\hat{\Omega}|\Lambda\rangle$ is truly null. This can be seen in the following way: we need to find the general form of $|\Lambda\rangle$ such that

$$\langle\Psi_{\Omega=0}|\hat{\Omega}|\Lambda\rangle = 0 \tag{4.300}$$

Next we need to find the possible forms of the physical states that we can reduce from the now validated equivalence

$$|\Psi_{\Omega=0}\rangle \sim |\Psi_{\Omega=0}\rangle + \hat{\Omega}|\Lambda\rangle \tag{4.301}$$

Writing

$$|\Psi_{\Omega=0}\rangle \sim \psi_{\Phi=\pi=0} + \psi^1_{\Phi=0}\eta_1 + \psi^0_{\pi=0}\eta_0 + \psi^{01}\eta_0\eta_1 \tag{4.302}$$

and

$$|\Lambda\rangle \sim \Lambda + \Lambda^0\eta_0 + \Lambda^1\eta_1 + \Lambda^{01}\eta_0\eta_1 \tag{4.303}$$

we demand that the above inner product vanish: $0 = \langle\Psi_{\Omega=0}|\hat{\Omega}|\Lambda\rangle$

$$= \Big(\psi_{\Phi=\pi=0} + \psi^1_{\Phi=0}\eta_1 + \psi^0_{\pi=0}\eta_0 + \psi^{01}\eta_0\eta_1\ ,\ \hat{\Omega}\,(\Lambda + \Lambda^0\eta_0 + \Lambda^1\eta_1 + \Lambda^{01}\eta_0\eta_1)\Big) \tag{4.304}$$

$$= \Big(\psi_{\Phi=\pi=0} + \psi^1_{\Phi=0}\eta_1 + \psi^0_{\pi=0}\eta_0 + \psi^{01}\eta_0\eta_1\ ,\ \hat{\Phi}\Lambda\eta_0 + \hat{\pi}\Lambda\eta_1 + (\hat{\pi}\Lambda^0 - \hat{\Phi}\Lambda^1)\eta_1\eta_0\Big) \tag{4.305}$$

$$= -\Big(\psi_{\Phi=\pi=0}\ ,\ \hat{\pi}\Lambda^0 - \hat{\Phi}\Lambda^1\Big) + \Big(\psi^0_{\pi=0}\ ,\ \hat{\pi}\Lambda\Big) - \Big(\psi^1_{\Phi=0}\ ,\ \hat{\Phi}\Lambda\Big) = 0 \tag{4.306}$$

which will happen if the Λ state is dual to the other one—for then the hermiticity properties of $\mathbf{\Omega}$ will be saved.

Thus, it is possible in general to write any state in the cohomology in the form

$$\Psi_{\Omega=0} = \psi_{\Phi=0=\pi} + \psi^1_{\Phi=0=\lambda}\eta_1 + \psi^0_{\chi=0=\pi}\eta_0 + \psi^{01}_{\chi=0=\lambda}\eta_0\eta_1 \tag{4.307}$$

Notice that duality of the sectors is lost if the constraint has more than one branch. We could have also obtained Fock space representations, if we had started with a different extended Hilbert space at the beginning. There would then be different solutions to the BRST equation—see section 3.2.4.

Notice that the original BRST inner product in extended space can be used for these states—there are no regularization problems! What has happened? Well, without the cluttering of the mutipliers we have written the cohomology as

$$|\varphi\rangle = |P\!=\!0\rangle + |Q\!=\!a\rangle\eta \qquad (4.308)$$

The BRST inner product forces the inner product to match the space and its dual—because of the role that the ghost plays in the inner product:

$$\langle\varphi|\varphi\rangle = \int d\eta\,(\langle P\!=\!0| + \langle Q\!=\!a|\eta)\,(|P\!=\!0\rangle + |Q\!=\!a\rangle\eta) \qquad (4.309)$$

$$= \int d\eta\,\eta\,2Re(\langle P\!=\!0|Q\!=\!a\rangle) = 2Re\langle P\!=\!0|Q\!=\!a\rangle \qquad (4.310)$$

We can consider other representations—use one ghost coordinate and one ghost momentum as it happens in the path integral.

4.4.2 Inner product in the zero ghost sector

So we have an inner product, what else do we want? *We want an inner product defined on the zero ghost sector, which yields positive definite norms.* For that we are going to need a map from one sector its dual sector, so that the above nice properties can be used. This is where the operator $\exp([\mathbf{K},\mathbf{\Omega}])$ enters. We already discussed how this inner product may be defined. The only problem was related to operator ordering. When the inner product computation was performed earlier, I ignored the fact that the operators in the exponent don't commute. However, as we will see, the result is still correct.

Consider the states that we discussed before in the cohomology discussion,

$$\Psi_{\Omega=0} = \psi_{\Phi=0=\pi} + \psi^{1}_{\Phi=0=\lambda}\eta_1 + \psi^{0}_{\chi=0=\pi}\eta_0 + \psi^{01}_{\chi=0=\lambda}\eta_0\eta_1$$

We will work with the states in the "middle sectors". Notice that inner product in the BRST enlarged space is well defined for these states:

$$\int d\eta_0 d\eta_1 \ \langle \psi^0_{\chi=0=\pi} | \psi^1_{\Phi=0=\lambda} \rangle \eta_0 \eta_1 \tag{4.311}$$

as we discussed earlier. These states have the same ghost number. Could we just define an inner product within one of these sectors? For definiteness let us work with the states $\psi^0_{\chi=0=\pi}\eta_0$ and see what we can do. These states satisfy

$$\hat{\pi}, \ \hat{\chi}, \ \hat{\eta}_0, \ \hat{\rho}_1 \ |\psi^0_{\chi=0=\pi=\eta_0}\rangle = 0 \tag{4.312}$$

Consider now the operator

$$e^{[\hat{K},\hat{\Omega}]} = e^{i\hat{\lambda}\hat{\Phi}+\hat{\rho}_0\hat{\eta}_1} \tag{4.313}$$

for $\hat{K} = \hat{\rho}_0\hat{\lambda}$.

It is easy to see that the new inner product is well defined for these states and matches the Dirac one:

$$(\psi, \psi') \equiv \langle \psi^0_{\chi=0=\pi=\eta_0} | \ e^{[\hat{K},\hat{\Omega}]} \ |\psi'^0_{\chi=0=\pi=\eta_0}\rangle$$

$$\int d\eta_0 d\eta_1 \ \langle \psi^0_{\chi=0=\pi} | \ e^{[\hat{K},\hat{\Omega}]} \ |\psi'^0_{\chi=0=\pi}\rangle \eta_0\eta_1 =$$

$$\int d\eta_0 d\eta_1 \ \langle \psi^0_{\chi=0=\pi} | \ e^{i\hat{\lambda}\hat{\Phi}+\hat{\rho}_0\hat{\eta}_1} |\psi'^0_{\chi=0=\pi}\rangle \eta_0\eta_1 = \langle \psi^0_{\chi=0} | \delta(\hat{\Phi}) | \psi'^0_{\chi=0}\rangle \tag{4.314}$$

This case is in fact going to reappear in the next chapter—these states are the ones we will use in the path integrals.

Consider, as another example, the states

$$|\psi_a(t,x), \ \Phi = 0 = \lambda = \eta_1 = \rho_0\rangle \sim \psi^1_{\Phi=0=\lambda}\eta_1 \tag{4.315}$$

which also satisfy the BRST condition, and the gauge-fixing function $\hat{K} = \hat{\rho}_1\hat{\chi}$, where $\chi = t - f(\tau)$, and the constraint is the one for the relativistic particle. It is easy to calculate

$$[\hat{K}, \hat{\Omega}] = \hat{\rho}_1\hat{\eta}_0[\hat{\chi}, \hat{\Phi}] + i\hat{\pi}\hat{\chi} \tag{4.316}$$

Let us calculate

$$\langle\psi_a(t,x),\ \Phi\!=\!0\!=\!\lambda\!=\!\eta_1\!=\!\rho_0|\ e^{\hat{\rho}_1\hat{\eta}_0[\hat{\chi},\hat{\Phi}]+i\hat{\pi}\hat{\chi}}|\psi_b(t,x),\ \Phi\!=\!0\!=\!\lambda\!=\!\eta_1\!=\!\rho_0\rangle = \tag{4.317}$$

$$\int d^4x\ \psi_a^*(t,x)\left[\int d\pi d\rho_1 d\eta_0\ e^{\rho_1\eta_0[\hat{\chi},\hat{\Phi}]+i\pi\hat{\chi}}\right]\psi_b(t,x)$$

Now,

$$[\hat{\chi}, \hat{\Phi}] = [\hat{t}, \hat{p}^2] = i2\hat{p}_t$$

To compute this integral we make use of the Campbell-Baker-Hausdorff theorem[19]. That is

$$ln(e^A e^B) = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] - \frac{1}{12}[B,[B,A]] + ... =$$

$$B + \int_0^1 dt\ \ g[e^{tAd\ A}\ e^{tAd\ B}](A) \tag{4.318}$$

where $Ad\ A \equiv [A,\ \ ]$ is an operator ($n^2 \times n^2$ matrix that acts on n^2-"vectors"), and

$$g(z) = \frac{\ln\ z}{z-1} = \sum_{j=0}^{\infty}\frac{(1-z)^j}{j+1} = 1 + \frac{1}{2}(1-z) + \frac{1}{3}(1-z)^3 + ... \tag{4.319}$$

Fortunately the series ends quickly here, and we have

$$e^{\rho_1\eta_0[\hat{\chi},\hat{\Phi}]+i\pi\hat{\chi}} = e^{\rho_1\eta_0[\hat{\chi},\hat{\Phi}]-i\rho_1\eta_0\pi}\ e^{i\pi\hat{\chi}} = \tag{4.320}$$

[19]MILLER, Symmetry Groups +....QA171M52

$$\left(1 + \rho_1\eta_0[\hat{\chi}, \hat{\Phi}] - i\rho_1\eta_0\pi\right) e^{i\pi\hat{\chi}}$$

The ghost integrations are now easily done, and yield

$$\int d\pi d\rho_1 d\eta_0 \; e^{\rho_1\eta_0[\hat{\chi},\hat{\Phi}]+i\pi\hat{\chi}} = i\hat{p}_t\delta(\hat{\chi}) + i\delta(\hat{\chi})\hat{p}_t \tag{4.321}$$

This is the Klein-Gordon inner product, since

$$\langle\psi_a|\delta(\hat{t} - f(\tau))\hat{p}_t|\psi_b\rangle = \int dxdt \, \langle\psi_a|x,t\rangle\langle x,t|\delta(\hat{t}-f(\tau))\hat{p}_t|\psi_b\rangle \tag{4.322}$$

which is just

$$\int dxdt \, \delta(t-f(\tau)) \, \langle\psi_a|x,t\rangle \left(-i\frac{\partial}{\partial t}\right) \langle x,t|\psi_b\rangle \tag{4.323}$$

Notice that what this inner product is doing is giving us the hermitized expression of the determinant operator—as discussed before. It does not yield an absolute value or anything else. From this point of view the Feyman amplitude is the more natural object in the theory.

Is this true for the general interacting case? Recall equation 2.94

$$\{t, \Phi_{EG}\} = 2g^{00}(p_0 - A_0) + 2g^{0i}(p_i - A_i) = 2g^{\mu 0}\Pi_\mu$$

so

$$[\hat{t}, \; [\hat{t}, \; \hat{\Phi}]] = 2\hat{g}^{00} \tag{4.324}$$

In general, things will not be so easy, because, for whatever ordering we choose, the series will not terminate that quickly.

We need

$$[\hat{g}^{00}, \; \widehat{g^{\mu 0}\Pi_\mu}] = 0 \tag{4.325}$$

for whatever ordering is used.

However, if there is no gravitational background—only an electromagnetic one—then the result is again immediate:

$$\int d\pi d\rho_1 d\eta_0 \; e^{\rho_1\eta_0[\hat{\chi},\hat{\Phi}]+i\pi\hat{\chi}} = i\hat{\Pi}_0\delta(\hat{\chi}) + i\delta(\hat{\chi})\hat{\Pi}_0 \tag{4.326}$$

The reason is that in such a case we have

$$[\hat{t},\; \hat{\Phi}_E] = 2i\hat{\Pi}_0 \tag{4.327}$$

and

$$[\hat{t},\; [\hat{t},\hat{\Phi}_E]] = -2 \tag{4.328}$$

which, again, commutes, and the series terminates.

What about using other gauge-fixing functions? We will consider this issue in the next section.

4.4.3 The Fradkin-Vilkovisky theorem

In agreement with the Fradkin-Vilkovisky theorem for the path integral we would expect that the defined inner product is invariant under changes of gauge-fixing. In essence we need to show that

$$\frac{\delta}{\delta\hat{K}}\langle\Psi|\; e^{[\hat{\Omega},\hat{K}]}|\Psi'\rangle = 0 \tag{4.329}$$

provided that the states satisfy $\hat{\Omega}|\Psi\rangle = 0$, or simply that to first order the amplitude does not depend on ε ,

$$\langle\Psi|\ e^{[\hat{\Omega},\hat{K}+\varepsilon\hat{J}]}|\Psi'\rangle = \langle\Psi|\ e^{[\hat{\Omega},\hat{K}]}|\Psi'\rangle + O(\varepsilon^2) \tag{4.330}$$

This will be a local statement—“large” gauge transformations are not covered.

Proposition 2 *For certain operators $\hat{K}$, $\hat{J}$ and for BRST invariant states we have:*

i) $(\Psi_a, \Psi_b) \equiv \langle\Psi_a|\ e^{[\hat{\Omega},\hat{K}]}|\Psi_b\rangle$ *exists, and*

ii) $\hat{\Omega}$ *is hermitean in the new inner product, and*

iii) $\frac{d}{d\epsilon}\langle\Psi_a|\ e^{[\hat{\Omega},\hat{K}+\epsilon\hat{J}]}|\Psi_b\rangle = 0$

This theorem was originally proved within the path integral context (see references in [19]).

As we saw before, as long as we choose the right gauge-fixing for the states we want to work with the above amplitude exists—the regularization works. So let us assume that we are taking the operator $\hat{K}$ as we defined it before. We can, in the same fashion, check that

$$(\Psi_a, \hat{\Omega}\Psi_b) = (\Psi_a, \hat{\Omega}^\dagger\Psi_b) = 0 \tag{4.331}$$

and moreover,

$$(\Psi_a,\ [\hat{M},\hat{\Omega}]\Psi_b) = ([\hat{M},\hat{\Omega}]\Psi_a,\ \Psi_b) = 0 \tag{4.332}$$

as long as $\hat{M}$ is regular. This follows, in part, from the fact that

$$[\hat{\Omega},\ [\hat{\Omega},\ \hat{A}]] = 0 \tag{4.333}$$

Finally, we also have that

$$[[\hat{\Omega},\ \hat{A}],\ [\hat{\Omega},\ \hat{B}]] = [\hat{\Omega},\ \hat{C}] \tag{4.334}$$

where $\hat{C} = [\hat{\Omega},\ [\hat{A},\ [\hat{\Omega}, \hat{B}]]]$

Using the Campbell-Baker-Hausdorff theorem mentioned above, it now follows that

$$e^{[\hat{\Omega},\ \hat{A}]+[\hat{\Omega},\ \hat{B}]} = e^{[\hat{\Omega},\ \hat{A}]}\ e^{[\hat{\Omega},\ \hat{B}']} \tag{4.335}$$

because all the extra terms in the CBH theorem can be written in the BRS- exact form, $[\hat{\Omega}, stuff]$. This simple result, applied to our situation implies that

$$e^{[\hat{\Omega},\ \hat{K}+\epsilon\hat{J}]} = e^{[\hat{\Omega},\ \hat{K}]}\ e^{[\hat{\Omega},\ \epsilon\hat{J}]} \approx e^{[\hat{\Omega},\ \hat{K}]}\ \Big(1 + [\hat{\Omega},\ \epsilon\hat{J}]\Big) \tag{4.336}$$

which finishes the argument (in fact, it is true that $\exp([\hat{\Omega},\ \hat{K}]) = 1 + [\hat{\Omega},\ \hat{L}]$ see exercise 14.13 in [2]).

4.4.4 The composition law and the BFV path integral

This inner product definition was of course inferred from the BFV path integral, from the term

$$e^{i\int d\tau\ \{K,\Omega\}} \tag{4.337}$$

In the path integral derivation we use a trick: we expand the exponent to first order, compute the expectation value, and re-exponentiate. That is, we can say that to compute the expectation value (for simplicity I use τ-independent gauge-fixing) of

$$\langle\Psi_\Phi|\, e^{[\hat{\mathcal{O}}_C,\hat{\Omega}]\Delta\tau}|\Psi'_\Phi\rangle = \tag{4.338}$$

$$\langle\psi_{\Phi=0},\lambda\!=\!\rho_0\!=\!\eta_1\!=\!0|\; e^{i\Delta\tau\left(-i\hat{\rho}_1\hat{\eta}_0[\hat{\chi},\hat{\Phi}]+\hat{\pi}\hat{\chi}+\hat{\lambda}\hat{\Phi}+\hat{\rho}_0\hat{\eta}_1\right)}|\psi'_{\Phi=0},\lambda\!=\!\rho_0\!=\!\eta_1\!=\!0\rangle =$$

$$\langle\psi_{\Phi=0},\lambda\!=\!\rho_0\!=\!\eta_1\!=\!0|$$
$$\lim_{N\to\infty}\left(1+\frac{i\Delta\tau}{N}\left(-i\hat{\rho}_1\hat{\eta}_0[\hat{\chi},\hat{\Phi}]+\hat{\pi}\hat{\chi}+\hat{\lambda}\hat{\Phi}+\hat{\rho}_0\hat{\eta}_1\right)\right)^N|\psi'_{\Phi=0},\lambda\!=\!\rho_0\!=\!\eta_1\!=\!0\rangle$$

Next, we insert the resolutions of the identity. Thus we obtain the path integral—in this case the one with "momentum" BRS-invariant boundary conditions. By the same technique it is easy to check that the naive result for the above amplitudes is indeed correct.

