Quantizations of conical symplectic resolutions II: category $\mathcal{O}$ and symplectic duality

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Abstract. We define and study category $\mathcal{O}$ for a symplectic resolution, generalizing the classical BGG category $\mathcal{O}$, which is associated with the Springer resolution. This includes the development of intrinsic properties paralleling the BGG case, such as a highest weight structure and analogues of twisting and shuffling functors, along with an extensive discussion of individual examples.

We observe that category $\mathcal{O}$ is often Koszul, and its Koszul dual is often equivalent to category $\mathcal{O}$ for a different symplectic resolution. This leads us to define the notion of a symplectic duality between symplectic resolutions, which is a collection of isomorphisms between representation theoretic and geometric structures, including a Koszul duality between the two categories. This duality has various cohomological consequences, including (conjecturally) an identification of two geometric realizations, due to Nakajima and Ginzburg/Mirković-Vilonen, of weight spaces of simple representations of simply-laced simple algebraic groups.

An appendix by Ivan Losev establishes a key step in the proof that $\mathcal{O}$ is highest weight.

1 Introduction

In this paper, we have two main goals:

- to introduce a version of category $\mathcal{O}$ attached to a symplectic variety with extra structure,

- to describe a conjectured relationship, which we call **symplectic duality**, between pairs of symplectic varieties. The most striking manifestation of this duality is a Koszul duality between the associated categories $\mathcal{O}$.

The motivating example is the classical BGG category $\mathcal{O}$, and the remarkable theorem of Beilinson, Ginzburg and Soergel [BGS96] showing that a regular integral block of category $\mathcal{O}$ is Koszul self-dual. In our formulation, this means that the Springer resolution of the nilpotent cone is self-dual as a symplectic variety.

Our perspective throughout is to study the geometry of symplectic varieties using deformation quantizations and their representation theory. The specific varieties that we want to study are called **conical symplectic resolutions**. The prequel to this paper [BPW] introduced these varieties, their quantizations, and the categories of modules over these quantizations. Here we will concentrate on a particular subcategory of this module category: category $\mathcal{O}$.

Versions of category $\mathcal{O}$ have appeared in many places in the literature: for representations of $U(\mathfrak{g})$ in [BGG76], for rational Cherednik algebras in [GGOR03], for W-algebras in [BGK08, Los12c], and for hypertoric enveloping algebras in [BLPW12]. Our general definition includes all of these particular examples as particular cases, and we are able to prove many basic facts about these categories in a unified way. We will discuss the details of their structure further below.

There is a one striking observation about these categories that we wish to give special prominence: they are often standard Koszul, and Koszul dual to the category $\mathcal{O}$ attached to a different variety. This is the heart of our definition of symplectic duality; much of this paper is concerned with fleshing out the structures surrounding this observation and explaining how it looks in the various examples where it is known to hold.

We interpret symplectic duality as evidence of a hidden mirror symmetry-like connection between the two varieties, though at the moment it is difficult to make the nature of this connection mathematically precise. However, the same pairs of examples have arisen in moduli spaces of vacua for certain S-dual pairs of field theories in physics, suggesting this is not pure coincidence.

BGG category $\mathcal{O}$. Let us discuss the content of the paper in more detail. As mentioned above, our motivating example is the representation theory of $U(\mathfrak{g})$, whose geometric avatar is the Springer resolution of the nilcone by the cotangent bundle $T^*(G/B)$. Fix a regular class $\lambda \in \mathfrak{h}^* \cong H^2(G/B; \mathbb{C})$ and let $\mathcal{O}_a$ be the subcategory of of BGG category $\mathcal{O}$ consisting of modules over $U(\mathfrak{g})$ with the same generalized central character as the simple highest weight module with highest weight $\lambda - \rho$. The subscript stands for *algebraic*, since $\mathcal{O}_a$ is defined as a category of modules over an algebra.

Let $\mathcal{O}_g$ be the category of finitely generated $(\lambda - \rho)$-twisted D-modules[5] on G/B that are smooth with respect to the Schubert stratification. Here the subscript stands for *geometric*, since $\mathcal{O}_g$ is defined as a category of sheaves. The following list gives some of the known

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[5]That is, modules over the sheaf of twisted differential operators denoted by $\mathscr{D}_\lambda$ in [BB81]; if λ is integral, this is simply the sheaf of differential operators on the line bundle with Euler class λ.

structures and properties of the categories $\mathcal{O}_a$ and $\mathcal{O}_g$. Our main goal in the paper will be to generalize these statements from $T^*(G/B)$ to arbitrary conical symplectic resolutions.

1. There exist localization and section functors relating $\mathcal{O}_a$ and $\mathcal{O}_g$. These functors are always inverse derived equivalences, and they are Abelian equivalences if λ is dominant [BB81].

2. The two categories are both highest weight [CPS88] and have graded lifts which are Koszul [BGS96].

3. If λ is integral, then the center of the Yoneda algebra of $\mathcal{O}_g$ is canonically isomorphic to $H^*(G/B; \mathbb{C})$ [Soe90].

4. The Grothendieck group $K(\mathcal{O}_g)$ is isomorphic, via the characteristic cycle map, to the top Borel-Moore homology group of the union of the conormal varieties to the Schubert strata on G/B. This isomorphism intertwines the Euler form with a geometrically-defined intersection form.

5. The group $K(\mathcal{O}_g)$ decomposes as a direct sum over all nilpotent orbits by looking at microlocal supports of D-modules. The top Borel-Moore homology group of the union of the conormal varieties to the Schubert strata on G/B decomposes as a direct sum over all nilpotent orbits via the Beilinson-Bernstein-Deligne (BBD) decomposition [BBD82, CG97]. If λ is integral and $G \cong \mathrm{SL}_r$, then these two decompositions agree.

6. There are two collections of derived auto-equivalences of $\mathcal{O}_a$, and of its graded lift, given by twisting and shuffling functors [AS03, Irv93]. These functors define two commuting actions of the Artin braid group of $\mathfrak{g}$ [AS03, MOS09, BBM04], and they categorify the left and right actions of the Weyl group on its group algebra.

7. The category $\mathcal{O}_a$ is Koszul self-dual [BGS96]. The induced derived auto-equivalence of the graded lift of $\mathcal{O}_a$ exchanges twisting and shuffling functors [MOS09, 6.5].

Category $\mathcal{O}$ in general. We now explain how these results generalize. Let $\mathfrak{M}_0$ be a Poisson cone, and let $\mathfrak{M} \to \mathfrak{M}_0$ be a symplectic resolution of $\mathfrak{M}_0$, equivariant with respect to the conical scaling action of $\mathbb{S} := \mathbb{C}^\times$. Let $\mathcal{D}$ be an $\mathbb{S}$-equivariant quantization of $\mathfrak{M}$, and let A be the ring of $\mathbb{S}$-invariant global sections of $\mathcal{D}$. Many rings of independent interest arise this way, such as spherical rational Cherednik algebras [EG02], central quotients of finite W-algebras [Pre02], central quotients of hypertoric enveloping algebras [BK12], and (conjecturally) quotients of shifted Yangians [KWWY14] (see Section 2.3 for more details).

Let $\mathbb{T} := \mathbb{C}^\times$ act on $\mathfrak{M}$ by Hamiltonian symplectomorphisms that commute with $\mathbb{S}$, and assume that the fixed point set $\mathfrak{M}^\mathbb{T}$ is finite. The action of $\mathbb{T}$ on $\mathfrak{M}$ lifts to $\mathcal{D}$ and induces

a $\mathbb{Z}$-grading on A. Let $A^+ \subset A$ be the non-negatively graded part. We define $\mathcal{O}_a$ to be the category of finitely generated A-modules that are locally finite with respect to A^+. Versions of this category have already been studied for Cherednik algebras [GGOR03, Rou08, GL14], for finite W-algebras [BGK08, Los12c, Web11, BG13], and for hypertoric enveloping algebras [BLPW10, BLPW12]. The classical case is where $\mathfrak{M} = T^*(G/B)$ and A is a central quotient of the universal enveloping algebra of $\mathfrak{g}$; if the period of the quantization is a regular element of $\mathfrak{h}^* \cong H^2(\mathfrak{M};\mathbb{C})$, then $\mathcal{O}_a$ is equivalent to the BGG category $\mathcal{O}_a$ (Remark 3.11).[6]

Let

$$\mathfrak{M}^+ := \{p \in \mathfrak{M} \mid \lim_{\mathbb{T} \ni t \to 0} t \cdot p \text{ exists}\}.$$

We define $\mathcal{O}_g$ to be the category of $\mathcal{D}$-modules that are set-theoretically supported on $\mathfrak{M}^+$ and admit a particularly nice lattice for a certain subalgebra $\mathcal{D}(0) \subset \mathcal{D}$; see Sections 2.5 and 3.15 for a precise definition. If $\mathfrak{M} = T^*(G/B)$ and $\mathbb{T}$ is a generic cocharacter of G, then $\mathfrak{M}^+$ is equal to the union of the conormal varieties to the Schubert strata, and $\mathcal{O}_g$ is equivalent to the category $\mathcal{O}_g$ above. The aforementioned results generalize as follows.

1. There exist localization and section functors relating $\mathcal{O}_a$ and $\mathcal{O}_g$ (Corollary 3.19). These functors are inverse derived equivalences for most quantizations (Theorem 2.9), and they are Abelian equivalences if λ is sufficiently positive (Theorem 2.8).

2. The category $\mathcal{O}_a$ is highest weight for most quantizations (Theorem 5.12[7]), and $\mathcal{O}_g$ is always highest weight (Proposition 5.17). We conjecture that both categories are Koszul (Conjectures 5.14 and 5.18). We can verify this conjecture in many examples, including cotangent bundles of partial flag varieties, S3-varieties, hypertoric varieties, Hilbert schemes on ALE spaces, and some quiver varieties (Section 9).

3. There is a natural graded ring homomorphism from $H^*(\mathfrak{M};\mathbb{C})$ to the Yoneda algebra of $\mathcal{O}_g$. We conjecture that, whenever $\mathcal{O}_g$ is indecomposable (this will depend on the choice of quantization), this homomorphism will be an isomorphism (Conjecture 5.23). We can prove this conjecture for cotangent bundles of partial flag varieties, S3-varieties in type A, and hypertoric varieties (Section 9). We also formulate a stronger version of Conjecture 5.23, relating the equivariant cohomology of $\mathfrak{M}$ to the center of the universal deformation of the Yoneda algebra (Conjecture 10.32), which we prove in the latter two cases.

4. The Grothendieck group $K(\mathcal{O}_g)$ is isomorphic, via the characteristic cycle map, to the top Borel-Moore homology group of $\mathfrak{M}^+$. This isomorphism intertwines the Euler form on the Grothendieck group with the equivariant intersection form defined using the localization formula (Theorem 6.5).

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[6]This statement really requires regularity of the period, otherwise it fails.

[7]The proof of this theorem relies heavily on an appendix by Ivan Losev.

5. The group $K(\mathcal{O}_\mathrm{g})$ decomposes as a direct sum over all symplectic leaves of $\mathfrak{M}_0$ by looking at supports of sheaves (Equation (5)). The top Borel-Moore homology group of $\mathfrak{M}^+$ decomposes via as a direct sum over all symplectic leaves of $\mathfrak{M}_0$ via the BBD decomposition (Equation (7)). Under special assumptions that are satisfied by hypertoric varieties and S3-varieties in type A, these two decompositions agree (Corollary 7.15). A weakening of this relationship holds more generally (Theorem 7.14).

Twisting and shuffling. To state the appropriate generalization of item 6, we need some more definitions. Let W be the Namikawa Weyl group of $\mathfrak{M}_0$; this is a finite group that acts faithfully on $H^2(\mathfrak{M}; \mathbb{R})$. Namikawa shows that there is a hyperplane arrangement $\mathcal{H}_\mathrm{tw}$ in $H^2(\mathfrak{M}; \mathbb{R})$ whose chambers are equal to the W-translates of the ample cones of the collection of symplectic resolutions of $\mathfrak{M}_0$ (Remark 2.4). Let $E_\mathrm{tw} \subset H^2(\mathfrak{M}; \mathbb{C})$ be the complement of the complexification of $\mathcal{H}_\mathrm{tw}$; this space may also be interpreted as the locus of points over which the universal deformation of $\mathfrak{M}$ is affine. In the special case where $\mathfrak{M}$ is the cotangent bundle of G/B, W is the Weyl group of G, and $\mathcal{H}_\mathrm{tw}$ is the Coxeter arrangement.

Next, let $\mathbb{W}$ be the Weyl group of the group of Hamiltonian symplectomorphisms of $\mathfrak{M}$ that commute with $\mathbb{S}$, and let T be a maximal torus. Let $\mathcal{H}_\mathrm{sh}$ be the arrangement in $\mathfrak{t}_\mathbb{R}$ whose hyperplanes describe the cocharacters of T with infinite fixed-point sets, and let $E_\mathrm{sh} \subset \mathfrak{t}$ be the complement of the complexification of $\mathcal{H}_\mathrm{sh}$. If $\mathfrak{M}$ is the cotangent bundle of G/B, then the group of Hamiltonian symplectomorphisms commuting with $\mathbb{S}$ is G, and everything is the same as in the previous paragraph. This example, however, is misleading; in general, W and $\mathbb{W}$ are unrelated, as are $\mathcal{H}_\mathrm{tw}$ and $\mathcal{H}_\mathrm{sh}$. For example, if $\mathfrak{M}$ is a crepant resolution of $\mathbb{C}^2/\Gamma$, then W is isomorphic to the Weyl group corresponding to Γ under the McKay correspondence, but $\mathbb{W}$ is trivial unless $\Gamma = \mathbb{Z}/2\mathbb{Z}$.

6. We construct two commuting collections of derived endomorphisms of $\mathcal{O}_\mathrm{a}$, called twisting and shuffling functors. We construct an action of $\pi_1(E_\mathrm{tw}/W)$ on $D^b(\mathcal{O}_\mathrm{a})$ via twisting functors (Theorem 8.3) and an action of $\pi_1(E_\mathrm{sh}/\mathbb{W})$ on $D^b(\mathcal{O}_\mathrm{a})$ via shuffling functors (Theorem 8.13).

Symplectic duality. Item 7 cannot generalize verbatim because, as mentioned above, the groups that act by twisting and shuffling functors are in general unrelated. The correct generalization involves two different symplectic resolutions, $\mathfrak{M} \to \mathfrak{M}_0$ and $\mathfrak{M}^! \to \mathfrak{M}_0^!$.

7. We define a symplectic duality between $\mathfrak{M}$ and $\mathfrak{M}^!$ to be a pair of isomorphisms

$$E_\mathrm{tw}/W \cong E_\mathrm{sh}^!/\mathbb{W}^! \qquad \text{and} \qquad E_\mathrm{sh}/\mathbb{W} \cong E_\mathrm{tw}^!/W^!$$

and a Koszul duality between $\mathcal{O}_\mathrm{a}$ and $\mathcal{O}_\mathrm{a}^!$ that exchanges twisting and shuffling functors (see Definition 10.1 for a more precise formulation). We have already seen that $T^*(G/B)$

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is self-dual (or, more naturally, dual to its Langlands dual). Furthermore, we show that every type A S3-variety is dual to a different type A S3-variety (Theorem 10.4), every hypertoric variety is dual to a different hypertoric variety (Theorem 10.8), and every affine type A quiver variety is dual to a different affine type A quiver variety (Theorem 10.9 and Corollary 10.11). We conjecture the existence of dualities between quiver varieties and slices in the affine Grassmannian (Remark 10.7) and between pairs of moduli spaces of instantons on ALE spaces (Remark 10.13).

The simplest examples of symplectic duality are between $T^*\mathbb{P}^{\ell-1}$ and a crepant resolution of $\mathbb{C}^2/(\mathbb{Z}/\ell\mathbb{Z})$. These are special cases of every class of examples mentioned above. Part of the interest in twisting and shuffling functors is that they can be used to construct braid group actions and homological invariants of knots. The exchange of twisting and shuffling functors under symplectic duality then provides an explanation for different geometric constructions of the same knot homology (see Section 10.7).

Symplectic duality appears to be closely related to a mirror duality in physics. Seiberg and Intrilligator [IS96] propose a notion of **mirror duality**[8] between three dimensional gauge theories which carry N=4 supersymmetry . Such a gauge theory has a moduli space attached to it with a number of different components, including two distinguished components called the **Higgs branch** and the **Coulomb branch**. Mirror duality exchanges these two components; that is, the Higgs branch of one theory is isomorphic to the Coulomb branch of the dual theory.

It was pointed out to us by Gukov and Witten that our list of known and conjectural examples of symplectic duality coincides almost perfectly with the known list of Higgs branches of mirror dual gauge theories (or, equivalently, with the known list of Higgs/Coulomb pairs for a single gauge theory). For example:

- Type A S3-varieties are mirror to other type A S3-varieties [dBHOO97, §3.3].

- Hypertoric varieties are mirror to other hypertoric varieties [dBHOO97, §4].

- Affine type A quiver varieties are mirror to other affine type A quiver varieties [dBHOO97, §3.3].

- An ALE space is mirror to the instanton moduli space for the corresponding simply-laced Lie group on $\mathbb{R}^2$ [IS96].

These examples strongly suggest that symplectic duality and mirror duality are two perspectives on the same phenomenon. Unfortunately, mirror duality and Coulomb branches do not yet have precise mathematical definitions, so there is not yet a rigorous mathematical

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[8]This duality should not be confused with the homological mirror symmetry of Calabi-Yau manifolds, which is perhaps better known to algebraic and symplectic geometers.

statement for us to propose in an attempt to relate symplectic duality to mirror duality in full generality. After the appearance of the first version of this paper, the authors became aware of work in progress of Nakajima, Braverman, and Finkelberg [Nak], as well as simultaneous work by Bullimore, Dimofte, and Gaiotto [BDG], which proposes a construction of the Coulomb branch of the gauge theory associated to a symplectic representation of a compact Lie group (for which the Higgs branch would be the hyperkähler quotient). One may therefore regard this construction as a conjectural construction of the symplectic dual of any conical symplectic resolution that arises via a hyperkähler quotient construction. Preliminary calculations suggest that their approach agrees with ours in the special cases which we understand well, but a precise comparison of these two theories will have to be left for future work.

Cohomology. A symplectic duality between $\mathfrak{M}$ and $\mathfrak{M}^!$ has two interesting cohomological implications. First, consider the decomposition of $K(\mathcal{O}_g)_{\mathbb{C}}$ from item 5 into direct summands indexed by symplectic leaves of $\mathfrak{M}_0$. A consequence of symplectic duality is that the summand indexed by a leaf in $\mathfrak{M}_0$ is canonically dual to the summand indexed by a corresponding leaf in $\mathfrak{M}_0^!$ (Proposition 10.22). In the case of type A S3-varieties, this duality of vector spaces is known as skew Howe duality. In the case of affine type A quiver varieties, it is rank-level duality. For hypertoric varieties, it is a reflection of the behavior of the Tutte polynomial under Gale duality (Example 10.24). When $\mathfrak{M}$ is a finite type ADE quiver variety and $\mathfrak{M}^!$ is a transverse slice in the affine Grassmannian, this duality relates Nakajima's geometric construction of weight spaces of simple representations to Ginzburg and Mirkovic-Vilonen's geometric construction of the same weight spaces (Example 10.27).

The second cohomological implication comes from the last sentence of item 3, in which we conjecture that $H_T^*(\mathfrak{M}; \mathbb{C})$ is isomorphic to the center of the universal deformation of the Yoneda algebra of $\mathcal{O}_g$. If this conjecture holds, then symplectic duality implies a relationship between the equivariant cohomology rings of $\mathfrak{M}$ and $\mathfrak{M}^!$ that was previously studied in several examples by Goresky and MacPherson [GM10] (Theorem 10.35). Thus, symplectic duality may be regarded as a categorification of many different previously studied dualities. Interestingly, neither of these two cohomological phenomena seems to have been familiar to physicists who study mirror duality.

Summary. The paper is structured as follows. Section 2 is a review of all of the relevant background on conical symplectic resolutions that do not involve choosing a Hamiltonian action of $\mathbb{T}$. Most of this material is taken from [BPW]. Section 3 is devoted to the definitions and basic properties of $\mathcal{O}_a$ and $\mathcal{O}_g$, including the localization and section functors that relate them. Section 4 is a review of the background material on Koszul, highest weight, and standard Koszul categories, which we apply to $\mathcal{O}_a$ and $\mathcal{O}_g$ in Section 5. Sections 6-8 deal with items 4-6 on our list. Section 9 consists of analyses of all of the structures that we have defined in the special cases of cotangent bundles of partial flag varieties, S3-varieties, hypertoric varieties,

Hilbert schemes on ALE spaces, quiver varieties, and slices in the affine Grassmannian. Finally, Section 10 is devoted to the definition, examples, and consequences of symplectic duality.

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2 Quantizations of conical symplectic resolutions

In this section we review the necessary background on conical symplectic resolutions. Roughly, the section is a summary of all of the definitions and constructions in this paper that do not involve choosing a Hamiltonian action of $\mathbb{T}$.[9] Most of the material that appears here is taken from [BPW]; the main exception is Section 2.4, which is new.

2.1 Conical symplectic resolutions

Let $\mathfrak{M}$ be a smooth, complex algebraic variety with an algebraic symplectic form ω. Suppose that $\mathfrak{M}$ is equipped with an action of the multiplicative group $\mathbb{S} \cong \mathbb{C}^\times$ such that $s^*\omega = s^n\omega$ for some integer $n \geq 1$. We will assume that $\mathbb{S}$ acts on the coordinate ring $\mathbb{C}[\mathfrak{M}]$ with only non-negative weights and that the trivial weight space $\mathbb{C}[\mathfrak{M}]^{\mathbb{S}}$ is 1-dimensional, consisting only of the constant functions. Geometrically, this means that the affinization $\mathfrak{M}_0 := \operatorname{Spec} \mathbb{C}[\mathfrak{M}]$ is contracted by the $\mathbb{S}$-action to a single cone point $o \in \mathfrak{M}_0$. We will assume that the minimal symplectic leaf of $\mathfrak{M}_0$ consists only of the point o, thus eliminating the possibility that $\mathfrak{M}_0$ contains a factor of a symplectic vector space.[10] Finally, we assume that the canonical map from $\mathfrak{M}$ to $\mathfrak{M}_0$ is a projective resolution of singularities (that is, it must be an isomorphism over the smooth locus of $\mathfrak{M}_0$). We will refer to this collection of data as a **conical symplectic resolution**.

Examples of conical symplectic resolutions include the following:

- $\mathfrak{M}$ is a crepant resolution of $\mathfrak{M}_0 = \mathbb{C}^2/\Gamma$, where Γ is a nontrivial finite subgroup of SL_2. The action of $\mathbb{S}$ is induced by the inverse of the diagonal action on $\mathbb{C}^2$, and $n = 2$.

- $\mathfrak{M}$ is the Hilbert scheme of a fixed number of points on the crepant resolution of $\mathbb{C}^2/\Gamma$, and $\mathfrak{M}_0$ is the symmetric variety of unordered collections of points on the singular space. Once again, $\mathbb{S}$ acts by the inverse diagonal action on $\mathbb{C}^2$, and $n = 2$.

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[9]This is not quite accurate, as twisting functors, which are not introduced until Section 8.1, also do not involve the torus $\mathbb{T}$. We wait until Section 8 to introduce twisting functors in order to emphasize the similarities between twisting functors and shuffling functors, which *do* involve the choice of $\mathbb{T}$.

[10]We did not include this condition as part of the definition of a conical symplectic resolution in [BPW], but it will be useful in the current work.

- $\mathfrak{M} = T^*(G/P)$ for a reductive algebraic group G and a parabolic subgroup P, and $\mathfrak{M}_0$ is the affinization of this variety (when $G = \mathrm{SL}_r$, this always be the closure of a nilpotent orbit in the Lie algebra of G). The action of $\mathbb{S}$ is the inverse scaling action on the cotangent fibers, and $n = 1$.

- $\mathfrak{M}$ is a hypertoric variety associated to a simple, unimodular, hyperplane arrangement in a rational vector space [BD00, Pro08], and $\mathfrak{M}_0$ is the hypertoric variety associated to the centralization of this arrangement. These varieties admit an action of $\mathbb{S}$ with $n = 1$ if and only if the arrangement has a bounded chamber; they always admit an action of $\mathbb{S}$ with $n = 2$.

- $\mathfrak{M}$ and $\mathfrak{M}_0$ are Nakajima quiver varieties [Nak94, Nak98]. These varieties admit an action of $\mathbb{S}$ with $n = 1$ if and only if the quiver has no loops; they always admit an action of $\mathbb{S}$ with $n = 2$.

- $\mathfrak{M}_0$ is a transverse slice to Gr^μ inside of Gr^λ, where Gr^μ and Gr^λ are Schubert varieties inside of the affine Grassmannian for a reductive group G. When λ is a sum of minuscule coweights for G, $\mathfrak{M}_0$ has a natural conical symplectic resolution constructed from a convolution variety; in most other cases, it seems to possess no such resolution. This example is discussed in greater generality in [KWWY14].

Remark 2.1 The fifth class of examples overlaps significantly with each of the others. The first two examples are special cases of quiver varieties, where the underlying graph of the quiver is the extended Dynkin diagram corresponding to Q. The third and sixth examples can be realized as quiver varieties if the group G is of type A. Finally, a hypertoric variety is a quiver variety if and only if the associated hyperplane arrangement is cographical.

Remark 2.2 All of these examples admit complete hyperkähler metrics, and in fact we know of no examples that do not admit complete hyperkähler metrics. (Such spaces do exist if we drop the hypothesis that $\mathfrak{M}$ is projective over $\mathfrak{M}_0$; some examples will appear in subsequent work by Arbo and the third author.) The unit circle in $\mathbb{S}$ acts by hyperkähler isometries, but is Hamiltonian only with respect to the real symplectic form. Our assumptions about the $\mathbb{S}$-weights of $\mathbb{C}[\mathfrak{M}]$ translate to the statement that the real moment map for the circle action is proper and bounded below.

2.2 Deformation theory and birational geometry

Let $\mathfrak{M}$ be a conical symplectic resolution. The following result is stated in [BPW, 2.7]; it is due in this form to Namikawa [Nam08], and is closely related to earlier results of Kaledin and Verbitsky [KV02].

Theorem 2.3 *The variety $\mathfrak{M}$ has a universal Poisson deformation $\pi\colon \mathscr{M} \to H^2(\mathfrak{M}; \mathbb{C})$ which is flat. The variety $\mathscr{M}$ admits an action of $\mathbb{S}$ extending the action on $\mathfrak{M} \cong \pi^{-1}(0)$, and π is $\mathbb{S}$-equivariant with respect to the weight $-n$ action on $H^2(\mathfrak{M}; \mathbb{C})$. This family is trivial in the category of smooth manifolds with circle actions.*

For any $\eta \in H^2(\mathfrak{M}; \mathbb{C})$, we will also be interested in the **twistor deformation**

$$\mathscr{M}_\eta := \mathscr{M} \times_{H^2(\mathfrak{M};\mathbb{C})} \mathbb{A}^1.$$

Let $\mathscr{M}_\eta(\infty) := (\mathscr{M}_\eta \smallsetminus \mathfrak{M})/\mathbb{S}$ be the generic fiber of $\mathscr{M}_\eta$. A fundamental result of Kaledin [Kal08, 2.5] says that, if η is the Euler class of an ample line bundle on $\mathfrak{M}$, then $\mathscr{M}_\eta(\infty)$ is affine. More generally, Namikawa [Nam] shows that there is a finite set $\mathcal{H}_{\text{tw}}$ of hyperplanes in $H^2(\mathfrak{M}; \mathbb{R})$ such that the union $\bigcup_{H \in \mathcal{H}_{\text{tw}}} H_\mathbb{C} \subset H^2(\mathfrak{M}; \mathbb{C})$ is equal to the locus over which the fibers of π fail to be affine.[11]

Namikawa constructs a universal Poisson deformation of $\mathfrak{M}_0$ over the base $HP^2(\mathfrak{M}_0)$ [Nam08]. Since $\operatorname{Spec} \mathbb{C}[\mathscr{M}]$ is itself a Poisson deformation of $\mathfrak{M}_0$, we obtain a map from $H^2(\mathfrak{M}; \mathbb{C})$ to $HP^2(\mathfrak{M}_0)$. Namikawa shows that this map is a quotient by a finite subgroup W of the general linear group of $H^2(\mathfrak{M}; \mathbb{C})$ [Nam10, 1.1]. In the case of the Springer resolution, $H^2(\mathfrak{M}; \mathbb{C})$ is isomorphic to the Cartan subalgebra and W is isomorphic to the Weyl group. For this reason, we refer to W more generally as the **Namikawa Weyl group**.

Remark 2.4 The Namikawa Weyl group in fact acts on $H^2(\mathfrak{M}; \mathbb{R})$, with a fundamental domain equal to the closure of the movable cone of $\mathfrak{M}$ [BPW, 2.17]. This movable cone can be further divided into chambers given by ample cones of various conical symplectic resolutions of $\mathfrak{M}_0$. (For any conical symplectic resolution $\mathfrak{M}'$, its second cohomology group and its movable cone are canonically identified with those of $\mathfrak{M}$.) Namikawa [Nam] proves that $\mathfrak{M}$ is a relative Mori dream space over $\mathfrak{M}_0$ in the sense of [AW, 2.4], and that the chambers of $\mathcal{H}_{\text{tw}}$ are exactly equal to the W-translates of the ample cones of the various resolutions of $\mathfrak{M}_0$. If $\mathfrak{M}$ is obtained as a symplectic quotient of a vector space by the action of a group G and the Kirwan map from $\chi(G)_\mathbb{R}$ to $H^2(\mathfrak{M}; \mathbb{R})$ is an isomorphism, these chambers coincide with the maximal cones of the GIT fan.

2.3 Quantizations

Let $\mathfrak{M}$ be a conical symplectic resolution. A **quantization** of $\mathfrak{M}$ is defined to be

- an $\mathbb{S}$-equivariant sheaf $\mathcal{Q}$ of flat $\mathbb{C}[[h]]$-algebras on $\mathfrak{M}$, complete in the h-adic topology, where $\mathbb{S}$ acts on h with weight n (see [BPW, §3.2] for a precise definition of $\mathbb{S}$-equivariance)

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[11]The subscript tw stands for "twisting", and is explained by Theorem 8.3. There will also be a "shuffling" arrangement $\mathcal{H}_{\text{sh}}$, and an analogous Theorem 8.13.

- an $\mathbb{S}$-equivariant isomorphism from $\mathcal{Q}/h\mathcal{Q}$ to the structure sheaf $\mathfrak{S}_\mathfrak{M}$ of $\mathfrak{M}$

satisfying the condition that, if f and g are functions over some open set and $\tilde{f}$ and $\tilde{g}$ are lifts to $\mathcal{Q}$, the image in $\mathfrak{S}_\mathfrak{X} \cong \mathcal{Q}/h\mathcal{Q} \cong h\mathcal{Q}/h^2\mathcal{Q}$ of the element $[\tilde{f}, \tilde{g}] \in h\mathcal{Q}$ is equal to the Poisson bracket $\{f, g\}$.

Using the work of Bezrukavnikov and Kaledin [BK04], who classify quantizations in a (much more general) non-equivariant setting, Losev [Los12b, 2.3.3] proves the following classification result (see also [BPW, 3.5]).

Theorem 2.5 *Quantizations of a conical symplectic resolution $\mathfrak{M}$ are in bijection with $H^2(\mathfrak{M}; \mathbb{C})$ via the period map of [BK04].*

Fix a quantization $\mathcal{Q}$ of $\mathfrak{M}$. Let $\mathcal{D}(0) := \mathcal{Q}[h^{1/n}]$, and let $\mathcal{D}(m) := h^{-m/n}\mathcal{D}(0)$ for all $m \in \mathbb{Z}$. Let

$$\mathcal{D} := \mathcal{Q}[h^{-1/n}] = \bigcup_{m=0}^{\infty} \mathcal{D}(m);$$

we will often abuse notation by referring to $\mathcal{D}$ as a quantization of $\mathfrak{M}$. Let

$$A := \Gamma_\mathbb{S}(\mathcal{D})$$

be the ring of $\mathbb{S}$-invariant sections of $\mathcal{D}$. This ring inherits an $\mathbb{N}$-filtration

$$A(0) \subset A(1) \subset \ldots \subset A$$

given by putting

$$A(m) := \Gamma_\mathbb{S}(\mathcal{D}(m)).$$

The associated graded of A may be canonically identified with $\mathbb{C}[\mathfrak{M}]$ as an $\mathbb{N}$-graded ring. Many of our examples of conical symplectic resolutions in the previous section admit quantizations for which the ring A is of independent interest.

- If $\mathfrak{M}$ is the Hilbert scheme of k points on a crepant resolution of $\mathbb{C}^2/\Gamma$, then is A is isomorphic to a spherical symplectic reflection algebra for the wreath product $S_k \wr \Gamma$ [EGGO07, 1.4.4].

- If $\mathfrak{M} = T^*(G/B)$ for a reductive algebraic group G and a Borel subgroup $B \subset G$, then A is a central quotient of the universal enveloping algebra $U(\mathfrak{g})$ [BB81, Lemma 3].

- If $\mathfrak{M}$ is the resolution of a Slodowy slice to a nilpotent orbit in $\mathfrak{g}$, then A is a central quotient of a finite W-algebra [Pre02, 6.4].

- If $\mathfrak{M}$ is a hypertoric variety, then A is a central quotient of a hypertoric enveloping algebra [BK12, §5], [BLPW12, 5.9].

- If $\mathfrak{M}_0$ is a slice to one affine Schubert variety inside another, then A is conjecturally isomorphic to a quotient of a shifted Yangian [KWWY14].

Note that $\mathcal{D}$ and A also carry a grading by the group $\mathbb{Z}/n\mathbb{Z}$, where $\mathcal{Q} \subset \mathcal{D}$ lies in degree $\bar{0}$ and $h^{1/n}$ has degree $\bar{1}$. The grading on A is compatible with the filtration and thus descends to a grading on $\operatorname{gr} A$, which is equal to the grading induced by the natural semigroup homomorphism from $\mathbb{N}$ to $\mathbb{Z}/n\mathbb{Z}$.

2.4   Integrality

We would like to have some notion of what it means for a quantization to be **integral**. Let $H^2(\mathfrak{M}; \mathbb{Z})_{\text{free}}$ be the quotient of $H^2(\mathfrak{M}; \mathbb{Z})$ by its torsion subgroup.[12] The naive definition would be that $\mathcal{Q}$ or $\mathcal{D}$ is integral if its period lies in the lattice $H^2(\mathfrak{M}; \mathbb{Z})_{\text{free}} \subset H^2(\mathfrak{M}; \mathbb{C})$, but this is not suitable for our purposes. For example, if $\mathfrak{M} = T^*X$ for a projective variety X and $\mathcal{D}$ is the quantization with period $\lambda \in H^2(\mathfrak{M}; \mathbb{C}) \cong H^2(X; \mathbb{C})$, then A is isomorphic to ring of differential operators on X, twisted by $\lambda + \frac{1}{2}\varpi_X$, where ϖ_X is the Euler class of the canonical bundle of X [BPW, 4.4]. In this case, we would like to say that $\mathcal{D}$ is integral if and only if $\lambda + \frac{1}{2}\varpi_X \in H^2(\mathfrak{M}; \mathbb{Z})_{\text{free}}$. More generally, the set of integral periods should be a coset Λ of $H^2(\mathfrak{M}; \mathbb{Z})_{\text{free}} \subset H^2(\mathfrak{M}; \mathbb{C})$ that satisfies the following properties.

- We have $\lambda \in \Lambda$ if and only if $-\lambda \in \Lambda$. Equivalently, 2Λ is contained in $H^2(\mathfrak{M}; \mathbb{Z})_{\text{free}}$. We include this condition because the quantization with period $-\lambda$ is the opposite ring of the quantization with period λ [BPW, 3.2], and the opposite of an integral quantization should be integral.

- If $X \subset \mathfrak{M}$ is a smooth Lagrangian subvariety, then the restriction of Λ to X is equal to $\frac{1}{2}\varpi_X + H^2(X; \mathbb{Z})_{\text{free}}$. In particular, this uniquely determines Λ if $\mathfrak{M}$ is a cotangent bundle.

- Suppose that G is a reductive group acting on a symplectic vector space V, and $\mathfrak{M}$ is a smooth symplectic quotient of V by G. (For example, all quiver varieties and smooth hypertoric varieties are of this form.) Given a Lagrangian G-subspace $L \subset V$, we may identify the Weyl algebra of V with the ring of differential operators on L. Consider the quantized moment map $\mu_L : U(\mathfrak{g}) \to \operatorname{Diff}(L)$ that takes an element of $\mathfrak{g}$ to the induced vector field on L, and consider the induced quantization $\mathcal{D}_L$ of $\mathfrak{M}$, as in [KR08, 2.8(i)]. The period of this quantization can be calculated from [BPW, 3.16]. We should choose Λ to be the coset of this period. Note that if L and L' are two different Lagrangian G-subspaces, then $\mathcal{D}_L$ and $\mathcal{D}_{L'}$ need not be equal, but [BPW, 3.16] shows

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[12]In the situation of greatest interest to us, when there is a Hamiltonian $\mathbb{C}^*$-action commuting with $\mathbb{S}$ that has isolated fixed points, there is no torsion in this group, as we show in Proposition 3.1.

that their periods will always differ by an element of $H^2(\mathfrak{M}; \mathbb{Z})_{\text{free}}$, corresponding to the determinant character of G acting on $L'/(L' \cap L)$.

Remark 2.6 By the first property above, the coset Λ is uniquely determined by the image c_Λ of 2Λ in $H^2(\mathfrak{M}; \mathbb{Z})_{\text{free}}/H^2(\mathfrak{M}; 2\mathbb{Z})_{\text{free}} \subset H^2(\mathfrak{M}; \mathbb{Z}/2\mathbb{Z})$. The second property above is equivalent to the statement that the restriction of c_Λ to any smooth Lagrangian subvariety should equal the second Stiefel-Whitney class of that subvariety. Unfortunately, this condition may not uniquely determine c_Λ, as it is possible that $\mathfrak{M}$ has no smooth Lagrangian subvarieties at all.

Very little of what we do in this paper depends on the notion of integrality. In Sections 2-9, we will only refer to integral quantizations in the context of cotangent bundles, hypertoric varieties, and quiver varieties, in which case the meaning is completely determined by the second and third conditions above. In Section 10, the notion of integrality will become important; in that section, we simply assume that every conical symplectic resolution comes with a choice of Λ that is consistent with our three conditions.

2.5 Sheaves of modules

Let $\mathfrak{M}$, $\mathcal{Q}$, and $\mathcal{D}$ be as in Section 2.3. A $\mathcal{D}(0)$-module $\mathcal{N}(0)$ is called **coherent** if it is a quotient of a sheaf which is locally free of finite rank. Setting $\mathcal{N}(m) := h^{-m/n}\mathcal{N}(0)$, Nakayama's lemma tells us that the following three conditions are equivalent:

- $\mathcal{N}(0)$ is coherent

- $\mathcal{N}(0)/\mathcal{N}(-1)$ is a coherent sheaf of modules over $\mathcal{D}(0)/\mathcal{D}(-1) \cong \mathfrak{S}_{\mathfrak{M}}$

- $\mathcal{N}(0)/\mathcal{N}(-n) = \mathcal{N}(0)/h\mathcal{N}(0)$ is a coherent sheaf of modules over $\mathcal{Q}/h\mathcal{Q} \cong \mathfrak{S}_{\mathfrak{M}}$.

An $\mathbb{S}$-equivariant $\mathcal{D}$-module $\mathcal{N}$ is called **good** if it admits a coherent $\mathbb{S}$-equivariant $\mathcal{D}(0)$-lattice $\mathcal{N}(0)$. We call a good $\mathcal{D}$-module **holonomic** if it has Lagrangian support.

Given a choice of lattice $\mathcal{N}(0)$, we will refer to the coherent sheaf $\mathcal{N}(0)/\mathcal{N}(-n)$ as the **big classical limit** of $\mathcal{N}$, and to $\mathcal{N}(0)/\mathcal{N}(-1)$ as the **small classical limit** of $\mathcal{N}$. Note that the big classical limit is an n-fold extension of the small classical limit, and this extension need not split.

2.6 Localization

Let $\mathfrak{M}$, $\mathcal{Q}$, $\mathcal{D}$, and A be as in Section 2.3. Let $\mathcal{D}$-mod denote the category of good $\mathbb{S}$-equivariant $\mathcal{D}$-modules. Note that the choice of lattice is not part of the data of an object of $\mathcal{D}$-mod.

Let A-mod be the category of finitely generated A-algebras. A **good filtration** of an A-module N is defined to be a filtration such that $\operatorname{gr} N$ is finitely generated over $\operatorname{gr} A$. For

any N, we can choose a good filtration by picking a finite generating set $Q \subset N$ and putting $N(m) := A(m) \cdot Q$.

We have a functor

$$\Gamma_{\mathbb{S}} : \mathcal{D}\text{-mod} \to A\text{-mod}$$

given by taking $\mathbb{S}$-invariant global sections. The left adjoint functor

$$\text{Loc} : A\text{-mod} \to \mathcal{D}\text{-mod}$$

is defined by putting $\text{Loc}(N) := \mathcal{D} \otimes_A N$. To see that $\text{Loc}(N)$ is indeed an object of $\mathcal{D}$-mod, choose a good filtration of N. We define the **Rees algebra** $R(A)$ to be the h-adic completion of

$$A(0)[[h^{1/n}]] + h^{1/n}A(1)[[h^{1/n}]] + h^{2/n}A(2)[[h^{1/n}]] + \ldots \subset A[[h^{1/n}]]$$

and the **Rees module** $R(N)$ to be the h-adic completion of

$$N(0)[[h^{1/n}]] + h^{1/n}N(1)[[h^{1/n}]] + h^{2/n}N(2)[[h^{1/n}]] + \ldots \subset N[[h^{1/n}]].$$

Note that $R(N)$ is a module over $R(A) \cong \Gamma(\mathcal{D}(0))$, and $\mathcal{D}(0) \otimes_{R(A)} R(N)$ is a coherent lattice in $\text{Loc}(N)$.

Remark 2.7 If N is an object of A-mod, we have shown that $\text{Loc}(N)$ always admits a coherent lattice, but the construction of that lattice depends on a choice of filtration of N. Conversely, any coherent lattice $\mathcal{N}(0)$ for an object $\mathcal{N}$ of $\mathcal{D}$-mod induces a filtration of $N := \Gamma_{\mathbb{S}}(\mathcal{N})$ by putting $N(m) := \Gamma_{\mathbb{S}}(\mathcal{N}(m))$.

If $\Gamma_{\mathbb{S}}$ and Loc are quasi-inverse equivalences of categories, we will say that **localization holds for $\mathcal{Q}$** or **localization holds for $\mathcal{D}$** or **localization holds at λ**, where λ is the period of $\mathcal{Q}$. If their derived functors induce quasi-inverse equivalences of derived categories, we say that **derived localization holds**. Localization and/or derived localization is known to hold in many special cases, including quantizations of the Hilbert scheme of points in the plane [KR08, 4.9], the cotangent bundle of G/P [BB81], resolved Slodowy slices [Gin09, 3.3.6] & [DK, 7.4], and hypertoric varieties [BK12, 5.8]. In [BPW, A & B.1], we have shown that localization and derived localization hold for "many" quantizations.

Theorem 2.8 *If η is the Euler class of an ample line bundle on $\mathfrak{M}$, then for any λ, localization holds at $\lambda + k\eta$ for sufficiently large integers k.*

Theorem 2.9 *If η is the Euler class of an ample line bundle on $\mathfrak{M}$, then for any λ, derived localization holds at $\lambda + k\eta$ for all but finitely many complex numbers k.*

Forthcoming work of McGerty and Nevins [MN] gives a considerable strengthening of Theorem 2.8, showing that the locus where localization fails is contained in countably many translates of hyperplanes from the discriminant locus. In earlier work [MN14], they also gave a cohomological criterion for when derived localization holds: when the section algebra has finite global dimension.

2.7 Modules with supports

Let $\mathfrak{M}$, $\mathcal{Q}$, $\mathcal{D}$, and A be as in Section 2.3. Let $\mathfrak{L}_0 \subset \mathfrak{M}_0$ be the subscheme defined by a graded ideal $J \subset \mathbb{C}[\mathfrak{M}_0]$, and let $\mathfrak{L} \subset \mathfrak{M}$ be the subscheme defined by a graded ideal sheaf $\mathcal{J} \subset \mathfrak{S}_{\mathfrak{M}}$. We will often assume that $\mathfrak{L}$ is the scheme-theoretic preimage of $\mathfrak{L}_0$, which is equivalent to saying that $\mathcal{J} = \mathfrak{S}_{\mathfrak{M}} \otimes_{\mathbb{C}[\mathfrak{M}_0]} J$ (see Propositions 2.13 and 2.14). We denote by $J + h \cdot R(A) \subset R(A)$ the preimage of $J \otimes \mathbb{C}[h^{1/n}]/\langle h \rangle$ under the natural map

$$R(A) \to \mathbb{C}[\mathfrak{M}_0] \otimes \mathbb{C}[h^{1/n}]/\langle h \rangle.$$

The following definitions appeared in [BPW, §6.1].

Definition 2.10 Let $C^{\mathfrak{L}_0}$ be the full subcategory of A-mod consisting of all modules N that admit good filtrations with either of the following two equivalent properties:

- Let $a \in A(k)$ be homogeneous of degree $\bar{k}$ for the $\mathbb{Z}/n\mathbb{Z}$ grading, and suppose that its symbol $\bar{a} \in A(k)/A(k-1) \cong \mathbb{C}[\mathfrak{M}_0]_k$ lies in J. Then $a \cdot N(m) \subset N(k+m-n)$.

- For any $a \in J + h \cdot R(A)$, we have $a \cdot R(N) \subset h \cdot R(N)$.

Let $D^b_{\mathfrak{L}_0}(A\text{-mod})$ be the full subcategory of $D^b(A\text{-mod})$ consisting of objects with cohomology in $C^{\mathfrak{L}_0}$.

Remark 2.11 Note that if N is an object of $C^{\mathfrak{L}_0}$, then the associated graded $\operatorname{gr} N$ will be killed by the ideal J, but the converse is not true unless $n = 1$.

Definition 2.12 Let $\mathcal{C}^{\mathfrak{L}}$ be the full subcategory of $\mathcal{D}$-mod consisting of modules with big classical limits that are scheme-theoretically supported on $\mathfrak{L}$. More precisely, a $\mathcal{D}$-module $\mathcal{N}$ is in $\mathcal{C}^{\mathfrak{L}}$ if it admits a lattice $\mathcal{N}(0)$ that is preserved by $h^{-1}\tilde{f}$ for any section $\tilde{f}$ of $\mathcal{Q}$ whose image in $\mathcal{Q}/h\mathcal{Q} \cong \mathfrak{S}_{\mathfrak{M}}$ lies in $\mathcal{J}$. Let $D^b_{\mathfrak{L}}(\mathcal{D}\text{-mod})$ be the full subcategory of $D^b(\mathcal{D}\text{-mod})$ consisting of objects with cohomology in $\mathcal{C}^{\mathfrak{L}}$.

Proposition 2.13 *If $\mathfrak{L}$ is the scheme-theoretic preimage of $\mathfrak{L}_0$, then* Loc *takes* $C^{\mathfrak{L}_0}$ *to* $\mathcal{C}^{\mathfrak{L}}$ *and* $\Gamma_{\mathbb{S}}$ *takes* $\mathcal{C}^{\mathfrak{L}}$ *to* $C^{\mathfrak{L}_0}$.

Proof: Let N be an object of $C^{\mathfrak{L}_0}$. Choose a filtration of N as in Definition 2.10, and let $\mathcal{N}(0) := \mathcal{D}(0) \otimes_{R(A)} R(N)$ be the induced lattice in $\mathcal{N} := \mathrm{Loc}(N)$. Let $\tilde{f}$ be a global section of $\mathcal{Q}$ whose image $f \in \mathcal{Q}/h\mathcal{Q} \cong \mathfrak{S}_\mathfrak{M}$ lies in $\mathcal{J}$. After decomposing $\tilde{f}$ into eigenvectors for the $\mathbb{S}$ action, we may assume that there exists an integer k such that $h^{-k/n}\tilde{f}$ is $\mathbb{S}$-invariant. Thus $h^{-k/n}\tilde{f} \in \Gamma_\mathbb{S}(\mathcal{D}) = A(k)$ is homogeneous of degree $\bar{k}$ for the $\mathbb{Z}/n\mathbb{Z}$ grading, so $h^{-k/n}\tilde{f} \cdot N(m) \subset N(k+m-n)$.

On any sufficiently small open subset U, we have $\mathcal{N}(0)(U) \cong \mathcal{D}(0)(U) \otimes_{R(A)} R(N)$; moreover,

$$\mathcal{N}(0)(U) = \sum_m \mathcal{D}(0)(U) \otimes h^{m/n} N(m),$$

where we write $\mathcal{D}(0)(U) \otimes h^{m/n} N(m)$ to denote the image of the tensor product over $\mathbb{C}$ inside of $\mathcal{N}(0)(U)$. Thus

$$\tilde{f} \cdot \mathcal{N}(0)(U) \subset \sum_m \left(h^{m/n}[\tilde{f}, \mathcal{D}(0)(U)] \otimes N(m) + \mathcal{D}(0)(U) \otimes h^{m+k/n} \cdot N(m+k-n) \right)$$

$$\subset h \cdot \mathcal{N}(0)(U).$$

Since the ideal sheaf $\mathcal{J}$ is generated by global sections, this suffices to show that $\mathrm{Loc}(\mathcal{N}) \in \mathcal{C}^\mathfrak{L}$.

For the opposite direction, let $\mathcal{N}$ be an object of $\mathcal{C}^\mathfrak{L}$, and let $\mathcal{N}(0)$ be a lattice preserved by $h^{-1}\tilde{f}$ for every section $\tilde{f}$ of $\mathcal{Q}$ whose image $f \in \mathcal{Q}/h\mathcal{Q} \cong \mathfrak{S}_\mathfrak{M}$ lies in $\mathcal{J}$. Let $N := \Gamma_\mathbb{S}(\mathcal{N})$, and let $N(m) := \Gamma_\mathbb{S}(\mathcal{N}(m))$ be the induced filtration. Let $a \in A(k)$ be homogeneous of degree $\bar{k}$ with symbol in J. Then $h^{k/n}a$ is a section of $\mathcal{Q}$ whose image lies in $\mathcal{J}$, so $h^{k/n-1}a \cdot \mathcal{N}(m) \subset \mathcal{N}(m)$, and therefore $a \cdot \mathcal{N}(m) \subset h^{1-k/n}\mathcal{N}(m) = \mathcal{N}(k+m-n)$. Applying $\Gamma_\mathbb{S}$, we see that $a \cdot N(m) \subset N(k+m-n)$, so N is an object of $C^{\mathfrak{L}_0}$. $\qquad\qquad\square$

Proposition 2.14 *If $\mathfrak{L}$ is the scheme-theoretic preimage of $\mathfrak{L}_0$ and derived localization holds at λ, then $\mathbb{L}\mathrm{Loc}$ takes $D^b_{\mathfrak{L}_0}(A\text{-mod})$ to $D^b_\mathfrak{L}(\mathcal{D}\text{-mod})$ and $\mathbb{R}\Gamma_\mathbb{S}$ takes $D^b_\mathfrak{L}(\mathcal{D}\text{-mod})$ to $D^b_{\mathfrak{L}_0}(A\text{-mod})$.*

Proof: Let $\mathcal{N}$ be an object in $\mathcal{C}^\mathfrak{L}$, and let $\mathcal{N}(0) \subset \mathcal{N}$ be a lattice satisfying the required condition. There is a spectral sequence (see [BPW, §6.1], particularly the proof of Theorem 6.5)

$$H^p(\mathfrak{M}; \mathcal{N}(0)/\mathcal{N}(-n)) \Rightarrow R\left(\mathbb{H}^p(\mathbb{R}\Gamma_\mathbb{S}(\mathcal{N}))\right)/hR\left(\mathbb{H}^p(\mathbb{R}\Gamma_\mathbb{S}(\mathcal{N}))\right).$$

Since the left-hand side is killed by the ideal J, the same is true of the right hand side, which implies that $\mathbb{H}^p(\mathbb{R}\Gamma_\mathbb{S}(\mathcal{N}))$ is in $C^{\mathfrak{L}_0}$, and is only non-zero in finitely many degrees since the map π is projective.

Now let N be an object of $C^{\mathfrak{L}_0}$ and put $\mathcal{N} := \mathbb{L}\mathrm{Loc}(N)$. This only has cohomology in finitely many degrees since A_λ has finite global dimension, by a result of McGerty and Nevins [MN14, §7.5]. A filtration of N induces a lattice in $\mathbb{H}^p(\mathcal{N})$. For any $a \in J + h \cdot R(A)$, we

16

have that $a \cdot R(N) \subset h \cdot R(N)$; thus, on any projective resolution, the map induced by a is null-homotopic mod h; this implies that our lattice in $\mathbb{H}^p(\mathcal{N})$ has the required property. $\quad \square$

Remark 2.15 If derived localization does not hold, then the functor $\mathbb{L}\mathrm{Loc}$ is not bounded. If we were to replace D^b by the bounded-above category D^-, then Proposition 2.14 would hold for arbitrary quantizations. This is discussed in greater detail in [BPW, §4.3].

2.8 Harish-Chandra bimodules and characteristic cycles

We continue with the notation $\mathfrak{M}$, $\mathcal{Q}$, and $\mathcal{D}$ from Section 2.3. The product $\mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}_0 \times \mathfrak{M}_0$ is a conical symplectic resolution with quantization $\mathcal{D} \boxtimes \mathcal{D}^{\mathrm{op}}$ (the tensor product is taken over $\mathbb{C}((h))$) and section ring $A \otimes A^{\mathrm{op}}$. Thus we can apply the previous definitions and results to A-bimodules and $\mathcal{D}$-bimodules.

Consider the diagonal $\mathfrak{Z}_0 \subset \mathfrak{M}_0 \times \mathfrak{M}_0$ (with its reduced scheme structure), and its preimage $\mathfrak{Z} := \mathfrak{M} \times_{\mathfrak{M}_0} \mathfrak{M}$, the **Steinberg** scheme (which may not be reduced).

Definition 2.16 A finitely generated A-bimodule (resp. $\mathcal{D}$-bimodule) is called **Harish-Chandra** if it lies in $C^{\mathfrak{Z}_0}$ (resp. $\mathcal{C}^{\mathfrak{Z}}$). We will use the notation

$$\mathbf{HC}^{\mathrm{a}} := C^{\mathfrak{Z}_0} \qquad \text{and} \qquad \mathbf{HC}^{\mathrm{g}} := \mathcal{C}^{\mathfrak{Z}}$$

for the abelian categories of algebraic and geometric Harish-Chandra bimodules, along with $D^b_{\mathbf{HC}^{\mathrm{a}}}(A\text{-mod-}A)$ and $D^b_{\mathbf{HC}^{\mathrm{g}}}(\mathcal{D}\text{-mod-}\mathcal{D})$ for the subcategories of the bounded derived categories of all bimodules consisting of objects with Harish-Chandra cohomology.

The following results appear in [BPW, §6.1].

Proposition 2.17 *The category* $\mathbf{HC}^{\mathrm{a}}$ *is a monoidal category under the operation of tensor product, and the category* $C^{\mathfrak{L}_0}$ *is a module category over* $\mathbf{HC}^{\mathrm{a}}$*; similarly, when A has finite global dimension,* $D^b_{\mathbf{HC}^{\mathrm{a}}}(A\text{-mod-}A)$ *has a monoidal structure induced by derived tensor product, and an action on* $D^b_{\mathfrak{L}_0}(A\text{-mod})$.

There is a geometric version of this derived tensor product, induced by convolution on $D^b_{\mathbf{HC}^{\mathrm{g}}}(\mathcal{D}\text{-mod-}\mathcal{D})$*, and the category* $D^b_{\mathfrak{L}}(\mathcal{D}\text{-mod})$ *is naturally a module category over* $D^b_{\mathbf{HC}^{\mathrm{g}}}(\mathcal{D}\text{-mod-}\mathcal{D})$*. These structures are compatible with the derived $\mathbb{S}$-invariant section functors.*

Let $\mathcal{H}$ be an object of $\mathbf{HC}^{\mathrm{g}}$ and let $\mathcal{N}$ be an object of $\mathcal{C}^{\mathfrak{L}}$. Let $d = \frac{1}{2} \dim \mathfrak{M}$. In [BPW,

§6.2] we constructed maps[13]

$$\text{CC} : K(\mathbf{HC}^{\mathrm{g}}) \to H_{\mathfrak{Z}}^{4d}(\mathfrak{M} \times \mathfrak{M}; \mathbb{Z}) \qquad \text{and} \qquad \text{CC} : K(\mathcal{C}^{\mathfrak{L}}) \to H_{\mathfrak{L}}^{2d}(\mathfrak{M}; \mathbb{Z}),$$

and we proved the following result [BPW, 6.15 & 6.16].

Proposition 2.18 *The map* CC *intertwines the monoidal structure on* $D_{\mathbf{HC}^{\mathrm{g}}}^{b}(\mathcal{D}\text{-mod-}\mathcal{D})$ *with the convolution product on* $H_{\mathfrak{Z}}^{4d}(\mathfrak{M} \times \mathfrak{M}; \mathbb{Z})$, *and it also intertwines the action of* $D_{\mathbf{HC}^{\mathrm{g}}}^{b}(\mathcal{D}\text{-mod-}\mathcal{D})$ *on* $D_{\mathfrak{L}}^{b}(\mathcal{D}\text{-mod})$ *with the convolution action of* $H_{\mathfrak{Z}}^{4d}(\mathfrak{M} \times \mathfrak{M}; \mathbb{Z})$ *on* $H_{\mathfrak{L}}^{2d}(\mathfrak{M}; \mathbb{Z})$.

Remark 2.19 The statements in [BPW, §6.2] are somewhat more technical than what we have stated above, because there we consider all quantizations at once. That is, for any pair $\lambda, \lambda' \in H^2(\mathfrak{M}; \mathbb{C})$, we define what it means for an $(A_\lambda, A_{\lambda'})$-bimodule or a $(\mathcal{D}_\lambda, \mathcal{D}_{\lambda'})$-bimodule to be Harish-Chandra, and so on. We will in fact need this stronger version when we discuss twisting functors in Section 8.1, but for the purposes of this summary we have elected to keep things clearer by fixing a particular quantization.

3 The categories $\mathcal{O}_{\mathrm{a}}$ and $\mathcal{O}_{\mathrm{g}}$

To define the categories $\mathcal{O}_{\mathrm{a}}$ and $\mathcal{O}_{\mathrm{g}}$ we need one more piece of geometric structure, namely a Hamiltonian action of the multiplicative group $\mathbb{T} \cong \mathbb{C}^\times$, commuting with the action of $\mathbb{S}$, such that $\mathfrak{M}^{\mathbb{T}}$ is finite. First, let us make some observations about the integral cohomology of such a symplectic resolution.

Proposition 3.1 *If* $\mathfrak{M}$ *is a conical symplectic resolution that admits a* $\mathbb{T}$-action *as above, then* $H^*(\mathfrak{M}; \mathbb{Z})$ *is torsion-free and concentrated in even degrees.*

Proof: By Poincaré duality, we can instead consider the Borel-Moore homology of the same variety. By [Nak01, 7.1.5], it suffices to show that the same is true of the smooth projective variety $\mathfrak{M}^{\mathbb{S}}$. The action of $\mathbb{T}$ preserves $\mathfrak{M}^{\mathbb{S}}$; thus, this projective variety has a torus action with isolated fixed points. The Białynicki-Birula decomposition of $\mathfrak{M}^{\mathbb{S}}$ shows that it has even torsion-free cohomology. $\qquad\qquad\square$

Remark 3.2 The analogous result is shown for quiver varieties even when they don't have a $\mathbb{T}$-action with finite fixed-point set in [Nak01, 7.3.5].

The action of $\mathbb{T}$ lifts canonically to an action on $\mathcal{Q}$, where $\mathbb{T}$ fixes h. By [BPW, 3.11], there exists an element ξ of $A(n) \subset A$, unique up to translation by $A(0) \cong \mathbb{C}$, such that the endomorphism of $\mathcal{D}$ induced by the generator of the Lie algebra $\mathfrak{t} := \text{Lie}(\mathbb{T})$ is given by conjugation with ξ.

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[13]We will review the definition of these maps in Section 6.1.

Remark 3.3 For any choice of ξ, the image $\bar{\xi}$ of ξ in $A(n)/A(n-1) \subset \operatorname{gr} A \cong \mathbb{C}[\mathfrak{M}]$ is the unique $\mathbb{S}$-equivariant moment map for the action of $\mathbb{T}$ on $\mathfrak{M}$. Another way to say this is to note that ξ induces a homomorphism from $U(\mathfrak{t})$ to A, and the associated graded of this homomorphism is the co-moment map.

3.1   The relative core

Choose an indexing set $\mathcal{I}$ for the $\mathbb{T}$-fixed points of $\mathfrak{M}$, so that $\mathfrak{M}^{\mathbb{T}} = \{p_\alpha \mid \alpha \in \mathcal{I}\}$. For each $\alpha \in \mathcal{I}$, let $X_\alpha \subset \mathfrak{M}$ be the closure of the set

$$X_\alpha^\circ := \big\{ p \in \mathfrak{M} \mid \lim_{\mathbb{T} \ni t \to 0} t \cdot p = p_\alpha \big\},$$

and let $\mathfrak{M}^+ := \bigcup X_\alpha$. The set $\mathfrak{M}^+$ is called the **relative core** of $\mathfrak{M}$.

The fact that the action of $\mathbb{T}$ preserves the symplectic form implies that each X_α is Lagrangian (though possibly singular), and the open subvariety X_α° is isomorphic to d-dimensional affine space.

In the affine variety $\mathfrak{M}_0$, let

$$\mathfrak{M}_0^+ := \big\{ p \in \mathfrak{M}_0 \mid \lim_{\mathbb{T} \ni t \to 0} t \cdot p = o \big\}$$

be the locus of points that limit to the unique $\mathbb{T}$-fixed point $o \in \mathfrak{M}_0$. Since $\mathfrak{M}$ is projective over $\mathfrak{M}_0$, a point in $\mathfrak{M}$ has a limit if and only if its image in $\mathfrak{M}_0$ does; it follows that $\mathfrak{M}^+ = \bigcup X_\alpha^\circ$ is the preimage of $\mathfrak{M}_0^+$.

Let $X_{\alpha,0}$ be the image of X_α in $\mathfrak{M}_0$.

Example 3.4 If $\mathfrak{M}$ is a crepant resolution of $\mathbb{C}^2/\Gamma$ with $\Gamma \cong \mathbb{Z}/k\mathbb{Z}$, the relative core components $\{X_\alpha \mid \alpha \in \mathcal{I}\}$ will consist of a chain of $k-1$ projective lines, along with a copy of $\mathbb{C}$ at one end of the chain. If $\mathfrak{M}$ is $T^*(G/P)$, they will be the conormal varieties to the Schubert strata of G/P. If $\mathfrak{M}$ is a hypertoric variety, they will all be toric varieties. The cotangent bundle of $\mathbb{P}^1$ is a special case of all three of these examples; in this case we have two subvarieties: the zero section and one of the fibers.

Remark 3.5 The preimage of $o \in \mathfrak{M}_0$ in $\mathfrak{M}$ is called the **core**, and is a subset of $\mathfrak{M}^+$ consisting of the union of all of the projective components. Our requirement that $\{o\}$ is a symplectic leaf of $\mathfrak{M}_0$ guarantees that the core is a Lagrangian subvariety of $\mathfrak{M}$. Note that the core is independent of the choice of $\mathbb{T}$-action, while the relative core depends on this choice.