However, we would also like to write a composition law exclusively in the physical subspace. For that we will have to develop an analog of the projector formalism in the Dirac quantization case. For the states we consider ($\chi = 0$ states) this operator is provided by (let here $[\hat{K},\hat{\Omega}] = i\hat{\lambda}\hat{\Phi} + \hat{\rho}_0\hat{\eta}_1$ for $\hat{K} = \hat{\rho}_0\hat{\lambda}$, and think $\chi_a = t - f(\tau_a)$, $Q \sim t$ being the gauge degree of freedom)

$$\hat{\mathcal{K}}_{\chi_J} = \delta(\hat{\eta}_0)\delta(\hat{\rho}_1)\delta(\hat{\pi})\delta(\hat{\chi}_J)(-i)[\hat{\chi}_J,\hat{\Phi}]e^{i[\hat{K},\hat{\Omega}]} \tag{4.339}$$

which satisfies

$$\langle\psi^0_{\chi_f=0=\pi=\eta_0=\rho_1}|\; e^{i[\hat{K},\hat{\Omega}]}|\psi^0_{\chi_i=0=\pi=\eta_0=\rho_1}\rangle = \tag{4.340}$$

$$\langle\psi^0_{\chi_f=0=\pi=\eta_0=\rho_1}|\; e^{i[\hat{K},\hat{\Omega}]}\hat{\mathcal{K}}_{\chi_J}|\psi^0_{\chi_i=0=\pi=\eta_0=\rho_1}\rangle =$$

$$\langle\psi^0_{\chi_f=0=\pi=\eta_0=\rho_1}|\, e^{i[\hat{K},\hat{\Omega}]}\delta(\hat{\eta}_0)\delta(\hat{\rho}_1)\delta(\hat{\pi})\delta(\hat{\chi}_J)(-i)[\hat{\chi}_J,\hat{\Phi}]e^{i[\hat{K},\hat{\Omega}]}|\psi^0_{\chi_i=0=\pi=\eta_0=\rho_1}\rangle =$$

$$\int dQdq\,\langle\psi^0_{\chi_f=0=\pi=\eta_0=\rho_1}|\,e^{i[\hat{K},\hat{\Omega}]}|\psi^0_{0=\pi=\eta_0=\rho_1}\rangle\,\langle\psi^0_{0=\pi=\eta_0=\rho_1}|\,\delta(\hat{\chi})(-i)[\hat{\chi}_J,\hat{\Phi}]\,e^{i[\hat{K},\hat{\Omega}]}|\psi^0_{\chi_i=0=\pi=\eta_0=\rho_1}\rangle$$

which after insertion of the full resolution of the identity yields the composition law,

$$\int dQdq\;\langle\psi^0_{\chi_f=0}|\;\delta(\hat{\Phi})|\psi^0,Q\rangle\langle\psi^0,Q|\;\delta(\hat{\chi}_J)(-i)[\hat{\chi}_J,\hat{\Phi}]\;\delta(\hat{\Phi})|\psi^0_{\chi_i=0}\rangle \tag{4.341}$$

We can obtain, for example, the Klein-Gordon composition law, since

$$\langle\psi_a|\delta(\hat{t}-f(\tau))\hat{p}_t|\psi_b\rangle = \int dxdt\;\langle\psi_a|x,t\rangle\langle x,t|\delta(\hat{t}-f(\tau))\hat{p}_t|\psi_b\rangle =$$

$$\int dxdt\delta(t-f(\tau))\;\langle\psi_a|x,t\rangle\;\left(-i\frac{\partial}{\partial t}\right)\;\langle x,t|\psi_b\rangle \tag{4.342}$$

just as we did previously in the Dirac quantization approach. We have to be careful about ordering for the jacobian—as usual—and for the relativistic case, for example, instead of the above the operator we should use—remember that we will get the on-shell or Feynman amplitudes—is

$$\hat{\mathcal{K}}_\chi = \delta(\hat{\eta}_0)\delta(\hat{\rho}_1)\delta(\hat{\pi})\left(\delta(\hat{\chi})(-i)[\hat{\chi}_J,\hat{\Phi}]+(-i)[\hat{\chi}_J,\hat{\Phi}]\delta(\hat{\chi})\right)e^{i[\hat{K},\hat{\Omega}]} \tag{4.343}$$

for the half range case, or

$$\hat{\mathcal{K}}_\chi = \delta(\hat{\eta}_0)\delta(\hat{\rho}_1)\delta(\hat{\pi})\delta(\hat{\chi})\;|(-i)[\hat{\chi}_J,\hat{\Phi}]|\;e^{i[\hat{K},\hat{\Omega}]} \tag{4.344}$$

for the full range one. Notice, though, that there is no way to get the absolute value for the jacobian in BFV. From this point of view the Feynman propagator is the more natural amplitude in the theory, as we already mentioned in section 4.4.2.

4.4.5 Other questions

How about the BRST observables, are they hermitean with respect to this inner product?

Recall that an observable is defined by demanding that it commute with the BRST generator. Now, if an observable $\hat{A}$ is to be hermitean it needs to be hermitean with respect to the full inner product and commute with the term $\{\hat{\mathcal{O}}, \hat{\Omega}\}$. Now

$$\{\hat{A}, \{\hat{\mathcal{O}}, \hat{\Omega}\}\} = -\{\hat{\Omega}, \{\hat{A}, \hat{\mathcal{O}}\}\} \tag{4.345}$$

by virtue of the Jacobi identity and the fact that $\hat{A}$ is an observable. Hence, for physical states we have, provided that we have hermicity of the BRST generator

$$\langle\Psi|\{\hat{A}, \{\hat{\mathcal{O}}, \hat{\Omega}\}\}|\Psi'\rangle = 0 \tag{4.346}$$

and we have hermiticity of the observables with respect to the regularized inner product[20].

Does the BRST formalism fix the arbitrariness in the quantum theory that arises from writing the constraint in different ways?

The inner product suffers from the same ambiguity we described before, since, indeed, it reduces to the Dirac inner product. How do the BRST observables behave under a rescaling of the constraint? Notice that no matter what we do we need to preserve the hermiticity of $\mathbf{\Omega}$. There is a cohomology definition for operators—just as there is one for the states.

[20] Again, hermiticity of $\mathbf{\Omega}$ has been assumed, i.e., we are using the above regularized inner product in general.

How about BRST-Fock?

There never was an inner product problem there.

4.5 Conclusions, summary

We started with the Dirac condition on the states, and we saw that unless the original gauge coordinate was bounded (compact gauge group) the Dirac states are not normalizable in the original bigger space. I emphasized that this is the "inner product problem": we cannot—in general—extract from the original big Hilbert space an inner product for the Dirac subspace.

As a cure we could use compact gauge coordinates—we could force something like periodic boundary conditions.

If we want to use a full coordinate space, we could use the "dual space trick" of Marnelius. I introduced instead a projector, and showed that this projection procedure leads to the Faddeev path integral, and is equivalent to the reduced phase space method for constraints that can be made into a momentum. It is also cumbersome because we don't really have what Marnelius calls an "inner product space".

This discussion was tied to the proper definition of the Dirac inner product, which I argued must be gauge-invariant and also yield states with real norms. This lead to the derivation of the Klein-Gordon inner product for the free relativistic case.

The projection procedure becomes unclear when the constraint is quadratic and unfactorizable. This is not surprising, because the existence of the projector implies a composition law, and in general there is no such composition law for the on-shell propagators.

We can also use a Fock representation. A weaker condition is imposed than the full constraint, so the physical states are regular in the original inner product, which can be used in the physical subspace. However, Fock space is compact—a discrete space. This corresponds to "half the constraints" being imposed, as described below. The gauge degrees of freedom disappear partly because of the appearance of null states. I showed that this is true of both the simple case in which the constraint can be made into a momentum and the case where we have a quadratic constraint that factorizes. If this condition is not met it is not clear at all that there are enough physical null states in the theory to reduce the physical subspace to the form $|vac_1\rangle \otimes |others\rangle + |vac_2\rangle \otimes |others\rangle$.

In the BRST formalism an even weaker condition is imposed on the states. I argued that in the cohomology discussion one must be careful about the allowed states, which must respect the hermicity properties of the BRST generator. The extended space inner product is then well defined for the states in this cohomology. These states are in fact the natural boundary conditions in the construction of the path integral (moreover, with them one can then derive the Klein-Gordon inner product, as described below). Another consequence of this restriction is that the usual duality statements of the different ghost sectors do not apply for the case of quadratic constraints.

However, and still with the intention of having an inner product space and because one wants to work with zero ghost states, a duality operator is used, the famous $\exp([\hat{K}, \hat{\Omega}])$ (the motivation for this factor comes from the BFV path integral, where it appears). This inner product, when used with the proper set of states in the above cohomology yields the Klein-Gordon inner product in the relativistic case, for example. I showed that this definition of inner product leads to the BFV path integral—modulo operator orderings—by simple insertions of the resolution of the identity in full space. BRST state space is not necessarily compact, unlike the Fock case—although one can also use the BRST-Fock representation.

I also developed a projector formalism in BRST akin to the one I constructed for the Dirac case, through which one can derive a composition law. Again, the Klein-Gordon case was given as an example.

Moreover, consistency of the quantum reduction leads to two different possible representations of the multipliers: fully-ranged real multipliers—this leads to states that satisfy the constraints—and half-ranged imaginary representations which lead to states satisfying the Fock condition—"half the constraints".

I also showed how to construct the BFV path integral strictly in the physical space, through the development of an analogous projector formalism to the one I developed for the Dirac/Faddeev case. This complements my earlier description of the construction of the path integral in the fully extended BRST space.

Chapter 5

Path integrals

In this section I will discuss the different path integral approaches to quantization of constrained systems.

We will discuss first the path integrals in phase space—the ones more directly connected with the Hilbert space description—and then the path integrals in configuration space, which should be derivable from the first by integration of the momenta.

I will not worry too much, for the most part, about the problems associated specifically with theories in curved backgrounds and how to skeletonize the path integrals (in phase-space or configuration space) in such cases, although I will review some important results in section 5.1.7.

5.1 Path integrals in phase space

We will look first at Faddeev's path integral [2, 18]; then we will study the more complex BFV path integral [2, 19]. These correspond to the reduced phase space and the BRST quantization schemes, respectively.

The non-relativistic case and single branched cases will be discussed first, and then the more complex interacting cases will be examined. We will pay particular attention to the problems arising from having a constraint that cannot be made into a momentum by a well-defined canonical transformation, and to possible path integral recipes in such a situation.

5.1.1 The Faddeev path integral: one branch

I will now introduce the Faddeev path integral [2, 18] from the point of view of reduced phase space quantization.

Consider starting from a well defined reduced phase space. Let this space be described by the coordinates q, p and some hamiltonian $h(q, p)$. The unphysical part of the phase space will be described by the variables Q, P and the constraint $P \approx 0$. The extended hamiltonian is then $H_E = h(q, p) + \lambda P$. In this simple situation the system effectively decouples into the physical and unphysical parts and we can drop the gauge part. With boundary conditions on the physical coordinates the path integral in this reduced phase space is then just like the unconstrained case in section 1, and we can write

$$\Gamma(q_i, q_f; \tau_i, \tau_f) = \int DqDp \; e^{i\int_{\tau_i}^{\tau_f} d\tau(p\dot{q} - h(q,p))} = \int DqDp \; e^{i\int_{\tau_i}^{\tau_f} pdq - h(q,p)d\tau} \tag{5.1}$$

where the Liouville measure is

$$DqDp = \frac{dp_0}{2\pi} \prod_{i=1}^{N} \frac{dq_i dp_i}{2\pi} \tag{5.2}$$

Notice that if $h(q,p) = 0$ we immediately have that $\Gamma(q_i, q_f; \tau_i, \tau_f) = \Gamma(q_i, q_f)$ only (indeed $\Gamma = \delta(q_f - q_i)$!!), since the integrand becomes independent of the parameter. *This illustrates the relationship between reparametrization invariance—or parametrization indifference—and a zero hamiltonian*, which Dirac described so well [1].

Now we can add the unphysical degrees of freedom Q, P by the use of the identity

$$1 = \int DQDP \; \delta(Q - Q(q,p,\tau))\delta(P) e^{i\int_{\tau_i}^{\tau_f} d\tau P\dot{Q}} \tag{5.3}$$

where the measure is defined[1] by

$$DQDP \; \delta(Q - Q(q,p,\tau))\delta(P) \equiv dP_0 \delta(P_0) \prod_{i=1}^{N} dQ_i dP_i \delta(Q_i - Q(q,p,\tau))\delta(P_i) \tag{5.4}$$

This "1" can then be inserted in the path integral above:

$$\Gamma(q_i, q_f; \tau_i, \tau_f) = \int DqDpDQDP\delta(Q - Q(q,p,\tau))\delta(P) e^{i\int_{\tau_i}^{\tau_f} d\tau(P\dot{Q} + p\dot{q} - h(q,p))} \tag{5.5}$$

$$= \int DqDpDQDPD\lambda\delta(Q - Q(q,p,\tau)) e^{i\int_{\tau_i}^{\tau_f} d\tau(P\dot{Q} + p\dot{q} - h(q,p) + \lambda P)} \tag{5.6}$$

[1] $DQDP \; \delta(Q - Q(q,p,\tau))\delta(P) \equiv \prod_{i=1}^{N} dQ_i dP_i \delta(Q_i - Q(q,p,\tau))\delta(P_i)$ also works

with

$$D\lambda = \frac{d\lambda_0}{2\pi} \prod_{i=1}^{N} \frac{d\lambda_i}{2\pi} \tag{5.7}$$

This is the simplest situation. One can also write[2]

$$\delta(Q - Q(q,p,\tau))\delta(P) \quad = \delta(f(Q,q,p,\tau))\ |\frac{\partial f}{\partial Q}|\ \delta(P) \tag{5.8}$$

$$= \delta(f(Q,q,p,\tau))\ |\{f,P\}|\ \delta(P), \tag{5.9}$$

so the path integral can be written as

$$\begin{aligned}
\Gamma(q_i, qf, \Delta\tau) \quad &= \int DqDpDQDP\delta(f(Q,q,p,\tau))\delta(P)|\{f,P\}|e^{i\int_{\tau_i}^{\tau_f} d\tau(P\dot{Q}+p\dot{q}-h(q,p))} \\
&= \int DqDpDQDP\delta(\chi)\delta(\Phi)|\{\chi,\Phi\}|e^{i\int_{\tau_i}^{\tau_f} d\tau(P\dot{Q}+p\dot{q}-h(q,p))} \\
&= \int DqDpDQDPD\lambda D\pi DcD\bar{c}\, e^{i\int_{\tau_i}^{\tau_f} d\tau(P\dot{Q}+p\dot{q}-h(q,p)+\lambda\Phi+\lambda\chi+c|\{\chi,\Phi\}|\bar{c})}
\end{aligned} \tag{5.10}$$

with the obvious identifications. To be perfectly clear, let us write out the full measures,

$$\begin{aligned}
D\mu \quad &\equiv DqDpDQDP\delta(\chi)\delta(\Phi)|\{\chi,\Phi\}| \\
&\equiv \left(\frac{dp_0 dP_0}{2\pi} \prod_{i=1}^{N} \frac{dQ_i dP_i dq_i dp_i}{2\pi}\right)\left(\delta_0(P)\prod_{i=1}^{N}\delta_i(Q - Q(q,p,\tau))\delta_i(P)\right) \qquad (5.11) \\
&= \left(\frac{dp_0 dP_0}{2\pi} \prod_{i=1}^{N} \frac{dQ_i dP_i dq_i dp_i}{2\pi}\right)\left(\delta_0(\Phi)\prod_{i=1}^{N}\delta_i(\chi)\delta_i(\Phi)|\{\chi,\Phi\}|_i\right)
\end{aligned}$$

and also

$$D\lambda D\pi DcD\bar{c} \equiv \frac{d\lambda_0}{2\pi} \prod_{i=1}^{N} \frac{d\pi_i}{2\pi} dc_i d\bar{c}_i \tag{5.12}$$

[2]note the absolute value!