Let $J \subset \mathbb{C}[\mathfrak{M}]$ be the ideal in the coordinate ring of $\mathfrak{M}$ generated by functions of non-negative $\mathbb{T}$-weight and $\mathbb{S}$-weight greater than or equal to n.

Lemma 3.6 *The relative core $\mathfrak{M}^+ \subset \mathfrak{M}$ is the vanishing locus of J.*

Proof: Let $f \in \mathbb{C}[\mathfrak{M}]$ be a function of non-negative $\mathbb{T}$-weight and $\mathbb{S}$-weight greater than or equal to n. Then f vanishes on $\mathbb{S}$-fixed points, and the core (being projective) contains at least one such point. Thus f vanishes on the entire core. For any $p \in \mathfrak{M}^+$,

$$f(p) = \lim_{t \to 0}(t \cdot f)(t \cdot p) = 0,$$

since $t \cdot f$ is approaching either f or 0, and $t \cdot p$ is approaching an element of the core. Thus f vanishes on all of $\mathfrak{M}^+$, so $\mathfrak{M}^+$ is contained in the vanishing locus of J.

Now suppose that $p \in \mathfrak{M} \setminus \mathfrak{M}^+$; we must produce an element of J that does not vanish at p. Let $p_0 \in \mathfrak{M}_0 \setminus \mathfrak{M}_0^+$ be the image of p. Since the limit as t goes to zero of $t \cdot p_0$ does not exist, there must exist a function $f \in \mathbb{C}[\mathfrak{M}_0] \cong \mathbb{C}[\mathfrak{M}]$ such that

$$\lim_{\mathbb{T} \ni t \to 0} f(t \cdot p_0) = \lim_{\mathbb{T} \ni t \to 0}(t^{-1} \cdot f)(p_0)$$

does not exist; if we require f to be a $\mathbb{S} \times \mathbb{T}$-weight function, this means that f has positive $\mathbb{T}$-weight and does not vanish at p_0. Since it has positive $\mathbb{T}$-weight, it is non-constant, and therefore has positive $\mathbb{S}$-weight; taking a power, we may assume its $\mathbb{S}$-weight is at least n. $\square$

Remark 3.7 Until now we have only defined $\mathfrak{M}^+ \subset \mathfrak{M}$ and $\mathfrak{M}_0^+ \subset \mathfrak{M}_0$ as subsets; we will now endow them with subscheme structures given by the ideal J, as suggested by Lemma 3.6.

Recall that

$$\bar{\xi} \in A(n)/A(n-1) \subset \operatorname{gr} A \cong \mathbb{C}[\mathfrak{M}]$$

is defined as the symbol of $\xi \in A(n)$. We define a $\bar{\xi}$-**equivariant coherent sheaf** on $\mathfrak{M}$ to be a coherent sheaf F along with an endomorphism $d : F \to F$ such that, for all locally defined sections v and functions f, we have $d(fv) = \{\bar{\xi}, f\}v + fd(v)$, where $\{,\}$ is the Poisson bracket on $\mathbb{C}[\mathfrak{M}]$. This definition is motivated by the following lemma.

Lemma 3.8 *Let $\mathcal{N}$ be a good $\mathcal{D}$-module with a coherent lattice $\mathcal{N}(0) \subset \mathcal{N}$ that is preserved by ξ.[14] Let $\bar{\mathcal{N}} := \mathcal{N}(0)/\mathcal{N}(-1)$ be the small classical limit. Then the action of ξ defines a $\bar{\xi}$-equivariant structure on $\bar{\mathcal{N}}$. Furthermore, the $\mathbb{C}[\mathfrak{M}]$-module $\Gamma(\bar{\mathcal{N}})$ is isomorphic to the associated graded of the filtered A-module $\Gamma_{\mathbb{S}}(\mathcal{N})$ as a module-with-endomorphism.*

Proof: The action of ξ clearly descends to an endomorphism $d : \bar{\mathcal{N}} \to \bar{\mathcal{N}}$. Let f be a function on $\mathfrak{M}$, and lift it to a section $\tilde{f}$ of $\mathcal{D}(0)$. Let v be a section of $\bar{\mathcal{N}}$, and lift it to a section $\tilde{v}$ of $\mathcal{N}(0)$. Then $d(fv)$ is the image in $\bar{\mathcal{N}}$ of the section $\xi\tilde{f}\tilde{v} = [\xi, \tilde{f}]\tilde{v} + \tilde{f}\xi\tilde{v}$ of $\mathcal{N}(0)$. *A priori*, $[\xi, \tilde{f}]$ is a section of $\mathcal{D}(n) = h^{-1}\mathcal{D}(0)$, but since it is a commutator, it in fact lies in $\mathcal{D}(0)$, and it descends to the function $\{\bar{\xi}, f\}$; this proves that $\bar{\mathcal{N}}$ is $\bar{\xi}$-equivariant. The statement about

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[14]Note that $\xi \in A(n)$, so *a priori* we only know that $\xi \cdot \mathcal{N}(0) \subset \mathcal{N}(n)$; here we are assuming that $\xi \cdot \mathcal{N}(0) \subset \mathcal{N}(0)$.

the associated graded follows formally. □

Lemma 3.9 *For any $\bar{\xi}$-equivariant coherent sheaf F on $\mathfrak{M}$ which is set-theoretically supported on $\mathfrak{M}^+$, the generalized $\bar{\xi}$-eigenspaces of $\Gamma(\mathfrak{M}; F)$ are finite dimensional and the real parts of the eigenvalues are bounded above.*

Proof: The condition holds for a sheaf if it holds for the successive quotients of a filtration of the sheaf, thus we may assume that F is scheme-theoretically supported on a single relative core component X_α. Since $X_{\alpha,0}$ is affine, the pushforward of F to $X_{\alpha,0}$ is a quotient of the tensor product of the structure sheaf of $X_{\alpha,0}$ with some finite-dimensional $\bar{\xi}$-module W. Thus we only need to prove that $\bar{\xi}$ acts on the coordinate ring of $X_{\alpha,0}$ with finite dimensional generalized eigenspaces, and that the eigenvalues that appear (all of which are integers, since the $\bar{\xi}$-action comes from the $\mathbb{T}$-action) are bounded above. This follows from the fact that every element of $X_{\alpha,0}$ limits to o under the $\mathbb{T}$-action. □

3.2 The category $\mathcal{O}_{\mathrm{a}}$

The action of $\mathbb{T}$ on $\mathcal{Q}$ induces an integer grading of A, where the kth graded piece is $A^k := \{a \in A \mid [\xi, a] = ka\}$. Let

$$A^+ := \bigoplus_{k \geq 0} A^k.$$

Definition 3.10 We define **algebraic category** $\mathcal{O}$ to be the full subcategory $\mathcal{O}_{\mathrm{a}}$ of finitely generated A-modules for which the subalgebra A^+ acts locally finitely. We define $D^b_{\mathcal{O}_{\mathrm{a}}}(A\text{-mod})$ to be the full subcategory of objects of $D^b(A\text{-mod})$ with cohomology in $\mathcal{O}_{\mathrm{a}}$.

Remark 3.11 Let $\mathfrak{g}$ be a simple Lie algebra with Borel subalgebra $\mathfrak{b}$ and Cartan subalgebra $\mathfrak{h}$. An infinitesimal block of the classical BGG category $\mathcal{O}$ is by definition the full subcategory of finitely generated $U(\mathfrak{g})$ modules for which $U(\mathfrak{b})$ acts locally finitely, $U(\mathfrak{h})$ acts semisimply, and the center of $U(\mathfrak{g})$ acts with a fixed generalized character. It is a theorem of Soergel [Soe86, Theorem 1] that, in the case of a regular character, this is equivalent to the category obtained by dropping the condition that $U(\mathfrak{h})$ acts semisimply but adding the condition that the center of $U(\mathfrak{g})$ acts with a fixed honest character.

In our setup, A is the analogue of a central quotient of $U(\mathfrak{g})$, A^+ is the analogue of $U(\mathfrak{b})$, and we have no analogue of $U(\mathfrak{h})$. When $\mathfrak{M} = T^*(G/B)$ and the period of the quantization is regular (that is, it has trivial stabilizer in the Namikawa Weyl group, which in this case is the same as the usual Weyl group), our category $\mathcal{O}_{\mathrm{a}}$ is equivalent to the corresponding infinitesimal

block of BGG category $\mathcal{O}$ by Soergel's theorem. If the period is not regular, then our category will be genuinely different from the corresponding infinitesimal block of BGG category $\mathcal{O}$.

Lemma 3.12 *For all integers k, A^k is finitely generated over A^0.*

Proof: This follows from the corresponding statement for gr $A \cong \mathbb{C}[\mathfrak{M}]$, which is a consequence of the fact that $\mathbb{C}[\mathfrak{M}]$ is finitely generated as a commutative algebra, and $\mathbb{C}[\mathfrak{M}]^{\mathbb{T}}$ is finitely generated since $\mathbb{T}$ is reductive. $\qquad\square$

We call an A-module N a **weight module** if it decomposes into generalized weight spaces for the action of $\xi \in A$. More precisely, for any $\ell \in \mathbb{C}$, let

$$N^\ell := \{x \in N \mid \text{ there exists } q \in \mathbb{Z} \text{ such that } (\xi - \ell)^q \cdot x = 0\}.$$

Then N is a weight module if and only if $N = \bigoplus_{\ell \in \mathbb{C}} N^\ell$. Note that for all $k \in \mathbb{Z}$ and $\ell \in \mathbb{C}$, $A^k \cdot N^\ell \subset N^{k+\ell}$.

Lemma 3.13 *A finitely generated A-module N lies in $\mathcal{O}_a$ if and only if N is a weight module, N^ℓ is finite dimensional for all ℓ, and $N^\ell = 0$ for all ℓ with sufficiently large real part.*

Proof: First suppose that the three conditions are satisfied. For any $x \in N^\ell$,

$$A^+ \cdot x \subset \bigoplus_{k \geq 0} N^{k+\ell},$$

which is finite dimensional. Thus A^+ acts locally finitely, and $N \in \mathcal{O}_a$.

Conversely, suppose that $N \in \mathcal{O}_a$. The fact that N is a weight module follows from the fact that $\xi \in A^+$ acts locally finitely. The fact that $N^\ell = 0$ for all ℓ with sufficiently large real part follows from the fact that N is finitely generated, thus the vector space obtained by applying A^+ to a generating set is finite dimensional. Finally, the fact that each generalized weight space of N is finite dimensional follows from Lemma 3.12 and the fact that N is locally finite for the action of $A^0 \subset A^+$. $\qquad\square$

Proposition 3.14 *The category $\mathcal{O}_a$ is equal to the category $C^{\mathfrak{M}_0^+}$ of Definition 2.10, where we use the scheme structure on $\mathfrak{M}^+$ coming from Lemma 3.6.*

Proof: First suppose that $N \in \mathcal{O}_a$. To show that $N \in C^{\mathfrak{M}_0^+}$, we must find a good filtration of N such for $k \geq n$, we have $A^+(k) \cdot N(m) \subset N(k+m-n)$. Choose a finite dimensional

subspace $S \subset N$ which is closed under A^+ and generates N, and define a filtration on N by putting $N(m) := A(m) \cdot S$. If $k \geq n$, then we have

$$A^+(k) \cdot N(m) \subset A(m)A^+(k) \cdot S + [A(m), A^+(k)] \cdot S \subset A(m) \cdot S + A(k+m-n) \cdot S = N(k+m-n).$$

Next, suppose that $N \in C^{\mathfrak{M}_0^+}$. By Definition 2.10, this means that we may choose a filtration of N such such that, for all $k \geq n$, $A_{\bar{k}}^+(k) \cdot N(m) \subset N(k + m - n)$. In particular, $\xi \cdot N(m) \subset N(m)$, and gr N is set-theoretically supported on $\mathfrak{M}_0^+$. Let F be the pullback of gr N from $\mathfrak{M}_0$ to $\mathfrak{M}$. By Lemma 3.9 applied to F, gr N has finite-dimensional $\bar{\xi}$ weight spaces, and the real parts of the eigenvalues are bounded above. Thus, the same holds for N, and Lemma 3.13 tells us that N is in $\mathcal{O}_a$. $\square$

3.3 The category $\mathcal{O}_g$

Definition 3.15 We define **geometric category** $\mathcal{O}$ to be the full subcategory $\mathcal{O}_g$ of $\mathcal{D}$-mod consisting of modules $\mathcal{N}$ such that

- the sheaf $\mathcal{N}$ is set-theoretically supported on the relative core $\mathfrak{M}^+$

- there exists a $\mathcal{D}(0)$-lattice $\mathcal{N}(0) \subset \mathcal{N}$ such that $\xi \cdot \mathcal{N}(0) \subset \mathcal{N}(0)$.

We define $D_{\mathcal{O}_g}^b(\mathcal{D}\text{-mod})$ to be the full subcategory of objects of $D^b(\mathcal{D}\text{-mod})$ with cohomology in $\mathcal{O}_g$.

Remark 3.16 Suppose that $\mathfrak{M} = T^*(G/P)$. By [BPW, 4.5], $\mathcal{D}$-mod is equivalent to the category of finitely generated twisted D-modules on G/P, where the twist is determined by the period of the quantization. Then $\mathcal{O}_g$ consists of regular twisted D-modules with microlocal supports in $\mathfrak{M}^+$.

Our first result is that, unlike $\mathcal{O}_a$, the category $\mathcal{O}_g$ depends only on the image of the period in the quotient $H^2(\mathfrak{M}; \mathbb{C})/H^2(\mathfrak{M}; \mathbb{Z})$.

Lemma 3.17 *Let* $\lambda, \lambda' \in H^2(\mathfrak{M}; \mathbb{C})$ *be two classes that differ by an element of* $H^2(\mathfrak{M}; \mathbb{Z})$. *Let* $\mathcal{D}$ *and* $\mathcal{D}'$ *be the quantizations with periods* λ *and* λ', *and let* $\mathcal{O}_g$ *and* $\mathcal{O}_g'$ *be the associated categories. Then* $\mathcal{O}_g$ *and* $\mathcal{O}_g'$ *are canonically equivalent.*

Proof: Let $\mathcal{L}$ be a line bundle with first Chern class $\lambda - \lambda' \in H^2(\mathfrak{M}; \mathbb{Z})$. In [BPW, §5.1] we construct a $\mathcal{Q} - \mathcal{Q}'$ bimodule $_{\lambda'}\mathcal{T}_\lambda$ which as, a left $\mathcal{Q}$-module, is a quantization of $\mathcal{L}$. Consider the functor $_{\lambda'}\mathcal{T}_\lambda[h^{-1/n}] \otimes_{\mathcal{D}} -$ from $\mathcal{D}$-mod to $\mathcal{D}'$-mod. Since $_{\lambda'}\mathcal{T}_\lambda[h^{-1/n}]$ is a quantization of a line bundle, this functor does not affect the support of an object. Furthermore, if $\mathcal{N}(0) \subset \mathcal{N}$ is a lattice satisfying the second condition of the definition of $\mathcal{O}_g$, then $_{\lambda'}\mathcal{T}_\lambda(0) \otimes_{\mathcal{D}(0)} \mathcal{N}(0)$ will

be such a lattice, as well. Thus our functor takes $\mathcal{O}_g$ to $\mathcal{O}'_g$. To show that it is a canonical equivalence, it is sufficient to show that $_\lambda \mathcal{T}_{\lambda'}[h^{-1/n}] \otimes_{\mathcal{D}'} {}_{\lambda'}\mathcal{T}_\lambda[h^{-1/n}] \cong \mathcal{D}$, where $_{\lambda'}\mathcal{T}_\lambda$ is a quantization of $\mathcal{L}^{-1}$. This follows from uniqueness of quantizations of line bundles [BPW, 5.2]. $\square$

Proposition 3.18 *The category $\mathcal{O}_g$ is equal to the category $\mathcal{C}^{\mathfrak{M}^+}$ of Definition 2.12.*

Proof: By Lemma 3.6, a $\mathcal{D}$-module $\mathcal{N}$ is in $\mathcal{C}^{\mathfrak{M}^+}$ if and only if it admits a lattice $\mathcal{N}(0)$ that is preserved by $h^{-1}\tilde{f}$ for any section $\tilde{f}$ of $\mathcal{Q}$ whose image $f \in \mathcal{Q}/h\mathcal{Q} \cong \mathfrak{S}_{\mathfrak{M}}$ has non-negative $\mathbb{T}$-weight and $\mathbb{S}$-weight at least n. In particular, $\xi \in \Gamma(h^{-1}\mathcal{Q})$ is of this form, and thus preserves this lattice. In addition, the big classical limit $\mathcal{N}(0)/\mathcal{N}(-n)$ is killed by an ideal whose vanishing set is $\mathfrak{M}^+$, and thus is set-theoretically supported on this locus. The same follows for $\mathcal{N}$, so we can conclude that $\mathcal{N} \in \mathcal{O}_g$. Thus $\mathcal{C}^{\mathfrak{M}^+} \subset \mathcal{O}_g$, and it remains to show the reverse inclusion.

Using Lemma 3.17 and the analogous statement for $\mathcal{C}^{\mathfrak{M}^+}$ (which can be proved in the same way), we may add a large multiple of an ample class to the period of our quantization. Then by Theorem 2.8, we may assume that localization holds. Let $\mathcal{N}$ be an object of $\mathcal{O}_g$, and let $N := \Gamma_{\mathbb{S}}(\mathcal{N})$. By Lemmas 3.8, 3.9, and 3.13, N is an object of $\mathcal{O}_a$, which is equal to $C^{\mathfrak{M}_0^+}$ by Proposition 3.14. Then by Proposition 2.13, $\mathcal{N} = \mathrm{Loc}(N)$ is in $\mathcal{C}^{\mathfrak{M}^+}$, and we are done. $\square$

The following corollary follows directly from Propositions 2.13, 2.14, 3.14, and 3.18.

Corollary 3.19 Loc *takes $\mathcal{O}_a$ to $\mathcal{O}_g$ and $\Gamma_{\mathbb{S}}$ takes $\mathcal{O}_g$ to $\mathcal{O}_a$. In particular, if localization holds for $\mathcal{D}$, then the geometric category $\mathcal{O}_g$ is equivalent to the algebraic category $\mathcal{O}_a$. Similarly, if derived localization holds, then $\mathbb{L}\mathrm{Loc}$ and $\mathbb{R}\Gamma_{\mathbb{S}}$ induce an equivalence between $D^b_{\mathcal{O}_a}(A\text{-}\mathrm{mod})$ and $D^b_{\mathcal{O}_g}(\mathcal{D}\text{-}\mathrm{mod})$.*

Remark 3.20 As discussed in Remark 2.15, a version of Corollary 3.19 holds in bounded above derived categories even if derived location fails.

Example 3.21 Combining Remarks 3.11 and 3.16 with Corollary 3.19, we obtain Beilinson and Bernstein's equivalence between an infinitesimal block of BGG category $\mathcal{O}$ and the category of finitely generated twisted D-modules on G/B, smooth with respect to the Schubert stratification.

Remark 3.22 It seems slightly dissatisfying to use the unreduced scheme structure of Remark 3.7 on $\mathfrak{M}^+$. One could also consider the category $\mathcal{C}^{\mathfrak{M}^+_{\mathrm{red}}}$ attached to the reduced scheme structure on the subset $\mathfrak{M}^+$; that is to say, the category of regular $\mathcal{D}$-modules that are set-theoretically supported on $\mathfrak{M}^+$. It is clear that $\mathcal{C}^{\mathfrak{M}^+_{\mathrm{red}}} \subset \mathcal{C}^{\mathfrak{M}^+} = \mathcal{O}_g$, but it is not clear whether or not this

containment is an equality. Since $\mathfrak{M}^+_{\mathrm{red}}$ is not the scheme-theoretic preimage of any subvariety of $\mathfrak{M}_0$, there is no obvious choice of a corresponding subcategory of $\mathcal{O}_\mathrm{a}$.

The following lemma provides an alternative formulation of the second condition of Definition 3.15; it will be used to prove Proposition 3.24.

Lemma 3.23 *A good $\mathcal{D}$-module $\mathcal{N}$ possesses a ξ-stable coherent lattice if and only if, for every finitely generated $\mathcal{D}(0)$-submodule $\mathcal{P}$, the sum*

$$\mathcal{P}_j := \mathcal{P} + \xi \cdot \mathcal{P} + \cdots + \xi^j \cdot \mathcal{P}$$

stabilizes for $j \gg 0$.

Proof: First suppose that the sum stabilizes for every finitely generated $\mathcal{P}$. If we take $\mathcal{P}$ to be a coherent lattice and take j in the stable range, then $\mathcal{P}_j$ is a ξ-stable lattice.

Now assume that $\mathcal{N}$ admits a ξ-stable coherent lattice $\mathcal{N}(0)$, and let $\mathcal{P} \subset \mathcal{N}$ be any finitely generated $\mathcal{D}(0)$-submodule. Then for some m, $\mathcal{P} \subset \mathcal{N}(m)$ and therefore $\mathcal{P}_j \subset \mathcal{N}(m)$ for all $j \geq 0$. The stabilization of $\mathcal{P}_j$ then follows from the fact that $\mathcal{N}(m)$ is a finitely generated module over a Noetherian ring. $\qquad\square$

Proposition 3.24 *The category $\mathcal{O}_\mathrm{g}$ is an Abelian Serre subcategory of $\mathcal{D}$-mod.*

Proof: In order to check that a full subcategory of an Abelian category is Serre and Abelian, we need only check that it is closed under quotients, submodules, and extensions. Obviously, all these are compatible with the support condition, so we need only consider the existence of a ξ-invariant lattice.

Let $\mathcal{N} \subset \mathcal{M}$ be objects of $\mathcal{D}$-mod. The image of a ξ-invariant lattice in $\mathcal{M}$ is such a lattice in $\mathcal{M}/\mathcal{N}$, so $\mathcal{M} \in \mathcal{O}_\mathrm{g} \Rightarrow \mathcal{M}/\mathcal{N} \in \mathcal{O}_\mathrm{g}$. Since any finitely generated $\mathcal{D}(0)$-module $\mathcal{N}$ is also a finitely generated submodule of $\mathcal{M}$, Lemma 3.23 shows that $\mathcal{M} \in \mathcal{O}_\mathrm{g} \Rightarrow \mathcal{N} \in \mathcal{O}_\mathrm{g}$.

Finally, assume that $\mathcal{N} \in \mathcal{O}_\mathrm{g}$ and $\mathcal{M}/\mathcal{N} \in \mathcal{O}_\mathrm{g}$. Let $\mathcal{P}$ be any finitely generated $\mathcal{D}(0)$-submodule of $\mathcal{M}$; by Lemma 3.23, it is sufficient to show that the sequence $\{\mathcal{P}_j\}$ of $\mathcal{D}(0)$-submodules stabilizes. Replacing $\mathcal{P}$ by some $\mathcal{P}_j$, we may assume that the image of $\mathcal{P}$ in $\mathcal{M}/\mathcal{N}$ is ξ-stable. Choose a finite generating set $\{m_i \in \Gamma(U_i; \mathcal{P})\}$ for $\mathcal{P}$, along with $\{a_i^j \in \Gamma(U_i; \mathcal{D}(0))\}$ such that for all i, $\chi_i := \xi m_i - \sum_j a_i^j m_j$ lies in $\Gamma(U_i; \mathcal{N})$. Let $\mathcal{R} \subset \mathcal{N}$ be the $\mathcal{D}(0)$-submodule generated by $\{\chi_i\}$.

For any section m of $\mathcal{P}$ on an open subset U, we may choose sections b_i of $\mathcal{D}(0)$ such that $m = \sum b_i m_i$ (perhaps after shrinking U), and therefore

$$\xi m = \sum \xi b_i m_i = \sum_i [\xi, b_i] m_i + \sum_i b_i \xi m_i = \sum_i [\xi, b_i] m_i + \sum_i b_i \chi_i + \sum_{i,j} b_i a_i^j m_j.$$

25

Thus, $\xi \cdot \mathcal{P} \subset \mathcal{P} + \mathcal{R}$; by induction, this implies that $\mathcal{P}_j \subset \mathcal{P} + \mathcal{R}_{j-1}$ for all j. Since $\mathcal{R}$ is a submodule of $\mathcal{N}$, the submodules $\{\mathcal{R}_j\}$ stabilize, and thus so do the submodules $\{\mathcal{P}_j\}$. $\qquad\square$

4 Categorical preliminaries

In this section we will collect various definitions and basic results about Koszul, highest weight, and standard Koszul categories, which we will apply to $\mathcal{O}_{\mathrm{a}}$ and $\mathcal{O}_{\mathrm{g}}$ in the next section.

4.1 Koszul categories

Much of the material in this section comes from the seminal work [BGS96], though our presentation follows more closely that of [MOS09], to which we refer the reader for further details.

Let $\tilde{\mathcal{C}}$ be a $\mathbb{C}$-linear Abelian category, which we will assume throughout is Noetherian and Artinian with enough projectives and finite projective dimension. Consider a choice of weight $\mathrm{wt}(L) \in \mathbb{Z}$ for each simple in $\tilde{\mathcal{C}}$, and assume further that there are finitely many simples of any given weight. The category $\tilde{\mathcal{C}}$ is said to be **mixed** if, whenever $\mathrm{Ext}^1(L, L') \neq 0$, we have $\mathrm{wt}(L') < \mathrm{wt}(L)$. A **Tate twist** on a mixed category is an autoequivalence $M \mapsto M(1)$ such that $\mathrm{wt}(L(1)) = \mathrm{wt}(L) - 1$.

Let $\tilde{\mathcal{C}}$ be a mixed category, and let $\tilde{\mathcal{C}}/\mathbb{Z}$ be the category whose objects are the same as those of $\tilde{\mathcal{C}}$, but whose morphism spaces are the graded vector spaces

$$\mathrm{Hom}_{\tilde{\mathcal{C}}/\mathbb{Z}}(M, M') := \bigoplus_{d \in \mathbb{Z}} \mathrm{Hom}_{\tilde{\mathcal{C}}}(M, M'(-d)).$$

Note that every object in $\tilde{\mathcal{C}}/\mathbb{Z}$ is isomorphic to all of its Tate twists.

If P is an indecomposable projective in $\tilde{\mathcal{C}}$, or more generally a projective whose head is concentrated in a single weight, then $\mathrm{End}_{\tilde{\mathcal{C}}/\mathbb{Z}}(P)$ is a mixed algebra in the sense of [BGS96, 4.1.5], and thus positively graded. In fact, any mixed category with Tate twist is equivalent to the category of finite dimensional graded modules over a projective limit of finite dimensional positively graded algebras with semi-simple degree zero part [BGS96, 4.1.6]; if we assume in addition that the are finitely many simples of weight 0, then the condition of having enough projectives guarantees that this algebra can be taken to be finite dimensional.

Let $\mathrm{Vect}_{\mathbb{C}}$ be the category of finite-dimensional complex vector spaces. Define the **degrading** $\mathcal{C}$ of $\tilde{\mathcal{C}}$ to be the category of additive functors $(\tilde{\mathcal{C}}/\mathbb{Z})^{\mathrm{op}} \to \mathrm{Vect}_{\mathbb{C}}$ for which the composition with the natural functor $\tilde{\mathcal{C}}^{\mathrm{op}} \to (\tilde{\mathcal{C}}/\mathbb{Z})^{\mathrm{op}}$ is left-exact[15]. This is the same as the category of

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[15]This is a degrading in the sense of [BGS96, §4.3], but it is somewhat stronger, since the condition $(*)_{M,N}^i$ is automatic from the Yoneda lemma.

additive functors from the opposite category of projectives in $\tilde{\mathcal{C}}$ to $\mathrm{Vect}_{\mathbb{C}}$, since every object in $\tilde{\mathcal{C}}$ can be presented as the cokernel of a map between projectives. In more concrete language, $\mathcal{C}$ is the category of finite dimensional right modules over the endomorphism ring in $\tilde{\mathcal{C}}/\mathbb{Z}$ of the sum of the projective covers of all simples of weight 0.

We say that $\tilde{\mathcal{C}}$ is a **graded lift** of $\mathcal{C}$. Similarly, we can define the degrading of a functor between mixed categories, and speak of graded lifts of functors. We call an object of $\mathcal{C}$ **gradeable** if it is representable, that is, if it is of the form $\mathrm{Hom}(M, -)$ for M in $\tilde{\mathcal{C}}/\mathbb{Z}$. If $\tilde{\mathcal{C}}$ is the category of finite dimensional graded modules over a positively graded finite dimensional algebra, then $\mathcal{C}$ is the category of finite dimensional ungraded modules.

We now define the category of **linear complexes** of projectives, which plays a central role in our discussion of Koszul duality below.

Definition 4.1 Let $\mathrm{LCP}(\tilde{\mathcal{C}})$ denote the category whose objects are complexes $X_\bullet$ of projective objects in $\tilde{\mathcal{C}}$ such that all summands of the head of X_j have weight j. This is "linear" in the sense that if we shifted every term to have head which is weight 0, then every differential would have "degree 1." The morphisms in $\mathrm{LCP}(\tilde{\mathcal{C}})$ are chain maps.

Remarkably, $\mathrm{LCP}(\tilde{\mathcal{C}})$ is an Abelian subcategory of the category of complexes in $\tilde{\mathcal{C}}$. The simple modules of $\mathrm{LCP}(\tilde{\mathcal{C}})$ are the complexes given by a single indecomposable projective in a single degree j; we can weight the category $\mathrm{LCP}(\tilde{\mathcal{C}})$ by endowing a complex concentrated in degree j with weight j. The Tate twist on this category is given by $[-1](-1)$. An indecomposable injective in this category arises as a quotient of a minimal projective resolution of a simple by the subcomplex consisting of objects with head in weight less than j in the jth term.

Remark 4.2 The category $\mathrm{LCP}(\tilde{\mathcal{C}})$ is canonically equivalent to the quadratic dual of the category $\tilde{\mathcal{C}}$ [MOS09, Theorem 12], even if $\tilde{\mathcal{C}}$ is not itself quadratic.

Definition 4.3 The category $\mathcal{C}$ is said to be **Koszul** if it admits a graded lift $\tilde{\mathcal{C}}$ with the property that the minimal projective resolution of every simple object in $\tilde{\mathcal{C}}$ is linear. If $\mathcal{C}$ is Koszul, then any two Koszul graded lifts are canonically equivalent as mixed categories [BGS96, 2.5.2].

Mazorchuk, Ovsienko, and Stroppel [MOS09, §5.1] define a pair of adjoint functors[16]

$$K_{\tilde{\mathcal{C}}} : D^b(\tilde{\mathcal{C}}) \longrightarrow D^b(\mathrm{LCP}(\tilde{\mathcal{C}})) \qquad \text{and} \qquad K'_{\tilde{\mathcal{C}}} : D^b(\mathrm{LCP}(\tilde{\mathcal{C}})) \longrightarrow D^b(\tilde{\mathcal{C}}).$$

We refer there for the complete definition; the facts we will need about these functors are summarized in the following result [MOS09, Theorem 30].

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[16]In [MOS09], a different finiteness condition is used on the derived category, but the Artinian and finite global dimension hypotheses guarantee that the functors are bounded.

Theorem 4.4 *The following are equivalent for a non-negatively graded category $\tilde{\mathcal{C}}$:*

1. *The degrading $\mathcal{C}$ is Koszul.*

2. *The functors $K_{\tilde{\mathcal{C}}}$ and $K'_{\tilde{\mathcal{C}}}$ above are mutually inverse equivalences of categories.*

3. *The functor $K_{\tilde{\mathcal{C}}}$ takes each indecomposable projective to the corresponding simple.*

4. *The functor $K'_{\tilde{\mathcal{C}}}$ takes each indecomposable injective to the corresponding simple.*

Proposition 4.5 *If $\mathcal{C}$ is Koszul, then so is $\mathrm{LCP}(\tilde{\mathcal{C}})$, and there is a canonical equivalence of categories between $\tilde{\mathcal{C}}$ and the double dual $\mathrm{LCP}\big(\mathrm{LCP}(\tilde{\mathcal{C}})\big)$.*

Proof: By [MOS09, Theorem 12], Koszulity of $\mathrm{LCP}(\tilde{\mathcal{C}})$ is equivalent to the quadratic dual of $\mathcal{C}$ being Koszul, which follows from [BGS96, 2.9.1].

Consider the composite equivalence

$$K_{\mathrm{LCP}(\tilde{\mathcal{C}})} \circ K_{\tilde{\mathcal{C}}} \colon D^b(\tilde{\mathcal{C}}) \to D^b(\mathrm{LCP}\big(\mathrm{LCP}(\tilde{\mathcal{C}})\big));$$

this is an equivalence of derived categories sending projectives to injectives. Composing with the inverse of the derived Nakayama functor, we obtain an equivalence of derived categories sending projectives to projectives, and thus inducing an equivalence of Abelian categories $\tilde{\mathcal{C}} \cong \mathrm{LCP}\big(\mathrm{LCP}(\tilde{\mathcal{C}})\big)$. $\square$

Remark 4.6 Though we have emphasized categories rather than algebras in the above definition of Koszul duality, it is sometimes convenient to reconsider the Koszul duality statements above from the point of view of algebras. Let P be the sum of the indecomposable projectives in $\mathcal{C}$; then $\mathcal{C}$ is equivalent to the category of finite-dimensional $\mathrm{End}_{\mathcal{C}}(P)^{\mathrm{op}}$-modules. Let L be the sum of the simples in $\mathcal{C}$, and choose a projective resolution $\Pi_{\bullet}$ of L. The algebra

$$\mathcal{E} := \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}(\Pi_{\bullet}, \Pi_{\bullet}[m])$$

is naturally a dg-algebra, which we call the **dg-Yoneda algebra** of $\mathcal{C}$; this algebra depends on the choice of $\Pi_{\bullet}$ only up to quasi-isomorphism. The cohomology ring

$$E := H^{\bullet}(\mathcal{E}) \cong \mathrm{Ext}_{\mathcal{C}}(L, L)$$

is the ordinary **Yoneda algebra** of $\mathcal{C}$. The algebra E carries a natural (cohomological) grading, and thus has a category of graded modules E-gmod.

Now assume that $\mathcal{C}$ is Koszul, and let $\tilde{\Pi}_{\bullet}$ be a linear projective resolution of the weight 0 graded lift $\tilde{L}$ of L. Then $\tilde{\Pi}_{\bullet}$ is an injective generator in $\mathrm{LCP}(\tilde{\mathcal{C}})$, and the algebra of

endomorphisms (of arbitrary degree) of $\tilde{\Pi}_\bullet$ in $\mathrm{LCP}(\tilde{\mathcal{C}})/\mathbb{Z}$ is a quotient dg-algebra of $\mathcal{E}$ with trivial differential, killing all elements of $\mathrm{Hom}(\tilde{\Pi}_\bullet, \tilde{\Pi}_\bullet[m](m))$ of positive degree. The quotient map is a quasi-isomorphism, so

$$E \cong \mathrm{End}_{\mathrm{LCP}(\tilde{\mathcal{C}})/\mathbb{Z}}(\tilde{\Pi}_\bullet, \tilde{\Pi}_\bullet) \cong \bigoplus_m \mathrm{End}_{\mathrm{LCP}(\tilde{\mathcal{C}})}\left(\tilde{\Pi}_\bullet, \tilde{\Pi}_\bullet[m](m)\right).$$

Thus E-gmod $\cong \mathrm{LCP}(\tilde{\mathcal{C}})$, and the algebra $\mathrm{End}_{\mathcal{C}}(P) = \mathrm{End}_{\tilde{\mathcal{C}}/\mathbb{Z}}(\tilde{P})$ is isomorphic as a graded algebra to the Yoneda algebra of E-gmod. This demonstrates explicitly that, if $\mathcal{C}$ is Koszul, it has a unique graded lift $\tilde{\mathcal{C}}$ (see Definition 4.3).

Remark 4.7 The Yoneda algebra of the category $\mathcal{O}_\mathrm{g}$ will be studied in Section 5.4.

For the purposes of Section 10, it will be convenient to introduce the following definition.

Definition 4.8 Let $\mathcal{C}$ and $\mathcal{C}^!$ be two Koszul categories, and let $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{C}}^!$ be their graded lifts. A **Koszul duality** from $\mathcal{C}$ to $\mathcal{C}^!$ is an equivalence of mixed categories

$$\psi : \mathrm{LCP}(\tilde{\mathcal{C}}) \to \tilde{\mathcal{C}}^!.$$

Taking derived functors and precomposing with $K_{\tilde{\mathcal{C}}}$, we also obtain an equivalence of triangulated categories

$$\Psi : D^b(\tilde{\mathcal{C}}) \to D^b(\tilde{\mathcal{C}}^!)$$

sending projective objects of $\tilde{\mathcal{C}}$ to simple objects of $\tilde{\mathcal{C}}^!$ and sending the Tate twist (1) to the functor $(-1)[-1]$. Conversely, any such equivalence of triangulated categories must induce an equivalence $\mathrm{LCP}(\tilde{\mathcal{C}}) \cong \tilde{\mathcal{C}}^!$, as these are the hearts of t-structures for which the functor is exact. Thus, we will also refer to Ψ as a Koszul duality from $\mathcal{C}$ to $\mathcal{C}^!$. We say that $\mathcal{C}^!$ is **Koszul dual** to $\mathcal{C}$ if there is a Koszul duality between them.

While we have not defined Koszul duality in a way which is obviously symmetric, the following observation demonstrates that it is a true duality.

Proposition 4.9 If $\psi : \mathrm{LCP}(\tilde{\mathcal{C}}) \to \tilde{\mathcal{C}}^!$ is a Koszul duality from $\mathcal{C}$ to $\mathcal{C}^!$, then the inverse of

$$\mathrm{LCP}(\psi) : \tilde{\mathcal{C}} \cong \mathrm{LCP}\big(\mathrm{LCP}(\tilde{\mathcal{C}})\big) \to \mathrm{LCP}(\tilde{\mathcal{C}}^!)$$

is a Koszul duality from $\mathcal{C}^!$ to $\mathcal{C}$.

Remark 4.10 We could also phrase Proposition 4.9 in terms of the derived equivalence $\Psi : D^b(\tilde{\mathcal{C}}) \to D^b(\tilde{\mathcal{C}}^!)$. The induced Koszul duality in the reverse direction is the precomposition of Ψ^{-1} with the right derived Nakayama functor.

4.2 Highest weight and standard Koszul categories

Let $\mathcal{C}$ be a $\mathbb{C}$-linear Abelian, Noetherian category with simple objects $\{L_\alpha \mid \alpha \in \mathcal{I}\}$, projective covers $\{P_\alpha \mid \alpha \in \mathcal{I}\}$, and injective hulls $\{I_\alpha \mid \alpha \in \mathcal{I}\}$. Let $\leq$ be a partial order on $\mathcal{I}$.

Definition 4.11 We call $\mathcal{C}$ **highest weight** with respect to this partial order if there is a collection of objects $\{\Delta_\alpha \mid \alpha \in \mathcal{I}\}$ and epimorphisms $P_\alpha \overset{\Pi_\alpha}{\twoheadrightarrow} \Delta_\alpha \overset{\pi_\alpha}{\twoheadrightarrow} L_\alpha$ such that for each $\alpha \in \mathcal{I}$, the following conditions hold:

1. The object $\ker \pi_\alpha$ has a filtration such that each sub-quotient is isomorphic to L_β for some $\beta < \alpha$.

2. The object $\ker \Pi_\alpha$ has a filtration such that each sub-quotient is isomorphic to Δ_γ for some $\gamma > \alpha$.

The objects Δ_α are called **standard objects**. Classic examples of highest weight categories in representation theory include integral blocks of parabolic BGG category $\mathcal{O}$ [FM99, 5.1].

In any highest weight category, we also have a notion of **costandard** objects.

Definition 4.12 Let ∇_α be the largest subobject of I_α whose composition factors are all isomorphic to L_β with $\beta \leq \alpha$. By [DR92, Theorem 1], the category $\mathcal{C}$ is highest weight if and only if I_α has a filtration by costandards ∇_γ analogous to the standard filtration on projectives.

If $\mathcal{C}$ admits a graded lift $\tilde{\mathcal{C}}$, then every standard object of $\mathcal{C}$ is gradeable. More precisely, if $\tilde{P}_\alpha$ is a graded lift of P_α, then we may define $\tilde{\Delta}_\alpha$ to be the largest quotient of $\tilde{P}_\alpha$ with no composition factors of the form $\tilde{L}_\beta$ for $\beta > \alpha$, and $\tilde{\Delta}_\alpha$ will be a graded lift of Δ_α. We refer to the graded lifts of standard objects of $\mathcal{C}$ as standard objects of $\tilde{\mathcal{C}}$.

Definition 4.13 A highest-weight category $\mathcal{C}$ is **standard Koszul** if it admits a graded lift $\tilde{\mathcal{C}}$ with the property that the minimal projective resolution of every standard object in $\tilde{\mathcal{C}}$ is linear. (Compare this definition to Definition 4.3, in which ordinary Koszulity is defined.)

The following result is the main theorem of [ÁDL03].

Theorem 4.14 *A highest-weight category $\mathcal{C}$ is standard Koszul with respect to a given partial order if and only if it is Koszul and its Koszul dual $\mathcal{C}^!$ is highest weight with respect to the opposite partial order.*

5 The structure of $\mathcal{O}_a$ and $\mathcal{O}_g$

In this section we fix a conical symplectic resolution and a Hamiltonian $\mathbb{T}$-action with isolated fixed points and we investigate the structure of the categories $\mathcal{O}_a$ and $\mathcal{O}_g$ for various different

quantizations. We prove that $\mathcal{O}_a$ is highest weight for most quantizations (Proposition 5.3 and Theorem 5.12), and that $\mathcal{O}_g$ is highest weight for all quantizations (Proposition 5.17). Furthermore, we conjecture that both categories are standard Koszul (Conjectures 5.14 and 5.18), based on the evidence from the theory of hypertoric varieties (Example 5.15) and from classical Lie theory (Example 5.16).

We also include a brief discussion of the Yoneda algebra of $\mathcal{O}_g$ (Section 5.4). In particular, we define a natural map from $H^*(\mathfrak{M}; \mathbb{C})$ to the center of the Yoneda algebra, and conjecture that this map is often an isomorphism (Conjecture 5.23).

5.1 The B algebra

For any $\mathbb{Z}$-graded ring A, let

$$B(A) := A^0 \Big/ \sum_{k>0} A^{-k} A^k.$$

Proposition 5.1 *If A is the $\mathbb{S}$-invariant section ring of a quantized conical symplectic resolution $\mathfrak{M}$, with grading induced by the action of $\mathbb{T}$, then $B := B(A)$ is finite dimensional as a vector space.*

Proof: Consider the associated graded ring $\operatorname{gr} B$; this ring admits a surjection from $B(\mathbb{C}[\mathfrak{M}])$, where $\mathbb{C}[\mathfrak{M}]$ is also graded by the action of $\mathbb{T}$. It therefore suffices to show that $B(\mathbb{C}[\mathfrak{M}])$ is finite dimensional.

Let p_0 be an element of $\mathfrak{M}_0$. In the proof of Lemma 3.6, we showed that if $p_0 \notin \mathfrak{M}_0^+$, then there exists a function of positive $\mathbb{T}$-weight that does not vanish at p_0. Let $\mathfrak{M}^-$ be the relative core for the opposite $\mathbb{T}$-action, and let $\mathfrak{M}_0^-$ be its image in $\mathfrak{M}_0$. By the same reasoning, if $p_0 \notin \mathfrak{M}_0^-$, then there exists a function of negative $\mathbb{T}$-weight that does not vanish at p_0. It follows that the set-theoretic vanishing locus in $\mathfrak{M}_0$ of the ideal I generated by all homogeneous elements of nonzero $\mathbb{T}$-weight is equal to $\mathfrak{M}_0^+ \cap \mathfrak{M}_0^- = \{o\}$, and therefore that this ideal has finite codimension in $\mathbb{C}[\mathfrak{M}_0] = \mathbb{C}[\mathfrak{M}]$. Thus $B(\mathbb{C}[\mathfrak{M}]) = (\mathbb{C}[\mathfrak{M}]/I)_0$ is finite dimensional. □

For each $\alpha \in \mathcal{I}$, let U_α be a formal neighborhood of p_α. Although the groups $\mathbb{S}$ and $\mathbb{T}$ do not act on U_α or on $\mathcal{D}|_{U_\alpha}$, their Lie algebras do, so we can make sense of the ring $\tilde{A}_\alpha := \Gamma_{\mathbb{S}}(\mathcal{D}|_{U_\alpha})$. Let A_α be the subring of $\tilde{A}_\alpha$ that is additively spanned by $\mathbb{T}$-weight vectors (this means passing from power series to polynomials), and let $B_\alpha := B(A_\alpha)$. Then A_α and B_α admit natural maps from A and B, respectively.

Lemma 5.2 *Let $d = \frac{1}{2} \dim \mathfrak{M}$. The algebra A_α is isomorphic to the ring of global differential operators on $\mathbb{C}^d$, and B_α is isomorphic to $\mathbb{C}$.*

Proof: There is only one quantization of the formal polydisk [BK04, 1.5], thus the ring of sections $\Gamma(\mathcal{D}|_{U_\alpha})$ must be isomorphic to the Weyl algebra

$$\mathbb{C}[[x_1, y_1, \ldots, x_d, y_d, h^{1/n}]](h^{-1/n}) \Big/ \Big\langle [h, x_i], [h, y_i], [x_i, x_j], [y_i, y_j], [x_i, y_j] - h\delta_{ij} \Big\rangle.$$

We may choose $x_1, y_1, \ldots, x_d, y_d, h$ to be simultaneous weight vectors for $\mathbb{S}$ and $\mathbb{T}$, with each $x_i y_i$ and h having $\mathbb{S}$ weight n and $\mathbb{T}$ weight 0. If χ_i is the $\mathbb{S}$ weight of x_i, let

$$z_i := h^{-\chi_i/n} x_i \qquad \text{and} \qquad w_i := h^{\chi_i/n - 1} y_i.$$

Then A_α is generated by

$$\{z_1, w_1, \ldots, z_d, w_d\},$$

subject to the relations

$$[z_i, z_j] = 0, \qquad [w_i, w_j] = 0, \qquad \text{and} \qquad [z_i, w_j] = \delta_{ij}.$$

The $\mathbb{C}$-vector space spanned by the z_i and w_i is isomorphic as a $\mathbb{T}$-space to the tangent space $T_{p_\alpha}\mathfrak{M}$. Since p_α is an isolated fixed point, none of the z_i or w_i can have zero weight. Without loss of generality suppose that the $\mathbb{T}$-weight of z_i is negative. The ring A_α has a PBW basis given by monomials of the form $z^a w^b$ for $a, b \in \mathbb{N}^d$. All such monomials with $a \neq 0$ are clearly in $A_\alpha^{-k} A_\alpha^k$ for some $k > 0$, so B_α is at most one-dimensional. On the other hand, the action of A_α^0 on the $\mathbb{T}$-invariant part of $A_\alpha / A_\alpha \langle w_1, \ldots, w_d \rangle$ descends to a nontrivial action of B_α, so B_α cannot be zero. $\qquad\square$

In what follows, fix an $\mathbb{S}$-equivariant line bundle $\mathcal{L}$, very ample over $\mathfrak{M}_0$, and let $\eta \in H^2(\mathfrak{M}; \mathbb{C})$ be its Euler class. Fix another class $\lambda \in H^2(\mathfrak{M}; \mathbb{C})$, and for all $k \in \mathbb{C}$, let $\mathcal{Q}_k$ be the quantization of $\mathfrak{M}$ with period $\lambda + k\eta$. Let

$$A_k := \Gamma_{\mathbb{S}}(\mathcal{D}_k) \qquad \text{and} \qquad B_k := B(A_k).$$

For each $\alpha \in \mathcal{I}$, define $A_{k,\alpha}$ and $B_{k,\alpha}$ as above.

Proposition 5.3 *The natural map $\varphi_k : B_k \to \bigoplus_{\alpha \in \mathcal{I}} B_{k,\alpha}$ is an isomorphism for all but finitely many values of $k \in \mathbb{C}$.*

Proof: Our plan is to construct a family of maps, parametrized by the affine line, such that the fiber over k is the map φ_k, and to show that the generic map φ_∞ is an isomorphism.

To accomplish this, we work with the twistor deformation $\mathscr{M}_\eta$ of $\mathfrak{M}$ over $\mathbb{A}^1$, introduced in 2.2. Let $\Delta := \operatorname{Spec} \mathbb{C}[[h]]$ be the formal disk, and let $\sigma_k : \Delta \to \Delta \times \mathbb{A}^1$ be the map that is

the identity on the first coordinate and pulls back the coordinate on $\mathbb{A}^1$ to kh. Following the argument in [BPW, 4.17], there exists a quantization $\mathscr{D}$ of $\mathscr{M}_\eta$ such that $\mathcal{D}_k$ is isomorphic to the pull-back of $\mathscr{D}$ along the map σ_k via the pull-back construction described in [BPW, §3.1]. The action of $\mathbb{T}$ extends to this situation by [Kal06b, 1.5].

Let $\mathscr{U}_\alpha \subset \mathscr{M}_\eta$ be a formal neighborhood of the component of $\mathscr{M}_\eta^{\mathbb{T}}$ corresponding to α, so that $\mathscr{U}_\alpha$ is a deformation of U_α over $\mathbb{A}^1$. Let π be the projection from $\mathscr{M}_\eta$ to $\mathbb{A}^1$. Let

$$\mathscr{B} := B(\pi_*\mathscr{D}) \qquad \text{and} \qquad \mathscr{B}_\alpha := B(\pi_*\mathscr{D}|_{\mathscr{U}_\alpha});$$

both are sheaves of algebras over $\mathbb{A}^1$, and we have a natural map $\varphi : \mathscr{B} \to \mathscr{B}_\alpha$ whose fiber over k is φ_k.

By a result of Kaledin [Kal08, 2.5], the generic fiber $\mathscr{M}_\eta(\infty)$ is affine. This tells us that the attracting sets to the fixed points are all closed affine spaces, so the associated graded algebras of $B(\infty)$ and $\oplus B_\alpha(\infty)$ are both isomorphic to the coordinate ring of $\mathscr{M}_\eta(\infty)^{\mathbb{T}}$. It follows that the map $\operatorname{gr}\varphi(\infty)$ is an isomorphism, and thus so is $\varphi(\infty)$. $\qquad\square$

Remark 5.4 If we use an action of $\mathbb{T}$ which does not have isolated fixed points, then these results proceed through in almost the same way, but with one important change: the algebras B_α should now be indexed by components of $\mathfrak{M}^{\mathbb{T}}$, and each one will be given by global sections of an induced quantization on the corresponding component.

5.2 The category $\mathcal{O}_a$ is highest weight (for most quantizations)

Throughout this section we will assume that the map $\varphi : B \to \oplus B_\alpha \cong \mathbb{C}^{\mathcal{I}}$ is an isomorphism. By Proposition 5.3, this is the case for "most" quantizations.

For each $\alpha \in \mathcal{I}$, let

$$\Delta_\alpha := A \otimes_{A^+} B_\alpha \qquad \text{and} \qquad \nabla_\alpha := \left(B_\alpha^* \otimes_{A^+} A\right)^\star,$$

where B_α is regarded as a quotient of B (and therefore also of A^+). Here $*$ denotes ordinary vector space duality and $\star$ denotes restricted duality: if N is a finitely generated right weight module (as defined in Section 3.2), then $N^\star := \oplus_{\ell\in\mathbb{C}}(N^\ell)^*$. We will refer to Δ_α and ∇_α as the **standard** and **costandard** modules indexed by α.

Lemma 5.5 *The modules Δ_α and ∇_α lie in $\mathcal{O}_a$.*

Proof: The fact that Δ_α lies in $\mathcal{O}_a$ follows from Lemmas 3.12 and 3.13, and the proof that ∇_α lies in $\mathcal{O}_a$ is identical. $\qquad\square$

Lemma 5.6 *Each standard object Δ_α has a unique simple quotient L_α. Furthermore, every simple object of $\mathcal{O}_a$ is isomorphic to a unique element of the set $\{L_\alpha \mid \alpha \in \mathcal{I}\}$.*

Proof: Let ℓ_α be the highest weight (measured by its real part) that appears in Δ_α. Then $\Delta_\alpha^{\ell_\alpha}$ is annihilated by A^k for all positive k, and is therefore naturally a B-module; it is isomorphic as a B-module to B_α. Let N_α be the sum of all submodules of Δ_α that do not contain $\Delta_\alpha^{\ell_\alpha}$. Then $L_\alpha := \Delta_\alpha/N_\alpha$ is evidently nonzero and simple. Furthermore, it is the only simple quotient of Δ_α, since Δ_α is generated by its highest weight space.

If $\alpha \neq \alpha'$, then the highest weight spaces of L_α and $L_{\alpha'}$ are not isomorphic as B-modules, therefore L_α and $L_{\alpha'}$ cannot be isomorphic as A-modules. Now suppose that L is an arbitrary simple object of $\mathcal{O}_a$. The highest weight space of L must be isomorphic as a B-module to B_α for some $\alpha \in \mathcal{I}$. We get a natural A^+-module homomorphism $B_\alpha \to L$, which induces an A-module homomorphism $\Delta_\alpha \to L$, which is a surjection since L is simple. Thus L is a quotient of Δ_α, so it is isomorphic to L_α. $\qquad\square$

Lemma 5.7 *All objects of $\mathcal{O}_a$ have finite length.*

Proof: Lemma 5.6 tells us that there are finitely many simple objects, so it is enough to prove that each simple object appears finitely many times in the composition series of any object of $\mathcal{O}_a$. This follows from Lemma 3.13, which says that each generalized weight space of an object of $\mathcal{O}_a$ is finite dimensional. $\qquad\square$

Lemma 5.8 *For all $\alpha \in \mathcal{I}$, $\mathrm{End}_{\mathcal{O}_a}(L_\alpha) = \mathbb{C}$.*

Proof: The natural maps $\mathbb{C} = \mathrm{End}_B(B_\alpha) \to \mathrm{End}_{\mathcal{O}_a}(\Delta_\alpha) \to \mathrm{End}_{\mathcal{O}_a}(L_\alpha)$ are isomorphisms. $\square$

For any subset $\mathcal{K} \subset \mathcal{I}$, let $\mathcal{O}_a(\mathcal{K})$ be the full subcategory of $\mathcal{O}_a$ consisting of objects whose simple subquotients all lie in the set $\{L_\alpha \mid \alpha \in \mathcal{K}\}$. Consider the partial order on $\mathcal{I}$ generated by putting $\alpha \leq \alpha'$ if L_α is isomorphic to a subquotient of $\Delta_{\alpha'}$ or of $\nabla_{\alpha'}$.

Remark 5.9 We will show in Corollary 6.4 below that Δ_α and ∇_α have the same composition series multiplicities for most quantizations, so in fact this partial order can also be defined using only one of these classes of objects.

Lemma 5.10 *Let $\mathcal{K} \subset \mathcal{I}$ be closed in the order topology (that is, $\alpha \leq \alpha' \in \mathcal{K} \Rightarrow \alpha \in \mathcal{K}$) and let $\alpha \in \mathcal{K}$ be a maximal element. Then the natural surjection $\Delta_\alpha \to L_\alpha$ is a projective cover in $\mathcal{O}_a(\mathcal{K})$ and the natural injection $L_\alpha \to \nabla_\alpha$ is an injective hull in $\mathcal{O}_a(\mathcal{K})$.*

Proof: Consider the functor from $\mathcal{O}_a(\mathcal{K})$ to the category of vector spaces taking N to $\mathrm{Hom}(\Delta_\alpha, N) \cong \mathrm{Hom}_{A^+}(B_\alpha, N)$. We wish to show that this functor is exact; it is obviously left-exact, so we need only show that it induces a surjection when applied to a surjection.

Assume not, and let $\phi \colon \Delta_\alpha \to N/N'$ be a homomorphism in $\mathcal{O}_a(\mathcal{K})$ which cannot be lifted to a map $\Delta_\alpha \to N$. Without loss of generality we can assume that ϕ is surjective, and that N

34

is generated as an A-module by a vector $v \in N^{\ell_\alpha}$ which lifts a nonzero vector in $\phi(B_\alpha)$. We can further assume that N/N' is isomorphic to L_α.

Let ℓ_β be the highest weight appearing in N. By adjunction we get a homomorphism $\psi \colon N \to \nabla_\beta$ which is an isomorphism on the ℓ_β-weight space. Since N is generated by v, it follows that $\psi(v) \neq 0$, and so L_α appears in a composition series of ∇_β. Thus $\alpha < \beta$, and L_β appears in a composition series of N, contradicting the fact that N lies in $\mathcal{O}_a(\mathcal{K})$.

Thus Δ_α is projective. That it is the projective cover of L_α follows from the fact that $\Delta_\alpha^{\ell_\alpha}$ is 1-dimensional. The second statement follows similarly from the fact that ∇_α corepresents the vector space dual of the same functor. $\qquad\square$

For all $\alpha \in \mathcal{I}$, let $\mathcal{K}_\alpha := \{\alpha' \in \mathcal{I} \mid \alpha' < \alpha\} = \overline{\{\alpha\}} \smallsetminus \{\alpha\}$.

Lemma 5.11 *For any $\alpha \in \mathcal{I}$, the kernel of $\Delta_\alpha \to L_\alpha$ and the cokernel of $L_\alpha \to \nabla_\alpha$ both lie in the subcategory $\mathcal{O}_a(\mathcal{K}_\alpha)$.*

Proof: It suffices to show that L_α appears in the composition series of both Δ_α and ∇_α with multiplicity exactly 1. This follows from the fact that $\dim \Delta_\alpha^{\ell_\alpha} = \dim \nabla_\alpha^{\ell_\alpha} = 1$. $\qquad\square$

Consider the set $\mathfrak{U} \subset H^2(\mathfrak{M}; \mathbb{C})$ consisting of periods of quantizations such that for all $\alpha, \alpha' \in \mathcal{I}$, $\mathrm{Ext}_A^k(\Delta_\alpha, \nabla_{\alpha'}) = 0$ for $k > 0$. Note that by [BGS96, 3.2.3], this further implies that for all $\alpha, \alpha' \in \mathcal{I}$, $\mathrm{Ext}^2_{\mathcal{O}_a}(\Delta_\alpha, \nabla_{\alpha'}) = 0$. By Losev's Theorem A.1, $\mathfrak{U}$ contains a non-empty Zariski open subset. and for every $\eta \in H^2(\mathfrak{M}; \mathbb{C})$ with $\mathscr{M}_\eta$ affine, we have $\kappa\eta + \lambda \in \mathfrak{U}$ for all but finitely many $\kappa \in \mathbb{C}$. The following theorem can be deduced from Theorem A.1 and Lemmas 5.5-5.11 via [BGS96, 3.2.1].

Theorem 5.12 *Assuming that the quantization $\mathcal{D}$ is chosen such that φ is an isomorphism and the period of $\mathcal{D}$ lies in $\mathfrak{U}$ (both generic conditions), the category $\mathcal{O}_a$ has enough projectives and is highest weight with respect to our partial order.*

The following corollary follows by an argument identical to that in [BGS96, 3.3.2].

Corollary 5.13 *For any $\lambda \in \mathfrak{U}$, the inclusion $D^b(\mathcal{O}_a) \to D^b_{\mathcal{O}_a}(A\text{-mod})$ is an equivalence of categories.*

Proof: These categories have a common generator as triangulated categories, given by a projective generator P or injective generator I. Thus, it suffices to check that the map $\mathrm{Ext}^i_{\mathcal{O}_a}(P, I) \to \mathrm{Ext}^i_{A_\lambda}(P, I)$ is an isomorphism for all i, and all projectives P and injectives I in $\mathcal{O}_a$. For $i = 0$, this is just the fact that $\mathcal{O}_a$ is a full subcategory. For $i > 0$, both sides are 0, since P is standard filtered and I costandard filtered. $\qquad\square$

Conjecture 5.14 *Whenever $\mathcal{O}_a$ is highest weight, it is also standard Koszul.*

Example 5.15 If $\mathfrak{M}$ is a hypertoric variety and $\mathcal{D}$ is chosen correctly, then $\mathcal{O}_a$ is standard Koszul by [BLPW12, 4.10]. The rings R and E are isomorphic to the rings A and B introduced in [BLPW10].

Example 5.16 If $\mathcal{O}_a$ is a regular infinitesimal block of BGG category $\mathcal{O}$ (see Remark 3.11), then it is known to be standard Koszul by [BGS96, RC80, ÁDL03]. (See [BLP⁺11, 9.2] for more details.)

5.3   The category $\mathcal{O}_g$ is highest weight

We begin by using Theorem 5.12 to prove that $\mathcal{O}_g$ is always highest weight.

Proposition 5.17 *For any choice of quantization, the category $\mathcal{O}_g$ is highest weight and the inclusion $D^b(\mathcal{O}_g) \to D^b_{\mathcal{O}_g}(\mathcal{D}\text{-mod})$ is an equivalence of categories.*

Proof: Let $\lambda \in H^2(\mathfrak{M}; \mathbb{C})$ be the period of the quantization, and let $\eta \in H^2(\mathfrak{M}; \mathbb{Z})$ be an ample class. By Lemma 3.17, we may replace λ by $\lambda + k\eta$ for any $k \in \mathbb{Z}$. By Proposition 2.8, we may choose k large enough so that localization holds. By Theorem 5.12, we may also choose k large enough so that $\mathcal{O}_a$ is highest weight. By Corollary 3.19, this implies that $\mathcal{O}_g$ is highest weight, and by Corollary 5.13, the full faithfulness follows as well. □

By the same argument, the following conjecture would be implied by Conjecture 5.14.

Conjecture 5.18 *For any choice of quantization, the category $\mathcal{O}_g$ is standard Koszul.*

Remark 5.19 Conjectures 5.14 and 5.18 will not come up again until Section 10, where they will play a central role in the definition of symplectic duality of conical symplectic resolutions.

In the remainder of this section we give an explicit construction for the standard modules in $\mathcal{O}_g$, which will be useful for our study of the Grothendieck group of this category in Section 6.2. Let $\Theta_\alpha := A_\alpha \otimes_{A_\alpha^+} B_\alpha$, regarded as a module over A. Note that the map of algebras $A \to A_\alpha$ descends to a map of A-modules $\Delta_\alpha \to \Theta_\alpha$.

Proposition 5.20 *Let $d = \frac{1}{2} \dim \mathfrak{M}$. The $\mathbb{T}$-character of Θ_α is $e^{w_\alpha} \prod_{i=1}^d (1 - e^{-\chi_i})^{-1}$, where $\chi_1, \ldots, \chi_d$ are the positive weights (with multiplicity) for the action of $\mathbb{T}$ on $T_{p_\alpha}\mathfrak{M}$ and w_α is the $\mathbb{T}$-weight of B_α.*

Proof: Lemma 5.2 tells us that A_α is isomorphic to the Weyl algebra for $\mathbb{C}^d$ with generators $z_1, w_1, \ldots, z_d, w_d$, where the z_i have positive $\mathbb{T}$-weight. Then, as a $\mathbb{T}$-vector space, Θ_α is isomorphic to $\mathbb{C}[w_1, \ldots, w_d] \otimes B_\alpha$. The result follows. □

The modules Θ_α and Δ_α have analogues which are families over the twistor deformation $\mathscr{M}_\eta$. Let $\tilde{\Delta}_{\eta,\alpha} := \mathscr{A}_\eta \otimes_{\mathscr{A}_\eta^+} \mathscr{B}_{\eta,\alpha}$ be the deformed standard module attached to η and α, and let $\tilde{\Theta}_\alpha$ be the restriction of $\mathscr{A}_{\eta,\alpha} \otimes_{\mathscr{A}_{\eta,\alpha}^+} \mathscr{B}_{\eta,\alpha}$ to $\mathscr{A}$. While $\tilde{\Delta}_{\eta,\alpha}$ is a more natural algebraic object, the family $\tilde{\Theta}_{\eta,\alpha}$ has a more regular structure, and in particular is flat over $\mathbb{A}^1$ by Proposition 5.20.