Let us do a *canonical* change of variables in this integral, $(q, p, Q, P) \to (z^\alpha, w_\alpha)$: a change of variables that is also a canonical transformation. This implies $dqdpdQdP = dz^\alpha dw_\alpha$, since the jacobian for the change of variables is the Poisson bracket of the new variables—which is one by definition of canonical transformation. After this change of variables the path integral becomes

$$\Gamma(q_i, qf; \tau_i, \tau_f) = \int Dz^\alpha Dw_\alpha \delta(\chi)\delta(\Phi)|\{\chi, \Phi\}|e^{i\int_{\tau_i}^{\tau_f} d\tau(\dot{z}^\alpha w_\alpha - H(z,w) + \frac{dW}{d\tau})} \tag{5.13}$$

where $Dz^\alpha Dw_\alpha \equiv DqDpDQDP$, and to be clear

$$Dz^\alpha Dw_\alpha = \frac{dP_0 dp_0}{2\pi}\prod_{i=1}^{N}\frac{dz_i^\alpha dw_{\alpha i}}{2\pi} \tag{5.14}$$

W is the generator of the canonical transformation, i.e.,

$$P\dot{Q} + p\dot{q} - h(q,p) = \dot{z}^\alpha w_\alpha - H(z,w) + \frac{dW}{d\tau} \tag{5.15}$$

It is important to note that the form of the identity one uses is not unique by any means, nor is unique what we define as the gauge momentum or coordinate. *The main point is that the path integral, to make sense in a quantum mechanical framework, must reduce to the original physical quantity* Γ*, which lives in a well-defined physical phase space.*

Let us apply these ideas mindlessly to the coordinate system of section 2.4.4. Recall that the hamiltonian is zero and the constraint $P \approx 0$, with

$$\begin{array}{lll} Q = t - t_0 & P = p_t + \dfrac{p_x^2}{2m} & \textit{gauge degrees of freedom,} \\ q = p_x(t - t_0) - mx & p = -\dfrac{p_x}{m} & \textit{physical degrees of freedom} \end{array} \tag{5.16}$$

We then obtain the result

$$\begin{aligned}\Gamma(q_i, q_f) &= \int DQDPDqDp\delta(Q - Q(q,p,\tau))\delta(P)e^{i\int_{\tau_i}^{\tau_f} d\tau(P\dot{Q}+p\dot{q})} \\ &= \int DqDpe^{i\int_{\tau_i}^{\tau_f} pdq} \qquad (5.17) \\ &= \delta(q_f - q_i)\end{aligned}$$

There is nothing wrong with this result, it's just that the action we are using is not the original one. As we saw in section 2.4.4 the action $\int_{\tau_i}^{\tau_f} d\tau(P\dot{Q} + p\dot{q})$ *differs from the original one by a surface term that depends on the gauge coordinates*, the $\int_{\tau_i}^{\tau_f} dW$ above. Indeed Γ' differs from Γ precisely by this gauge dependent boundary term in the action, where

$$\Gamma' = \int DtDxDp_tDp_x\delta(\chi)\delta(\Phi)|\{\chi,\Phi\}|e^{i\int_{\tau_i}^{\tau_f} d\tau(p_x\dot{x}+P_t\dot{t})} \qquad (5.18)$$

with

$$DtDxDp_tDp_x \equiv dp_{t0}dp_{x0}\prod_{i=1}^{N}\frac{dt_i dp_{ti} dx_i dp_{xi}}{(2\pi)^2} \qquad (5.19)$$

and

$$\delta(\chi)\delta(\Phi)|\{\chi,\Phi\}| \equiv \delta_0(\Phi)\prod_{i=1}^{N}\delta_i(\chi)\delta_i(\Phi)|\{\chi,\Phi\}|_i \qquad (5.20)$$

since the measure " boundary effect"

$$dP_0 dp_0 = dp_{t0}dp_{x0}\frac{\partial(P_0,p_0)}{\partial(p_{t0},dp_{x0})} = -\frac{1}{m}dp_{t0}dp_{x0}$$

is absent except for the constant factor $(-1/m)$—which we will ignore— and in fact, using the τ-dependent gauge $t = t(\tau)$, $\{\chi,\Phi\} = 1$, we obtain the blatantly " gauge dependent" result

$$\Gamma' = \left(\frac{m}{2\pi i \Delta t}\right)^{\frac{1}{2}} e^{i\frac{\Delta x^2}{2\Delta t}} \tag{5.21}$$

where gauge dependence comes in through the factor $\Delta t = t(\tau_f) - t(\tau_i)$—the boundary conditions. With the measure I defined above the dependence comes in from the boundary conditions, not the gauge fixing condition, since there is no gauge fixing at the boundaries. Please note that this is just a matter of semantics. We could have indeed defined the measure to be [14]

$$\begin{aligned} D\mu^* &\equiv DqDpDQDP\delta(\chi)\delta(\Phi)|\{\chi,\Phi\}| \\ &\equiv \left(\frac{dp_0 dP_0}{2\pi}\prod_{i=1}^{N}\frac{dQ_i dP_i dq_i dp_i}{2\pi}\right)\cdot \\ &\left(\delta_0(P)\prod_{i=1}^{N}\delta_i(Q-Q(q,p,\tau))\delta_i(P)\right) dQ_0\delta_0(Q-Q(q,p,\tau)) \\ &= \left(\frac{dp_0 dP_0}{2\pi}\prod_{i=1}^{N}\frac{dQ_i dP_i dq_i dp_i}{2\pi}\right)\left(\delta_0(\Phi)\delta_0(\Phi)|\{\chi,\Phi\}|_0\prod_{i=1}^{N}\delta_i(\chi)\delta_i(\Phi)|\{\chi,\Phi\}|_i\right) \end{aligned} \tag{5.22}$$

which in the present case would add the integration $dt_0\ \delta(t_0 - t(\tau_0))$, a term that overrides the boundary conditions (or a term that implements them!) This is the point of view of reference [14], which implicitly uses this measure $D\mu^*$. The result would be the same as long as the gauge matches the intended boundary conditions. *The advantadge of the measure $D\mu$ I previously defined is that it makes the result gauge-fixing invariant, unlike $D\mu^*$.* But this is all a matter of semantics. If you know what the boundary conditions are for the path integral, and wish to use the measure

$D\mu^*$, just choose the gauge-fixing function to match the boundary conditions, that's all.

Regardless of the measure one uses, Γ' cannot be reduced to the gauge invariant reduced phase space path integral, as their actions differ by the variant surface term. However it is easy to see that Γ' can be reduced to something else. Let us use the measure defined just above, $D\mu^*$, and let us perform the " gauge" integrations; then

$$\begin{aligned} \Gamma' &= \int DxDp_x e^{i\int_{\tau_i}^{\tau_f} d\tau p_x\dot{x} - t\frac{p_x^2}{2m}\Big|_{\tau_i}^{\tau_f}} \\ &= \int DxDp_x e^{i\int_{\tau_i}^{\tau_f} d\tau(p_x\dot{x} - \frac{p_x^2}{2m}\frac{dt}{d\tau}) - (t-t(\tau))\frac{p_x^2}{2m}\Big|_{\tau_i}^{\tau_f}} \end{aligned} \tag{5.23}$$

With gauge and boundary choices satisfying $t - t(\tau)|_{\tau_i}^{\tau_f} = 0$, we have

$$\begin{aligned} \Gamma' &= \int DxDp_x e^{i\int_{\tau_i}^{\tau_f} d\tau(p_x\dot{x} - \frac{p_x^2}{2m}\frac{dt}{d\tau})} \\ &= \int DxDp_x e^{i\int p_x dx - \frac{p_x^2}{2m}dt} \end{aligned} \tag{5.24}$$

so the system reduces to the unconstrained case. It can also formally look like the reduced phase space of section 2.4—in the gauge $t = f(\tau)$:

$$\begin{aligned} H_\alpha &= \alpha\frac{p_x^2}{2m} = H_1 + \frac{\partial W_\alpha}{\partial t}, \\ \Gamma' &= \int DxDp_x e^{i\int_{\tau_i}^{\tau_f} d\tau(p_x\dot{x} - \alpha\frac{p_x^2}{2m})} \end{aligned} \tag{5.25}$$

We thus see here the same phenomenon as before: changing gauge choices corresponds to canonical transformations in RPS^*, with the generator of the canonical transformation playing its basic role as a surface term in the action. This is no longer surprising: *all the gauge dependence of the action comes from a surface term, and we know that actions that differ by a surface term can be understood to differ by a canonical transformation.*

Continuing with the measure $D\mu^*$, if on the other hand we don't have $t - t(\tau)|_{\tau_i}^{\tau_f} = 0$, then we can just eliminate that surface term—as is done in reference [14]—substracting it off the full action:

$$\begin{aligned}\Gamma'' &= \int DtDxDp_tDp_x\delta(\chi)\delta(\Phi)|\{\chi,\Phi\}|e^{i\int_{\tau_i}^{\tau_f} d\tau(p_x\dot{x}+p_t\dot{t})+(t-t(\tau))\frac{p_x 2}{2m}\big|_{\tau_i}^{\tau_f}} \\ &= \int DxDp_x e^{i\int_{\tau_i}^{\tau_f} d\tau(p_x\dot{x}-\frac{p_x^2}{2m}\frac{dt}{d\tau})} \qquad (5.26) \\ &= \int DxDp_x e^{i\int_{\tau_i}^{\tau_f} p_x dx-\frac{p_x^2}{2m}dt}\end{aligned}$$

This is the point of view taken in reference [14]. The reason for adding such a term should be clear now: to get rid of the gauge dependence of the action. Let us take a closer look at this action:

$$A'' = \int d\tau(p_x\dot{x} + p_t\dot{t} + \lambda\Phi) + (t - t(\tau))\left.\frac{p_x^2}{2m}\right|_{\tau_i}^{\tau_f} \qquad (5.27)$$

Recall that in the gauge invariant variables of section 2.4.4 we saw that

$$\begin{aligned}&\int_{\tau_i}^{\tau_f}(P\dot{Q} + p\dot{q} - w(\tau)P)d\tau \\ =\ &\int_{\tau_i}^{\tau_f}(p_x\dot{x} + p_t\dot{t} - w(\tau)\Phi)d\tau - (t-t_0)\left.\frac{p_x^2}{2m}\right|_{\tau_i}^{\tau_f} = A' - B|_{\tau_i}^{\tau_f}\end{aligned} \qquad (5.28)$$

where $B = (t - t_0)p_x^2/2m = QP^2m/2$ is the generator of the canonical transformation.

Now, we can therefore *split the original action into a gauge invariant and a gauge dependent part*:

$$\begin{aligned}&\int_{\tau_i}^{\tau_f}(p_x\dot{x}+p_t\dot{t}-w(\tau)\Phi)d\tau = \\ &\int_{\tau_i}^{\tau_f}(p_x\dot{x}+p_t\dot{t}-w(\tau)\Phi)d\tau-(t-t_0)\left.\frac{p_x^2}{2m}\right|_{\tau_i}^{\tau_f}+(t-t_0)\left.\frac{p_x^2}{2m}\right|_{\tau_i}^{\tau_f}\end{aligned} \tag{5.29}$$

The last term is the one that will change when we change gauge fixings. We can substract it off, and add it already implementing the boundary conditions: we finally have the gauge-fixing-at-the-boundaries-independent action

$$\int_{\tau_i}^{\tau_f}(p_x\dot{x}+p_t\dot{t}-w(\tau)\Phi)d\tau-(t-t_0)\left.\frac{p_x^2}{2m}\right|_{\tau_i}^{\tau_f} \tag{5.30}$$

This is the action of reference [14], and it is to be employed if one wants to gauge-fix at the boundaries, i.e., use the measure $D\mu^*$:

$$DtDxDp_tDp_x = dp_{t0}dp_{x0}\prod_{i=1}^{N}\frac{dt_idp_{ti}dx_idp_{xi}}{(2\pi)^2} \tag{5.31}$$

with

$$\delta(\chi)\delta(\Phi)=\delta_0(\chi)\delta_0(\Phi)\prod_{i=1}^{N}\delta_i(\chi)\delta_i(\Phi). \tag{5.32}$$

If one uses the measure $D\mu*$, however, there is no need to modify the action, as the formalism already is gauge-fixing invariant. The merit of proceeding as in reference [14] and using $D\mu^*$, is that it appears that one is closer to the reduced phase space approach— after all, the unphysical degrees of freedom are taken care of by delta functions at all "τimes".

It can be criticized that the surface term that one adds to take care of the gauge fixing dependence is not unique, as one could add a surface term of the form $\gamma(\tau)p_x^2/2m|_{\tau_i}^{\tau_f}$—i.e., subtract the term $(t(\tau)+\gamma(\tau)-t)p_x^2/2m|_{\tau_i}^{\tau_f}$ instead—but one could also add such a term in the other approach. The action is determined classically only up to some surface terms, but what we are doing here is more than that, as it affects the dynamics, and the path integral as well.

But are we closer to the reduced phase space philosophy? The bottom line is that there is no absolute reduced phase space to frame a quantization scheme if we insist on using gauge-dependent boundary conditions. But now we recognize the gauge-fixing dependence as being equivalent to "time" dependent canonical transformations in the reference reduced phase space of our choice—and the existence of different pictures in quantum mechanics. If we are able to define quantum mechanics for the true observables of the system, i.e., constants of the motion, we are in business. But how does one do that? For the cases at hand it seems that such a world is pretty boring!

At any rate, we have seen that the one-branch Faddeev approach yields a path integral that is equivalent to the one in the reduced phase space approach. We can obtain, for example, the propagator for the Schrödinger equation for constraints of the form $\Phi \approx p_t - H$, with $\hat{H}$ the hamiltonian operator in some ordered form. Ordering is equivalent to a choice of $H(q,p) = \langle q|\hat{H}|p\rangle$, as usual. Again, by SchrÖdinger equation I mean any equation of the form $i\partial_t\psi = \hat{H}\psi$. This includes the non-relativistic particle and the relativistic square-root forms, after picking a branch of the quadratic constraint.

5.1.2 The Faddeev path integral: multiple branches

In the previous section we discussed the Faddeev path integral in the simplest case. In fact it appears that only in such a case the path integral is unambiguoulsy defined. When there is more than one branch in the RPS the simple recipe

$$\begin{aligned}\Gamma(q_i, qf, \Delta\tau) &= \int DqDpDQDP\delta(f(Q,q,p,\tau))\delta(P)|\{f,P\}|e^{i\int_{\tau_i}^{\tau_f} d\tau(P\dot{Q}+p\dot{q}-h(q,p))} \\ &= \int DqDpDQDP\delta(\chi)\delta(\Phi)|\{\chi,\Phi\}|e^{i\int_{\tau_i}^{\tau_f} d\tau(P\dot{Q}+p\dot{q}-h(q,p))} \\ &= \int DqDpDQDPD\lambda D\pi DcD\bar{c}\, e^{i\int_{\tau_i}^{\tau_f} d\tau(P\dot{Q}+p\dot{q}-h(q,p)+\lambda\Phi+\lambda\chi+c|\{\chi,\Phi\}|\bar{c})}\end{aligned} \tag{5.33}$$

cannot be mindlessly applied.

There are several pitfalls in the mindless approach. We can compute the above path integral with or without an absolute value on the determinant. Or we can not use a "hermitean" skeletonization for it.... We will illustrate below.

Having two branches, in the simplest interpretation, means having two distinct reduced phase spaces. A more general way to write the path integral follows from the study of quantum mechanics with complex topologies:

$$\begin{aligned}\Gamma_{top}(q_i, qf, \Delta\tau) &= A_+\int DqDpDQDP\delta(\chi_+)\delta(\Phi_+)|\{\chi_+,\Phi_+\}|e^{i\int_{\tau_i}^{\tau_f} d\tau(P\dot{Q}+p\dot{q}-h(q,p))} \\ &\quad +A_-\int DqDpDQDP\delta(\chi_-)\delta(\Phi_-)|\{\chi_-,\Phi_-\}|e^{i\int_{\tau_i}^{\tau_f} d\tau(P\dot{Q}+p\dot{q}-h(q,p))}\end{aligned} \tag{5.34}$$

Included in this prescription is the possibility of using one branch only. The philosophy of this path integral is that the two worlds described by the branches don't "talk" to each other. In fact, it is not hard to see that consistency of the formalism will demand decoupling for the above prescription—the composition law.

What we have is essentially two quantum systems that don't interact. Another way of looking at this is to say that the number of degrees of freedom doubles.

This is the simplest interpretation—two disconnected worlds. Interesting things will happen when we try to connect these two worlds. Whether they are or are not connected depends on the inner product we bring into the theory, as we discussed earlier. As we will see next, this is in turn reflected in the properties of the determinant in the path integral.

Unitarity will depend at the end on the inner product we choose—within each branch and across branches—and on whether we connect the two worlds consistently. We expect that this will put serious constraints on the inner product between branches. The symmetries of the theory may make things easier.

Let us look at the determinant. For the full form of the constraint the determinant is not a simple numerical operator. It becomes a full-fledged operator, and the usual issues of ordering come up. For example, for the free case this determinant is

$$|\{\chi, \Phi\}| = 2|E| \tag{5.35}$$

where I have included the absolute value. With the skeletonization in the path integral—which corresponds to the operator ordering in the canonical formalism—this can mean different things; for example $E = E_i$, or $(E_i + E_{i-1})/2$ with or without absolute values. The guiding theme here has to be hermiticity, but with respect to what inner product? The problem is that in the physical space the operator E is not hermitean, as we saw.

Let us see how things can go wrong. Let us take the mindless route of just applying blindly the Faddeev prescription. Here are two examples of strange results (which will cease to strange in a moment).

If we just use E_i for the determinant, the path integral yields an oscillating result—depends on the number of bones in the skeletonization.

If we use $|E_i|$ the result is

$$A = \int \frac{dEdp_x}{2\pi} \delta(E^2 - p_x^2 - m^2) e^{ip_x\,\Delta x}$$

As for a nice result, if we still use the mindless approach but use the "hermitean skeletonization" with the absolute value, det= $|E_i + E_{i+1}|/2$ we get [43]

$$A = \int \frac{dEdp_x}{2\pi} \delta(E^2 - p_x^2 - m^2) e^{iE\,\Delta t - ip_x \Delta x} = \Gamma_{top}$$

just as defined above. The reason is that this choice of determinant decouples the branches—it corresponds to the Klein-Gordon inner product, as we will discuss later. Without the absolute value the branches still decouple, but the "negative" branch part changes sign with each iteration in the skeletonization.

We already provided an interpretation for the existence of the two branches. The basic idea is that the two branches represent the two possible time orientations of the paths: positive energy means positive orientation, negative energy means negative orientation—going back in time. In fact, it is not hard to realize that the only way in which we can represent a particle going back in time in this theory is by associating to it a negative hamiltonian.

Now, the above decomposition of the path integral, although a priori consistent, lacks imagination. As mentioned, it corresponds to taking the *direct sum* of two Hilbert spaces, with an inner product that decouples them. The most general way to write the path integral is, schematically

$$\Gamma_{\uparrow\downarrow} = a_{\nearrow}\int(\nearrow) + a_{\searrow}\int(\searrow) + a_{\nearrow\searrow}\int(\nearrow\searrow) + a_{\searrow\nearrow}\int(\searrow\nearrow) + ...$$

Consistency (the composition law) will fix these coefficients. We need to show that a positive definite inner product is in general not compatible with this path integral. Let the composition law be given by

$$\Gamma(x_f, t_f; x_i, t_i) = \int dxdt\, \Gamma(x_f, t_f; x_a, t_a)\, \hat{O}\, \Gamma(x_a, t_a; x_i, t_i)$$

Now, let us expand—writing this in a very economic fashion

$$= \int_a \left(\int_{fa}(\nearrow) + \int_{fa}(\searrow) + \int_{fa}(\nearrow\searrow) + \cdots \right) \hat{O} \left(\int_{ai}(\nearrow) + \int_{ai}(\searrow) + \int_{ai}(\nearrow\searrow) + \cdots \right)$$

$$= \int_a \left(\int_{fa}(\nearrow)\hat{O}\int_{ai}(\searrow) + \int_{fa}(\nearrow\searrow)\hat{O}\int_{ai}(\nearrow) + \int_{fa}(\nearrow)\hat{O}\int_{ai}(\searrow\nearrow) + \right.$$
$$\left. \cdots + (\nearrow \text{ exchange } \searrow) \right)$$

As was already discussed in the previous chapter, it isn't clear that one can make sense of the Faddeev path integral when there is no decoupling, as an operator constructive interpretation is lacking.In general,we have a consistent situation, as follows.