As usual, we use the subscript k below to denote the period $\lambda + k\eta$. The following lemma says that the two modules are isomorphic if the period is sufficiently large.

Lemma 5.21 *The natural homomorphism $\tilde{\Delta}_{\eta,\alpha} \to \tilde{\Theta}_{\eta,\alpha}$ induces an isomorphism of A_k-modules $\Delta_{k,\alpha} \cong \Theta_{k,\alpha}$ for k sufficiently large.*

Proof: The non-zero fibers of the twistor deformation are affine, so the intersection of $\mathscr{M}_\eta^+$ with such a fiber is a union of affine spaces, one for each fixed point.

The result [BPW, 4.15] applied to the kernel and cokernel of the map shows that there is a polynomial q such that $q(h^{-1}t)$ acts trivially on the kernel and cokernel. Thus, as in the proofs of [BPW, 4.17 & 5.16] for $k \gg 0$, we avoid the all roots of this polynomial, and after specialization, we have an isomorphism. $\square$

In order to describe the sheaves in $\mathcal{O}_{\mathrm{g}}$ which give standard objects we recall a construction from [BPW]. Proposition 5.2 from that paper shows that for every pair of integers k and m there is a good, $\mathbb{S}$-equivariant $(\mathcal{Q}_k, \mathcal{Q}_m)$-bimodule ${}_k\mathcal{T}_m$ (unique up to canonical equivalence) with ${}_k\mathcal{T}_m/h\,{}_k\mathcal{T}_m \cong \mathcal{L}^{k-m}$. Let

$$
{}_kZ_m := \Gamma_{\mathbb{S}}({}_k\mathcal{T}_m[h^{1/n}]) \qquad \text{and} \qquad Z := \bigoplus_{k \geq m \geq 0} {}_kZ_m.
$$

This is a $\mathbb{Z}$-algebra in the sense of Gordon and Stafford [GS05, §5], with multiplication given by tensor products of sections. We have a localization functor $\mathrm{Loc}^{\mathbb{Z}} : Z\text{-mod} \to \mathcal{D}\text{-mod}$ given by $\mathrm{Loc}^{\mathbb{Z}}(N) := \left(\bigoplus_{k \geq 0} {}_0\mathcal{T}_k[h^{-1/n}] \right) \otimes_Z N$; it becomes an isomorphism after modding out by Z-modules which are bounded above.

Let ${}_k(Z_\alpha)_m$ be the space of $\mathbb{S}$-invariant and $\mathbb{T}$-finite vectors in the completion of ${}_k\mathcal{T}_m[h^{-1/n}]$ at the point p_α, and let

$$
{}_k(\Theta_\alpha^{\mathbb{Z}})_m := {}_k(Z_\alpha)_m \otimes_{A_{m,\alpha}^+} B_{m,\alpha} \qquad \text{and} \qquad \Theta_\alpha^{\mathbb{Z}} := \bigoplus_{k \geq 0} {}_k(\Theta_\alpha^{\mathbb{Z}})_0;
$$

then $\Theta_\alpha^{\mathbb{Z}}$ is a module over Z. Let $\mho_\alpha := \mathrm{Loc}^{\mathbb{Z}}(\Theta_\alpha^{\mathbb{Z}})$.

Proposition 5.22 *The sheaf $\mho_\alpha$ is the standard object of $\mathcal{O}_{\mathrm{g}}$ corresponding to α. In particular, if localization holds at λ, then $\mho_\alpha = \mathrm{Loc}(\Delta_\alpha)$.*

Note that this result, together with Corollary 3.19 shows that $\mho_\alpha$ has a unique simple quotient, which we denote by Λ_α.

Proof: By Lemma 5.21, we have $\mho_\alpha = \mathrm{Loc}(\Delta_\alpha)$ for the quantization with period $\lambda + k\eta$ when k is sufficiently large. If localization holds at λ, then consider the following commuting square of equivalences.

$$
\begin{array}{ccc}
\mathcal{O}_{\mathrm{g}}^{\lambda} & \longrightarrow & \mathcal{O}_{\mathrm{g}}^{\lambda+k\eta} \\
\uparrow & & \uparrow \\
\mathcal{O}_{\mathrm{a}}^{\lambda} & \longrightarrow & \mathcal{O}_{\mathrm{a}}^{\lambda+k\eta}
\end{array}
$$

The vertical arrows are given by localization, the top horizontal arrow is given by Lemma 3.17, and the bottom horizontal arrow is given by tensor product with $_{\lambda+k\eta}T_\lambda$. We know that the vertical arrow on the right takes Δ_α to $\mho_\alpha$, and it is easy to check that the two horizontal arrows take $\mho_\alpha$ to $\mho_\alpha$ and Δ_α to Δ_α. The proposition follows. $\qquad\square$

We can also construct deformed versions $\tilde{Z}_{\eta,\alpha}$ and $\tilde{\Theta}_{\eta,\alpha}^{\mathbb{Z}}$. This allows us to construct a deformed standard object $\tilde{\mho}_{\eta,\alpha} := \mathrm{Loc}^{\mathbb{Z}}(\tilde{\Theta}_{\eta,\alpha}^{\mathbb{Z}})$ on $\mathscr{M}_\eta$, with the property that $\tilde{\mho}_{\eta,\alpha}$ is flat over $\mathbb{A}^1$ (by Proposition 5.20) and $\tilde{\mho}_{\eta,\alpha}|_{\mathfrak{M}} \cong \mho_\alpha$. This construction will be used in the proof of Theorem 6.5 below.

5.4   The center of the Yoneda algebra of $\mathcal{O}_{\mathrm{g}}$

Consider the Hochschild cohomology ring

$$
HH^*(\mathcal{D}) := \mathrm{Ext}^\bullet_{\mathcal{D}\otimes\mathcal{D}^{\mathrm{op}}}(\mathcal{D}, \mathcal{D}).
$$

Here the Ext algebra is computed in the bounded below derived category of sheaves of modules over the sheaf $\mathcal{D} \otimes \mathcal{D}^{\mathrm{op}}$; that is, by taking an injective resolution of the left-hand term. (The existence of such an injective resolution follows by mimicking the argument of [Har77, 2.2] with $\mathcal{D}(0)_x$ in place of $\mathfrak{S}_{\mathfrak{M},x}$.) By the usual formalism, there is a spectral sequence

$$
H^i(\mathfrak{M}; \mathscr{E}xt^j_{\mathcal{D}\otimes\mathcal{D}^{\mathrm{op}}}(\mathcal{D}, \mathcal{D})) \Rightarrow HH^{i+j}(\mathcal{D}).
$$

By [WX98, 3.1], the Hochschild cohomology of the Weyl algebra vanishes in all higher degrees, so the spectral sequence collapses at the E_2 page and we have

$$
HH^*(\mathcal{D}) \cong H^*(\mathfrak{M}; Z(\mathcal{D})) \cong H^*(\mathfrak{M}; \mathbb{C}((h))).
$$

Thus, for any object in $D^+(\mathcal{D}\text{-mod})$, we obtain a map from $H^*(\mathfrak{M}; \mathbb{C}) = H^*(\mathfrak{M}; \mathbb{C}((h)))^{\mathbb{S}}$ to the center of its Ext-algebra. In particular, $H^*(\mathfrak{M}; \mathbb{C})$ maps to the Yoneda algebra E of $\mathcal{O}_g$, as defined in Section 4.1.

This map need not be an isomorphism; for example, if the period of $\mathcal{D}$ is as generic as possible, then $\mathcal{O}_g$ will be semisimple, and its Yoneda algebra will be concentrated in degree zero. However, we make the following conjecture, which essentially says that this is the only thing that can go wrong.

Conjecture 5.23 *If the category $\mathcal{O}_g$ is indecomposable (that is, if it has no proper block decomposition), then the map $H^*(\mathfrak{M}; \mathbb{C}) \to Z(E)$ is an isomorphism.*

Remark 5.24 Conjecture 5.23 holds for cotangent bundles of partial flag varieties (Proposition 9.1), quiver varieties in finite type ADE and affine type A [Webb, 3.5], and for hypertoric varieties (part (vi) of Section 9.3).

Remark 5.25 In Section 10.6 we formulate a stronger version of Conjecture 5.23 which relates the equivariant cohomology of $\mathfrak{M}$ to the center of the universal deformation of E.

6 The Grothendieck group of $\mathcal{O}_g$

We continue to let $d = \frac{1}{2} \dim \mathfrak{M}$. In this section, we show that the Grothendieck group $K(\mathcal{O}_g)$ is canonically isomorphic by the characteristic cycle map to the cohomology $H^{2d}_{\mathfrak{M}^+}(\mathfrak{M}; \mathbb{Z})$ with support in $\mathfrak{M}^+$ as lattices with inner products (Theorem 6.5). This is accomplished by studying the characteristic cycles of standard objects, but we also give some partial results about the images of simple objects under this isomorphism (Section 6.3).

6.1 Characteristic cycles revisited

In Section 2.8 we alluded to a characteristic cycle map $CC : K(\mathcal{C}^{\mathfrak{L}}) \to H^{2d}_{\mathfrak{L}}(\mathfrak{M}; \mathbb{Z})$ that was studied in [BPW, §6.2], following ideas of Kashiwara and Schapira [KS12]. In this section we review this construction and study it in greater detail.

Let $\mathcal{N}$ be an object of $D^b(\mathcal{D}\text{-mod})$. We have isomorphisms

$$\mathcal{H}om^{\bullet}_{\mathcal{D}}(\mathcal{N}, \mathcal{N}) \cong \mathcal{H}om^{\bullet}_{\mathcal{D}}(\mathcal{N}, \mathcal{D}) \overset{L}{\otimes}_{\mathcal{D}} \mathcal{N} \cong \mathcal{D}_{\Delta} \overset{L}{\otimes}_{\mathcal{D} \boxtimes \mathcal{D}^{\mathrm{op}}} \left(\mathcal{N} \boxtimes \mathcal{H}om^{\bullet}_{\mathcal{D}}(\mathcal{N}, \mathcal{D}) \right),$$

and evaluation defines a canonical map to the Hochschild homology

$$\mathcal{H}\mathcal{H}(\mathcal{D}) := \mathcal{D}_{\Delta} \overset{L}{\otimes}_{\mathcal{D} \boxtimes \mathcal{D}^{\mathrm{op}}} \mathcal{D}_{\Delta}.$$

All this is completely general, and holds in both the Zariski and the classical topology. In the classical topology, we also have an isomorphism $\mathcal{H}\mathcal{H}(\mathcal{D}^{\mathrm{an}}) \cong \mathbb{C}_{\mathfrak{M}_{\Delta}}((h))[2d]$ by [KS12, 6.3.1].

We define the **characteristic cycle**

$$\mathrm{CC}(\mathcal{N}) \in H^0(\mathcal{HH}(\mathcal{D}^{\mathrm{an}})) \cong H^{2d}\big(\mathfrak{M}; \mathbb{C}((h))\big)$$

to be the image of id $\in H^0(\mathcal{H}om_{\mathcal{D}}^\bullet(\mathcal{N}^{\mathrm{an}}, \mathcal{N}^{\mathrm{an}}))$ along this map. More generally, if $\mathcal{N}$ is supported on a subvariety $j\colon \mathfrak{L} \hookrightarrow \mathfrak{M}$, then we may consider the identity map of $\mathcal{N}^{\mathrm{an}}$ to be a section of $j^!\mathcal{H}om_{\mathcal{D}}^\bullet(\mathcal{N}^{\mathrm{an}}, \mathcal{N}^{\mathrm{an}})$. Applying our map then gives us a class

$$\mathrm{CC}(\mathcal{N}) \in H^0(j^!\mathcal{HH}(\mathcal{D}^{\mathrm{an}})) \cong H^{2d}_{\mathfrak{L}}\big(\mathfrak{M}; \mathbb{C}((h))\big).$$

(Our abuse of the notation $\mathrm{CC}(\mathcal{N})$ is justified by the fact that this class is functorial for inclusions of subvarieties.) By Poincaré-Verdier duality, this can also be considered as a Borel-Moore homology class on $\mathfrak{L}$.

If $\mathfrak{L}$ is Lagrangian, then Kashiwara and Schapira [KS12, 7.3.5] show that $\mathrm{CC}(\mathcal{N})$ actually lies in $H^{2d}_{\mathfrak{L}}(\mathfrak{M}; \mathbb{Z})$; more precisely, if $\mathcal{N}(0)$ is a good lattice, then

$$\mathrm{CC}(\mathcal{N}) = \sum_{i=1}^{r} \mathrm{rk}_{\mathfrak{L}_i}\big(\mathcal{N}(0)/\mathcal{N}(-1)\big) \cdot [\mathfrak{L}_i] \in H^{2d}_{\mathfrak{L}}(\mathfrak{M}; \mathbb{Z}) \subset H^{2d}_{\mathfrak{L}}\big(\mathfrak{M}; \mathbb{C}((h))\big),$$

where $\{\mathfrak{L}_1, \ldots, \mathfrak{L}_r\}$ are the components of $\mathfrak{L}$ and $\mathrm{rk}_{\mathfrak{L}_i}$ is the rank at the generic point of $\mathfrak{L}_i$.

We can also take characteristic cycles in families for modules on twistor deformations $\mathcal{M}_\eta \to \mathbb{A}^1$. Let $\mathcal{N}$ be such a module, and consider the image of the identity via the natural morphisms

$$\mathcal{H}om_{\mathscr{D}}^\bullet(\mathcal{N}, \mathcal{N}) \cong \mathcal{H}om_{\mathscr{D}}^\bullet(\mathcal{N}, \mathscr{D}) \overset{L}{\otimes}_{\mathscr{D}} \mathcal{N} \cong \mathscr{D}_\Delta \overset{L}{\otimes}_{\mathscr{D} \boxtimes_{\mathbb{A}^1} \mathscr{D}^{\mathrm{op}}} \big(\mathcal{N} \boxtimes_{\mathbb{A}^1} \mathcal{H}om_{\mathscr{D}}^\bullet(\mathcal{N}, \mathscr{D})\big)$$

$$\to \mathscr{D}_\Delta^{\mathrm{an}} \otimes_{\mathscr{D}^{\mathrm{an}} \boxtimes_{\mathbb{A}^1} \mathscr{D}^{\mathrm{an,op}}} \mathscr{D}_\Delta^{\mathrm{an}} \cong \pi^{-1}\mathfrak{S}_{\mathbb{A}^1}[2d]((h)). \quad (1)$$

This defines a class in relative cohomology $\mathrm{CC}(\mathcal{N}) \in H^{2d}_{\mathscr{L}}(\mathcal{M}_\eta/\mathbb{A}^1; \mathbb{C}((h)))$ for any Lagrangian $\mathscr{L} \supset \mathrm{Supp}(\mathcal{N})$. If we let $\mathfrak{L} := \mathfrak{M} \cap \mathscr{L}$, then we have a natural restriction map

$$H^{2d}_{\mathscr{L}}(\mathcal{M}_\eta/\mathbb{A}^1; \mathbb{C}((h))) \to H^{2d}_{\mathfrak{L}}\big(\mathfrak{M}; \mathbb{C}((h))\big)$$

given by dividing by the coordinate t on $\mathbb{A}^1$. We also have a natural functor of restriction from $\mathscr{D}$-mod $\to \mathcal{D}$-mod given by $\mathcal{N}|_{\mathfrak{M}} := \mathcal{N} \overset{L}{\otimes}_{\mathbb{C}[t]} \mathbb{C}$. The following lemma says that these operations are compatible.

Lemma 6.1 If $\mathcal{N}$ is a good $\mathscr{D}$-module, then $\mathrm{CC}(\mathcal{N}|_{\mathfrak{M}}) = \mathrm{CC}(\mathcal{N})|_{\mathfrak{M}}$.

Proof: Consider the complex (1) of $\pi^{-1}\mathfrak{S}_{\mathbb{A}^1}$ modules, and take the derived tensor product with $\mathbb{C}$ over $\mathbb{C}[t]$. We claim that we obtain a corresponding complex for $\mathcal{N}|_{\mathfrak{M}}$. That is, we

obtain

$$\mathcal{H}om_{\mathscr{D}}^{\bullet}(\mathscr{N}|_{\mathfrak{M}}, \mathscr{N}|_{\mathfrak{M}}) \cong \mathcal{H}om_{\mathscr{D}}^{\bullet}(\mathscr{N}|_{\mathfrak{M}}, \mathcal{D}) \overset{L}{\otimes}_{\mathcal{D}} \mathscr{N}|_{\mathfrak{M}} \cong \mathcal{D}_{\Delta} \overset{L}{\otimes}_{\mathcal{D} \boxtimes \mathcal{D}^{\mathrm{op}}} \left(\mathscr{N}|_{\mathfrak{M}} \boxtimes \mathcal{H}om_{\mathcal{D}}^{\bullet}(\mathscr{N}|_{\mathfrak{M}}, \mathcal{D}) \right)$$

$$\to \mathcal{D}_{\Delta}^{\mathrm{an}} \otimes_{\mathcal{D}^{\mathrm{an}} \boxtimes \mathcal{D}^{\mathrm{an,op}}} \mathcal{D}_{\Delta}^{\mathrm{an}} \cong \mathbb{C}_{\mathfrak{M}}[2d]((h)). \quad (2)$$

It suffices to prove this for $\mathscr{N}$ locally free. In this case, $\mathcal{H}om^{\bullet}(\mathscr{N}, \mathscr{D})$ is concentrated in degree 0 and is itself locally free, so the statement is clear.

Thus $\mathrm{CC}(\mathscr{N})|_{\mathfrak{M}}$ can be obtained as the image of the identity under the map (2). By definition $\mathrm{CC}(\mathscr{N}|_{\mathfrak{M}})$ is the image of the identity under (2), so we are done. $\square$

6.2 Intersection forms for category $\mathcal{O}$

We now turn our attention to the subcategory $\mathcal{O}_{\mathrm{g}} = \mathcal{C}^{\mathfrak{M}^+} \subset \mathcal{D}$-mod, so that the characteristic cycle map goes from $K(\mathcal{O}_{\mathrm{g}})$ to $H_{\mathfrak{M}^+}^{2d}(\mathfrak{M}; \mathbb{Z})$. We first need to reinterpret the group $H_{\mathfrak{M}^+}^{2d}(\mathfrak{M}; \mathbb{Z}) = H^{2d}(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+; \mathbb{Z})$ equivariantly, so that we can apply localization.

Lemma 6.2 *The forgetful homomorphism*

$$H_{\mathbb{T}}^{2d}(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+; \mathbb{Z}) \to H^{2d}(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+; \mathbb{Z})$$

is an isomorphism. The localization map

$$H_{\mathbb{T}}^{2d}(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+; \mathbb{Z}) \to H_{\mathbb{T}}^{2d}(\mathfrak{M}^{\mathbb{T}}; \mathbb{Z})$$

is an injection; a class lies in the image if and only if its restriction to p_α is a $\mathbb{Z}$-multiple of the equivariant Euler class $e_{\mathbb{T}}(T_{p_\alpha}\mathfrak{M})$, which is the product of the negative weights of the action of $\mathbb{T}$ on $T_{p_\alpha}\mathfrak{M}$.

Proof: Choose an ordering $\alpha_1, \ldots, \alpha_r$ of the index set $\mathcal{I}$ refining the closure order $\leftharpoonup$, so that $\mathfrak{M}_k^+ := \bigcup_{i=1}^k X_i^\circ$ is closed for all k. Let $U_k = \mathfrak{M} \setminus \mathfrak{M}_k^+$ and $U_0 = \mathfrak{M}$. Then for $1 \leq k \leq r$, the cohomology $H^*(U_{k-1}, U_k; \mathbb{Z})$ is isomorphic to the Borel-Moore homology $H_{4d-*}^{BM}(X_k^\circ)$, so it is isomorphic to $\mathbb{Z}$ in degree $2d$ and 0 in all other degrees. It follows that $H_{\mathbb{T}}^*(U_{k-1}, U_k; \mathbb{Z})$ is a free $H_{\mathbb{T}}^*(pt)$-module generated by $H_{\mathbb{T}}^{2d}(U_{k-1}, U_k; \mathbb{Z}) \cong H^{2d}(U_{k-1}, U_k; \mathbb{Z})$. In addition, the restriction of a generator of $H_{\mathbb{T}}^{2d}(U_{k-1}, U_k; \mathbb{Z})$ to $p_k = p_{\alpha_k}$ is the equivariant Euler class of $T_{p_k} X_{p_k}^\circ$.

The result now follows by an easy induction using the exact sequence

$$H_{\mathbb{T}}^*(U_0, U_{k-1}; \mathbb{Z}) \to H_{\mathbb{T}}^*(U_0, U_k; \mathbb{Z}) \to H_{\mathbb{T}}^*(U_{k-1}, U_k; \mathbb{Z})$$

which is short exact since the left and right terms vanish in odd degrees. $\square$

From the first part of the lemma, we have a canonical map

$$H^{2d}_{\mathfrak{M}^+}(\mathfrak{M}; \mathbb{Z}) = H^{2d}(\mathfrak{M}^+; j^! \mathbb{Z}_{\mathfrak{M}}) \to H^{2d}_{\mathbb{T}}(\mathfrak{M}^{\mathbb{T}}; \mathbb{Z}).$$

For all $\gamma \in H^{2d}_{\mathfrak{M}^+}(\mathfrak{M}; \mathbb{Z})$ and $\alpha \in \mathcal{I}$, we will write $\gamma|_\alpha$ to denote the image of γ in $H^{2d}_{\mathbb{T}}(p_\alpha; \mathbb{Z})$. The second part of the lemma implies that The lattice $H^{2d}_{\mathfrak{M}^+}(\mathfrak{M}; \mathbb{Z})$ is freely generated by the classes $\{v_\alpha \mid \alpha \in \mathcal{I}\}$, where $v_\alpha|_\alpha$ is the product of the negative weights of the action of $\mathbb{T}$ on $T_{p_\alpha}\mathfrak{M}$ and $v_\alpha|_\beta = 0$ for $\beta \neq \alpha$.

The classes v_α form an orthonormal basis for the **equivariant intersection form**

$$\langle \beta, \gamma \rangle := (-1)^d \sum_{\alpha \in \mathcal{I}} \frac{\beta|_\alpha \cdot \gamma|_\alpha}{e(\alpha)},$$

where $e(\alpha) \in H^{4d}_{\mathbb{T}}(p_\alpha; \mathbb{Z})$ is the product of all of the weights of the action of $\mathbb{T}$ on $T_{p_\alpha}\mathfrak{M}$.

On $K(\mathcal{O}_g)$ we have the **Euler form** given by the formula

$$\Big\langle [\mathcal{M}], [\mathcal{N}] \Big\rangle := \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{Ext}_{\mathcal{D}\text{-mod}}(\mathcal{M}, \mathcal{N}).$$

Proposition 6.3 *The classes $\{[\mho_\alpha] \mid \alpha \in \mathcal{I}\}$ form an orthonormal basis for $K(\mathcal{O}_g)$. In particular, the Euler form on $K(\mathcal{O}_g)$ is symmetric.*

Proof: This follows from the universal coefficient theorem applied to $\mathrm{Ext}^*(\tilde{\mho}_{\alpha,\eta}, \tilde{\mho}_{\beta,\eta})$. Generically on $\mathbb{A}^1$, the supports of $\tilde{\mho}_{\alpha,\eta}$ and $\tilde{\mho}_{\beta,\eta}$ are distinct if $\alpha \neq \beta$, so $\mathrm{Ext}^*(\tilde{\mho}_{\alpha,\eta}, \tilde{\mho}_{\beta,\eta})$ is supported on $\{0\} \subset \mathbb{A}^1$. The universal coefficient theorem shows that

$$\mathrm{Ext}^i(\mho_\alpha, \mho_\beta) \cong (\mathrm{Ext}^{i+1}(\tilde{\mho}_{\alpha,\eta}, \tilde{\mho}_{\beta,\eta}) \oplus \mathrm{Ext}^i(\tilde{\mho}_{\alpha,\eta}, \tilde{\mho}_{\beta,\eta})) \otimes_{\mathbb{C}[t]} \mathbb{C}.$$

Thus, obviously, the Euler characteristic of this complex is 0, and the classes $[\mho_\alpha]$ and $[\mho_\beta]$ are orthogonal.

On the other hand, we know from the standard property that

$$\mathrm{Ext}^i(\mho_\alpha, \mho_\alpha) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i \neq 0 \end{cases}$$

so this establishes orthonormality. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 6.4 *If the period of a quantization is chosen so that localization holds and the hypotheses of Theorem 5.12 are satisfied, then for each α we have $[\nabla_\alpha] = [\Delta_\alpha]$ in $K(\mathcal{O}_a)$; in*

particular, the multiplicities of any simple in ∇_α and Δ_α are the same.

Proof: Since Δ_α is sent to $\mho_\alpha$ under localization, the proposition implies that the standards Δ_α give an orthonormal basis of $K(\mathcal{O}_\alpha)$ under the Euler form. But in any highest weight category the classes of standards are (left) orthogonal to the classes of costandards, so we must have $[\Delta_\alpha] = [\nabla_\alpha]$. $\qquad\square$

Theorem 6.5 *The map* $\mathrm{CC} : K(\mathcal{O}_{\mathrm{g}}) \to H^{2d}_{\mathfrak{M}^+}\big(\mathfrak{M}; \mathbb{Z}\big)$ *is an isomorphism that intertwines the Euler form with the equivariant intersection form.*

Proof: Since the standard modules $\{\mho_\alpha \mid \alpha \in \mathcal{I}\}$ form an orthonormal basis for $K(\mathcal{O}_{\mathrm{g}})$, it suffices to show that $\mathrm{CC}(\mho_\alpha) = v_\alpha$ for all $\alpha \in \mathcal{I}$. Consider the sheaf $\tilde{\mho}_{\eta,\alpha} = \mathrm{Loc}^{\mathbb{Z}}(\tilde{\Theta}^{\mathbb{Z}}_\alpha)$ on $\mathscr{M}_\eta$, which we introduced at the end of the previous section, along with its Euler class $\mathrm{CC}(\tilde{\mho}_{\eta,\alpha}) \in H^{2d}_{\mathscr{M}^+_\eta}(\mathscr{M}_\eta/\mathbb{A}^1; \mathbb{C}(\!(h)\!))$. Since $\mathscr{M}^{\mathbb{T}}_\eta$ is isomorphic to a disjoint union of $|\mathcal{I}|$ copies of $\mathbb{A}^1$ and the space $\mathscr{M}^+_\eta$ is an $\mathbb{A}^d$-bundle over this space, the group $H^{2d}_{\mathscr{M}^+_\eta}(\mathscr{M}_\eta/\mathbb{A}^1; \mathbb{C}(\!(h)\!))$ is a $\mathbb{C}(\!(h)\!)$-vector space of dimension $|\mathcal{I}|$. Let

$$\{\tilde{v}_\alpha \mid \alpha \in \mathcal{I}\} \subset H^{2d}_{\mathscr{M}^+_\eta}(\mathscr{M}_\eta/\mathbb{A}^1; \mathbb{C}(\!(h)\!))$$

be the $\mathbb{C}(\!(h)\!)$-basis indexed by the components of the fixed point set, so that $\tilde{v}_\alpha|_{\mathfrak{M}} = v_\alpha$.

Over a generic element of $\mathbb{A}^1$, the restriction of $\tilde{\mho}_{\eta,\alpha}$ is simply the structure sheaf of the locus of points whose $\mathbb{T}$-limit is equal to the fixed point labeled by α. This implies that $\mathrm{CC}(\tilde{\mho}_{\eta,\alpha}) = \tilde{v}_\alpha$, and therefore that

$$\mathrm{CC}(\mho_\alpha) = \mathrm{CC}(\tilde{\mho}_{\eta,\alpha}|_{\mathfrak{M}}) = \mathrm{CC}(\tilde{\mho}_{\eta,\alpha})|_{\mathfrak{M}} = \tilde{v}_\alpha|_{\mathfrak{M}} = v_\alpha.$$

This completes the proof. $\qquad\square$

Remark 6.6 While the hypothesis that $\mathbb{T}$ has isolated fixed points was used in an essential way here, it should be possible to generalize this result to more general $\mathbb{T}$-actions, at the cost of downgrading from an isomorphism to an injection. A forthcoming result of Baranovsky and Ginzburg [BG] shows that the map CC is injective in the case where $\mathbb{T}$ is trivial. In this case, CC takes values in the top degree homology group of the core (the preimage of $o \in \mathfrak{M}_0$), and it intertwines the Euler form with the ordinary intersection form on $\mathfrak{M}$ by [KS12, 6.5.4]. This map can be extremely far from surjective, though; for generic periods, category $\mathcal{O}_{\mathrm{g}}$ for a trivial action has no non-zero objects. A recent preprint of Bezrukavnikov and Losev shows how complicated this dependence can be in the case of certain quiver varieties [BL]. We expect

that this result should extend to arbitrary $\mathbb{T}$ as a mix of these two situations. See Remark 5.4 for a related discussion.

We conclude this section by noting that we can specify a geometrically-defined partial order with respect to which the category $\mathcal{O}_{\mathrm{g}}$ is highest weight. (We already know that it is highest weight by Proposition 5.17, but the relation between the partial order $\leq$ we used there and the geometry of $\mathfrak{M}$ is not clear.) Define a partial order $\hookleftarrow$ on $\mathcal{I}$ by putting $\alpha \hookleftarrow \beta$ if $p_\alpha \in X_\beta$ (or equivalently $\overline{X_\alpha^\circ} \cap X_\beta^\circ \neq \emptyset$) and then taking the transitive closure.

Proposition 6.7 *The support of $\mho_\alpha$ is contained in $\bigsqcup_{\beta \hookleftarrow \alpha} X_\beta$. In particular, $\mathcal{O}_{\mathrm{g}}$ is highest-weight with respect to the partial order $\hookleftarrow$.*

Proof: This follows immediately from the structure on the fixed point classes, since the change of basis matrix between the bases $\{v_\alpha \mid \alpha \in \mathcal{I}\}$ and $\{[X_\alpha] \mid \alpha \in \mathcal{I}\}$ is triangular with ones on the diagonal with respect to this partial order. Since the simple Λ_α defined after Proposition 5.22 has non-trivial support on X_α, if it occurs in $\mho_\beta$, the standard $\mho_\beta$ must have X_α in its support, so $\alpha \hookleftarrow \beta$. $\qquad\square$

It is worth noting that $\hookleftarrow$ is not necessarily the weakest partial order with respect to which $\mathcal{O}_{\mathrm{g}}$ is highest weight. For example, $\mathcal{O}_{\mathrm{g}}$ may be semi-simple, and thus highest weight for the trivial partial order.

6.3 Supports of simples

The key to the previous section was the computation of the characteristic cycles of standard objects. It is also interesting to consider the characteristic cycles of simple objects, though they are much more difficult to understand. In this section we will obtain some partial results about their set-theoretic supports that will be useful in later sections.

We call an A-module N **holonomic** if its derived localization $\mathbb{L}\mathrm{Loc}(N)$ on any resolution has Lagrangian support. Note that this is independent of the choice of resolution, since the functors $\mathbb{L}\mathrm{Loc}$ for different resolutions are related by convolution with a Harish-Chandra bimodule, which preserves holonomicity.

For any simple A-module L, let $\mathfrak{M}_L \subset \mathfrak{M}_0$ be the subscheme defined by the ideal $\mathrm{gr}\,\mathrm{Ann}(L) \subset \mathrm{gr}\,A \cong \mathbb{C}[\mathfrak{M}_0]$. This subscheme is always the closure of a symplectic leaf [Gin03]; in particular, it is a subvariety. A leaf that arises in this way will be called **special**, in analogy with the existing terminology for nilpotent orbits. We let $\mathscr{S}$ denote the set of all symplectic leaves of $\mathfrak{M}_0$, and $\mathscr{S}^{\mathrm{sp}}$ denote the subset of leaves which are special for a fixed quantization $\mathcal{D}$.

Theorem 6.8 *If L is holonomic, the support of the sheaf $\mathrm{gr}\,L$ on $\mathfrak{M}_0$ is contained in $\mathfrak{M}_L$, and it intersects the dense leaf of $\mathfrak{M}_L$ nontrivially; equivalently, a symplectic leaf closure contains $\mathrm{Supp}(\mathrm{gr}\,L)$ if and only it contains $\mathfrak{M}_L$.*

Before proving Theorem 6.8 we establish a pair of lemmas. Consider the Rees algebra $R(A) \cong \Gamma(\mathcal{D}(0))$. Following Losev, we wish to consider the completion of this algebra at a maximal ideal in $\mathbb{C}[\mathfrak{M}_0] \cong R(A)/h \cdot R(A)$. For $s \in \mathfrak{M}_0$, we let $R(A)_s^{\wedge}$ be the completion of $R(A)$ in the topology induced by the maximal ideal $\mathfrak{m}_s + h \cdot R(A)$. Let S be the symplectic leaf containing s and, following Kaledin [Kal06a, 2.3], let $\mathcal{Y}_s$ denote the formal slice to S inside of $\mathfrak{M}_0$.

Lemma 6.9 *The completion $R(A)_s^{\wedge}$ is isomorphic to the tensor product $W_0^{\wedge} \hat{\otimes}_{\mathbb{C}[[h]]} C$ where W is the Weyl algebra on the symplectic vector space $T_s^* S$ and C is a quantization of $\mathcal{Y}_s$.*

Proof: The algebra $R(A)_s^{\wedge}$ is a quantization of a formal neighborhood of the fiber over s in $\mathfrak{M}$; this formal scheme is isomorphic as a Poisson scheme to the product of the completion of S at s with a symplectic resolution of $\mathcal{Y}_s$. By the classification of quantizations in [BK04], any quantization of the latter will have sections of the form $W_0^{\wedge} \hat{\otimes} C$, so we are done. $\square$

Consider a holonomic A-module N with a fixed good filtration, and choose a point u which is a smooth point of $\mathrm{Supp}(\mathrm{gr}\, L)$ and which is in a symplectic leaf S of maximal dimension amongst those intersecting $\mathrm{Supp}(\mathrm{gr}\, L)$. Now, we may form the completion $R(N)_s^{\wedge}$, which is a module over $R(A)_s^{\wedge}$.

Lemma 6.10 *The tangent space $T_s \mathrm{Supp}(\mathrm{gr}\, N)$ is Lagrangian in the symplectic space $T_s S$.*

Proof: The component Y of $\mathrm{Supp}(\mathrm{gr}\, N)$ which contains s must be the image of a component Y' of $\mathrm{Supp}\,\mathbb{LLoc}(N)$, which is Lagrangian by the assumption of holonomicity. Let S' be the preimage of S in $\mathfrak{M}$; this is a coisotropic subvariety of $\mathfrak{M}$ whose closure contains Y'. Since $Y' \cap S'$ is Lagrangian, it must be a union of the leaves of the null-foliation of S', which is given by the fibers of the projection to S. Thus, $Y \cap S$, which is the image of $Y' \cap S'$, is Lagrangian in S. The result follows. $\square$

The elements of $T_s^* S$ that are conormal to $\mathrm{Supp}(\mathrm{gr}\, N)$ topologically generate a left ideal in $W_0^{\wedge}$; we call the quotient by this ideal a **vacuum representation** $\mathcal{V}$ of $W_0^{\wedge}$. The following lemma is based on the argument in [Los10, 3.3.4].

Lemma 6.11 *The module $R(N)_s^{\wedge}$ decomposes as the tensor product of $\mathcal{V}$ with a C-module which is free of finite rank over $\mathbb{C}[[h]]$.*

Proof: Note that completing the coherent sheaf $R(N)/h \cdot R(N)$ at s produces a finite rank locally free sheaf on the completion of $\mathrm{Supp}(\mathrm{gr}\, N)$, since s is a generic point of this support. This implies that $R(N)_s^{\wedge}$ is finitely generated over $W_0^{\wedge}$. Thus, it is the completion of a finitely generated module over W which is free over $\mathbb{C}[[h]]$, and has associated graded supported on

45

$T_s \operatorname{gr} N \subset T_s S$. We can think of the same object as a D-module on $T_s \operatorname{gr} N$ which is coherent as a $\mathfrak{S}_{T_s \operatorname{gr} N}$-module, so it is just a sum of copies of the structure sheaf. Taking completion, this means that as a $W_0^\wedge$-module, $R(N)_s^\wedge \cong \operatorname{Hom}_{W_0^\wedge}(\mathcal{V}, R(N)_s^\wedge) \otimes_{\mathbb{C}[[h]]} \mathcal{V}$. If we let C act on $\operatorname{Hom}_{W_0^\wedge}(\mathcal{V}, R(N)_s^\wedge)$ in the obvious manner and trivially on $\mathcal{V}$, this becomes an isomorphism of $W_0^\wedge \hat{\otimes} C$ modules. This completes the proof. $\qquad\square$

Proof of Theorem 6.8: The theorem can be reformulated as saying that $\mathfrak{M}_L$ is the union of the closures of those symplectic leaves that intersect the support of $\operatorname{gr}(\operatorname{Ann} L)$. Since $\operatorname{gr}(\operatorname{Ann} L)$ kills $\operatorname{gr} L$, we have $\bar{S} \subset \mathfrak{M}_L$ for any leaf S intersecting the support of L; this makes one of the two containments clear. For the reverse inclusion, it suffices to show that, for s and S as above, we have $\mathfrak{M}_L = \bar{S}$.

Let $I := \operatorname{Ann}(R(L)) \subset R(A)$; then $\mathfrak{M}_L$ is defined by the ideal $I/hI = \operatorname{gr} \operatorname{Ann}(L)$. We have an injective map $R(L) \to R(L)_s^\wedge$ (by the simplicity of L), and thus an injective map

$$R(A)/I \to W_0^\wedge \hat{\otimes} \operatorname{End}\left(\operatorname{Hom}_{W_0^\wedge}(\mathcal{V}, R(N)_s^\wedge)\right).$$

The latter algebra has the property that its quotient modulo h^n for any n is of GK dimension $\dim S$. Thus, the GK dimension of the former modulo h^n is at most $\dim S$. The same is thus true of the coordinate ring of the associated variety $V(I/hI)$. Thus, the variety must have dimension at most that of S, but it also contains S. By results of Ginzburg [Gin03], $V(I/hI)$ must be the closure of a single leaf of this dimension, and thus, we must have $V(I/hI) = \bar{S}$. This completes the proof. $\qquad\square$

We now consider some consequences of Theorem 6.8. Recall that Λ_α is the simple object of $\mathcal{O}_{\mathrm{g}}$ indexed by α. Let $\mathfrak{M}_{\alpha,0}$ be the union of the closures of the symplectic leaves that intersect the image in $\mathfrak{M}_0$ of $\operatorname{Supp} \Lambda_\alpha$. By Theorem 6.8, $\mathfrak{M}_{\alpha,0}$ is equal to the closure of a single leaf, which we denote $\mathring{\mathfrak{M}}_{\alpha,0}$. Furthermore, if $\lambda \in H^2(\mathfrak{M}; \mathbb{C})$ is the period of $\mathcal{D}$, this leaf depends only on the coset of λ in $H^2(\mathfrak{M}; \mathbb{C})/H^2(\mathfrak{M}; \mathbb{Z})$.

Corollary 6.12 *Choose any $\lambda' \in H^2(\mathfrak{M}; \mathbb{C})$ such that $\lambda - \lambda' \in H^2(\mathfrak{M}; \mathbb{Z})$ and localization holds at λ'. Let $\mathcal{D}'$ be the quantization with period λ', and let $A' := \Gamma_{\mathbb{S}}(\mathcal{D}')$. Then the A'-module $L := \Gamma_{\mathbb{S}}(_{\lambda'}\mathcal{T}_\lambda \otimes \Lambda_\alpha)$ is simple, and $\mathfrak{M}_{\alpha,0} = \mathfrak{M}_L$.*

Proof: Simplicity of L follows from the fact that $_{\lambda'}\mathcal{T}_\lambda \otimes \Lambda_\alpha$ is a simple $\mathcal{D}'$-module and local- ization holds at λ'. The support of Λ_α is equal to that of $_{\lambda'}\mathcal{T}_\lambda \otimes \Lambda_\alpha$, so the image in $\mathfrak{M}_0$ of the support of Λ_α is equal to the support of the associated graded of L. Applying Theorem 6.8, we are done. $\qquad\square$

Corollary 6.13 *If localization holds for $\mathcal{D}$, then for all $\alpha \in \mathcal{I}$, $\mathring{\mathfrak{M}}_{\alpha,0}$ is a special leaf.*

46

The support of Λ_α always contains the relative core component X_α, but it may contain other components of $\mathfrak{M}^+$, as well. For example, if $\mathfrak{M}$ is a hypertoric variety and the period of $\mathcal{D}$ is integral in the sense of Section 2.4, then the support of Λ_α is equal to X_α [BLPW12, §6.3]. On the other hand, if $\mathfrak{M} = T^*(G/B)$, then there exists α for which the support of Λ_α has multiple components unless $G = \mathrm{SL}_r$ for $r \le 7$. Thus $\mathfrak{M}_{\alpha,0}$ always contains $X_{\alpha,0}$, but it is possible that $X_{\alpha,0}$ is contained in a smaller leaf closure.

Definition 6.14 We call the pair $(\mathfrak{M}, \mathcal{D})$ **interleaved** if localization holds for $\mathcal{D}$ and, for all $\alpha \in \mathcal{I}$, $\mathfrak{M}_{\alpha,0}$ is the smallest special leaf closure that contains $X_{\alpha,0}$. If we have a notion of integrality (Section 2.4) and the pair $(\mathfrak{M}, \mathcal{D})$ is interleaved for some (equivalently any) integral quantization $\mathcal{D}$ for which localization holds, then we will simply say that $\mathfrak{M}$ is interleaved.

Example 6.15 As mentioned above, if $\mathfrak{M}$ is a hypertoric variety and the period of $\mathcal{D}$ is integral, then the support of Λ_α is equal to X_α, thus $\mathfrak{M}$ is interleaved by Theorem 6.8.

Example 6.16 Finite and affine type A quiver varieties (which include finite type A Slodowy slices) are interleaved; this follows from Theorem 7.14 and [Webd, §5].

Example 6.17 If G is the adjoint group of type F_4, then $T^*(G/B)$ is **not** interleaved. We will deduce this from Theorem 7.14; see Remark 7.17 for details.

The property of being interleaved will be used in the form of the following lemma, which will be one of the main ingredients in the proof of Theorem 7.14.

Lemma 6.18 *For all $\alpha \in \mathcal{I}$, there exist unique integers $\{\eta_{\alpha\beta} \mid \beta \in \mathcal{I}\}$ such that*

$$\mathrm{CC}\,\Lambda_\alpha = [X_\alpha] + \sum \eta_{\alpha\beta}[X_\beta],$$

where $\eta_{\alpha\beta}$ can only be nonzero if $\beta \leftharpoonup \alpha$. In addition, if $(\mathfrak{M}, \mathcal{D})$ is interleaved, then $\eta_{\alpha\beta} \ne 0$ also implies $\mathfrak{M}_{\beta,0} \subset \mathfrak{M}_{\alpha,0}$.

Proof: The existence and uniqueness of $\{\eta_{\alpha\beta} \mid \beta \in \mathcal{I}\}$ follows from the fact that the classes $\{[X_\beta] \mid \beta \in \mathcal{I}\}$ form a basis for $H^{2d}_{\mathfrak{M}^+}(\mathfrak{M}; \mathbb{Z})$. Suppose that $\eta_{\alpha\beta} \ne 0$. Since $[X_\beta]$ appears in $\mathrm{CC}(\Lambda_\alpha)$ and Λ_α is a quotient of $\mathfrak{V}_\alpha$, $[X_\beta]$ must also appear in $\mathrm{CC}(\mathfrak{V}_\alpha)$. By Proposition 6.7, this implies that $\beta \leftharpoonup \alpha$. Furthermore, X_β is contained in the support of Λ_α, and therefore $X_{\beta,0}$ is contained in $\mathfrak{M}_{\alpha,0}$. If $(\mathfrak{M}, \mathcal{D})$ is interleaved, this implies that $\mathfrak{M}_{\beta,0} \subset \mathfrak{M}_{\alpha,0}$. $\qquad\square$

7 Categorical filtrations

In this section we define categorical filtrations of $\mathbf{HC}^{\mathrm{a}}$, $\mathbf{HC}^{\mathrm{g}}$, $\mathcal{O}_{\mathrm{a}}$, $\mathcal{O}_{\mathrm{g}}$, and their derived categories. These induce decompositions on their Grothendieck groups, and in Theorem 7.14

and Corollary 7.15 we relate the decomposition of $K(\mathcal{O}_{\mathrm{g}})$ to the Beilinson-Bernstein-Deligne (BBD) decomposition of $H^{2d}_{\mathfrak{M}^+}(\mathfrak{M}; \mathbb{C})$, using the characteristic cycle map. We also relate this decomposition to a generalization of Lusztig's theory of two-sided cells (Remark 7.33).

7.1 Filtration on Harish-Chandra bimodules

Let $S \in \mathscr{S}$ be a symplectic leaf of $\mathfrak{M}_0$, and let $\bar{S}_\Delta := (\bar{S} \times \bar{S}) \cap \mathfrak{Z}_0 \subset \mathfrak{M}_0 \times \mathfrak{M}_0$. Recall that $\mathfrak{Z}_0 \cong \mathfrak{M}_0$ is the diagonal, so that $\bar{S}_\Delta \cong \bar{S}$.

Definition 7.1 Let $\mathbf{HC}^{\mathrm{a}}_S \subset \mathbf{HC}^{\mathrm{a}}$ be the full subcategory of algebraic Harish-Chandra bimodules H such that for some (equivalently any) filtration of H, the coherent sheaf $\mathrm{gr}\, H$ on

$$\mathfrak{Z}_0 \subset \mathfrak{M}_0 \times \mathfrak{M}_0$$

is set-theoretically supported on $\bar{S}_\Delta$. Let $\mathbf{HC}^{\mathrm{a}}_{\partial S} \subset \mathbf{HC}^{\mathrm{a}}_S$ be the full subcategory supported on leaves strictly smaller than S. Let $D^b_{\mathbf{HC}^{\mathrm{a}}_S}(A\text{-mod-}A)$ (respectively $D^b_{\mathbf{HC}^{\mathrm{a}}_{\partial S}}(A\text{-mod-}A)$) be the full subcategory of $D^b_{\mathbf{HC}^{\mathrm{a}}}(A\text{-mod-}A)$ consisting of objects with cohomology in $\mathbf{HC}^{\mathrm{a}}_S$ (respectively $\mathbf{HC}^{\mathrm{a}}_{\partial S}$).

Proposition 7.2 *Any simple module H in $\mathbf{HC}^{\mathrm{a}}_S \setminus \mathbf{HC}^{\mathrm{a}}_{\partial S}$ has support equal to $\bar{S}_\Delta$.*

Proof: By Proposition 6.8, there is a unique minimal symplectic leaf of $\mathfrak{M}_0 \times \mathfrak{M}_0$ whose closure contains $\mathrm{Supp}\, H$, given by the vanishing locus of the annihilator of H as an $A \otimes A^{\mathrm{op}}$-module. This must be of the form $S' \times S'$ for some $S' \in \mathscr{S}$, since

$$(\bar{S}' \times \bar{S}'') \cap \mathfrak{Z}_0 = ((\bar{S}' \cap \bar{S}'') \times (\bar{S}' \cap \bar{S}'')) \cap \mathfrak{Z}_0.$$

Furthermore, we must have $\bar{S}' \supset S$, since part of the support of H must intersect $S \times S$ (or we would have $H \in \mathbf{HC}^{\mathrm{a}}_{\partial S}$). On the other hand, the dimension of the support of H must be at least half of the dimension of $\bar{S}' \times \bar{S}'$, which implies $\dim S' = \dim S$, so $S' = S$. Thus, the support of H is contained in the irreducible variety $\bar{S}_\Delta$. Since they have the same dimension, we are done. $\qquad\square$

Proposition 7.3 *The left annihilator of a Harish-Chandra bimodule is a primitive ideal.*

Proof: Note that the left annihilator I of any simple bimodule H over any ring is prime, since if $J_1, J_2 \not\subset I$, we have that $J_1 \cdot J_2 \cdot H = J_1 \cdot H = H$, so $J_1 \cdot J_2 \not\subset I$.

Now let H be a filtered Harish-Chandra A-bimodule. For any element $h \in H$, let $\bar{h} \in \mathrm{gr}\, H$ be its symbol, which we regard as a section of a sheaf on $\mathfrak{Z}_0 \cong \mathfrak{M}_0$. Choose an h such that the section $\bar{h}$ is non-zero on a leaf $S \subset \mathrm{Supp}\, H$ of maximal dimension (here we take the support of H as a left module). Consider the filtered left submodule $A \cdot h \subset A$. Using the

Noetherian property, we can find a simple quotient L of $A \cdot h$ which is supported on S. Thus, there is a simple subquotient L of H whose support has non-trivial intersection with S. Let $I := \mathrm{Ann}(H)$ and $J := \mathrm{Ann}(L)$. Then $I \subset J$, so $\bar{S} = V(\mathrm{gr}\, J) \subset V(\mathrm{gr}\, I)$. However, I is prime and $\dim V(\mathrm{gr}\, I) = \dim S = \dim V(\mathrm{gr}\, J)$, thus $I = J$ [BK76, Korollar 3.6]. $\qquad\square$

Remark 7.4 By Proposition 7.3, we have that $\mathbf{HC}_S^{\mathrm{a}} = \mathbf{HC}_{\partial S}^{\mathrm{a}}$ unless S is special. If S is special, then let I be a primitive ideal whose associated variety is $\bar{S}$. In this case, A/I is an object in $\mathbf{HC}_S^{\mathrm{a}}$, but not $\mathbf{HC}_{\partial S}^{\mathrm{a}}$.

Definition 7.5 Let $\mathbf{HC}_S^{\mathrm{g}} \subset \mathbf{HC}^{\mathrm{g}}$ be the full subcategory of geometric Harish-Chandra bimodules $\mathcal{H}$ that are set-theoretically supported on the preimage in $\mathfrak{Z} \subset \mathfrak{M} \times \mathfrak{M}$ of $\bar{S}_\Delta$. Let $\mathbf{HC}_{\partial S}^{\mathrm{g}} \subset \mathbf{HC}_S^{\mathrm{g}}$ be the full subcategory supported on preimages of leaves strictly smaller than S. Let $D_{\mathbf{HC}_S^{\mathrm{g}}}^b(\mathcal{D}\text{-mod-}\mathcal{D})$ (respectively $D_{\mathbf{HC}_{\partial S}^{\mathrm{g}}}^b(\mathcal{D}\text{-mod-}\mathcal{D})$) be the full subcategory of $D_{\mathbf{HC}^{\mathrm{g}}}^b(\mathcal{D}\text{-mod-}\mathcal{D})$ consisting of objects with cohomology in $\mathbf{HC}_S^{\mathrm{g}}$ (respectively $\mathbf{HC}_{\partial S}^{\mathrm{g}}$).

Let $\mathfrak{J}_S$ be the localization of $\mathbf{HC}_S^{\mathrm{g}}$ at $\mathbf{HC}_{\partial S}^{\mathrm{g}}$. The Grothendieck group $K(\mathbf{HC}^{\mathrm{g}})$ is filtered by the poset $\mathscr{S}$, with $K(\mathbf{HC}_S^{\mathrm{g}})/K(\mathbf{HC}_{\partial S}^{\mathrm{g}}) \cong K(\mathfrak{J}_S)$. Using the Euler form on $K(\mathbf{HC}^{\mathrm{g}})$, we may split this filtration to obtain a direct sum decomposition

$$K(\mathbf{HC}^{\mathrm{g}}) \cong \bigoplus_{S \in \mathscr{S}} K(\mathfrak{J}_S). \tag{3}$$

See Remark 7.11 for a more categorical interpretation of this decomposition.

If derived localization holds at λ, then the functors $\mathbb{L}\mathrm{Loc}$ and $\mathbb{R}\Gamma_{\mathbb{S}}$ induce an equivalence of categories between $D_{\mathbf{HC}_S^{\mathrm{g}}}^b(\mathcal{D}\text{-mod-}\mathcal{D})$ and $D_{\mathbf{HC}_S^{\mathrm{a}}}^b(A\text{-mod-}A)$ for all S. In particular, as long as derived localization holds at λ, the category $\mathfrak{J}_S$ will be non-trivial if and only if the leaf S special, just as in $\mathbf{HC}^{\mathrm{a}}$; this is false if derived localization doesn't hold at λ.

Remark 7.6 Losev and Ostrik prove that, if $\mathfrak{M} \to \mathfrak{M}_0$ is the Springer resolution of the nilpotent cone of a simple Lie algebra, then $\mathfrak{J}_S$ is an indecomposable multi-fusion category [LO14, 5.4 & 5.5]. It would be interesting to know if this result holds for more general conical symplectic resolutions.

Remark 7.7 Let ν be the map from $\mathfrak{M}$ to $\mathfrak{M}_0$. For any symplectic leaf S, let ϕ_S be the local system obtained by restricting $R^{\mathrm{codim}\, S} \nu_! \mathbb{C}_{\mathfrak{M}}$ to S. In other words, the stalks of ϕ_S are the top cohomology groups of the fibers of ν. By [CG97, 8.9.8], there is an isomorphism

$$H_{\mathfrak{Z}}^{4d}(\mathfrak{M} \times \mathfrak{M}; \mathbb{C}) \cong \bigoplus_{S \in \mathscr{S}} \mathrm{End}\, \phi_S. \tag{4}$$

The ring homomorphism CC : $K(\mathbf{HC}^g)_{\mathbb{C}} \to H^{4d}_{\mathfrak{Z}}(\mathfrak{M} \times \mathfrak{M}; \mathbb{C})$ of Proposition 2.18 induces a map from $K(\mathfrak{J}_S)_{\mathbb{C}}$ to $\mathrm{End}\, \phi_S$. If $\mathfrak{M}_0$ is an S3-variety of type A, then both $K(\mathfrak{J}_S)_{\mathbb{C}}$ and $\mathrm{End}\, \phi_S$ are matrix algebras of the same rank, thus the map is an isomorphism. In other types, this will not be the case; in particular, $\mathfrak{J}_S$ will be trivial unless S is special.

7.2   Filtration on category $\mathcal{O}$

Definition 7.1 (respectively 7.5) gives us a filtration of the monoidal category $\mathbf{HC}^a$ (respectively $\mathbf{HC}^g$) by sub-monoidal categories indexed by the poset of symplectic leaves. By Propositions 2.17, 3.14, and 3.18, $\mathcal{O}_a$ (respectively $\mathcal{O}_g$) is a module category over $\mathbf{HC}^a$ (respectively $\mathbf{HC}^g$). In this section we will define the analogous filtrations of the module categories.

Definition 7.8 Let $\mathcal{O}_a^S \subset \mathcal{O}_a$ be the full subcategory consisting of modules N such that for some (equivalently any) filtration of N, the coherent sheaf $\mathrm{gr}\, N$ on $\mathfrak{M}_0$ is set-theoretically supported on the closure of S. Let $\mathcal{O}_a^{\partial S} \subset \mathcal{O}_a^S$ be the full subcategory supported on leaves strictly smaller than S. Let $D^b_{\mathcal{O}_a^S}(A\text{-mod})$ (respectively $D^b_{\mathcal{O}_a^{\partial S}}(A\text{-mod})$) be the full subcategory of $D^b(A\text{-mod})$ consisting of objects whose cohomology lies in $\mathcal{O}_a^S$ (respectively $\mathcal{O}_a^{\partial S}$).

Definition 7.9 Let $\mathcal{O}_g^S \subset \mathcal{O}_g$ be the full subcategory of objects $\mathcal{N}$ that are set-theoretically supported on the preimage in $\mathfrak{M}$ of the closure of S. Let $\mathcal{O}_g^{\partial S} \subset \mathcal{O}_g^S$ be the full subcategory supported on preimages of leaves strictly smaller than S. Let $D^b_{\mathcal{O}_g^S}(\mathcal{D}\text{-mod})$ (respectively $D^b_{\mathcal{O}_g^{\partial S}}(\mathcal{D}\text{-mod})$) be the full subcategory of $D^b(\mathcal{D}\text{-mod})$ consisting of objects whose cohomology lies in $\mathcal{O}_g^S$ (respectively $\mathcal{O}_g^{\partial S}$).

The following straightforward proposition asserts that the above filtrations interact well with the structures that we have already defined.

Proposition 7.10 *Consider the tensor product and convolution actions of Proposition 2.17.*

1. *The functor $\Gamma_{\mathbb{S}}$ takes $\mathbf{HC}_S^g$ to $\mathbf{HC}_S^a$ and $\mathcal{O}_g^S$ to $\mathcal{O}_a^S$.*

2. *If $H \in \mathbf{HC}_S^a$ and $N \in \mathcal{O}_a$, then $H \otimes N \in \mathcal{O}_a^S$.*

3. *If $\mathcal{H} \in \mathbf{HC}_S^g$ and $\mathcal{N} \in \mathcal{O}_g$, then $\mathcal{H} \star \mathcal{N} \in \mathcal{O}_g^S$.*

4. *Each of the above statements holds when S is replaced with ∂S.*

5. *Each of the above statements holds in the derived setting.*

Let $\mathfrak{P}_S$ denote the quotient of $\mathcal{O}_g^S$ by $\mathcal{O}_g^{\partial S}$. Using the Euler form on $K(\mathcal{O}_g)$, we obtain an orthogonal decomposition

$$H^{2d}_{\mathfrak{M}^+}(\mathfrak{M}; \mathbb{Z}) \cong K(\mathcal{O}_g) \cong \bigoplus_{S \in \mathscr{S}} K(\mathfrak{P}_S), \tag{5}$$

completely analogous to that of Equation (3).

Remark 7.11 This decomposition can be given a categorical interpretation as follows. For simplicity, we will consider a single step in the filtration. Let $(\mathcal{O}_g^{\partial S})^\perp$ denote the full subcategory of $\mathcal{O}_g^S$ consisting of objects X such that, for every object $Y \in \mathcal{O}_g^{\partial S}$, $\mathrm{Ext}^k(X, Y) = 0$ for all k; the fact that $\mathcal{O}_g$ is Artinian with a projective generator guarantees that the same is true of $\mathcal{O}_g^S, \mathcal{O}_g^{\partial S}$ and $(\mathcal{O}_g^{\partial S})^\perp$.

Proposition 7.12 $(\mathcal{O}_g^{\partial S})^\perp$-proj $\cong \mathfrak{P}_S$-proj.

Proof: A projective module P lives in $(\mathcal{O}_g^{\partial S})^\perp$ if and only if it has $\mathrm{Hom}(P, Y) = 0$ for every $Y \in \mathcal{O}_g^{\partial S}$. On the other hand, quotient to $\mathfrak{P}_S$ followed by its left adjoint sends each module N to its minimal submodule N_S such that $N/N_S \in \mathcal{O}_g^{\partial S}$. By assumption, the natural map $P \to P/P_S$ is 0, so $P \cong P_S$. This shows that the object $\bar{P}$ in the quotient category satisfies $\mathrm{Hom}(\bar{P}, \bar{N}) \cong \mathrm{Hom}(P, N)$ for all modules N. In particular, the quotient induces a fully faithful functor $(\mathcal{O}_g^{\partial S})^\perp$-proj $\to \mathfrak{P}_S$-proj.

The exactness of the quotient functor means that its left adjoint sends projectives to projectives; since this left adjoint lands in $(\mathcal{O}_g^{\partial S})^\perp$, this provides a splitting to the fully faithful functor of reduction, and thus induces an equivalence of categories. $\square$

By Proposition 7.12, the decomposition

$$K(\mathcal{O}_g^S) \cong K(\mathcal{O}_g^{\partial S}) \oplus K(\mathfrak{P}_S)$$

may be identified with the decomposition

$$K(\mathcal{O}_g^S) = K(\mathcal{O}_g^{\partial S}) \oplus K\big((\mathcal{O}_g^{\partial S})^\perp\big),$$

which is categorified by the semiorthogonal decomposition of $\mathcal{O}_g^S$ into $\mathcal{O}_g^{\partial S}$ and $(\mathcal{O}_g^{\partial S})^\perp$.

7.3 Relation with the BBD filtration

Let $\mathcal{F}$ be the derived pushforward of the constant sheaf from $\mathfrak{M}$ to $\mathfrak{M}_0$, and let $\mathcal{F}_S := \mathrm{IC}^\bullet(\phi_S)[-\mathrm{codim}\, S]$. By the BBD decomposition theorem, we have a canonical direct sum decomposition [CG97, 8.9.3]

$$\mathcal{F} \cong \bigoplus_{S \in \mathscr{S}} \mathcal{F}_S. \tag{6}$$

Define a functor $\Sigma \colon D_{\mathscr{S}}^b(\mathfrak{M}_0) \to \mathbb{C}$-mod by

$$\Sigma(-) = H_{\mathfrak{M}_0^+}^{2d}(\mathfrak{M}_0; -);$$

applying it to both sides of (6) we obtain isomorphisms

$$H^{2d}_{\mathfrak{M}+}\big(\mathfrak{M};\mathbb{C}\big) \cong \Sigma(\mathcal{F}) \cong \bigoplus_{S\in\mathscr{S}} \Sigma(\mathcal{F}_S). \tag{7}$$

The isomorphism (4) is a consequence of an isomorphism of the convolution algebra $H^{4d}_{\mathfrak{Z}}(\mathfrak{M}\times\mathfrak{M};\mathbb{C})$ with $\mathrm{End}(\mathcal{F})$, so (7) is compatible with the action of $H^{4d}_{\mathfrak{Z}}(\mathfrak{M}\times\mathfrak{M};\mathbb{C})$.

Lemma 7.13 *The direct sum decomposition of Equation* (7) *is orthogonal with respect to the equivariant intersection form introduced in Section 6.2.*

Proof: We will use the fact that the equivariant intersection form is compatible with the action of the convolution algebra in the sense that, for all $a\in H^{4d}_{\mathfrak{Z}}(\mathfrak{M}\times\mathfrak{M};\mathbb{C})$ and $b,c\in\Sigma(\mathcal{F})$, we have $(a\star b, c) = (b, a^*\star c)$, where $a\mapsto a^*$ is the anti-automorphism of $H^{4d}_{\mathfrak{Z}}(\mathfrak{M}\times\mathfrak{M};\mathbb{C})$ given by flipping the two factors of $\mathfrak{M}\times\mathfrak{M}$. To see this, note that $H^{4d}_{\mathfrak{Z}}(\mathfrak{M}\times\mathfrak{M};\mathbb{C})$ and $\Sigma(\mathcal{F}) = H^{2d}_{\mathfrak{M}+}\big(\mathfrak{M};\mathbb{C}\big)$ are isomorphic to the T-equivariant versions of these groups, since the cohomology in lower degrees vanishes. We can localize the equivariant groups to the T-fixed points, and using the projection formula for a proper pushforward, we get

$$(a\star b)|_\alpha = \sum_{\beta\in I}\frac{a|_{(\alpha,\beta)}\cdot b_\beta}{e(\beta)}$$

for all $\alpha\in I$. Since $a^*|_{(\alpha,\beta)} = a|_{(\beta,\alpha)}$, the result follows.