In we restrict the theory to one branch, forcing decoupling, sometimes the theory will also be consistent with other requirements, like space-time covariance, and sometimes it won't—as we have discussed. The results in reference [45] (see also reference [46] for the configuration space version of this path integral) are another example of this conflict between unitarity and *space-time covariance* when one tries to stay in one branch (the gravitational background case). Staying in one branch means that we have a unitary theory (if we order the hamiltonian to be hermitean) and an inner product with yields positive definite norms for the states. But we may have to give up some symmetries to do this. This is really an issue regarding the ordering requirements that we start with. Decoupling means unitarity within the one branch sector—i.e., the one particle sector.

For example, we know that the Faddeev path integral for the one-branch square-root relativistic particle hamiltonian solves the corresponding Schrödinger equation, and hence, when there is no electric field and the space-time is flat (decoupling) it also solves the Klein-Gordon equation (this is a general feature of the decoupling case)—a covariant equation.

If we try to work with both branches, in general we do not have decoupling. Then the interpretation of the Faddeev path integral is lost. As far as I know, this path integral is an object that properly belongs to reduced phase space quantization, or to Dirac quantization when it is equivalent to reduced pahse space quantization. It could be that one can show that there is a proper place for this path integral in a more general setting, though. From our understanding of the free case we can say that the hermitized form of the determinant and its absolute value will be necessary.

We studied all these cases in the operator formalism already—we know how to derive these path integrals from reduced phase space quantization and from Dirac quantization as well.

5.1.3 The Faddeev-Fock path integral

We have already remarked that one can also obtain an amplitude by using the Fock representation. Recall equation 4.259:

$$\mathcal{A} = \langle t_f, x_f, \pi = 0 | e^{i\hat{\lambda}\hat{\Phi}} | t_i x_i, \pi = 0\rangle = \langle t_f, x_f | \frac{1}{\hat{\Phi} + i\epsilon} | t_i x_i\rangle \tag{5.36}$$

This is easy to see if one uses the resolution of the identity—equation 4.216,

$$\hat{I} = \int_{-i\varepsilon}^{i\infty} dQ^1 \int_{-\infty - i\epsilon}^{\infty - i\epsilon} dP_2 |Q^{1*} P_2^*\rangle\langle Q^1 P_2|$$

Here $Q_1 = \lambda$, and $P_2 = p_t$ are the proper identifications.

Let us look at the easier case first $P_2 = P \approx 0$. Consider ($P_1 = \pi$, $Q^1 = \lambda$, $Q^2 = Q$)

$$\langle Q_f, \pi = 0 |\; e^{i\hat{\lambda}\hat{P}} | Q_i, \pi = 0\rangle =$$

$$\int_{-i\varepsilon}^{i\infty} d\lambda \int_{-\infty - i\epsilon}^{\infty - i\epsilon} dP\; \langle Q_f, \pi = 0 |\; e^{i\lambda P} | \lambda, P\rangle\langle \lambda^*, P^* | Q_i, \pi = 0\rangle =$$

$$\int_{-\infty-i\epsilon}^{\infty-i\epsilon} dP \; e^{iQ_fP} \frac{1}{P+i\epsilon} e^{-iQ_iP}$$

As mentioned, this amplitude is the causal amplitude—leads to the Feynman propagator for the full relativistic case, for example, and it will lead us to the causal amplitude, the Feynman amplitude. An important question arises. Can we interpret this amplitude within the Fock quantization approach? We did use the Fock space representation, in which the multiplier appears with half the range and is imaginary. But where is the Fock quantization condition? Can we interpret it, for instance, as

$$\langle Q_f, \pi = 0| \; \delta(\hat{a}) |Q_i, \pi = 0\rangle$$

or

$$\langle Q_f, \pi = 0|vac\rangle\langle vac|Q_i, \pi = 0\rangle$$

perhaps?

At the moment it seems that the most natural interpretation for this amplitude is: it is the amplitude that corresponds to the BRST amplitude (see equation 4.314) when the Fock representation is used:

$$(\psi, \psi') \equiv \langle \psi^0_{\chi=0=\pi=\eta_0}| \; e^{[\hat{K},\hat{\Omega}]} \; |\psi'^0_{\chi=0=\pi=\eta_0}\rangle$$

$$\int d\eta_0 d\eta_1 \;\; \langle \psi^0_{\chi=0=\pi}| \; e^{[\hat{K},\hat{\Omega}]} \; |\psi'^0_{\chi=0=\pi}\rangle \eta_0\eta_1 =$$

$$\int d\eta_0 d\eta_1 \;\; \langle \psi^0_{\chi=0=\pi}| \; e^{i\hat{\lambda}\hat{\Phi}+\hat{\rho}_0\hat{\eta}_1} |\psi'^0_{\chi=0=\pi}\rangle \eta_0\eta_1 = \langle \psi^0_{\chi=0}| \frac{1}{\hat{\Phi}+i\epsilon} |\psi'^0_{\chi=0}\rangle$$

i.e.,*it is the amplitude that appears in the BRST-Fock approach (recall section 3.2.4).* It can also be said that this object follows from an action that carries a *trivial* representation of the disconnected Z_2 part of the reparametrization group. Paths going back are weighted the same as their forward going counterparts. The sign of the lapse carries this information—it is the only one that can in these path integrals.

This amplitude should thus be called the "BRST-Fock amplitude" (and path integral).

This path integral is otherwise very similar to the earlier ones. The only difference is in the contours of integration—which are those associated to the Fock space inner product.

5.1.4 The BFV path integral: one branch

As explained at the end of section 3.2.3, the path integral in the BRST extended phase space [2, 19, 21] is given by

$$\Gamma' = \int DtDp_tDxDp_xD\lambda D\pi D\eta_0 D\rho_0 D\eta_1 D\rho_1 e^{iS} \tag{5.37}$$

where S is given by

$$S = \int_{\tau_i}^{\tau_f} (\dot{t}p_t + \dot{x}p_x + \dot{\lambda}\pi + \dot{\eta}_0\rho_0 + \dot{\eta}_1\rho_1 - \{\mathcal{O}, \Omega\})d\tau \tag{5.38}$$

where $\mathcal{O}$ is the gauge fixing function. The boundary conditions are that at $\tau = \tau_i, \tau_f$

$$\pi = \rho_1 = \eta_0 = 0, \tag{5.39}$$

and as usual we fix x and t as well[3].

[3] In the literature the variables $C = \eta_0, i\bar{C} = \rho_1, \bar{P} = \rho_0, -iP = \eta_1$ are often used. Notice that this is a canonical transformation.

The Fradkin-Vilkovisky theorem assures us that the path integral is independent of the choice of this function $\mathcal{O}$, although there are some caveats (see reference [20] for example). Let us look at two different types of gauge fixing.

Non-canonical gauge fixing: $\mathcal{O}_{NC} = \rho_1 f(\lambda) + \rho_0 \lambda$

With this choice we have

$$\{\mathcal{O}_{NC}, \Omega\} = \{\rho_1 f(\lambda) + \rho_0\lambda, \eta_0\Phi + \eta_1\pi\} = \rho_1\eta_1 f'(\lambda) + \pi f(\lambda) + \lambda\Phi + \rho_0\eta_1$$

and the action S is given by

$$S = \int_{\tau_i}^{\tau_f} d\tau(\dot{t}p_t + \dot{x}p_x + \dot{\lambda}\pi + \dot{\eta}_0\rho_0 + \dot{\eta}_1\rho_1 - \rho_1\eta_1 f'(\lambda) - \pi f(\lambda) - \lambda\Phi - \rho_0\eta_1) \quad (5.40)$$

For simplicity let us look at the non-relativistic free case using the gauge $f(\lambda) = 0$ (but see ref. [21]).

Result:

$$\Gamma' = (\frac{m}{(2\pi i\Delta t)}))^{\frac{1}{2}} e^{\frac{i(\Delta x)^2}{2\Delta t}} \quad (5.41)$$

when the range of the lapse is the full one [14]—just as for the unconstrained particle of section 1. If the range is $(0, \infty)$ we get the Green's function for the Schrödinger equation [21]. The last λ integration is indeed

$$\frac{\Delta\tau}{2\pi}\int dp_{x0} d\lambda_0 \delta(\Delta t - 2m\Delta\tau\lambda_0) e^{i(p_{x0}\Delta x - \lambda_0 p_{x0}^2\Delta\tau)} \quad (5.42)$$

Hence, with the $(0, \infty)$ range for λ we get the usual propagator times a Heaviside function, $\Theta(\Delta t)$,

$$\Gamma'_G = \frac{\Theta(\Delta t)}{4\pi m}\int dp_{x0} e^{i(p_{x0}\Delta x - \Delta t\frac{p_{x0}^2}{2m})} = \Theta(\Delta t)\Gamma' \quad (5.43)$$

It isn't hard to see that this is a Green's function for the Schrödinger operator—see section 1. It is interesting to note that we can do all but the x, p_x integrations to yield the unconstrained path integral once more,

$$\Gamma' = \int DxDp_x e^{i\int_{\tau_i}^{\tau_f} d\tau (p_x \dot{x} - \frac{p_x^2 \Delta t}{\Delta\tau 2m})} \tag{5.44}$$

or

$$\Gamma'_G = \frac{\Theta(\Delta t)}{2m} \int DxDp_x e^{i\int_{\tau_i}^{\tau_f} d\tau (p_x \dot{x} - \frac{p_x^2 \Delta t}{\Delta\tau 2m})} \tag{5.45}$$

if we have chosen the half range for the "lapse" λ.

In fact, this type of result is fairly general. Consider the general constraint case in the simple gauge above. $\Gamma'_{f=0} =$

$$\int DtDp_tDxDp_xD\lambda D\pi D\eta_0 D\rho_0 D\eta_1 D\rho_1 \; e^{i\int_{\tau_i}^{\tau_f} d\tau (\dot{t}p_t + \dot{x}p_x + \dot{\lambda}\pi + \dot{\eta}_0\rho_0 + \dot{\eta}_1\rho_1 - \lambda\Phi - \rho_0\eta_1)} =$$

$$\int d\lambda_0 \; \Delta\tau \int DtDp_tDxDp_x \; e^{i\int_{\tau_i}^{\tau_f} d\tau (\dot{t}p_t + \dot{x}p_x) - \lambda_0 \int_{\tau_i}^{\tau_f} d\tau (\Phi)}$$

which, as I will discuss in the next section, is just

$$\langle x_\mu | \; \delta(\hat{\Phi}) | y_\mu \rangle = \langle x_\mu, \pi = 0 | \; e^{i\hat{\lambda}\hat{\Phi}} | y_\mu, \pi = 0 \rangle$$

which are the Dirac (Fock) amplitudes—full range case (half-range) of the previous chapter. Thus, for the one-branch case we have a solution to the (general, modulo ordering) Schrödinger equation.

Canonical gauge fixing: $\mathcal{O}_C = \rho_1 \chi'(t, x, p_x, p_t, \tau) + \rho_0 \lambda$

Here we have

$$\begin{aligned}\mathcal{H}_C &= \{\mathcal{O}_C, \Omega\} \\ &= \{\rho_1\chi' + \rho_0\lambda, \eta_0\Phi + \eta_1\pi\} = \rho_1\eta_0\{\chi', \Phi\} + \pi\chi' + \lambda\Phi + \rho_0\eta_1\end{aligned} \tag{5.46}$$

Result:

Let $\chi' \equiv \chi/\epsilon$. Taking the limit $\epsilon \to 0$ the path integral reduces to the Faddeev path integral of section 5.1.1, as we expect by now, with our new understanding of the BRST operator formalism. If the limit of integration is taken at the end, the result is still the same.

Let us see how this works. With this choice of gauge fixing the action S_C is

$$S_C = \int d\tau(\dot{t}p_t + \dot{x}p_x + \dot{\lambda}\pi + \dot{\eta}_0\rho_0 + \dot{\eta}_1\rho_1 - \rho_1\eta_0\{\frac{\chi}{\epsilon}, \Phi\} - \pi\frac{\chi}{\epsilon} - \lambda\Phi - \rho_0\eta_1) \tag{5.47}$$

Now consider the change of variables

$$\begin{aligned}\pi &\longrightarrow \pi\epsilon \\ \rho_1 &\longrightarrow \rho_1\epsilon\end{aligned} \tag{5.48}$$

Notice that the measure is unaffected by this change of variables, since the boundary conditions on the path integral mean that there are as many π as ρ_1 integrations: the jacobian for this change of variables is thus one.

Now let us take the limit $\epsilon \to 0$:

$$S \longrightarrow \int d\tau(\dot{t}p_t + \dot{x}p_x + \dot{\eta}_0\rho_0 - \rho_1\eta_0\{\chi, \Phi\} - \pi\chi - \lambda\Phi - \rho_0\eta_1) \tag{5.49}$$

After the ghost integrations we formally obtain the Faddeev path integral—for the full range of the lapse case—the only tricky points being that there is no chance that the jacobian $\{\chi, \Phi\}$ will get an absolute value, and that it isn't clear, of course, that one can interchange these limits in general—although things work here. At any rate, this gauge fixing is not really canonical as it isn't operating at the boundaries: we are really using the measure $D\mu$ as the boundary conditions imply that there is no gauge fixing at the boundaries.

We can also think about more general gauge fixings and the resulting path integrals in configuration space after the momenta are integrated. This will be discussed shortly.

5.1.5 The BFV path integral: multiple branches

The path integral is again given by

$$\Gamma' = \int_{\tau_i}^{\tau_f} Dt Dp_t Dx Dp_x D\lambda D\pi D\eta_0 D\rho_0 D\eta_1 D\rho_1 e^{iS} \tag{5.50}$$

where S is given by

$$S = \int_{\tau_i}^{\tau_f} (\dot{t}p_t + \dot{x}p_x + \dot{\lambda}\pi + \dot{\eta}_0\rho_0 + \dot{\eta}_1\rho_1 - \{\mathcal{O}, \Omega\})d\tau \tag{5.51}$$

where $\mathcal{O}$ is the gauge fixing function, just as in the previous section. In fact, since we did not specify right away the form of the constraint, a good number of the equations in the previous section hold here, but for convenience I will write them again. The boundary conditions are that at $\tau = \tau_i, \tau_f$

$$\pi = \rho_1 = \eta_0 = 0, \tag{5.52}$$

and as usual we fix x and t as well.

Non-canonical gauge fixing: $\mathcal{O}_{NC} = \rho_1 f(\lambda) + \rho_0 \lambda$

With this choice we have

$$\{\mathcal{O} - NC, \Omega\} = \{\rho_1 f(\lambda) + \rho_0\lambda, \eta_0\Phi + \eta_1\pi\} = \rho_1\eta_1 f'(\lambda) + \pi f(\lambda) + \lambda\Phi + \rho_0\eta_1$$

and the action S is given by

$$S = \int_{\tau_i}^{\tau_f} d\tau(\dot{t}p_t + \dot{x}p_x + \dot{\lambda}\pi + \dot{\eta}_0\rho_0 + \dot{\eta}_1\rho_1 - \rho_1\eta_1 f'(\lambda) - \pi f(\lambda) - \lambda\Phi - \rho_0\eta_1) \quad (5.53)$$

For simplicity let us use the gauge $f(\lambda) = 0$ (but see ref. [21]).

Result: when the full range of the lapse is used, we get the Hadamard Green function,

$$\Delta_1(x-y) = \frac{1}{(2\pi)^3}\int d^4k\ \delta(k^2 - m^2)\ e^{ik(x-y)} = \quad (5.54)$$
$$\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d\lambda \int d^4k\ e^{ik(x-y)+i\lambda(k^2-m^2)}$$

which is a solution to the Klein-Gordon equation. If we use instead the half-range for the lapse we get the Feynman amplitude,

$$i\Delta_F(x-y) = \frac{1}{(2\pi)^4}\int d^4k\ \frac{e^{-ik(x-y)}}{k^2 - m^2 + i\epsilon} = \frac{-i}{(2\pi)^4}\int_O^{\infty} d\lambda \int d^4k\ e^{-ik(x-y)-\lambda(k^2-m^2+i\epsilon)}$$
$$\sim \langle x_\mu|\ \frac{1}{\hat{\Phi} + i\epsilon}\ |y_\mu\rangle \quad (5.55)$$

which satisfies the Klein-Gordon compostion law. This corresponds to an action that carries a *trivial* representation of the Z_2 part of the reparametrization group, as opposed to the previous one, which is faithful. The half range case can be pictured by a lapse with an absolute value—providing a trivial representation (the lapse changes sign, but the absolute value destroys this effect.)

It is not possible to get all the other Green functions we discussed, because it is not possible to write some of those amplitudes in the form

$$\langle x_\mu, \pi = 0|\, e^{i\hat{\lambda}\hat{G}}|y_\mu, \pi = 0\rangle \tag{5.56}$$

for some version of the constraint. We can get the one branch cases, of course.

Canonical gauge fixing: $\mathcal{O}_C = \rho_1\chi'(t, x, p_x, p_t, \tau) + \rho_0\lambda$

Here we have

$$\begin{aligned} \mathcal{H}_C &= \{\mathcal{O}_C, \Omega\} \\ &= \{\rho_1\chi' + \rho_0\lambda, \eta_0\Phi + \eta_1\pi\} = \rho_1\eta_0\{\chi', \Phi\} + \pi\chi' + \lambda\Phi + \rho_0\eta_1 \end{aligned} \tag{5.57}$$

Result:

Let $\chi' \equiv \chi/\epsilon$. As before, taking the limit $\epsilon \to 0$ the path integral formally reduces to the Faddeev path integral of section 5.1.1. However, notice that there isn't an absolute value on the determinant. Recall that the Faddeev path integral did not converge in such a situation. Thus, once we have taken this limit, this path integral converges only with a half-range of the lagrange multiplier, and then it yields the Feynman propagator.

What this is telling us is that the Fradkin-Vilkovisky theorem conditions do not allow for this "degenerate" gauge-fixing, if the full range of the lapse representation is desired. See reference [20] for extensive discussions on this topic.

Finally, let me point out that using the non-canonical gauge-fixing, and using the special case $f(\lambda) = 0$, the action in the path integral is

$$S = \int_{\tau_i}^{\tau_f} d\tau(\dot{t}p_t + \dot{x}p_x + \dot{\lambda}\pi + \dot{\eta}_0\rho_0 + \dot{\eta}_1\rho_1 - \lambda\Phi - \rho_0\eta_1) \tag{5.58}$$

and the path integral can be considerably simplified ([27]): $\Gamma'_{f=0} =$

$$\int DtDp_tDxDp_xD\lambda D\pi D\eta_0 D\rho_0 D\eta_1 D\rho_1 \; e^{i\int_{\tau_i}^{\tau_f} d\tau(\dot{t}p_t + \dot{x}p_x + \dot{\lambda}\pi + \dot{\eta}_0\rho_0 + \dot{\eta}_1\rho_1 - \lambda\Phi - \rho_0\eta_1)} =$$

$$\int d\lambda_0 \; \Delta\tau \int DtDp_tDxDp_x \; e^{i\int_{\tau_i}^{\tau_f} d\tau(\dot{t}p_t + \dot{x}p_x) - \lambda_0 \int_{\tau_i}^{\tau_f} d\tau\, \Phi} =$$

$$\Delta\tau \int DtDp_tDxDp_x \; e^{i\int_{\tau_i}^{\tau_f} d\tau(\dot{t}p_t + \dot{x}p_x)} \; \delta(\int_{\tau_i}^{\tau_f} d\tau\, \Phi) \tag{5.59}$$

What is this expression? The answer is simply—for the general constraint case

$$\Gamma'_{f=0} = \langle x_\mu| \; \delta(\hat{\Phi})|y_\mu\rangle = \langle x_\mu, \pi{=}0| \; e^{i\hat{\lambda}\hat{\Phi}}|y_\mu, \pi{=}0\rangle =$$

$$\Delta\tau \int_{-\infty}^{\infty} d\lambda \; \lim_{N\to\infty} \langle x_\mu| \; \left(1 + i\frac{\Delta\tau}{N}\hat{\Phi}\right)^N |y_\mu\rangle =$$

$$\Delta\tau \int_{-\infty}^{\infty} d\lambda \int dp_0^4 \left(\prod_{i=1}^N dx_i^4 dp_i^4\right) \; \lim_{N\to\infty} \langle x_\mu| \; \left(1 + i\epsilon_1\hat{\Phi}\right)|p_{\mu 0}\rangle\langle p_{\mu 0}|\cdot$$

$$|x_{\mu 1}\rangle\langle x_{\mu 1}|... \left(1 + i\epsilon_N\hat{\Phi}\right)|y_\mu\rangle =$$

$$\int d\lambda \; \Delta\tau \int DtDp_tDxDp_x \; e^{i\int_{\tau_i}^{\tau_f} d\tau(\dot{t}p_t + \dot{x}p_x) - \lambda\int_{\tau_i}^{\tau_f} d\tau\, \Phi} \tag{5.60}$$

Indeed, we recover our old Dirac amplitude, which of course satisfies the constraints for the full range of the lapse (see [27, 28]), since

$$\langle x_\mu| \; \hat{\Phi} \; \delta(\hat{\Phi})|y_\mu\rangle = 0 \tag{5.61}$$

For the half-ranged case, the BRST-Fock representation, we get the amplitude

$$\Gamma_F = \langle x_\mu | \frac{1}{\hat{\Phi} + i\epsilon} | y_\mu \rangle \tag{5.62}$$

5.1.6 Constraint rescalings and canonical transformations in BRST extended phase space

This section clarifies the issue of constraint rescalings, interpreted within the BRST formalism. One would expect by now that rescaling the constraint

$$\Phi \longrightarrow \Omega^2 \Phi \equiv \Phi' \tag{5.63}$$

may affect the path integral, as we saw a boundary effect in the Faddeev path integral, even though the constraint surface is unaffected. We already interpreted this effect as being equivalent to performing a transformation in the state space. Let us see what happens in BRST.

Indeed, consider again the simple situation in section 5.1.1. Since the gauge degrees of freedom are added inserting an identity like

$$1 = \int DQDP\ \delta(Q - Q(q,p,\tau))\delta(P) e^{i\int_{\tau_i}^{\tau_f} d\tau P\dot{Q}}$$

where the measure is defined by

$$DQDP\delta(Q - Q(q,p,\tau))\delta(P) = dP_0\delta(P_0) \prod_{i=1}^{N} dQ_i dP_i \delta(Q_i - Q(q,p,\tau))\delta(P_i)$$

in the physical reduced phase space path integral, using a rescaled constraint would have not affected things: we would have used instead

$$1 = \int DQDP\delta(Q - Q(q,p,\tau))\delta(\Omega^2 P)\Omega^2 e^{i\int_{\tau_i}^{\tau_f} d\tau P\dot{Q}}$$

with the measure

$$DQDP\delta(Q - Q(q,p,\tau))\delta(P)\Omega^2 =$$

$$dP_0\delta(\Omega^2 P_0)\Omega^2 \prod_{i=1}^{N} dQ_i dP_i \delta(Q_i - Q(q,p,\tau))\delta(\Omega^2 P_i)\Omega^2$$

In fact, the safest thing to do seems to be to use the measure

$$\begin{aligned} D\mu^* &\equiv DqDpDQDP\delta(\chi)\delta(\Phi)|\{\chi,\Phi\}| \\ &\equiv \left(\frac{dp_0 dP_0}{2\pi}\prod_{i=1}^{N} dQ_i dP_i \frac{dq_i dp_i}{2\pi}\right)\cdot \qquad (5.64) \\ &\left(\delta(P_0)\prod_{i=1}^{N}\delta(Q_i - Q(q,p,\tau))\delta(P_i)\right) dQ_0\ \delta(Q_0 - Q(q,p,\tau)) \end{aligned}$$

as it is impervious to constraint or gauge fixing rescalings. However, as remarked in section 5.1.4, the measure that appears in the BFV formalism—with the boundary conditions we used there!—is not $D\mu^*$ but $D\mu$. So the path integral there is seen to depend on Ω. In fact, it isn't hard to see that under the rescaling $\Phi \to \Omega^2\Phi \equiv \Phi'$ the path integral changes by the rule

$$\Gamma'_\Phi \longrightarrow \Omega^{-2}\Gamma'_\Phi = \Gamma'_{\Omega^2\Phi} \qquad (5.65)$$

There is, as we mentioned, an interpretation for this effect within BFV, however. If in the path integral one performs the canonical transformation (notice that the

Poisson bracket is unaffected)

$$\lambda \longrightarrow \lambda' \equiv \Omega^{-2}\lambda,$$

and

$$\pi \longrightarrow \pi' \equiv \Omega^{2}\pi$$

the action—as we will now show—doesn't change, except that the constraint and the gauge fixing function are rescaled. Here Ω has to be a constant, otherwise other brackets will be affected.

Consider the more general gauge-fixing

$$\mathcal{O}_{NC} = \rho_1\chi(\lambda, x^\mu) + \rho_0\lambda \tag{5.66}$$

With this choice we have

$$\{\mathcal{O} - NC, \Omega\} = \{\rho_1\chi(\lambda, x^\mu) + \rho_0\lambda, \eta_0\Phi + \eta_1\pi\} =$$

$$\rho_1\eta_1\{\pi, \chi(\lambda, x^\mu)\} + \rho_1\eta_0\{\Phi, \chi(\lambda, x^\mu)\} + \pi\chi(\lambda, x^\mu) + \lambda\Phi + \rho_0\eta_1 \tag{5.67}$$

and the action is given by

$$\begin{aligned} S &= \int_{\tau_i}^{\tau_f} d\tau \Big[\dot{t}p_t + \dot{x}p_x + \dot{\lambda}\pi + \dot{\eta}_0\rho_0 + \dot{\eta}_1\rho_1 - \rho_1\eta_1\{\pi, \chi(\lambda, x^\mu)\} - \\ &\quad \rho_1\eta_0\{\Phi, \chi(\lambda, x^\mu)\} - \pi\chi(\lambda, x^\mu) - \lambda\Phi - \rho_0\eta_1] \end{aligned} \tag{5.68}$$

Now, let us perform the above transformation. The action becomes

$$\begin{aligned} S' &= \int_{\tau_i}^{\tau_f} d\tau \Big[\dot{t}p_t + \dot{x}p_x + \dot{\lambda}\pi + \dot{\eta}_0\rho_0 + \dot{\eta}_1\rho_1 - \rho_1\eta_1\{\pi, \chi(\Omega^{-2}\lambda, x^\mu)\}\Omega^2 - \\ &\quad \rho_1\eta_0\{\Phi, \chi(\Omega^{-2}\lambda, x^\mu)\} - \Omega^2\pi\chi(\Omega^{-2}\lambda, x^\mu) - \Omega^{-2}\lambda\Phi - \rho_0\eta_1\Big] \end{aligned} \tag{5.69}$$

Now let us define a new constraint $\Phi' = \Omega^{-2}\Phi$ and a new gauge-fixing function $\chi' = \Omega^2\chi(\Omega^{-2}\lambda, x^\mu)$. We see that the have our old action with these new constraint and gauge-fixing:

$$S' = \int_{\tau_i}^{\tau_f} d\tau \left[\dot{t}p_t + \dot{x}p_x + \dot{\lambda}\pi + \dot{\eta}_0\rho_0 + \dot{\eta}_1\rho_1 - \rho_1\eta_1\{\pi, \chi'\} - \right. \\ \left. \rho_1\eta_0\{\Phi', \chi'\} - \pi\chi' - \lambda\Phi' - \rho_0\eta_1\right] \qquad (5.70)$$

Thus, the result of this active canonical transformation on the action is to rescale the constraint and the gauge-fixing function. Now consider the path integral, and recall that we have boundary conditions on π. Let us now perform the above canonical transformation on the action. We can now

a) obtain an equivalent path integral with a rescaled constraint and gauge-fixing, or

b) absorb the transformation on the measure, up to a boundary term, because there is one more λ integration than π.

Since the Fradkin-Vilkovisky theorem [2, 19] says that the action is invariant under gauge- fixing changes (and this is obvious in our path integrals), *all that the canonical transformation does in* a) *is to rescale the constraint.* Notice that this is essentially due to the boundary conditions on π. If we didn't have such boundary condition the measure would be $D\mu^*$, and there wouldn't be such an effect.

Thus, a canonical transformation in the action is equivalent, via the Fradkin-Vilkovisky (FV) theorem, to a rescaling of the constraint and of the amplitude.

This transformation is not unitary in the physical space, but as I explained earlier (Chapter 1), this effect can be understood as a simple change of normalization of the basis kets (e.g., the Newton-Wigner states versus the covariant ones for the relativistic case), or is a manifestation of the fact that one must choose the inner product and observables in the theory by using some external input—like physics!

To summarize: we started with an action that is not gauge invariant at the boundaries, so if we insist on gauge-fixing invariance (FV theorem) we cannot gauge-fix at the boundaries, and as a result we have this rescaling of the constraint effect.

If we don't insist on gauge-fixing invariance, but demand "covariance" only—understanding the gauge-fixing dependence as an effect of canonical transformations in the RPS^*—then the measure one uses is $D\mu^*$, and there won't be such a rescaling effect (however, the ambiguities in the quantized reduced phase space inner product are always going to be there.)

5.1.7 Skeletonizations and curved space-time

In this subsection I want to quickly review some of the work that has already been done on the issue of how to skeletonize path integrals in the case where there is a curved background [29, 33, 45–47].

Consider first the case of a path integral in configuration space. Without the guidance of a canonical (operator) scheme one needs some other principles to skeletonize the action. In this case this requirement is space-time covariance.

Two ingredients enter the path integral: the skeletonized action and the skeletonized measure. For the first it is natural to use the action of the classical path that connects the endpoints of the skeletonization. This is the Hamilton-Jacobi function, and it is a scalar in both its arguments, the endpoints—provided we started with an invariant action, of course.

Of the measure one requires space-time covariance. Unfortunately, as we will now see, this does not fix the measure uniquely.

Parker [46] provides us with a beautiful connection between this path integral and the wave equation, which I will summarize here.

Consider the path integral

$$\langle x, s|x', 0\rangle = \int d[x(s')]\,[\Delta^p]\ \exp\left(\frac{i}{\hbar}\int_0^s d\tau\ \{\frac{1}{2}mg_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta - \frac{\hbar^2}{2m}[\xi + \frac{1}{3}(p-1)]R(x)\}\right) \tag{5.71}$$

Parker shows that this amplitude—which I will define more precisely in a moment—satisfies the equation

$$i\hbar\frac{\partial}{\partial s}\langle x, s|x', 0\rangle = [-\frac{\hbar^2}{2m}g^{\alpha\beta}(x)\nabla_\alpha\nabla_\beta + \frac{\hbar^2}{2m}\xi R(x)]\ \langle x, s|x', 0\rangle \tag{5.72}$$

and moreover

$$\lim_{s\to 0}\langle x, s|x', 0\rangle = [g(x)]^{-1/2}\delta(x - x') \tag{5.73}$$

The measure in the path integral is given by

$$\left(\frac{m}{2\pi i\epsilon}\right)^{n/2}\prod_{j=1}^{N} d^n x_j\sqrt{g(x_j)}\,[\Delta(x_{j+1}, x_j)]^p\left(\frac{m}{2\pi i\epsilon}\right)^{n/2} \tag{5.74}$$

and where

$$\Delta(x_{j+1}, x_j) = [g(x_{j+1})]^{-1/2} det[-\frac{\partial^2 \sigma(x_{j+1}, x_j)}{\partial x_{j+1} \partial x_j}]\ [g(x_j)]^{-1/2} \tag{5.75}$$

is easily seen to be scalar, since $\sigma(x_{j+1}, x_j)$ is defined to be the proper length—a scalar.

Parker uses a geodesic to compute the skeletonized action.

The main point is that there is an ambiguity in the measure that is associated with the ambiguity in the transition from classical to quantum—ordering ambiguities. In this case this is reflected by the appearance (or modification) of the curvature term in the wave-equation.

Similarly, there is an ambiguity in the phase-space constructions. The measure in such a case is uniquely defined, but there is is again a one parameter family of skeletonizations of the action [29, 33].

This ambiguity is troublesome, and there have been attempts to fix it. For example, Halliwell [27, 29], in the context of minisuperspace, has proposed that the conformal value of the curvature parameter is the right one because it ensures invariance under constraint rescalings. Indeed, consider rescaling the constraint—classically—and then quantizing it. How do we order it? We can interpret the new constraint as having a new metric and a new potential and order covariantly with respect to this new metric, as Halliwell assumes we will do. Then, he shows, the conformal value for the curvature term insures that this paradigm is invariant under such rescalings. This argument assumes that one will take the factor in the quadratic term of the constraint and call it the supermetric.

The choice of the conformal value in the Klein-Gordon equation is interesting because it implies that if the metric is rescaled and the equation is rewritten covariantly with respect to the new metric, the new solution is related in a simple way to the old one—by a rescaling. There is nothing in the theory so far that demands the use of conformally covariant amplitudes, though. Perhaps conformal transformations are ultimately tied to the concept of time and the probabilistic interpretation.

5.2 Path integrals in configuration space (PICSs)

In this section we wil first review some of the path integral approaches in configuration space. Then we will compare them with the ones that we will obtain from integrating out the momenta in the phase space path integrals.

5.2.1 Review of path integral formalisms in configuration space

Fadeev-Popov

Consider the path integral in configuration space

$$A = \int Dz \; e^{iS[z(\tau)]} \tag{5.76}$$

where we assume that both the action and the measure are invariant under gauge-transformations. What does this statement mean for the measure? It means that for an arbitrary functional F we have[4]

$$\int Dz \; F[z^{\mathcal{G}}] = \int Dz \; F[z] \tag{5.77}$$

or

$$D(z^{\mathcal{G}^{-1}}) = Dz \tag{5.78}$$

where $z^{\mathcal{G}}$ is the result of operating on the coordinates with the group. This is also called the Haar masure.

[4]This will hold if it holds in the skeletonized sense, of course.

Now consider the identity

$$1 = \int D\omega \; \delta(F[z^\omega(\tau)])\Delta_{FP} \tag{5.79}$$

where

$$\Delta_{FP} = |\frac{\delta F[z^\omega(\tau)]}{\delta\omega}| \tag{5.80}$$

Notice that for this to be an identity we have to be careful that $D\omega \sim \delta\omega$. Both are invariant measures expressed in terms of whatever parameter we use to describe the group. This equation means that $\delta(F[z^\omega(\tau)])\Delta_{FP}$ is invariant, because $D\omega$ and 1 are. It also means that $\Delta_{FP}|_{F=0} = \Delta_{FP}|_{\omega_0}$ is also invariant, since we have

$$1 = \Delta_{FP}|_{\omega_0} \int D\omega \; \delta(F[z^\omega(\tau)]) \tag{5.81}$$

because any change in ω_0 can be compensated by the measure.

We now insert this identity in the above path integral and change the orders of integration,

$$A = \int D\omega \int Dz \; \delta(F[z^\omega(\tau)])\Delta_{FP} \; e^{iS[z(\tau)]} \tag{5.82}$$

It is not too hard to see now that

$$\mathcal{A} = \int Dz \; \delta(F[z^\omega(\tau)])\Delta_{FP} \; e^{iS[z(\tau)]} \tag{5.83}$$

does not depend on ω. This is because Dz, $\delta(F[z^\omega(\tau)])\Delta_{FP}$, and $S[z(\tau)]$ are all invariant. One can further rewrite this as

$$\mathcal{A} = \int DzD\pi DcD\bar{c} \; \delta(F[z^\omega(\tau)]) \; e^{iS[z(\tau)]+i\int d\tau \; (\pi F[z^\omega(\tau)]+c\,\Delta_{FP}\,\bar{c})} \tag{5.84}$$

where—have to be careful about $d\tau$ factors

$$D\pi = \prod d\pi_i d\tau_i, \quad DcD\bar{c} = \prod dc_i d\bar{c}_i d\tau_i \tag{5.85}$$

This is the Faddeev-Popov path integral.