Now let e_S be the central idempotent in $H^{4d}_{\mathfrak{Z}}(\mathfrak{M}\times\mathfrak{M};\mathbb{C})$ which projects $\Sigma(\mathcal{F})$ onto $\Sigma(\mathcal{F}_S)$. Then e_S^* is again a central idempotent; we will prove by induction on $\mathscr{S}$ that $e_S^* = e_S$. Assume that $e_{S'}^* = e_{S'}$ for all $S' < S$. It is clear that e_S is represented entirely by cycles on $\mathfrak{Z}$ that live over $\bar{S}\subset\mathfrak{Z}_0\cong\mathfrak{M}_0$, therefore the image of e_S^* is contained in $\bigoplus_{S'\leq S}\Sigma(\mathcal{F}_{S'})$. For all $S' < S$, our inductive hypothesis tells us that $\Sigma(\mathcal{F}_{S'})$ is equal to the image of $e_{S'}^*$, and is therefore disjoint from the image of e_S^*. Since the image of e_S^* is invariant under the convolution algebra and complementary to the sum of the images of $e_{S'}^*$ for $S' < S$, it must be equal to $\Sigma(\mathcal{F}_{S'})$. Thus $e_S^* = e_S$.

Suppose that $b\in\Sigma(\mathcal{F}_S)$ and $c\in\Sigma(\mathcal{F}_{S'})$ for some $S\neq S'$. Then

$$(b,c) = (e_S b, c) = (b, e_S c) = (b, 0) = 0.$$

This establishes the result. $\qquad\qquad\Box$

It is tempting to guess that two decompositions of $H^{2d}_{\mathfrak{M}+}\big(\mathfrak{M};\mathbb{C}\big)$ given in Equations (5) and (7) coincide. This cannot be correct in general, however, because $\mathfrak{P}_S$ is trivial unless S is special at a parameter where derived localization holds, while $\Sigma(\mathcal{F}_S)$ is always nontrivial. The next natural guess is that the appropriately coarsened statement holds for special leaves; that

is, if S is special, then the subspace

$$K(\mathcal{O}_{\mathrm{g}}^S)_{\mathbb{C}} \subset K(\mathcal{O}_{\mathrm{g}})_{\mathbb{C}} \cong H_{\mathfrak{M}^+}^{2d}(\mathfrak{M}; \mathbb{C})$$

should coincide with the sum of the subspaces

$$\Sigma(\mathcal{F}_{S'}) \subset \Sigma(\mathcal{F}) \cong H_{\mathfrak{M}^+}^{2d}(\mathfrak{M}; \mathbb{C})$$

corresponding to leaves S' that are contained in the closure of S. Even this statement fails in general (see Remark 7.17), but it holds if $(\mathfrak{M}, \mathcal{D})$ is interleaved (Definition 6.14). We state this result below, and prove it at the end of this section.

Theorem 7.14 *Suppose that localization holds for $\mathcal{D}$. Then $(\mathfrak{M}, \mathcal{D})$ is interleaved if and only if for every special symplectic leaf $S \in \mathscr{S}^{\mathrm{sp}}$, the image of $K(\mathcal{O}_{\mathrm{g}}^S)_{\mathbb{C}} \subset K(\mathcal{O}_{\mathrm{g}})_{\mathbb{C}}$ under* CC *is equal to*

$$\bigoplus_{S' \leq S} \Sigma(\mathcal{F}_{S'}).$$

Corollary 7.15 *Suppose that $(\mathfrak{M}, \mathcal{D})$ is interleaved and all symplectic leaves are special. Then the characteristic cycle isomorphism takes the categorical decomposition of $K(\mathcal{O}_{\mathrm{g}})$ from Equation (5) to the BBD decomposition of $H_{\mathfrak{M}^+}^{2d}(\mathfrak{M}; \mathbb{C}) = \Sigma(\mathcal{F})$ from Equation (7).*

Proof: By definition, the categorical decomposition is orthogonal with respect to the Euler pairing. By Lemma 7.13, the BBD decomposition is orthogonal with respect to the intersection pairing. The result then follows from Theorems 6.5 and 7.14. □

Example 7.16 If $\mathfrak{M}$ is a hypertoric variety or a finite type A quiver variety and $\mathcal{D}$ is an integral quantization for which localization holds, then $(\mathfrak{M}, \mathcal{D})$ is interleaved and all symplectic leaves are special, so the corollary applies.

Remark 7.17 One can use Theorem 7.14 to show that $(\mathfrak{M}, \mathcal{D})$ is not interleaved by finding a special leaf for which the two vector spaces in question have different dimensions. In the case where $\mathfrak{M} = T^*(G/B)$ and the quantization is integral, the subspaces $K(\mathcal{O}_{\mathrm{g}}^S)_{\mathbb{C}}$ and $\bigoplus_{S' \leq S} \Sigma(\mathcal{F}_{S'})$ are sums of isotypic components for the action of the Weyl group on $K(\mathcal{O}_{\mathrm{g}})_{\mathbb{C}} \cong \Sigma(\mathcal{F})$; thus, we need only consider which simple representations appear in this subspace. Using the notation of Carter's book [Car93], when S is the nilpotent orbit A_2, the subspace $K(\mathcal{O}_{\mathrm{g}}^S)_{\mathbb{C}}$ is the sum of the isotypic components for the families

$$\{\phi_{1,24}\} \qquad \{\phi_{2,16}'', \phi_{4,13}, \phi_{2,16}'\} \qquad \{\phi_{9,10}\} \qquad \{\phi_{8,9}''\};$$

these families are listed on [Car93, pg. 414]. The first index is the dimension of the representation, so $\dim K(\mathcal{O}_{\mathrm{g}}^S)_{\mathbb{C}} = 170$.

However, the corresponding piece of the BBD filtration $\bigoplus_{S' \leq S} \Sigma(\mathcal{F}_{S'})$ is larger; by the chart on [Car93, pg. 428], it also includes the isotypic component for $\phi''_{1,12}$, which corresponds under the Springer correspondence to the unique non-trivial local system on A_2; thus its dimension is 171. This shows that $T^*(G/B)$ is not interleaved for G of type F_4 (we thank Victor Ostrik for pointing out this example to us). Inspection of the charts in Carter's book also shows that the same is true for types E_7 and E_8.

In types B, C, and D, it is also easy to find examples where these two filtrations do not match. For example, in C_4, the representations associated to the pairs of partitions $((2, 1), (1))$ and $((2, 2), \emptyset)$ are both associated to the nilpotent orbit with Jordan type $(4, 2, 2)$. However, only the former is in the 2-sided cell of this special orbit; the latter is in the 2-sided cell for $(4, 4)$. Outside of a few cases of small rank, the variety $T^*(G/B)$ will be interleaved only in type A.

To prove Theorem 7.14, we need a generalization of the functor Σ. For all $S \in \mathscr{S}$, define a functor $\Sigma_S : D^b_{\mathscr{S}}(\mathfrak{M}_0) \to \mathbb{C}$-mod by

$$\Sigma_S := H^{2d}_{\mathfrak{M}_0^+ \cap \bar{S}}(\mathfrak{M}_0; -).$$

For every S, the inclusion of $\mathfrak{M}_0^+ \cap \bar{S}$ into $\mathfrak{M}_0^+$ induces a natural transformation $\Sigma_S \to \Sigma$.

Lemma 7.18 *For any $S \in \mathscr{S}$, the map $\Sigma_S(\mathcal{F}) \to \Sigma(\mathcal{F})$ is injective, and the image has basis $\{[X_\alpha] \mid X_{\alpha,0} \subset \bar{S}\}$.*

Proof: Let $\mathfrak{M}^+_{\bar{S}} := \mathfrak{M}^+ \cap \nu^{-1}(\bar{S})$. Then Poincaré duality gives an isomorphism

$$\Sigma_S(\mathcal{F}) \cong H^{2d}(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+_{\bar{S}}; \mathbb{C}) \cong H^{BM}_{2d}(\mathfrak{M}^+_{\bar{S}}; \mathbb{C}).$$

Since $\mathfrak{M}^+_{\bar{S}}$ is purely d-dimensional, this last group is the subgroup of $H^{BM}_{2d}(\mathfrak{M}^+; \mathbb{C})$ spanned by the classes $[X_\alpha]$ for all $X_\alpha \subset \mathfrak{M}^+_{\bar{S}}$. $\qquad \square$

Let $\mathcal{F}_{\leq S} := \bigoplus_{S' \leq S} \mathcal{F}_{S'}$.

Lemma 7.19 *The image of the natural injection $\Sigma(\mathcal{F}_{\leq S}) \to \Sigma(\mathcal{F})$ is the same as the image of $\Sigma_S(\mathcal{F}) \to \Sigma(\mathcal{F})$.*

Proof: It is clear that $\Sigma_S(\mathcal{F}_{\leq S}) \to \Sigma(\mathcal{F}_{\leq S})$ is an isomorphism, so it is enough to show that $\Sigma_S(\mathcal{F}_{\leq S}) \to \Sigma_S(\mathcal{F})$ is an isomorphism, or equivalently that $\Sigma_S(\mathcal{F}_{S'}) = 0$ if $S' \not\leq S$. Let $j \colon \bar{S} \hookrightarrow \mathfrak{M}_0$ be the inclusion; then

$$\Sigma_S(\mathcal{F}_{S'}) = H^{2d}_{\mathfrak{M}_0^+ \cap \bar{S}}(\bar{S}; j^! \mathcal{F}_S).$$

54

Since $\mathcal{F}_{S'}$ is an intersection cohomology complex and $S' \cap \bar{S} = \emptyset$, we have

$$j^! \mathcal{F}_{S'} \in {}^p D_c^{\geq 2d+1}(\bar{S});$$

see, for example, [HTT08, 8.2.5]. This implies that $H^k_{\mathfrak{M}_0^+ \cap \bar{S}}(\bar{S}; j^! \mathcal{F}_S)$ vanishes for $k \leq \dim \mathfrak{M}$ [HTT08, 8.1.24], so $\Sigma_S(\mathcal{F}_{S'}) = 0$, as desired. $\qquad\square$

Proof of Theorem 7.14: First suppose that $(\mathfrak{M}, \mathcal{D})$ is interleaved. The image of $K(\mathcal{O}_{\mathrm{g}}^S)_{\mathbb{C}}$ is spanned by $\{\mathrm{CC}\,\Lambda_\alpha \mid \mathfrak{M}_{\alpha,0} \subset \bar{S}\}$. By Lemma 6.18, it is spanned by $\{[X_\alpha] \mid \mathfrak{M}_{\alpha,0} \subset \bar{S}\}$. On the other hand, Lemmas 7.18 and 7.19 imply that $\bigoplus_{S' \leq S} \Sigma(\mathcal{F}_{S'})$ is spanned by $\{[X_\alpha] \mid X_{\alpha,0} \subset \bar{S}\}$. Since S is special, $\mathfrak{M}_{\alpha,0} \subset \bar{S}$ if and only if $X_{\alpha,0} \subset \bar{S}$, so the two vector spaces agree.

Now suppose that $(\mathfrak{M}, \mathcal{D})$ is not interleaved. This means that there exists an element $\alpha \in \mathcal{I}$ and a special leaf S such that $X_{\alpha,0} \subset \bar{S} \subsetneq \mathfrak{M}_{\alpha,0}$. Lemma 6.18 says that the basis $\{\mathrm{CC}\,\Lambda_\beta\}_{\beta \in \mathcal{I}}$ is triangular with respect to the order $\hookleftarrow$, so if we write $[X_\alpha]$ in this basis, $\mathrm{CC}\,\Lambda_\alpha$ must occur with non-zero coefficient. It follows that $[X_\alpha]$ lies in $\bigoplus_{S' \leq S} \Sigma(\mathcal{F}_{S'})$ but not in $K(\mathcal{O}_{\mathrm{g}}^S)_{\mathbb{C}}$. $\qquad\square$

7.4   The extreme pieces

All of the structures discussed in this section are of particular interest when S is equal to either the point stratum $\{o\}$ or the dense stratum $\overset{\circ}{\mathfrak{M}}_0$. We begin with the point stratum.

The category $\mathcal{O}_{\mathrm{a}}^{\{o\}}$ is equal to the category of finite-dimensional A-modules. Therefore, if localization holds, we have

$$K(\mathfrak{P}_{\{o\}}) = K(\mathcal{O}_{\mathrm{g}}^{\{o\}}) \cong K(\mathcal{O}_{\mathrm{a}}^{\{o\}}) = \mathbb{Z}\{[L_\alpha] \mid \dim L_\alpha < \infty\}.$$

By Lemmas 7.18 and 7.19, the map CC takes $K(\mathfrak{P}_{\{o\}})_{\mathbb{C}}$ to $\Sigma(\mathcal{F}_{\{o\}}) \cong H^{2d}(\mathfrak{M}; \mathbb{C})$. This map may or may not be an isomorphism. The following theorem is an alternate version of Theorem 7.14 that we may apply if we only care about the point stratum; the proof is clear.

Theorem 7.20 *Suppose that localization holds for $\mathcal{D}$. Then the natural map from $K(\mathfrak{P}_{\{o\}})_{\mathbb{C}}$ to $H^{2d}(\mathfrak{M}; \mathbb{C})$ is always injective, and it is an isomorphism if and only if $\dim L_\alpha < \infty$ for all $\alpha \in \mathcal{I}$ such that $X_{\alpha,0} = \{o\}$.*

Definition 7.21 *If $(\mathfrak{M}, \mathcal{D})$ satisfies hypotheses of Theorem 7.20, we will refer to the pair as* **fat-tailed**.

Next, we turn our attention to the dense stratum. A simple object of $\mathcal{O}_{\mathrm{a}}$ lies in $\mathcal{O}_{\mathrm{a}}^{\partial \overset{\circ}{\mathfrak{M}}_0}$ if and only if its annihilator is nonzero, so if localization holds, we have

$$K(\mathfrak{P}_{\overset{\circ}{\mathfrak{M}}_0}) \cong \mathbb{Z}\{[L_\alpha] \mid \mathrm{Ann}(L_\alpha) = 0\}.$$

55

By Lemmas 7.18 and 7.19, the map CC takes $K(\mathcal{O}_{\mathrm{a}}^{\partial \mathring{\mathfrak{M}}_0})_{\mathbb{C}}$ to $\bigoplus_{\bar{S} \neq \mathfrak{M}_0} \Sigma(\mathcal{F}_S)$, so there is a naturally induced map of quotient spaces from $K(\mathfrak{P}_{\mathring{\mathfrak{M}}_0})$ to $\Sigma(\mathcal{F}_{\mathring{\mathfrak{M}}_0})$. Furthermore, we have a nice interpretation of the vector space $\Sigma(\mathcal{F}_{\mathring{\mathfrak{M}}_0})$, as described in the following lemma.

Lemma 7.22 *There is a canonical isomorphism* $\Sigma(\mathcal{F}_{\mathring{\mathfrak{M}}_0}) \cong IH_{\mathbb{T}}^{2d}(\mathfrak{M}_0; \mathbb{C})$.

Proof: We have $IH^*(\mathfrak{M}_0, \mathfrak{M}_0 \setminus \mathfrak{M}_0^+; \mathbb{C}) \subset H^*(\mathfrak{M}, \mathfrak{M} \setminus \mathfrak{M}^+; \mathbb{C})$. Since the second group is zero except in degree $2d$, so is the first. It follows that the forgetful map induces an isomorphism

$$IH_{\mathbb{T}}^{2d}(\mathfrak{M}_0, \mathfrak{M}_0 \setminus \mathfrak{M}_0^+; \mathbb{C}) \cong IH^{2d}(\mathfrak{M}_0, \mathfrak{M}_0 \setminus \mathfrak{M}_0^+; \mathbb{C}) = \Sigma(\mathcal{F}_{\mathring{\mathfrak{M}}_0}).$$

Since $\dim \mathbb{T} = 1$ and $\mathfrak{M} \setminus \mathfrak{M}_0^+$ contains no $\mathbb{T}$-fixed points, the total dimension of $IH_{\mathbb{T}}^*(\mathfrak{M}_0 \setminus \mathfrak{M}_0^+; \mathbb{C})$ is finite. The result now follows by applying the long exact sequence in $IH_{\mathbb{T}}$ for the pair $(\mathfrak{M}_0, \mathfrak{M}_0 \setminus \mathfrak{M}_0^+)$ together with the fact that $IH_{\mathbb{T}}^*(\mathfrak{M}_0; \mathbb{C})$ is generated as an $H_{\mathbb{T}}^*(pt)$-module by the part in degrees $< 2d$. $\qquad \square$

The following theorem is the alternate version of Theorem 7.14 that we may apply if we only care about the dense stratum; again, the proof is clear.

Theorem 7.23 *Suppose that localization holds for* $\mathcal{D}$. *Then the natural map from* $K(\mathfrak{P}_{\mathring{\mathfrak{M}}_0})_{\mathbb{C}}$ *to* $IH_{\mathbb{T}}^{2d}(\mathfrak{M}_0; \mathbb{C})$ *is always surjective, and it is an isomorphism if and only if* $\mathfrak{M}_{\alpha,0} \neq \mathfrak{M}_0$ *for all* $\alpha \in \mathcal{I}$ *such that* $X_{\alpha,0}$ *is contained in the closure of a non-dense leaf.*

Definition 7.24 If $(\mathfrak{M}, \mathcal{D})$ satisfies hypotheses of Theorem 7.23, we will refer to the pair as **light-headed**.

Example 7.25 If $\mathfrak{M}$ is a hypertoric variety or a finite type A quiver variety and $\mathcal{D}$ is an integral quantization for which localization holds, then $(\mathfrak{M}, \mathcal{D})$ is interleaved (Example 7.16), and therefore both fat-tailed and light-headed.

Example 7.26 If $\mathfrak{M}$ is a quiver variety of finite simply-laced type and $\mathcal{D}$ is an integral quantization for which localization holds, then $(\mathfrak{M}, \mathcal{D})$ is fat-tailed [BL, 1.2].

Conjecture 7.27 *If* $\mathfrak{M}$ *is a resolution of a transverse slice in the affine Grassmannian of finite simply-laced type and* $\mathcal{D}$ *is an integral quantization for which localization holds, then* $(\mathfrak{M}, \mathcal{D})$ *is light-headed.*

Remark 7.28 In type A, transverse slices in the affine Grassmannian coincide with quiver varieties [MV], so Conjecture 7.27 follows from Example 7.25; the open cases are in types D and E. We will revisit this conjecture in Example 10.27.

7.5 Cells

Throughout this section we will assume that localization holds. Consider a pair of indices $\alpha, \beta \in \mathcal{I}$.

Definition 7.29 We say that $\alpha \overset{L}{\leq} \beta$ if $\operatorname{Ann} L_\beta \subset \operatorname{Ann} L_\alpha$. We say that $\alpha \overset{R}{\leq} \beta$ if there exists a Harish-Chandra bimodule H such that L_α is a subquotient of $H \otimes L_\beta$. We define a third pre-order on $\mathcal{I}$ by putting $\alpha \overset{2}{\leq} \beta$ if $\alpha \overset{L}{\leq} \beta$ or $\alpha \overset{R}{\leq} \beta$, and then taking the transitive closure.

If $\alpha \overset{2}{\leq} \beta$ and $\beta \overset{2}{\leq} \alpha$, we say that α and β lie in the same **two-sided cell** of $\mathcal{I}$.

Proposition 7.30 *If* $\alpha \overset{2}{\leq} \beta$, *then* $\mathfrak{M}_{\alpha,0} \subset \mathfrak{M}_{\beta,0}$.

Proof: It is enough to show that either $\alpha \overset{L}{\leq} \beta$ or $\alpha \overset{R}{\leq} \beta$ implies that $\mathfrak{M}_{\alpha,0} \subset \mathfrak{M}_{\beta,0}$. By Corollary 6.12 and the fact that localization holds, $\mathfrak{M}_{\alpha,0} = \mathfrak{M}_{L_\alpha}$ and $\mathfrak{M}_{\beta,0} = \mathfrak{M}_{L_\beta}$. If $\alpha \overset{L}{\leq} \beta$, then $\operatorname{Ann} L_\beta \subset \operatorname{Ann} L_\alpha$, so by definition of $\mathfrak{M}_L$, we have $\mathfrak{M}_{\alpha,0} \subset \mathfrak{M}_{\beta,0}$.

If $\alpha \overset{R}{\leq} \beta$, then there exists an algebraic Harish-Chandra bimodule H such that L_α is a subquotient of $H \otimes L_\beta$. Localizing, we obtain a geometric Harish-Chandra bimodule $\mathcal{H}$ such that Λ_α is a subquotient of $\mathcal{H} \otimes \Lambda_\beta$. This implies that the support of Λ_α is contained in the support of Λ_β. By definition of $\mathfrak{M}_{\alpha,0}$, this implies that $\mathfrak{M}_{\alpha,0} \subset \mathfrak{M}_{\beta,0}$. $\qquad\square$

Remark 7.31 By Proposition 7.30, we have a surjective map from the set of two-sided cells to the set of special leaves that takes the cell containing α to the special leaf whose closure is $\mathfrak{M}_{\alpha,0}$. If $\mathfrak{M}$ is a hypertoric variety, this map is a bijection [BLPW12, 7.14]; the same is true if $\mathfrak{M} = T^*(G/B)$ and the period of $\mathcal{D}$ is integral. However, it need not be a bijection when $\mathfrak{M} = T^*(G/B)$ and the period is non-integral.

For example, let $G = G_2$, let $\alpha_1, \alpha_2 \in \mathfrak{h}^* \cong H^2(\mathfrak{M}; \mathbb{C})$ be the short and long simple roots, and consider the quantization with period $\alpha_1 + \alpha_2/2$. In this case, the simple modules with highest weights $\alpha_1 - \alpha_2/2$ and $-11\alpha_1 - 5\alpha_2/2$ are both associated with the sub-regular nilpotent orbit.

Conjecture 7.32 *Suppose that* $\mathcal{D}$ *is an integral quantization in the sense of Section 2.4. Then the map from the set of two-sided cells to* $\mathscr{S}^{\mathrm{sp}}$ *taking the equivalence class of* α *to* $\mathfrak{M}_{\alpha,0}$ *is an isomorphism of posets.*

Remark 7.33 If Conjecture 7.32 holds, then the summands of $H^{2d}_{\mathfrak{M}^+}(\mathfrak{M}; \mathbb{Z})$ in Equation (5) are simply the spans of the classes of the simple elements in each two-sided cell.

Remark 7.34 One may refine the set of two-sided cells in two different ways: by left cells (using the preorder $\overset{L}{\leq}$) or by right cells (using the preorder $\overset{R}{\leq}$). By considering the left preorder,

we can construct a surjective map from the set of left cells to the set of irreducible components of preimages in $\mathfrak{M}$ of special leaves in $\mathfrak{M}_0$. If $\mathfrak{M}$ is a hypertoric variety, then this map is a bijection by [BLPW12, 7.2]. If $\mathfrak{M} = T^*(G/B)$ and the period of $\mathcal{D}$ is integral, the same is true by the circle on the first page of the book of Borho, Brylinski and MacPherson [BBM89] (also based on work of Joseph, Kashiwara and Duflo, among others). As in the case of two-sided cells, the non-integral case is more subtle.

By considering the right preorder, we can construct a surjective map from the set of right cells to the set of irreducible components of varieties of the form $\mathfrak{M}^+_{\alpha,0} := \mathfrak{M}_{\alpha,0} \cap \mathfrak{M}^+$; such components are called **orbital varieties**. In the hypertoric case, this map is a bijection by [BLPW12, 7.11].

8 Twisting and shuffling functors

The purpose of this section is to introduce two commuting collections of endofunctors of $\mathcal{O}_a$, called twisting and shuffling functors. The twisting functors, which operate by varying the period of the quantization, act on the entire category A-mod, taking the subcategory $\mathcal{O}_a$ to itself. These functors were introduced in [BPW, §6.4], and they generalize Arkhipov's twisting functors on BGG category $\mathcal{O}$ (see Remark 8.4). The shuffling functors operate by varying the choice of $\mathbb{T}$, and therefore can only be defined on the category $\mathcal{O}_a$. These functors generalize Irving's shuffling functors on BGG category $\mathcal{O}$ (see Proposition 8.14).

8.1 Twisting functors

Let $\lambda, \lambda' \in H^2(\mathfrak{M}; \mathbb{C})$ be a pair of classes such that $\lambda - \lambda' \in H^2(\mathfrak{M}; \mathbb{Z})$. Let A_λ and $A_{\lambda'}$ denote the algebras of $\mathbb{S}$-invariant global sections of quantizations with periods λ and λ', and let $\mathcal{O}_a \subset A$-mod and $\mathcal{O}'_a \subset A_{\lambda'}$-mod be the associated categories.

In [BPW, 6.21], we define an $(A_{\lambda'}, A_\lambda)$-bimodule ${}_{\lambda'}T_\lambda$. In the most general situation, ${}_{\lambda'}T_\lambda$ is defined as a specialization of the space of sections of a quantized line bundle on the universal deformation $\mathscr{M}$ of $\mathfrak{M}$. If localization holds at λ', then ${}_{\lambda'}T_\lambda$ can be described more simply as the bimodule of $\mathbb{S}$-invariant global sections of ${}_{\lambda'}\mathcal{T}_\lambda[h^{-1/n}]$ [BPW, 6.26]. If $\mathfrak{M}$ is constructed as the symplectic quotient of a symplectic vector space, then ${}_{\lambda'}T_\lambda$ can also be realized as a weight space in a quotient of the Weyl algebra of the vector space [BPW, 6.28].

Let
$$\Phi^{\lambda',\lambda} \colon D(A_\lambda\text{-mod}) \to D(A_{\lambda'}\text{-mod})$$

be the functor obtained by derived tensor product with ${}_{\lambda'}T_\lambda$.

Let $\Pi \subset H^2(\mathfrak{M}; \mathbb{R})$ be the set of λ such that

- λ does not lie on the complexification of any of the hyperplanes of $\mathcal{H}_{\mathrm{tw}}$ (Section 2.2), and

- there exists some conical symplectic resolution $\mathfrak{M}'$ of $\mathfrak{M}_0$ and some element w of the Namikawa Weyl group such that localization holds at $w\lambda$ on $\mathfrak{M}'$.

We prove in [BPW, 6.32] that, if $\lambda, \lambda' \in \Pi$, then the functor $\Phi^{\lambda',\lambda}$ is an equivalence, and it preserves bounded derived categories; we will use the symbol $\Phi^{\lambda',\lambda}$ for the induced functor in this case. By [BPW, 6.37], in this case, $\Phi^{\lambda',\lambda}$ also takes the subcategory $D^b_{\mathcal{O}_a}(A\text{-mod})$ to $D^b_{\mathcal{O}_{a'}}(A\text{-mod})$.

Remark 8.1 By Remark 2.4, the chambers of $\mathcal{H}_{\mathrm{tw}}$ are equal to W-translates of ample cones of conical symplectic resolutions of $\mathfrak{M}_0$. Thus Π is the set of W-translates of classes that are not only ample on some resolution, but deep enough in the ample cone so that localization holds on that resolution. By Theorem 2.8, the intersection of Π with any chamber of $\mathcal{H}_{\mathrm{tw}}$ is nonempty (that is, it is always possible to go deep enough into the ample cone so that localization holds).

We define a **pure twisting functor** to be an auto-equivalence of $D(A_\lambda\text{-mod})$ obtained by composing functors of the form $\Phi^{\lambda'',\lambda'}$ with $\lambda', \lambda'' \in \Pi$. Such compositions go through module categories for many different quantizations; we require that they pass only through elements of Π, and that they begin and end at a single parameter λ. To define twisting functors in general, we incorporate the action of the Namikawa Weyl group.

For any $w \in W$, the rings A_λ and $A_{w\lambda}$ are canonically isomorphic [BPW, 3.10]. Let

$$\Phi^\lambda_w \colon A_{w\lambda}\text{-mod} \to A_\lambda\text{-mod}$$

be the equivalence induced by this isomorphism; we will use the same symbol to denote the induced functor on the derived category.

Proposition 8.2 *The functor Φ^λ_w takes $\mathcal{O}_a$ to itself.*

Proof: Recall that the isomorphism in [BPW, 3.10] that we use to define Φ^λ_w arises from a W-action on the universal deformation $\mathscr{M}$. This action commutes with that of $\mathbb{T}$ since the isomorphism $A_\lambda \cong A_{w\lambda}$ induces the identity map on $A_\lambda(n)/A_\lambda(n-1) \cong A_{w\lambda}(n)/A_{w\lambda}(n-1)$ and preserves the grading by $\mathbb{Z}/n\mathbb{Z}$. Thus it sends a non-commutative moment map for $\mathbb{T}$ in A_λ to one in $A_{w\lambda}$. Thus, the functor Φ^λ_w preserves category $\mathcal{O}_a$. $\qquad\qquad\square$

We define a **twisting functor** to be a composition of functors of the form $\Phi^{\lambda'',\lambda'}$ and their inverses (passing only through elements of Π), beginning at $D(A_\lambda\text{-mod})$ and ending at $D(A_{w\lambda}\text{-mod})$, followed by the functor Φ^λ_w. Note that by [BPW, 6.25] there is a natural isomorphism $\Phi^{\lambda',\lambda} \circ \Phi^\lambda_w \cong \Phi^{\lambda'}_w \circ \Phi^{w\lambda',w\lambda}$, so the set of twisting functors is closed under composition.

Let $E_{\mathrm{tw}} := H^2(\mathfrak{M}; \mathbb{C}) \smallsetminus \bigcup_{H \in \mathcal{H}_{\mathrm{tw}}} H_{\mathbb{C}}$. The following theorem is proven in [BPW, 6.35 & 6.37].

Theorem 8.3 *There is a natural homomorphism from $\pi_1(E_{\mathrm{tw}}/W, [\lambda])$ to the group of twisting functors on $D^b(A_\lambda\text{-mod})$, preserving the full subcategory $D^b_{\mathcal{O}_{\mathrm{a}}}(A\text{-mod})$. The subgroup $\pi_1(E_{\mathrm{tw}}, \lambda)$ maps to the group of pure twisting functors.*

Remark 8.4 In the case of hypertoric varieties, twisting functors are studied in detail in [BLPW10, §6] and [BLPW12, §8.2]. In the case of the Springer resolution, we show in [BPW, 6.38] that they coincide with the twisting functors defined by Arkhipov [AS03] (thus justifying the name).

Remark 8.5 On the level of the Grothendieck group, pure twisting functors act trivially [BPW, 6.39], therefore we obtain an action of W on $K(\mathcal{O}_{\mathrm{a}}) \cong K(\mathcal{O}_{\mathrm{g}}) \cong H^{2d}_{\mathfrak{M}^+}(\mathfrak{M}; \mathbb{Z})$. This action coincides with the one arising from the natural map from $\mathbb{C}[W]$ to the convolution algebra $H^{4d}_{\mathfrak{Z}}(\mathfrak{M} \times \mathfrak{M}; \mathbb{C})$ [BPW, 6.40].

8.2 Shuffling functors

In this section we discuss shuffling functors, which are in a certain sense "dual" to twisting functors (this will be explained in Section 10). Unlike twisting functors, these are unavoidably tied to the subcategory $\mathcal{O}_{\mathrm{a}} \subset A\text{-mod}$, since they are constructed by varying the action of $\mathbb{T} \cong \mathbb{C}^\times$. We will fix a single quantization throughout this section, and we will assume that its period lies in $\Pi \cap \mathfrak{U}$.[17]

Lemma 8.6 *Let G be the full group of Hamiltonian symplectomorphisms of $\mathfrak{M}$ that commute with $\mathbb{S}$. A maximal torus $T \subset G$ containing the image of $\mathbb{T}$ is unique up to conjugation by the largest unipotent subgroup commuting with $\mathbb{T}$.*

Proof: Let $C = C_G(\mathbb{T})$ be the centralizer of this cocharacter. We wish to show that this group is an extension of a torus by a unipotent subgroup. That is, we wish to show that any reductive subgroup of C is a torus. Any such subgroup must be contained in a maximal reductive subgroup G^r. Let T be a maximal torus in G^r containing $\mathbb{T}$. For all $\alpha \in \mathcal{I}$, consider the map $\mathfrak{g}^r \to T_{p_\alpha}\mathfrak{M}$, and let $\mathfrak{k} \subset \mathfrak{g}^r$ denote the intersection over all α of the kernels of these maps. Since the fixed points are isolated, we have $\mathfrak{t} \subset \mathfrak{c} \cap \mathfrak{g}^r \subset \mathfrak{k}$. We will show that $\mathfrak{k} = \mathfrak{t}$. This will imply that $\mathfrak{t} = \mathfrak{c} \cap \mathfrak{g}^r$, and therefore that T is the identity component of $C \cap G^r$ and a maximal reductive subgroup of C. Since every torus containing $\mathbb{T}$ is conjugate to T under C by the uniqueness of maximal tori in C, the proof will be complete.