For the situation at hand, the only tricky point is to find the correct group measure, to compute the determinant correctly. Let the reparametrizations be generated by $\omega(\tau)$:

$$z(\tau) \longrightarrow z(\tau + \omega(\tau)) \approx z(\tau) + \omega \, \frac{d}{d\tau} z(\tau) \tag{5.86}$$

Under reparametrization changes, the quantity $\frac{d}{d\tau}z$ acts as a vector (of the form "v_μ"—a covariant vector). Since z is a scalar, it follows that ω behaves as a vector of the other kind ("v^μ"—contravariant—with the index fixed here, $\mu = 1$ only). Now we need to know what the proper measure for these objects is. Consider[5] the quantity

$$\int \left(\prod_a du_a \right) e^{i \sum_a u_a u_a} = const \tag{5.87}$$

which is a number—no matter how u_a tranform. Let $u_a = v^\mu_a \lambda^{3/2}_a d\tau^{1/2}_a$. Then we have

$$\int \left(\prod_a v^\mu_a \lambda^{3/2}_a d\tau^{1/2}_a \right) e^{i \sum_a v^\mu_a \lambda^2_a v^\mu_a \ \lambda_a d\tau_a} = const \tag{5.88}$$

Now, the integrand is a scalar ($\int \lambda d\tau \ v^2 \lambda^2$), the whole integral is also a scalar, so the measure must be as well. Thus, a proper measure for ω is

$$D\omega = \prod_a d(\omega_a \lambda^{3/2}_a d\tau^{1/2}_a) \tag{5.89}$$

[5] See references [17] for more on this approach to defining path integrals.

We also will need

$$D\lambda = \prod d(\lambda_i^{-1/2} d\tau_i^{1/2} \lambda_i) \tag{5.90}$$

which we can obtain from

$$const = \int \prod d(\lambda_i^{-1/2} d\tau_i^{1/2} \lambda_i) e^{i\int (\lambda_i^{-1/2} d\tau_i^{1/2} \lambda_i)^2} = \int \prod d(\lambda_i^{-1/2} d\tau_i^{1/2} \lambda_i) e^{i\int \lambda\, d\tau} \tag{5.91}$$

For a scalar z, we can form the invariant $\int d\tau\ \lambda z^2$, from which we infer the measure $d(z\lambda^{1/2}\, d\tau^{1/2})$.

For the free relativistic particle, for example, we can now form the path integral

$$\mathcal{A} = \int Dx^\mu D\lambda D\pi Dc D\bar{c}\; e^{i\int_{\tau_i}^{\tau_f} d\tau \{(\frac{1}{2}\frac{\dot{x}\cdot\dot{x}}{\lambda} + \lambda m^2) + \pi(\dot{\lambda} - \chi) + c\,\Delta_{FP}\,\bar{c}\}} \tag{5.92}$$

where the gauge-fixing is $\dot{\lambda} - \chi(\lambda, x^\mu) = 0$, and

$$\Delta_{FP} = \frac{\delta_\omega}{\delta(\omega_a \lambda_a^{3/2} d\tau_a^{1/2})} (\dot{\lambda} - \chi) \tag{5.93}$$

now, to calculate this let us review the following transformation laws: under a change of parametrization generated by $\omega(\tau)$ we have

$$x(\tau) \longrightarrow x(\tau + \omega) \approx x(\tau) + \dot{x}\omega \tag{5.94}$$

$$\lambda(\tau) \longrightarrow \lambda(\tau + \omega)(1 + \dot{\omega}) \approx \lambda(\tau) + \frac{d}{d\tau}(\lambda\omega)$$

$$\dot{\lambda} \longrightarrow \lambda + \frac{d^2}{d\tau^2}(\lambda\omega)$$

Thus

$$\delta_\omega(\dot{\lambda} - \chi) = \delta\dot{\lambda} - \frac{\partial\chi}{\partial\lambda}\delta\lambda - \frac{\partial\chi}{\partial x^\mu}\delta x^\mu = \tag{5.95}$$

$$\frac{d^2}{d\tau^2}(\lambda\omega) - \frac{\partial\chi}{\partial\lambda}\frac{d}{d\tau}(\lambda\omega) - \frac{\partial\chi}{\partial x^\mu}\dot{x}^\mu\omega$$

Hence

$$\Delta_{FP} = \frac{\delta_\omega}{\delta(\omega_a\lambda_a^{3/2}d\tau_a^{1/2})}(\dot{\lambda} - \chi) = \tag{5.96}$$

$$\frac{d^2}{d\tau^2}\frac{\circ}{\lambda^{1/2}d\tau^{1/2}} - \frac{\partial\chi}{\partial\lambda}\frac{\circ}{\lambda^{1/2}d\tau^{1/2}} - \frac{\partial\chi}{\partial x^\mu}\frac{\circ}{\lambda^{3/2}d\tau^{1/2}}$$

Finally ($\epsilon_i = d\tau_i$)

$$\mathcal{A} = \int \epsilon_0 d\lambda_0 \prod_i^{N-1}\left((\lambda_i\epsilon_i)^{d/2}d\lambda_i\epsilon_i d^dx_i d\pi_i dc_i d\bar{c}_i\right) e^{i\int_{\tau_i}^{\tau_f} d\tau\, \{(\frac{1}{2}\frac{x\cdot x}{\lambda}+\lambda m^2)+\pi(\dot{\lambda}-\chi)+c\,\Delta_{FP}\,\bar{c}\}} \tag{5.97}$$

This is essentially the BFV path integral, again up to a normalization constant.

Notice that we can extract factors of $\lambda_i\epsilon_i$, as we defined the gauge transformations and the skeletonizations, ϵ_i, so that these factors are constant. Hence, up to this gauge-invariant constant the path integral reduces to

$$\mathcal{A} = \int \epsilon_0 d\lambda_0 \prod_i^{N-1}\left(d\lambda_i\epsilon_i d^dx_i d\pi_i dc_i d\bar{c}_i\right) e^{i\int_{\tau_i}^{\tau_f} d\tau\, \{(\frac{1}{2}\frac{x\cdot x}{\lambda}+\lambda m^2)+\pi(\dot{\lambda}-\chi)+c\,\Delta_{FP}\,\bar{c}\}} \tag{5.98}$$

This is the path integral that will reappear below.

The geometric path integral

I will now review the geometric path integral construction [16, 17] and show that it is equivalent to the Faddeev-Popov one, up to a numerical, gauge invariant measure factor.

We will discuss the relativistic free particle. The amplitude we want to compute is

$$A(x_i^\mu, x_f^\mu) = \int [d\lambda][dx^\mu] \exp(-S[\lambda, x]) \tag{5.99}$$

where the action is that of the free relativistic particle witha lagrange multiplier,

$$S = \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau \, [\lambda^{-1}\dot{x}^2 + \lambda m^2] \tag{5.100}$$

in imaginary time (here I follow closely the discussion in Cohen et al, in reference [16]).

The path integral is based on a (super)geometric definition of the measure. The basic idea in this construction is illustrated by the following example, in which the measure for the coordinate functions for the particle is defined.

The measure Dx, is defined by first constructing an invariant inner product for $\delta x(\tau) = \delta x_\tau$, which is a vector in the tangent space to the (super)manifold of all functions $x(\tau)$:

$$< \delta z(\tau), \delta y(\tau) >_P \equiv \int_{\tau_i}^{\tau_f} d\tau \, \delta z(\tau) \, \delta y(\tau) \, \lambda(\tau) \tag{5.101}$$

P denotes a point in the (super)manifold, $P = (x(\tau), t(\tau), \lambda(\tau))$. It isn't hard to see that this inner product is gauge invariant[6]. Then the path integral measure in the tangent space is defined by

$$1 \equiv \int D(\delta x(\tau)) \, e^{i<\delta x(\tau), \delta x(\tau)>_P} \tag{5.102}$$

[6]Under a transformation $\tau \to f(\tau)$, we have that $\delta z(\tau) \to \delta z(f(\tau))$, the same transformation law as for $z(\tau)$, and λ transforms as $\lambda(\tau) \to \lambda(f(\tau)) df/d\tau$.

which forces the measure to be a gauge-invariant object. Notice that with a skeletonization τ_i this just means

$$D(\delta x(\tau)) = \prod_{i=1}^{N} d(\delta x(\tau))_i \left(\frac{d\tau_i \lambda(\tau_i)}{\pi}\right)^{d/2} \tag{5.103}$$

where d is the number of space-time dimensions. From a mathematical point of view (again, see [16]) one is defining an inner product (metric) in the tangent space to a (super)manifold. Using this inner product one then defines a volume form in the tangent space (a measure). This in turn is also used for the definition of a volume form in the manifold itself. If the measure in the tangent space is of the form $f(q)da^1 \wedge \cdots \wedge da^n$, where q denotes a point in the manifold, then the form $f(q)dq^1 \wedge \cdots \wedge dq^n$ is a well-behaved volume form for the manifold.

Similarly, the invariant measure for the lagrange multiplier is defined by the following inner product:

$$\langle \delta\lambda_1(\tau), \delta\lambda_2(\tau)\rangle_P = \int_{\tau_i}^{\tau_f} d\tau \, \frac{\delta\lambda_1(\tau)\, \delta\lambda_2(\tau)}{\lambda(\tau)} \tag{5.104}$$

and then by demanding

$$1 = \int [d(\delta\lambda)] \, \exp(-\frac{1}{2} \parallel \delta\lambda \parallel^2) \tag{5.105}$$

It is not too hard to see that λ can be alternatively described by a pure gauge part and a pure non-gauge part, $\lambda \sim (c, f(\tau))$, where $c = \int_{\tau_i}^{\tau_f} d\tau \, \lambda$ is an invariant. Indeed, any λ can be written in the form $\lambda = c\dot{f}$, where $f(\tau)$ satisfies $f(\tau_i) = \tau_i$ and $f(\tau_f) = \tau_f$ (or viceversa).

An important point in this approach [16] is that one can show that the measure for λ can then be split into a gauge dependent and a gauge invariant part—with a gauge invariant jacobian.

Indeed, one finds that

$$\| \delta\lambda \|^2 = \frac{\delta c^2}{c} - \int_{\tau_i}^{\tau_f} d\tau \sqrt{g}\, g\, \xi \Delta \xi =$$

$$\frac{\delta c^2}{c} - \int_{\tau_i}^{\tau_f} d\tau \sqrt{g}\, g\, \xi \left(c^{-2} \lambda^{-1} \frac{d^2}{d\tau^2} \lambda \right) \xi \tag{5.106}$$

where $g = \lambda^2$ is the metric on the line,

$$\Delta = g^{-1} \frac{d}{d\tau} \frac{1}{\sqrt{g}} \frac{d}{d\tau} \sqrt{g} \tag{5.107}$$

is the laplacian associated with this metric, and the corresponding covariant derivative—which on vectors is $(\xi^i)_{;j} = \frac{d\xi}{d\tau} + \frac{1}{2} g^{-1} \frac{dg}{d\tau}$, for example, and the quantity $\xi = \delta f \circ f^{-1}$ is the vector field associated with the diffeomorphism $\tau \to f(\tau)$.

The norm (or, with the obvious generalization,inner product) for ξ is now defined to be

$$\| \xi \|^2 = \int_{\tau_i}^{\tau_f} d\tau \sqrt{g}\, \xi\, g\, \xi \tag{5.108}$$

We can now rewrite equation 5.105 by using equation 5.106 and by writing

$$[d(\delta\lambda)] = J(\lambda)\, d(\delta c)\, [d(\delta\xi)] \tag{5.109}$$

i.e., $1 = \int [d(\delta\lambda)]\ \exp(-\frac{1}{2} \| \delta\lambda \|^2) =$

$$\int J(\lambda)\, d(\delta c)\, [d(\delta\xi)]\ \exp\left(-\frac{1}{2} \frac{\delta c^2}{c} + \frac{1}{2} \int_{\tau_i}^{\tau_f} d\tau \sqrt{g}\, g \xi \Delta \xi \right) =$$

$$J(\lambda)\int d(\delta c)\,[d(\delta\xi)]\;\exp\left(-\frac{1}{2}\frac{\delta c^2}{c}+\frac{1}{2}\int_{\tau_i}^{\tau_f}d\tau\sqrt{g}\,g\xi\Delta\xi\right)=$$

$$J(\lambda)c^{1/2}\mathrm{det}^{-1/2}\left(-c^{-2}\Delta\tau^2\frac{d^2}{d\tau^2}\right)\tag{5.110}$$

as well as equation 5.108, the norm definition for the diffeomorphisms.

We now have to compute the determinant, det' $\left(-c^{-2}\Delta\tau^2\frac{d^2}{d\tau^2}\right)$—where the prime stands for "zero mode excluded". The boundary conditions for the functions into which the operator acts are that they vanish at the boundaries. The eigenfunctions of this operator are $\sin(n\pi\tau/\Delta\tau)$, with eigenvalues $\frac{n^2\pi^2}{c^2}$. To compute the determinant we next use the trick $\log\det A = \mathrm{tr}\log A$—i.e., the logarithm of the product of the eigenvalues is the sum of the logarithms of the eigenvalues,

$$\log\det' = \sum_{n>0}\log(\frac{n^2\pi^2}{c^2}) = -\left.\frac{d}{ds}\right|_{s=0}\sum_{n>0}\left(\frac{n^2\pi^2}{c^2}\right)^{-s}\tag{5.111}$$

since $-\left.\frac{d}{ds}\right|_{s=0}\alpha^{-s}=\log\alpha$, and now this is just

$$-2\zeta(0)\log c+\mathrm{const}=\log(c)+\mathrm{const}\tag{5.112}$$

where

$$\zeta(s)=\sum_{n>0}\left(\frac{1}{n}\right)^s\tag{5.113}$$

is Riemann's ζ-function.

Thus, we find that, up to a normalization constant the determinant equals

$$\det'\left(-c^{-2}\Delta\tau^2\frac{d^2}{d\tau^2}\right)=c\tag{5.114}$$

So we have the measure

$$J = c^{-1/2}\text{det}^{1/2}\left(-c^{-2}\Delta\tau^2\frac{d^2}{d\tau^2}\right) = 1 \tag{5.115}$$

(up to a normalization constant), which is gauge invariant as promised as it depends only on c—trivially in this case.

Consider now the original amplitude that we wanted to compute. The action itself is already invariant, so the gauge volume neatly separates and can be thrown out. Indeed,

$$A(x_i^\mu, x_f^\mu) = \int [d\lambda][dx^\mu]\ \exp(-S[\lambda, x]) =$$

$$\int J\, d(\delta c)\, [d(\delta\xi)]\ [dx^\mu]\ \exp(-S[c, x]) \tag{5.116}$$

with ($\lambda = c/\Delta\tau$)

$$S[c, x] = \frac{1}{2}\int_{\tau_i}^{\tau_f} d\tau\, [\left(\frac{c}{\Delta\tau}\right)^{-1}\dot{x}^2 + \frac{c}{\Delta\tau}m^2] \tag{5.117}$$

Factorizing the gauge group out we end up with

$$A \sim \int d(\delta c)\, [dx^\mu]\ \exp(-S[c, x]) \tag{5.118}$$

The $[dx^\mu]$ integrations can now be performed to yield again a power of our famous determinant and a gauge-invariant exponent (what else could they yield?),

$$\int dc \int [dx^\mu]\ \exp(-S[c, x]) = \int dc\ \exp\left(-\frac{(\Delta x)^2}{2c} - \frac{m^2 c}{2}\right)\left(-c^{-2}\frac{d^2}{d\tau^2}\right)^{-d/2} \tag{5.119}$$

The final "lapse" integrations yield the Feynman propagator or the on-shell amplitude—and no other Green functions—depending on whether we use half the range for c or the full one.

The ambiguities in this method lie in the definition of: a) the inner product, and b) in equation (5.102), as instead of the "1" we could actually use any gauge invariant quantity. At the end we will regularize, at any rate, so as long as the constant is invariant it doesn't matter. But we will see that the BRST phase space path integral needs no regularization.

For more on this approach see the references in [16].

5.2.2 PICS from the Faddeev path integral

For simplicity, let us consider the non-relativistic case. The main lesson from this example is that the phase space integral takes care of regularization automatically, and this yields a non-trivial measure in coordinate space. If we integrate the momenta in the Faddeev path integral—with the measure $D\mu$, say—we obtain

$$\Gamma' = \int DtDx(\frac{2\pi dt}{m})^{\frac{1}{2}} \prod_{i=1}^{N} \{\delta_i(t - f(\tau))(\frac{2\pi dt}{m})^{-\frac{1}{2}}\} e^{i\int_{\tau_i}^{\tau_f} d\tau(m\frac{\dot{x}^2}{2\dot{t}})} \tag{5.120}$$

where $dt \equiv t_i - t_{i-1}$ is the skeletonization, etc. We do obtain here a "gauge-fixed" configuration space path integral.

5.2.3 PICS from the BFV path integral

Here I will relate the BFV path integral to the described configuration space path integrals—by integrating the momenta out. Consider, as an introduction, the non-relativistic case. Performing the momenta integrations in the BFV path integral we get $\Gamma' =$

$$\int DtDxD\pi DcD\bar{c}(8\pi imdt)^{-\frac{1}{2}}\prod_{i=1}^{N}(8\pi imdt)^{-\frac{1}{2}}\cdot$$
$$\exp\left(i\int_{\tau_i}^{\tau_f}d\tau(m\frac{\dot{x}^2}{2\dot{t}}+\pi(\frac{\ddot{t}}{2m}-f(\frac{\dot{t}}{2m}))+ic(-\frac{d^2}{d\tau^2}+f'(\frac{\dot{t}}{2m})\frac{d}{d\tau})\bar{c})\right) \tag{5.121}$$

where the more general non-canonical gauge fixing

$$\mathcal{O}_{NC}=i\bar{C}f(\lambda)+\bar{P}\lambda$$

has been used (recall that $C=\eta_0, i\bar{C}=\rho_1, \bar{P}=\rho_0, -iP=\eta_1$). This is essentially the Faddeev-Popov path integral above, up to a gauge-invariant normalization constant.

Let me now describe two related points. I discussed ealier that the BFV path integral reduces—after a smart gauge choice—to the form (equation 5.60)

$$\Gamma'_{f=0}=\int d\lambda\,\Delta\tau\int DtDp_tDxDp_x\;e^{i\int_{\tau_i}^{\tau_f}d\tau(\dot{t}p_t+\dot{x}p_x)-\int_{\tau_i}^{\tau_f}d\tau\,\lambda\,\Phi}$$

Incidentally, this is the form advocated by Claudio Teitelboim many years ago [24], and traces its origins to earlier work by Feynman and Nambu [25].

This expression needs no regularization. What is the induced measure in configuration space? It is easy to see that after the (gaussian) momentum integrations we are left with

$$\Gamma'_{f=0}=\int dc\,\Delta\tau\int\mathcal{D}t\mathcal{D}x\;e^{\frac{i}{2}\int_{\tau_i}^{\tau_f}d\tau(\frac{\dot{x}^\mu\dot{x}_\mu}{c}+cm^2)} \tag{5.122}$$

where the measure is

$$\mathcal{D}^dx=\prod_{i=1}^{N}d^dx(\tau)_i\,(2\pi id\tau_i\lambda(\tau_i))^{-d/2} \tag{5.123}$$

i.e., “the Feynman and Hibbs measure”, [5], which we can compare with geometric measure, equation 5.103

$$D(x(\tau)) = \prod_{i=1}^{N} d(x(\tau))_i \left(\frac{d\tau_i \lambda(\tau_i)}{\pi}\right)^{d/2}$$

As I claimed, these are the same up to a gauge-invariant constant.

Let me now derive the Faddeev-Popov path integral for the relativistic case. Let us work with the non-canonical gauge-fixing $\mathcal{O}_{NC} = \rho_1 \chi(t, x, \lambda) + \rho_0 \lambda$

With this choice we have $\{\mathcal{O}_{NC}, \Omega\} =$

$$\{\rho_1 f(\lambda) + \rho_0 \lambda, \eta_0 \Phi + \eta_1 \pi\} = \rho_1 \eta_1 \partial_\lambda \chi + 2\rho_1 \eta_0 p_t \partial_t \chi - 2\rho_1 \eta_0 p_x \partial_x \chi \pi \chi + \lambda \Phi + \rho_0 \eta_1$$

and the action S is given by

$$\int_{\tau_i}^{\tau_f} d\tau (\dot{x}^\mu p_\mu + \dot{\lambda}\pi + \dot{\eta}_0 \rho_0 + \dot{\eta}_1 \rho_1 - \rho_1 \eta_1 \partial_\lambda \chi - 2\rho_1 \eta_0 p_t \partial_t \chi + 2\rho_1 \eta_0 p_x \partial_x \chi - \pi \chi - \lambda \Phi - \rho_0 \eta_1) \tag{5.124}$$

It is not hard to perform the gaussian momentum integrations. The result is that one gets the Faddeev-Popov path integral: the action becomes

$$\int_{\tau_i}^{\tau_f} d\tau \left(\frac{\dot{x}^\mu \dot{x}_\mu}{2\lambda} + \lambda m^2/2 + \pi(\dot{\lambda} - \chi) + i\eta_0 \left(-\frac{d^2}{d\tau^2} + \partial_\lambda \chi \frac{d}{d\tau} + \frac{\dot{t}}{\lambda}\partial_t \chi + \frac{\dot{x}}{\lambda}\partial_x \chi\right)\rho_1\right) \tag{5.125}$$

and the measure is ($\epsilon_i = d\tau_i$ are both notations I have employed for the skeletonizations)

$$\prod_{i=1}^{N} d^d x \, \epsilon_i d\lambda \, d\pi \, d\eta_0 \, d\rho_1 \left(\frac{\lambda \epsilon_i}{\pi}\right)^{-d/2} \tag{5.126}$$

Again, measures in the different approaches agree up to gauge-invariant normalization.

5.3 The class of paths that contribute

Let us first study the situation for the case of the non-relativistic parametrized particle. To begin with, we are describing the paths by writing them in the form $x(\tau), t(\tau)$, so any paths are allowed that can be written this way. Unlike the unconstrained case we see that paths going back in time are included in the path integral. However, the action in the path integral has the final say on the matter. Indeed, after momentum integrations we end up with the parametrized action,

$$S = \int_{\tau_i}^{\tau_f} d\tau L = \int_{\tau_i}^{\tau_f} d\tau \, m\frac{\dot{x}^2}{2\dot{t}}$$

Now, to see the path going back and forth in time—turning around- -we need

$$\frac{dt}{dx} = 0 = \frac{dt}{d\tau}\frac{d\tau}{dx};$$

with regular parametrizations this implies $dt/d\tau = 0$ ($dx/d\tau \neq 0$). Clearly the action blows up for such paths. The two worlds $dt/d\tau > 0$ and $dt/d\tau < 0$ are separated and cannot "talk" to each other. We need to choose one, and the boundary conditions force one of them to be realized (one can picture operator insertions like $\Theta(\Delta t) = \Theta(t(\tau_f) - t(\tau_i))$ to forever settle the choice). In this case it is easy to see in the BFV path integral that λ is essentially $dt/d\tau$, as follows from the p_t integration; this is why the λ integration is related to the appearance of the Heaviside theta function. For this reason, paths going back and forth in time are not allowed—i.e., will not contribute in the path integral. The path integral really divides into two: paths going back and paths going forward in time. There will be no particle creation in this formalism unless the action is modified so that it doesn't blow up when $dt/d\tau = 0$ (notice that we are taking the point of view that only differentiable paths contribute).

Notice that the original unconstrained action already did not allow such paths. In the original $(DxDt)$ unconstrained path integral such paths could not even be described; however, going to the parametrized version of the path integral brings in principle all the possible paths of the form $x(\tau), t(\tau)$. If the action doesn't eliminate them—as it does in this case by uncontrolled oscillatory cancellation—they will contribute.

Notice that this discussion also applies to the electromagnetic interaction case. It is also the situation for more general hamiltonians—as in the square-root cases.

Next consider the relativistic *unconstrained* particle (see [22, 50], for example), which we saw is equivalent—up to normalization—to the reduced phase space quantization of one branch at the time. Here we are integrating over paths that go forward in time—even in the parametrized case, as discussed above. That it is possible to construct such a single particle theory and still be consistent with special relativity indeed follows from considering what type of trajectories one is summing over in the path integral and how these change from observer to observer [32]. A Lorentz invariant classification of trajectories in configuration space is provided by the light-cone structure. If we say a trajectory stayed within the light-cone and then integrate over all such trajectories in the path integral, then we are constructing a path integral independently of the choice of Lorentz frame—as long as we are using a Lorentz-invariant action. If, on the other hand, we allow trajectories outside of the light cone, but we ask that they move forward in time, our class of paths will be different than if we had started in a different frame and chosen the paths with the same scheme, as a space-like section of a path will be moving back in time in some other frames. Of course, in the free case the end result will be the same despite this choice of different sets of paths in different frames, because one can set up a one-to-one correspondence between the sets of paths that wonder out of the light-cone in two frames and that have the same action. This, however, is a peculiarity of the simplest case.

Taking only paths that go forward in time corresponds to taking one branch only. Only when the Klein-Gordon equation decouples—we saw—does this yield a covariant result. Thus, in general it is not possible to set up such a one-to-one correspondence between the sets of paths that different observers use, *because the action gets in the way.* For the electromagnetic case with non-trivial electric field, for example, the action is sensitive to changes in time direction. The free case, on the other hand, is characterized by its Z_2 invariance: a path and its time-reversed one have the same action (recall section 2.2).

Let us look carefully at this logic for the free case. We are going to consider two observers, O_A and O_B. They will both start with the same path integral—the free relativistic one with one branch, say, the Faddeev path integral. They will both consider paths that go forward in time (in their respective frames) only, not necessarily causal. Thus, they will agree on all their paths except for some of the *acausal* ones. Not all of them—this depends on the relative velocities of the observers.

Consider one of this problematic acausal paths—in frame O_A, say—and let it be infinitesimal. Thus, this path is going forward in time at a constant velocity grater than c, and as far as observer O_B is concerned it is a bad path: it is going back in time. However, if we change the orientation of this path in the space-time diagram, it is going forward in time *at a speed faster than light*. So this path, after this transformation, is one of the *acausal paths of* O_B. If the action is invariant under Z_2, the observers will agree on the computation of the path integral, and the amplitude will be Lorentz invariant.

What about the interacting case? Let us consider the electromagnetic interaction—in a flat background. There is no longer a saving symmetry. We saw that the breaking term is simply

$$e\,\delta x^{\mu} A_{\mu} \tag{5.127}$$

Our observers cannot match each others' paths anymore. Recall, however, that an acausal path that goes forward in time for one observer becomes a path going back in time for another. It is not too hard to see that a sufficient condition for this case is the vanishing of

$$e \oint dx^\mu A_\mu \tag{5.128}$$

for space-time loops that lay locally on planes with $dt \wedge dx^i$ area elements, because in such a situation we can set up the one-to-one correspondence of paths: again match a path in one coordinate frame to its Lorentz boosted and "inverted" one in the other frame. Part of the action is still invariant under Z_2. So when the observers compare their actions for their respective paths they will find that the difference is indeed $e \oint dx^\mu A_\mu$.

This condition is equivalent to the condition of vanishing curvature we found before, which leads to the decoupling of the Klein-Gordon equation,

$$F_{0i} = [D_0,\, D_i] \tag{5.129}$$

(use Stokes' theorem to see this).

The bad guys in this picture are the acausal paths. If we eliminate them we will always obtain covariant results. What does this mean? Can we insert a Heaviside function at each τ slice restricting $(\Delta x^\mu)^2 > 0$?

As pointed out in reference [22], paths that do not respect causality contribute in the "square-root" action path integral, which is not surprising: all that happens when a path goes faster than light is that the action becomes imaginary. Thus, such paths contribute exponentially rather than in an oscillatory way.

In the case where both branches enter in the path integral we are dealing with paths that go back and forth in time. Our observers will always agree on their path integrals as long as the actions are Lorentz invariant. Causality can be enforced by insertion of a Heaviside function, or by the use of representations in which the lapse is half-ranged.

How about having a curved background? Ferraro [45] shows, for example, that when there is a Killing time-like vector field in the theory, it is possible to take the square-root of the Klein-Gordon equation: the resulting hamiltonian is hermitean. *Remember that when taking the square-root it is crucial to get a hermitean, cavariant hamiltonian.* The hermicity part is not the problem for the electromagnetic case, but it is here. As for space-time covariance, it is not clear what one should demand of comparisons by different observers, as, in general, there are no isometries of the metric analogous to the ones given by Poincaré group.

5.4 Conclusions, summary

There are two main aspects of the path integrals we discussed that are worth noting. One is that one has to be aware of the fact that one may not be using actions that are fully invariant—there may be some residual gauge dependence at the boundaries. When this happens the connection with a quantized phase space is lost, strictly speaking, although for the cases at hand we provided a connection to the RPS^*.

The other tricky aspect is in the determinant in the Faddeev approach. This determinant is directly tied to the composition law and inner product in the physical space. Do we need an absolute value? Although, from the point of view of the quantized reduced phase space it needs to be there, in practice it is relevant only if the determinant changes sign. This means that it vanishes, and for the cases at hand, this is also related to the fact that the reduced phase space is split—in essence that there really wasn't a quantized reduced phase space to begin with. If there existed a quantized phase space then the absolute value would not be necessary.

This determinant then determines the coupling between the branches—for the split RPS case. There are many choices for it, starting from the ambiguities resulting from how to write the constraint to ordering ones. To resolve these ambiguities one needs extra input.

We have succeeded in putting all these path integrals in phase space in some Hilbert space construction. Those we have then related to the path integrals in configuration space. They can all be put in three categories: reduced phase space, Dirac, or Dirac-Fock quantization. The bottom line of our analysis is that *if we are careful we can make sense of all the path integrals.* Then, at the end, we will get the (Dirac) amplitude

$$A_D = \langle x^\mu | \, \delta(\hat{\Phi}) \, | y^\mu \rangle$$

which satisfies the constraints (i.e., the Schrödinger equation or the Klein-Gordon equation), or the (Fock) amplitude

$$A_F = \langle x^\mu | \, \frac{1}{\hat{\Phi} + i\epsilon} \, | y^\mu \rangle$$

which gives us a Green's function. *The only important exception is the Faddeev path integral in the presence of interactions that do not allow for "decoupling".* It is not clear how to make sense of it, and this is tied up to the fact that the on-shell ampltudes do not satisfy in general a composition law. *The Faddeev path integral was designed and is as an object that properly belongs in the quantized reduced phase space approach, and there is no clear generalization of it other than the BVF path integral when there isn't a well-defined quantized reduced phase space.*

I have also reviewed the path integral constructions in configuration space and showed that they are equivalent to the phase space ones up to normalization—up to gauge-invariant normalization constants. An important point is that the BFV amplitude requires no regularization, unlike the configuration space path integrals—Faddeev-Popov and the geometric one—the aformentioned normalization constant gets chosen just “right”.

Finally, I connected the discussion on the factorization of the Klein-Gordon equation with the discussion on the paths that contribute in the path integrals. I showed that the group Z_2, the disconnected part of the diffeomorphism group plays an important role in determining when it is possible to restrict the theories to one branch ans mantain space-time covariance.

Chapter 6

Second quantization and quantum gravity

What is "second quantization"? This terminology comes from the early days of quantum mechanics and the first attempts to quantize relativistic wave equations—such as the Klein-Gordon equation. In this context "second quantization" stands for the idea of find a lagrangian $\mathcal{L}$ that has as the equation of motion the Klein-Gordon equation. The Klein-Gordon wave-function then becomes a field ϕ. This process of second quantization solves two problems associated with the use of the Klein-Gordon equation as a wave-function equation [6].

Firstly, the Klein-Gordon equation is second order in time. This is already a serious fault, as the initial value problem now requires the specification of both the wave-function and its time derivative. The fact that the Schrödinger equation is first order in time is central to quantum mechanics and its interpretation. For the free case, however, we can separate the equation into two parts that are first order in time and that don't "talk" to each other, the positive and negative energy sectors, and that have associated with them unitary quantum theories.

Another problem is that in general one cannot construct a *conserved* and *positive* probability density with the wave-function. A conserved inner product exists, but one that doesn't yield a positive probability.

To sum up: there are two branches, but one cannot in general restrict the theory to one of them without losing unitarity in the positive sector and/or space-time covariance.

A pretty way to understand what is going on is to first rewrite the equation as a first order matrix equation, with two components. The two components are essentially the wave-function and its first derivative, the two independent initial values (like a coordinate and a momentum), or some combination of the two. This equation then looks like a Schrödinger equation but with a hamiltonian that is not hermitean in general when we restrict to one branch (see Baym's book—reference [6]—for a delightful discussion on these issues, as well as [13]).

6.1 Minisuperspace and (2+1)-dimensional quantum gravity

What can we say about the issue of unitarity within the "one-universe" sector? As explained, minisuperspace models are mathematically very similar to the relativistic particle in a curved background. Indeed, one can write an action for full gravity which is very reminiscent of the particle [31]

$$S[g_{ab}, N^a] = \int d\tau \int_\Sigma d^3x \sqrt{gR(U_{ab}U^{ab} - U^2)} \tag{6.1}$$

where $U_{ab} \equiv \dot{g}_{ab} - N_{a;b} - N_{b;a}$, and where g_{ab} is the spatial metric on the spacelike hypersurface Σ labeled by the time τ, $g = det(g_{ab})$, and R is the spatial scalar curvature on Σ. Unitarity within the one branch sector will depend on the supermetric: does it have "time-like" Killing vector (super) fields—can we factorize the Wheeler-DeWitt equation? In general the answer is no for the particle and for minisuperspace [35]. For example, consider the Robertson-Walker model described by the metric

$$ds^2 = \frac{-N^2(\tau)}{q(\tau)} dt^2 + q(\tau) d\Omega_3^2 \tag{6.2}$$

where $d\Omega_3^2$ is the metric on the unit three-sphere [29]. The hamiltonian constraint that corresponds to the appropiate Einstein-Hilbert action with a cosmological term then is [30]

$$\mathcal{H} = \frac{1}{2}(-4p^2 + \Lambda q - 1). \tag{6.3}$$

Whether we can factorize this equation or not depends on the ordering needed for space-time covariance [27]. In fact, in this case there is no ambiguity [29], and the Wheeler-DeWitt equation is

$$\frac{1}{2}\left(4\frac{d^2}{dq^2}+\Lambda q-1\right)\,\Psi=0 \tag{6.4}$$

Decoupling occurs only for the zero cosmological constant case.

The authors in reference [34] have proposed a solution to this problem—in the minisuperspace context—based on another definition of time (unlike the one considered here, "deparametrization", which as we have seen brakes space-time covariance in the general case.)

The idea is to start from the Wheeler-DeWitt equation—with a covariant ordering, of course—and select a subset of its solution space that is positive definite with respect to the Klein-Gordon inner product. This subset is chosen as the set of solutions that are positive frequency with respect to conformal transformations—which are a symmetry in the distant conformal past. In essence, we are picking a subset of solutions that are positive definite in the "past" in a covariant way[1].

A problem of this approach—and of any square-root one—is that of causality with respect to the metric. This is a problem for the particle, but not for the universe, as Wald explains.

[1]Incidentally, we see here another point of view for wanting "time-like" Killing vector field. If there is one it means that we have a "time-like" operator that commutes with the constraint. This means that the Klein-Gordon equation decouples.

Next consider (2+1)-gravity. This is a remarkable system, because it is described by a finite number of degrees of freedom. The constraints eliminate almost all the dynamics.

As discussed by Carlip [38, 37], the different quantization schemes available seem to yield substantially different theories. For example [40, 39], it is possible, through a proper choice of "extrinsic time", to explicitely reduce the theory classically and the quantize the remaining, finite, degrees of freedom. The Chern-Simons approach to the theory is also of the "constraint then quantize" kind, and Carlip [39] has shown their equivalence, at least for simple topologies.

On the other hand we have the Wheeler-DeWitt approach [37], which is much less understood. However, and for whatever that is worth, the results of this work lead me to believe that the correct physics will be described by a theory in which this equation will play a central role. Linear constraints (constraints that can be made globally into a momentum) can probably always be treated in the reduced phase space context. However, the appearance of a quadratic constraint signals a serious departure from the reduced phase space quantization philosophy—as we have seen. When the Klein-Gordon equation is second quantized we can say that a constraint becomes a dynamical equation—one that belongs in a field theory. There is no more talk of constraints or invariances, either. Perhaps this is the way to go for quantum gravity as well.

In terms of the ideas of the previous chapters, these two approaches correspond to the quantized reduced phase space approach and to the Dirac approach. The fact that the constraints become much simpler in the case of a specific foliation (a choice of gauge) can be used to solve classically part of the constraints—the momentum constraints. The rest of the constraints can then be solved classically, or through the Dirac/Wheeler-DeWitt approach. Also, it isn't clear either what happens—in this context—to the case of non-zero cosmological constant. It seems, at least for some minisuperspace models, that the zero Λ case is degenerate—akin to $P^2 \approx 0$.

At any rate, it seems that the conformal factor is the analog in quantum gravity of the time coordinate in the particle case.

(2+1)-dimensional quantum gravity is definitely a promising area for future research.

Chapter 7

Conclusion

The goal of this research was to examine whether it is possible to quantize simple parametrized systems. The answer is yes, one can quantize these systems consistently. However, the development of a probabilistic interpretation is not always possible, and even when it is possible there are some features in these theories that make the quantization process difficult. Let me review what was achieved in the present work.

First I pointed out that even in the unconstrained non-relativistic case, the quantization process contain some ambiguities: there are many choices for the inner product and the observables. I also studied the issue of space-time covariance of the "Schrödinger square-root" equation, and showed that covariance can be mantained if there is a frame in which the electric field is zero—a result that ties nicely with the associated field theory of a Klein-Gordon field in an electromagnetic background.

In the next chapter I studied the various systems classically, specially the behavior of the Dirac reduced phase spaces under changes of gauge-fixing. One of the particular problems of these constrained systems is that the actions are not gauge invariant at the boundaries—this due to the boundary conditions one wants to use or to the non-linearity of the constraint, your choice—and therefore the definition of a reduced phase space is trickier than usual. I showed that changes of the gauge-fixing can be understood in terms of "time"-dependent canonical transformations in the reduced phase space. The reduced phase space for the single branch cases is isomorphic to the unconstrained space of chapter 1, although depending on the gauge fixing it may come in some "time"-dependent coordinates. The reduced phase space for the two branches case is in fact doubled.

The next step was canonical quantization. The fact that a gauge fixing can be regarded as a canonical transformation allowed us as well to define the quantized reduced phase space, where the different gauge fixings play the role of unitary transformations—or "pictures". Quantization of both branches of the reduced phase space makes only sense if the two decouple—if the theory is to preserve unitarity. This, we concluded, is a nice method, but one that has a very small range of applicability.

I then discussed Dirac quantization. The main problem in this approach is that the states selected by the "physicality condition" do not really belong in the original unconstrained inner product space. One of the solutions is to introduce a regularized projector. This is not unique, but I reasoned that as in the unconstrained case, it doesn't have to be unique. The inner product in constrained theories is always defined up to such ambiguities. One can choose to fix it by demanding that certain operators one deems important be hermitean, for example, but the ambiguities are there. To fix them one needs exterior input, like a classical limit or other experimental facts.

For the one branch systems I found no differences between the Dirac quantization approach and the reduced phase space quantization. The constrain/reduce order is not important: Dirac's method yields the same quantum mechanics as the reduced phase space quantization. Both suffer from the same inner product and ordering ambiguities.

In the two branch theories, the above equivalence was also found to hold when decoupling occured in the Dirac quantization. If decoupling does not occur, Dirac quantization does not produce a unitary theory in the one particle sector, while reduced phase space quantization breaks the required invariances by forcing decoupling.

I also showed that requiring that the Dirac inner product be well behaved (real norms) leads to the Klein-Gordon inner product in the relativistic case.

We studied some of the circumstances under which decoupling occurs in the

Dirac covariant formulation. A sufficient condition was (equation 3.44)

$$[D_0,\ g^{ij}D_iD_j + \xi R]$$

where the derivative operators are the fully covariant ones. As an example, with no gravitational background, the electric field has to be null

$$[D_0,\ D_i] = 0$$

which is also the condition for no particle creation in the corresponding second quantized theory.

The Fock space approach is equivalent to the above except in the theories we study in this paper—those that are not fully invariant. I discussed the coordinate representation and found that it leads to the causal amplitudes. The states do not satisfy the constraints. This representation leads, in the coordinate representation, to a lagrange multiplier with half the range. This approach is essentially gauge-fixed from the beginning and it needs no further regularization.

I also showed that for the relativistic free case it yields two branches.

Then the BRST approach was discussed. It is equivalent to either the Dirac (or RPS for the simplest situations) approaches, or to Fock, depending on the representation we choose. It is also a more elegant treatment of the inner product problem. However, it doesn't seem to contain any new physics. I described the state cohomology, paying careful attention to regularization issues. I developed a well defined inner product and a physical projector formalism just as in the Dirac case, from which the path integral representation was derived. For example, I derived the Klein-Gordon inner product within this formlism, and the associated composition law. I was able to prove the Fradkin-Vilkovisky theorem at the operator level.

When the Fock representation of the states was employed, I showed that the BRST-Fock amplitude leads naturally to the Feynman propagator—in the relativistic case, and more generally to amplitudes of the form

$$\langle \frac{1}{\hat{\Phi} + i\epsilon} \rangle$$

where $\hat{\Phi}$ is the constraint.

Path integrals in phase space were derived from all the above approaches, and from those I showed how to derive the path integrals in configuration space: the ambiguities that exist in the Hilbert space description do not disappear (though sometimes they will hide under assumptions like ultralocality of the supermetric).

For the one branch case all the path integrals reduce to the unconstrained case—up to coordinate systems in the Faddeev $D\mu^*$ case. For the two branches case, the determinant that appears in the Faddeev path integral determines the coupling between the branches, and it is essentially the inner product in disguise.

I concluded that the Faddeev path integral is an object that belongs only in the quantized reduced phase space approach—it ceases to make sense when the constraint does not decouple as an operator. For the free case it yields the on-shell propagator if the absolute value of a carefully ordered determinant is used,

$|E_i + E_{i+1}|$. The ordering leads to decoupling of the branches, while the absolute value is just what one expects from the reduced phase space approach—essentially, it is a rule for the Dirac delta function. Thus, this path integral is an artifact: one is really doing two reduced phase space quantizations at the same time. If the constraint is more complicated decoupling can still be imposed by a choice of ordering. But the equation that the amplitude will satisfy is not one that implements other symmetries that may be relevant (like space-time covariance).

The BRST path integral is instead the most general one: it yields reasonable results in the fully interacting cases—the amplitudes are a solution of the constraint or its causal Green function. Neither of the phase space path integrals need regularization—unlike the configuration space path integrals. Other than differences in gauge-invariant normalizations, though, the Faddeev-Popov and geometric path integrals are equivalent to the BFV path integral—and explicitely equivalent to the path integral that results from integrating out the momenta.

The range of the lapse λ is related to the class of paths allowed. For example, causal paths only will contribute in the half-ranged case.

We also saw the role that the nontrivial element of Z_2 in the reparametrization group, the disconnected, path-reversing part, plays in determining whether it is possible to select a branch and mantain space-time covariance in the flat case. In the electromagnetic case the condition that this imposes on the actions is as the condition for decoupling of the covariant Klein-Gordon equation.

Z_2 also plays another role. If the actions carry a faithful representation of the full reparametrization group we get on-shell physics (Dirac representation). If the representation is trivial we get causal amplitudes (Fock representation).

Finally, I discussed the issues of unitarity and the probabilistic interpretation. Unitarity can always be achieved, as long as we have a hermitean hamiltonian. The inner product will in general not be one that leads to a positive definite inner product space for the case of two branches with interactions, so the probability interpretation will be lost. If decoupling is forced some other features of the theory may be lost, like space-time covariance in the particle case. Minisuperspace models are most like the relativistic case in a gravitational background where there is particle creation. What this means in the particle case is that the Klein-Gordon equation fails to decouple in such a background. If decoupling were to be forced, space-time covariance would be broken. The analogy holds here for minisuperspace, and in general it may not be possible to avoid the jump into "third quantization", i.e., the developement and quantization of superactions that yield the constraint as an equation of motion. There are, however, promising alternatives to full decoupling [34].

Acknowledgments

I want to thank my advisors (I am fortunate enough to have two of them), Ling-Lie Chau and Emil Mottola, for their patience, help and support. I also thank my teachers Jackie Barab and Rod Reid, as well as Steven Carlip and Joe Kiskis for help at odd, desperate hours. I dedicate this work to all of them and especially to my son Cristian and the rest of my family.

Bibliography

[1] Dirac, P.A.M., *Lectures in Quantum Mechanics*, Belfer Graduate School of Sciences (Yeshiva U. NY, 1964)
Dirac, P.A.M., Can J Math 2, p 129, 1950

[2] Henneaux, M., Teitelboim, C., *Quantization of Gauge Systems*, Princeton University Press, 1992

[3] Dirac, P.A.M., *Principles of Quantum Mechanics*, 4th ed., p. 103-107, Oxford, 1958

[4] Shankar, R. , *Principles of Quantum Mechanics*, Plenum Press, 1980

[5] Feynman, R P, Hibbs, A R, *Quantum Mechanics and Path Integrals*, New York, McGraw-Hill, 1965

[6] Baym, Gordon, *Lectures on Quantum Mechanics*, Benjamin/Cummings,1979

[7] Lanczos, Cornelius *The Variational Principles of Mechanics*, Dover 1986

[8] Misner, W., Thorne, Wheeler, J A, *Gravitation*, Freeman, 1973

[9] Kuchăr, K. V.,"Canonical Methods of Quantization", in *Quantum Gravity 2*, ed. C. J. Isham, R. Penrose, D. W. Sciama, Oxford: Clarendon Press, 1981

[10] Wald, R M, *General Relativity*, U. Chicago Press, 1984

[11] Samarov, K.L., Sov. Phys. Dokl. 29(11), p 909, November 1984

[12] Sucher, J. , J Math Phys 4, p 17–23, 1963

[13] Feshbach, H., Villars, V., Rev. Mod. Phys. 30, p 24–45, 1958

[14] Henneaux, M., Teitelboim, C., Vergara, JD., Nuclear Physics B 387 (1992) 391-418

[15] Evans, JM, Phys Lett B256, p 245, 1991
Evans, JM, Tuckey, PA, Int J of Mod Phys A8, p 4055, 1993

[16] Cohen, A, Moore, G, Nelson, P, Polchinski, J, Nucl Phys B267, p 143, 1987
Moore, G, Nelson, P, Nucl Phys B266, p 58, 1986
Mazur, P, Mottola, E, Nucl Phys B341, p 187, 1990
Bern, Z, Blau, SK, Mottola, E, Phys Rev D43, p 1212, 1991

[17] Fujikawa, K, Yasuda, O, Nucl. Phys. B245, p 436, 1984
Fujikawa, K, "Path Integral Quantization of Gravitational Interactions—Local Symmetry Properties", in *Summer Kyoto Institute*, 8th, 1986, p 107

[18] Faddeev, LD, Theoret Math Phys 1, p 1, 1970
Faddeev, LD, Slavnov, *Gauge Fields: an Introduction to Quantum Theory*, Frontiers in Physics V 83, Addison-Wesley, 1991
Faddeev, LD, Popov, VN, Sov. Phys.–Usp., Vol. 16, No. 6, May–June 1974
Senjanovic, P, Ann. Phys. 100, p 227, 1976
Itzyknson, C, Zuber, JB, *Quantum Field Theory*, McGraw-Hill, 1980, section 9-3

[19] Fradkin, ES, Vilkovisky, GA, Phys Lett 55B, p 224, 1975
Batalin, IA, Vilkovisky, GA, Phys Lett 69B, p 309, 1977
Fradkin, ES, Fradkina, TE, Phys Lett 72B, p 343, 1978
Batalin, IA, Fradkin, ES, Yad iz 39, p 231, 1984
Batalin, IA, Fradkin, ES, Riv. Nuovo Cimento v. 9, p 1, 1966
Henneaux, M., Teitelboim, C., *Quantization of Gauge Systems*, Princeton University Press, 1992
Henneaux, M, Phys Rep 126, p 1, 1985
Fradkin, ES, Vilkovisky, GA, CERN report TH 2332-CERN (1977)

[20] Govaerts, J, Int. J. Mod. Phys. A4, p 173, 1989
Govaerts, J, Int. J. Mod. Phys. A4, p 173, 1989
Govaerts, J, Int. J. Mod. Phys. A5, p 3625, 1990
Govaerts, J, Troost, W, Class. Quantum Grav. 8, p 1723, 1991
Govaerts, J, *Hamiltonian Quantisation and Constrained Dynamics*, Leuven Notes in Mathematical Physics, v. 4, Leuven U. Press, 1991

[21] Batlle, C, Gomis, J, Roca, J, Class Quantum Grav 5, p1663, 1988
Gomis, J, Roca, J, Phys Lett B207, p 309, 1988
Gomis, J, Paris, J, Roca, J, Class Quantum Grav 8, p 1053,1991

[22] Redmount, IH, Suen, W, Int J of Mod Phys A8, P 1629, 1993

[23] Marnelius, R, Nucl.Phys. B418:353-378,1994 (hep-th/9309002)
Marnelius, R, Phys.Lett.B318:92-98,1994
Marnelius, R, Nucl.Phys. B391:621-650,1993

Marnelius, R, Goteborg-93-17, Aug 93, (hep-th/9309004)
Marnelius, R, Nucl.Phys. B351:474-490,1991

[24] Teitelboim, C, Phys. Rev. D25, p 3159–3179, 1982
Teitelboim, C, Phys. Rev. D28, p 297–309, 1982

[25] Feynman, R.P., Phys. Rev. 80, p 440, 1950
Feynman, R.P., Phys. Rev. 76, p 749 and p 769, 1949
Nambu, Y, Prog. Theor. Phys. vol V, n.1, p 82, 1950

[26] Halliwell, J J, Ortiz, M E, "Sum-Over-Histories Origin of the Composition Laws of Relativistic Quantum Mechanics", CPT#2134, Imperial–TP/92–93/06, gr-qc 9211004, Oct 1992

[27] Halliwell, J J, "The Wheeler-DeWitt Equation and the Path Integral in Minisuperspace and Quantum Cosmology", in *Conceptual Problems of Quantum Gravity*, A Ashtekar, J Stachel eds., Einstein Studies v.2, 1991

[28] Halliwell, J J, Hartle, J B, Phys Rev D43, p 1170, 1991

[29] Halliwell, J J, Phys Rev D38, p 2468-2481, 1988

[30] Louko, J., Class. Quantum Grav. 4, p 581, 1987

[31] Hartle, J. B., Kuchăr, K. V., Phys Rev D34, p 2323–2331, 1986

[32] Hartle, J.B., Phys Rev D37, p 2818, 1988

[33] Kuchăr, K. V., J. Math. Phys. 24, p 2122, 1983

[34] Wald, R., "A Proposal for Solving the 'Problem of Time' in Quantum Gravity", gr-qc/9305024, Phys Rev D48, 2377, 1993
Higuchi, A, Wald, R, "Applications of a new proposal for solving the 'problem of time' to some simple quantum cosmological systems.", PACS# : 04.60.+n, 03.70.+k, 04.20.Cv

[35] Hajicek, P, Phys Rev D34, p 1040–1048, 1986

[36] Carlip, S, "Time in (2+1)-Dimensional Quantum Gravity", gr-qc/9405043

[37] Carlip, S, Class. Quantum Grav. 11 , 31–39, 1994

[38] Carlip, S, "Six Ways to Quantize (2+1)-Dimensional Gravity", gr-qc/9305020

[39] Carlip, S, Phys Rev D42, p 2647–2654, 1990

[40] Moncrief, V, J. Math. Phys. 30, p 2907–2914, 1989

[41] Nelson, J E, “The Constraints of 2+1 Gravity”, DFTT 53/93

[42] González, G, Pullin, J, Phys Rev D42, p 3395–3400, 1990

[43] Ikemori, H., Phys Rev D40, p 3512–3519, 1989

[44] Rudolph, O, ”Sum-over-histories representation for the causal Green function of free scalar field theory”, DESY 93–145, hep-th/9311018

[45] Ferraro, R, Phys Rev D45, p 1198–1209, 1992

[46] Parker, L, Phys Rev D19, p 438–441, 1979

[47] Parker, L, Phys Rev D23, p 2850–2869, 1981

[48] Ashtekar, A, Tate, R, CGPG-94/6-1, gr-qc/9405073, (1994)
Rendall, A.D., gr-qc/9403001, (1994)

[49] Guven, J, Vergara, J.D., Phys Rev D44, p 1050-1058, 1991

[50] Ruffini, Giulio, “Path Integrals and the Relativistic Particle”, 1992, *unpublished*. See arxiv.org grqc/9806958.

[51] Ruffini, Giulio, “Classical aspects and quantization of the parametrized non-relativistic particle”, 1994, *unpublished*. See arxiv.org grqc/9806958.

[52] Ruffini, Giulio, “Classical aspects and quantization of the relativistic particle”, 1994, *unpublished*. See arxiv.org grqc/9806958.

[53] Ruffini, Giulio, “Four approaches to quantization of the relativistic particle, arxiv.org grqc/9806958.