Since $\mathfrak{k}$ is invariant under the adjoint action of $\mathfrak{t}$, it must be a sum of $\mathfrak{t}$ and some root spaces of $\mathfrak{g}^r$. Assume for the same of contradiction that there exists a root γ such that $\mathfrak{g}_\gamma \subset \mathfrak{k}$,

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[17] In fact, it is easy to show, using twisting functors, that $\mathfrak{U} \subset \Pi$.

and let $G' \subset G^r$ be a semisimple subgroup whose Lie algebra $\mathfrak{g}'$ contains $\mathfrak{g}_\gamma$. Consider a projective orbit $X \subset \mathfrak{M}$ of G'. By assumption, for all $\alpha \in \mathcal{I}$, the map $\mathfrak{g}' \to T_{p_\alpha} X$ kills $\mathfrak{g}_\gamma$. By the classification of projective homogeneous spaces, this is only possible if X is a point, thus all projective G'-orbits in $\mathfrak{M}$ are trivial.

Consider the action of G' on the core $\mathfrak{X} \subset \mathfrak{M}$ (Remark 3.5). If G' acted nontrivially on $\mathfrak{X}$, then it would contain a nontrivial closed orbit [BB80, Corollary 2], which we have seen is not possible. Thus G' fixes all of $\mathfrak{X}$. Any Hamiltonian action of a reductive group on a connected symplectic variety that fixes a Lagrangian subvariety must be trivial, thus we obtain a contradiction. $\qquad\square$

Let $\zeta : \mathbb{T} \to T$ be the cocharacter of T by which $\mathbb{T}$ acts. In this section we will vary ζ, and thereby vary the action of $\mathbb{T}$. We call a cocharacter **generic** if $\mathfrak{M}^{\mathbb{T}}$ is finite. For any generic ζ, we will write $\mathcal{O}_a^\zeta$ for the corresponding algebraic category $\mathcal{O}$. The set of non-generic cocharacters is equal to the intersection of the cocharacter lattice of T with the union of a finite set $\mathcal{H}_{\mathrm{sh}}$ of hyperplanes in $\mathfrak{t}_\mathbb{R}$.

Let $D_h^b(\mathcal{D}\text{-mod})$ be the full subcategory of $D^b(\mathcal{D}\text{-mod})$ consisting of complexes with holonomic cohomology, and let $D_h^b(A\text{-mod})$ be the full subcategory of $D^b(A\text{-mod})$ that is taken to $D_h^b(\mathcal{D}\text{-mod})$ by $\mathbb{L}\mathrm{Loc}$. Also let $\iota^\zeta : D_{\mathcal{O}_a^\zeta}^b(A\text{-mod}) \to D_h^b(A\text{-mod})$ be the inclusion functor; it is full and faithful by definition of $\mathfrak{U}$ (Section 5.2).

Proposition 8.7 *The functor ι^ζ has left and right adjoints*

$$^L\pi^\zeta : D_h^b(A\text{-mod}) \to D_{\mathcal{O}_a^\zeta}^b(A\text{-mod}) \qquad and \qquad {}^R\pi^\zeta : D_h^b(A\text{-mod}_h) \to D_{\mathcal{O}_a^\zeta}^b(A\text{-mod}).$$

Proof: If P is projective generator and I an injective generator of $\mathcal{O}_a^\zeta$, then the functor $\mathrm{Hom}(P, -)$ induces an equivalence $D_{\mathcal{O}_a^\zeta}^b(A\text{-mod}) \cong D^b(\mathrm{End}(P)^{\mathrm{op}}\text{-mod})$, and similarly $\mathrm{Hom}(-, I)$ induces an equivalence with $D^b(\mathrm{End}(I)\text{-mod})^{\mathrm{op}}$. In fact, there is a richer structure here: replacing P with a projective resolution of P as an A-module, for any object N of $D_{\mathcal{O}_a^\zeta}^b(A\text{-mod})$, we can think of $\mathrm{Ext}(P, N)$ as an object in $D^b(\mathrm{End}(P)\text{-mod})^{\mathrm{op}}$. Similarly, we can think of $\mathrm{Ext}(N, I)$ as an object in $D^b(\mathrm{End}(I)\text{-mod})$. (These are the Hom-spaces in the usual dg-enhancement of A-mod.) Note that the hypothesis that $\mathbb{L}\mathrm{Loc}(N)$ is holonomic guarantees that these complexes are finite dimensional. We can define

$$^L\pi^\zeta(N) := \mathrm{Ext}_{\mathrm{End}(I)}(\mathrm{Ext}_{\mathcal{O}_a^\zeta}(N, I), I) \qquad and \qquad {}^R\pi^\zeta(N) := P \otimes_{\mathrm{End}(P)^{\mathrm{op}}} \mathrm{Ext}_{\mathcal{O}_a^\zeta}(P, N).$$

This completes the proof. $\qquad\square$

Remark 8.8 One can think of these two adjoints as "projections" onto $D_{\mathcal{O}_a^\zeta}^b(A\text{-mod})$. The functor $^L\pi^\zeta$ is the derived functor of taking the largest quotient of a module that lies in $\mathcal{O}_a^\zeta$,

and $^R\pi^\zeta$ is the derived functor of taking the largest such submodule. It's clear that these functors are left/right exact, respectively.

Given two different generic cocharacters ζ and ζ' of T, let

$$\Psi^{\zeta',\zeta} := {}^L\pi^{\zeta'} \circ \iota^\zeta : D^b_{\mathcal{O}^\zeta_a}(A\text{-mod}) \to D^b_{\mathcal{O}^{\zeta'}_a}(A\text{-mod}),$$

and let $\Xi^{\zeta,\zeta'} = {}^R\pi^\zeta \circ \iota^{\zeta'}$ be its right adjoint. The following result, which should be regarded as an analogue of [BPW, 6.32], was conjectured in a previous draft of this paper, and has recently been proved by Losev [Los, 7.3].

Proposition 8.9 *The functor $\Psi^{\zeta',\zeta}$ is an equivalence.*

We define a **pure shuffling functor** to be an endofunctor of $D^b_{\mathcal{O}^\zeta_a}(A\text{-mod})$ obtained by composing functors of the form $\Psi^{\zeta'',\zeta'}$ for various generic cocharacters, beginning and ending at a single generic cocharacter ζ. To define shuffling functors in general, we incorporate the action of the Weyl group.

Let $\mathbb{W} := N_G(T)/T$ be the Weyl group of G. We use the blackboard-bold font to distinguish $\mathbb{W}$ from the Namikawa Weyl group W, which is typically different. For example, if $\mathfrak{M}$ is a crepant resolution of $\mathbb{C}^2/\Gamma$, then W is isomorphic to the Weyl group corresponding to Γ under the McKay correspondence, but $\mathbb{W}$ is trivial unless $\Gamma = \mathbb{Z}/2\mathbb{Z}$.

The action of G on $\mathbb{C}[\mathfrak{M}]$ lifts canonically to an action on A. The Weyl group $\mathbb{W}$ acts on the cocharacter lattice of T, and on the subset of generic cocharacters. For all $w \in W$ and all generic ζ, let $\zeta_w := w\zeta w^{-1}$. Define a functor $\Psi_{\bar{w}} : D^b(A\text{-mod}) \to D^b(A\text{-mod})$ taking an A-module N to the A-module with the same underlying vector space, but with action

$$a \cdot x = (\bar{w}a)x \text{ for all } x \in N \text{ and } a \in A,$$

for any $\bar{w} \in N_G(T)$ where the action on the left side is the action on $\Psi_{\bar{w}}(N)$, and the action on the right is the original action on N. If w is in the image of $\bar{w}$ in $\mathbb{W}$, we let $\Psi^\zeta_w : D^b_{\mathcal{O}^{\zeta_w}_a}(A\text{-mod}) \to D^b_{\mathcal{O}^\zeta_a}(A\text{-mod})$ denote the functor obtained by restricting $\Psi_{\bar{w}}$. Our notation for this functor is justified by the following lemma.

Lemma 8.10 *Up to natural isomorphism, the functor Ψ^ζ_w is independent of the choice of $\bar{w}$.*

Proof: It suffices to prove that, for any $t \in T$, the functor from $\mathcal{O}^\zeta_a$ to itself given by twisting the module structure by the action of t is isomorphic to the identity functor.

By [BPW, 3.11], the action of T on $\mathfrak{M}$ admits a quantized moment map $\eta : U(\mathfrak{t}) \to A$. The element $\xi \in A$ is the image of a generator of $\mathfrak{t}$ under a quantized moment map for the $\mathbb{T}$-action, so we can assume that η extends this map. In particular, η induces an action of $\mathfrak{t}$ on

any A-module N in $\mathcal{O}_a^\zeta$ which commutes with the action of ξ; since the ξ-weight spaces are finite-dimensional the $\mathfrak{t}$-action is semi-simple.

We can assume that N is indecomposable, and so all the $\mathfrak{t}$ weights lie in the same coset of the weight lattice of T inside of $\mathfrak{t}^*$. Thus, there is a character $\lambda \in \mathfrak{t}^*$ such that the action of $\mathfrak{t}$ on N via $\eta'(x) := \eta(x) - \lambda(x)$ integrates to an action $\rho \colon T \to \mathrm{End}(N)$.

Since $\lambda(x)$ is a scalar, we have

$$[\eta'(X), a]y = [\eta(X), a]y = \left.\frac{d}{ds}(e^{sX} \cdot a)y\right|_{s=0}$$

for any $y \in N$. Integrating, we get an equality $\rho(t)a\rho(t^{-1}) = t \cdot a$ of operators in $\mathrm{End}(N)$ for any $t \in T$. Thus for all $a \in A$ and $y \in N$, we have $(t \cdot a)\rho(t)y = \rho(t)ay$. In other words, the map $\rho(t) \colon N \to N$ intertwines the t-twisted action with the original action. $\qquad\square$

We define a **shuffling functor** to be a composition of of functors of the form $\Psi^{\zeta'',\zeta'}$ and their inverses, beginning at $\mathcal{O}_a^\zeta$ and ending at $\mathcal{O}_a^{\zeta_w}$, followed by the functor Ψ_w^ζ.

With twisting functors, we have a result [BPW, 6.33] that says that twisting between parameters that all lie within a fixed chamber of $\mathcal{H}_{\mathrm{tw}}$ is trivial. We now establish the analogous result for shuffling functors, which we will need in Section 10. For convenience, we will assume that derived localization holds.

Lemma 8.11 *Assume that derived localization holds. Suppose that ζ and ζ' lie in the same chamber of $\mathcal{H}_{\mathrm{sh}}$. Then the subcategories $D_{\mathcal{O}_a^\zeta}^b(A\text{-}\mathrm{mod}), D_{\mathcal{O}_a^{\zeta'}}^b(A\text{-}\mathrm{mod}) \subset D^b(A\text{-}\mathrm{mod})$ are equal, and $\Psi^{\zeta',\zeta}$ is the identity functor.*

Proof: Since derived localization holds, it is sufficient to prove that $\mathcal{O}_g^\zeta = \mathcal{O}_g^{\zeta'}$. By the definition of geometric category $\mathcal{O}$, it is sufficient to prove that ζ and ζ' induce the same relative core $\mathfrak{M}^+ \subset \mathfrak{M}$.

Suppose not; this means that there exists $\alpha \in \mathcal{I}$ such that the relative core components X_α and X_α' defined by ζ and ζ' (Section 3.1) are different. This in turn means that there is a character χ of T such that the χ-weight space of $T_{p_\alpha}\mathfrak{M}$ is nonzero and χ has opposite signs on ζ and ζ'. The vanishing set of χ is a hyperplane of $\mathcal{H}_{\mathrm{sh}}$ that separates ζ from ζ'; this contradicts the fact that ζ and ζ' lie in the same chamber. $\qquad\square$

Lemma 8.12 *There is a natural isomorphism $\Psi_w^{\zeta'} \circ \Psi^{\zeta'_w,\zeta_w} \cong \Psi^{\zeta',\zeta} \circ \Psi_w^\zeta$.*

Proof: By definition, we have $\Psi_{\bar{w}} \circ \iota^{\zeta_w} \cong \iota^\zeta \circ \Psi_w^\zeta$ for any $\bar{w} \in N_G(T)$. Then by adjointness, we have $^L\pi^{\zeta'} \circ \Psi_{\bar{w}} \cong \Psi_w^{\zeta'} \circ {}^L\pi^{\zeta'_w}$, and the result follows. $\qquad\square$

We are now ready to state the analogue of Theorem 8.3. Let $E_{\mathrm{sh}} := \mathfrak{t} \smallsetminus \bigcup\limits_{H \in \mathcal{H}_{\mathrm{tw}}} H_{\mathbb{C}}$.

Theorem 8.13 *There is a natural homomorphism from $\pi_1(E_{\mathrm{sh}}/\mathbb{W}, [\zeta])$ to the group of shuffling functors on $D^b_{\mathcal{O}^\zeta_{\mathrm{a}}}(A\text{-}\mathrm{mod})$. The subgroup $\pi_1(E_{\mathrm{sh}}, \zeta)$ maps to the group of pure shuffling functors.*

In fact, there are two natural such actions, intertwined by the automorphism on $\pi_1(E_{\mathrm{sh}}/\mathbb{W}, [\zeta])$ induced by complex conjugation. One sends a minimal length oriented path in the Deligne quiver to the functor $\Psi^{\zeta,\zeta'}$, and the second sends such a path to $\Xi^{\zeta,\zeta'}$. The second is the one that will appear in the definition of symplectic duality (Definition 10.1).

Proof: Here we follow the structure of the proof of Theorem 8.3 in [BPW, 6.35]. We model the fundamental group of E_{sh} using the Deligne groupoid, which is equivalent to $\pi_1(E_{\mathrm{sh}})$. The fundamental group of E_{sh}/W is thus equivalent to the semi-direct product of W with the Deligne groupoid.

The result [Los, 7.3] establishes that we have an action of the Deligne groupoid, and Lemma 8.12 shows that this action is compatible with the action of W on the Deligne groupoid. Thus we have an action of the semi-direct product, and therefore of $\pi_1(E_{\mathrm{sh}}/\mathbb{W}, [\zeta])$. □

We have chosen to call the functors defined in this section "shuffling functors" because they coincide with Irving's shuffling functors [Irv93] in the case of the Springer resolution. More precisely, let $\mathfrak{M} = T^*(G/B)$. The group of Hamiltonian symplectomorphisms that commute with $\mathbb{S}$ is isomorphic to G itself, and its Weyl group $\mathbb{W}$ is the usual Weyl group of G. (This example is unusual in that the Weyl group and the Namikawa Weyl group are isomorphic.) Let $T \subset B \subset G$ be the unique maximal torus of B, and let $\zeta : \mathbb{T} \to T$ be a generic cocharacter with non-negative weights on $\mathfrak{b}$. Consider the shuffling functor

$$\Psi_w := \Psi^\zeta_w \circ \Psi^{\zeta_w,\zeta} : D^b_{\mathcal{O}^\zeta_{\mathrm{a}}}(A\text{-}\mathrm{mod}) \to D^b_{\mathcal{O}^\zeta_{\mathrm{a}}}(A\text{-}\mathrm{mod}).$$

As noted in [BPW, §6.4], the category $\mathcal{O}^\zeta_{\mathrm{a}}$ for the period $\lambda + \rho$ (λ a dominant integral weight) is equivalent to a regular infinitesimal block of BGG category $\mathcal{O}$ via an equivalence defined by Soergel [Soe86].

Proposition 8.14 *For each $w \in \mathbb{W}$, the derived version of Soergel's equivalence takes Ψ_w to Irving's shuffling functor $C^{-1}_{w_0 w w_0}[\ell(w)]$.*

Remark 8.15 To prove Proposition 8.14, we will make use of the fact that our twisting and shuffling functors commute, which we will prove in the next section (Theorem 8.19).

We first show that Proposition 8.14 is correct when applied to any Verma module. For any $v \in W$, we denote by $M(v)$ the Verma module with highest weight $v(w_0\lambda - \rho) - \rho$. It is an object in $\mathcal{O}_a^\zeta$.

Lemma 8.16 *For any Verma module $M(v)$, there is an isomorphism between the image $\Psi_w(M(v))$ and the image under the shuffling functor $C_{w_0ww_0}^{-1}(M(v))[\ell(w)]$.*

Proof: We know from [BPW, 6.38] that Soergel's equivalence takes the twisting functors defined in the previous section to Arkhipov's twisting functors. Since all Verma modules are related by twisting functors, this and Theorem 8.19 imply that if Ψ_w and the shuffling functor $C_{w_0ww_0}^{-1}[\ell(w)]$ act the same way on one Verma module, then they have the same action on all of them.

The category $D^b_{\mathcal{O}_a^\zeta}(A\text{-mod})$ can be identified, via localization and [BPW, 4.5], with the derived category of λ-twisted D-modules on G/B which are smooth along the Schubert cells $X_w := BwB/B$. By tensoring with a line bundle, we can further identify it with the derived category of untwisted D-modules which are smooth along the Schubert cells. Via this identification, the standard objects of $\mathcal{O}_a^\zeta$ correspond to the D-modules $\mho_v := (j_{v^{-1}})_! \mathfrak{S}_{X_{v^{-1}}}$, and the functor Ψ_w^ζ corresponds to the pullback along the map $(\bar{w}\cdot) : G/B \to G/B$.

Soergel's equivalence takes the Verma module $M(w_0vw_0)$ to $\mho_v$. For any $w, v \in \mathbb{W}$, we have

$$\mathrm{Ext}^\bullet(\Psi_w\mho_e, \mho_v^\star) \cong \mathrm{Ext}^\bullet(\bar{w}^*\Psi^{\zeta_w,\zeta}\mho_e, \mho_v^\star) \cong \mathrm{Ext}^\bullet(\Psi^{\zeta_w,\zeta}\mho_e, \bar{w}_*\mho_v^\star)$$
$$\cong \mathrm{Ext}^\bullet(\iota^\zeta\mho_e, \iota^{\zeta_w}\bar{w}_*\mho_v^\star) \cong \mathrm{Ext}^\bullet(\mho_e, \bar{w}_*\mho_v^\star).$$

The D-module $\mho_e$ is supported at the point B/B and $\bar{w}_*\mho_v^\star = j_*\mathfrak{S}_{wBv^{-1}B}$. Thus, we get trivial Exts unless $e \in wBv^{-1}B$, which only happens when $v = w$. When $v = w$, then $wBv^{-1}B/B$ is an affine space of dimension $\ell(w)$. Thus we are reduced to a computation over the Weyl algebra, and we obtain

$$\mathrm{Ext}^\bullet(\Psi_w\mho_e, \mho_v^\star) \cong \begin{cases} 0 & v \neq w \\ \mathbb{C}[-\ell(w)] & v = w. \end{cases}$$

This implies that $\Psi_w\mho_e \cong \mho_w[\ell(w)]$.

Now that we know how Ψ_w acts on $\mho_e$, it remains to compute the action of $C_{w_0ww_0}^{-1}[\ell(w)]$ on the corresponding Verma module $M(e)$. We wish to show that $C_{w_0ww_0}^{-1}[\ell(w)]$ takes $M(e)$ to $M(w_0ww_0)[\ell(w)]$, or equivalently that C_w takes $M(w)$ to $M(e)$. If s is a simple reflection and $ws < w$, then by [Irv93, 3.1] (using the notation of that paper), we have

$$C_sM(w) \cong C_sM(w_0, w_0w) \cong M(w_0, w_0ws) \cong M(ws).$$

We can now prove the desired isomorphism by induction on length. □

Proof of Proposition 8.14: The usual t-structure on the derived BGG category $\mathcal{O}$ is induced by the exceptional collection of Verma modules as in [Bez03, Prop. 1]. Since the composition

$$F := \Psi_w \circ C_{w_0 w w_0}[-\ell(w)]$$

sends Vermas to Vermas, it induces an auto-equivalence of the abelian category $\mathcal{O}$ which sends every simple to itself.

Consider the trivial module $\mathbb{C}$ over $\mathfrak{g}$. The functor Ψ_w sends $\mathbb{C}$ to itself, inducing the identity on $\mathrm{Ext}_{\mathcal{O}}(\mathbb{C}, \mathbb{C}) \cong H^*(G/B)$, since $\mathbb{C}$ lies in category $\mathcal{O}$ for every Borel. The same is true of $C_{w_0 w w_0}^{-1}[\ell(w)]$ since $\mathbb{C}$ is killed by translation to any wall. Thus, we have an isomorphism $\mathbb{C} \cong F(\mathbb{C})$ which induces the same isomorphism $\mathrm{Ext}_{\mathcal{O}}(\mathbb{C}, \mathbb{C}) \cong \mathrm{Ext}_{\mathcal{O}}(F(\mathbb{C}), F(\mathbb{C}))$ as the functor F. It follows that for any simple L in $\mathcal{O}_{\mathrm{a}}^{\zeta}$, we have a canonical isomorphism $f \colon \mathrm{Ext}^{\bullet}(\mathbb{C}, L) \cong \mathrm{Ext}^{\bullet}(\mathbb{C}, F(L))$ of $H^*(G/B)$-modules induced by the functor F.

The Koszul dual form of Soergel's Endomorphismensatz [Soe90] states that for any two simple modules, we have an isomorphism

$$\mathrm{Ext}^{\bullet}(L, L') \cong \mathrm{Hom}_{H^*(G/B)}(\mathrm{Ext}^{\bullet}(\mathbb{C}, L), \mathrm{Ext}^{\bullet}(\mathbb{C}, L')).$$

This shows, in particular, that

$$\mathrm{Ext}^{\bullet}(L, F(L)) \cong \mathrm{Hom}_{H^*(G/B)}(\mathrm{Ext}^{\bullet}(\mathbb{C}, L), \mathrm{Ext}^{\bullet}(\mathbb{C}, F(L))),$$

so the isomorphism f induces an isomorphism $L \cong F(L)$. The Endomorphismensatz similarly shows that this isomorphism induces the same isomorphism of Yoneda algebras $\mathrm{Ext}^{\bullet}(\oplus L, \oplus L) \cong \mathrm{Ext}^{\bullet}(\oplus F(L), \oplus F(L))$ as the functor F. Thus, it induces an isomorphism between F and the identity functor. □

We conclude this section by discussing the action of shuffling functors on the Grothendieck group. Just as we saw for twisting functors in Remark 8.5, we will find that the pure shuffling functors act trivially, and we are left with an action of the Weyl group $\mathbb{W}$.

Recall from Theorem 6.5 that we have

$$\mathrm{CC} : K(\mathcal{O}_{\mathrm{a}}^{\zeta}) \cong K(\mathcal{O}_{\mathrm{g}}^{\zeta}) \xrightarrow{\cong} H^{2d}_{\mathfrak{M}_{\zeta}^+}(\mathfrak{M}; \mathbb{Z}),$$

and that this isomorphism intertwines the Euler form with the equivariant intersection form. Furthermore, we have

$$H^{2d}_{\mathfrak{M}^+_\zeta}(\mathfrak{M};\mathbb{Z}) \cong H^{2d}_{\mathfrak{M}^+_\zeta,T}(\mathfrak{M};\mathbb{Z}) \hookrightarrow H^{2d}_T(\mathfrak{M}^T;\mathbb{Z}),$$

with image independent of ζ. In particular, this gives us a canonical way to identify the lattices $K(\mathcal{O}^\zeta_a) \cong H^{2d}_{\mathfrak{M}^+_\zeta}(\mathfrak{M};\mathbb{Z})$ and $K(\mathcal{O}^{\zeta'}_a) \cong H^{2d}_{\mathfrak{M}^+_{\zeta'}}(\mathfrak{M};\mathbb{Z})$ for any two generic cocharacters ζ and ζ'. One can check that this identification sends v_α to $\pm v'_\alpha$; the sign is given by the parity of the codimension of the space of points that flow in to p_α for both ζ and ζ' inside the space of points that flow in for ζ (note that this is symmetric under switching ζ and ζ').

This identification agrees with the map $K(\mathcal{O}^\zeta_a) \to K(\mathcal{O}^{\zeta'}_a)$ induced by $\Psi^{\zeta',\zeta}$. We omit a full proof of this fact, since it will not be used later in the paper. The proof is similar to Theorem 6.5; the key is to show that the Euler pairing between $K(\mathcal{O}^\zeta_a)$ and $K(\mathcal{O}^{\zeta'}_a)$ inside of $K(A\text{-mod}_h)$ agrees with the equivariant intersection pairing on $H^{2d}_T(\mathfrak{M}^T;\mathbb{Z})$. Deformation arguments show that it suffices to do this on a generic fiber of a twistor deformation. Thus, we are reduced to calculating the Exts between modules M_L, M'_L over the Weyl algebra deforming the structure sheaves of Lagrangian subspaces L, L':

$$\text{Ext}^i(M_L, M_{L'}) = \begin{cases} \mathbb{C} & i = \dim(L/(L \cap L')) \\ 0 & i \neq \dim(L/(L \cap L')). \end{cases}$$

This implies the following proposition, which is an analogue of [BPW, 6.39] (see Remark 8.5).

Proposition 8.17 *Pure shuffling functors act trivially on the Grothendieck group of $\mathcal{O}^\zeta_a$.*

Impure shuffling functors, however, act in an interesting way. Consider the impure shuffling functor $\Psi_w = \Psi^\zeta_w \circ \Psi^{\zeta w,\zeta}$. We know from the above discussion that $\Psi^{\zeta w,\zeta}$ induces the aforementioned canonical isomorphism from $H^{2d}_{\mathfrak{M}^+_\zeta}(\mathfrak{M};\mathbb{Z})$ to $H^{2d}_{\mathfrak{M}^+_{\zeta w}}(\mathfrak{M};\mathbb{Z})$. The map induced by Ψ^ζ_w is given by choosing a lift $\bar{w} \in N(T) \subset G$ and considering the automorphism of $\mathfrak{M}$ induced by $\bar{w}^{-1}$. This automorphism intertwines the action of $\mathbb{T}$ by ζw with the action of $\mathbb{T}$ by ζ, and therefore induces an isomorphism from $H^{2d}_{\mathfrak{M}^+_{\zeta w}}(\mathfrak{M};\mathbb{Z})$ to $H^{2d}_{\mathfrak{M}^+_\zeta}(\mathfrak{M};\mathbb{Z})$. This isomorphism is different from the canonical one; in other words, the automorphism of $K(\mathcal{O}^\zeta_a) \cong H^{2d}_{\mathfrak{M}^+_\zeta}(\mathfrak{M};\mathbb{Z})$ induced by Ψ_w is non-trivial. These automorphisms are compatible with multiplication in the Weyl group, so we obtain a shuffling action of $\mathbb{W}$ on $K(\mathcal{O}^\zeta_a)$.

Remark 8.18 In the case of the Springer resolution, both W and $\mathbb{W}$ are isomorphic to the ordinary Weyl group. Since pure twisting and shuffling functors act trivially on the Grothendieck group (Remark 8.5 and Proposition 8.17) we obtain both a twisting and a shuffling action of W on $K(\mathcal{O}^\zeta_a)$. Furthermore, the two actions commute with each other by

67

Theorem 8.23, which we will prove in the next section. Indeed, what we obtain is isomorphic to the canonical action of $W \times W$ on $\mathbb{C}[W]$, with one factor acting by left multiplication and the other by inverse right multiplication.

8.3 Twisting and shuffling commute

The purpose of this section is to show that twisting and shuffling functors commute. We begin with the pure ones.

Lemma 8.19 *Let $\lambda, \lambda' \in \Pi$ be parameters with $\lambda - \lambda' \in H^2(\mathfrak{M}; \mathbb{Z})$, and ζ, ζ' two generic cocharacters of T. Then we have a natural isomorphism of functors*[18]

$$\Psi^{\zeta',\zeta} \circ \Phi^{\lambda',\lambda} \cong \Phi^{\lambda',\lambda} \circ \Psi^{\zeta',\zeta}.$$

Proof: First, we observe that the four functors above are not affected if we replace $\mathfrak{M}$ with some other conical symplectic resolution $\mathfrak{M}'$ of $\mathfrak{M}_0$ [BPW, 3.9 & 6.24]. By definition, localization holds for *some* resolution at every element of Π, thus we may assume that localization holds at λ'. By [BPW, 6.31], the functor $\Phi^{\lambda',\lambda}$ can be written as the composition of the localization functor $\mathbb{L}\mathrm{Loc}$ at λ, the "geometric twist" $_{\lambda'}\mathcal{T}[h^{-1/n}]_\lambda \otimes_{\mathcal{D}_\lambda} -$, and the derived sections functor $\mathbb{R}\Gamma_\mathbb{S}$ at λ'. Similarly, the functor $\Psi^{\zeta',\zeta}$ (at either λ or λ') can be written as the composition of $\mathbb{L}\mathrm{Loc}$, the "geometric shuffle" (defined in a way completely analogous to that of $\Psi^{\zeta',\zeta}$), and $\mathbb{R}\Gamma_\mathbb{S}$. Thus it suffices to show that geometric twists commute with geometric shuffles. This follows immediately from the fact that $_{\lambda'}\mathcal{T}[h^{-1/n}]_\lambda \otimes_{\mathcal{D}_\lambda} -$ and its adjoint commute with the inclusion of $\mathcal{O}_\mathrm{g}$ into $\mathcal{D}\text{-mod}_h$. □

We next move on to the various impure cases.

Lemma 8.20 *Let $\lambda \in \Pi$ and $w \in W$ be such that $w \cdot \lambda - \lambda \in H^2(\mathfrak{M}; \mathbb{Z})$, and let ζ, ζ' be generic cocharacters of T. Then we have a natural isomorphism of functors*

$$\Psi^{\zeta',\zeta} \circ \Phi^\lambda_w \cong \Phi^\lambda_w \circ \Psi^{\zeta',\zeta}.$$

Proof: It is clear from the definition of Φ^λ_w that it commutes with inclusion functors and their adjoints, and therefore with $\Psi^{\zeta',\zeta}$. □

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[18]Here we interpret the first $\Psi^{\zeta',\zeta}$ for the quantization with period λ', and the second $\Psi^{\zeta',\zeta}$ for the quantization with period λ. We do not need to say anything about $\Phi^{\lambda',\lambda}$, since this functor is defined without reference to the choice of cocharacter. Similar comments apply to the statements of Lemmas 8.22, 8.20, and 8.21.

Lemma 8.21 *Let $\lambda \in \Pi$ and $w \in W$ be such that $w \cdot \lambda - \lambda \in H^2(\mathfrak{M}; \mathbb{Z})$. Let ζ be generic, and let $v \in \mathbb{W}$ be arbitrary. Then we have a natural isomorphism of functors*

$$\Psi_v^\zeta \circ \Phi_w^\lambda \cong \Phi_w^\lambda \circ \Psi_v^\zeta.$$

Proof: This follows immediately from the fact that the canonical isomorphism $A_\lambda \cong A_{w \cdot \lambda}$ that was used to define the functor Φ_w^λ is G-equivariant. $\qquad\square$

Lemma 8.22 *Let $\lambda, \lambda' \in \Pi$ be parameters with $\lambda - \lambda' \in H^2(\mathfrak{M}; \mathbb{Z})$, and ζ a generic cocharacter of T, and $w \in \mathbb{W}$. Then we have a natural isomorphism of functors*

$$\Psi_w^\zeta \circ \Phi^{\lambda',\lambda} \cong \Phi^{\lambda',\lambda} \circ \Psi_w^\zeta.$$

Proof: For any object N of $\mathcal{O}_a^{\zeta_w}$ (with period λ), we have

$$\Psi_w^\zeta \circ \Phi^{\lambda',\lambda}(N) = \Psi_w^\zeta \left({}_{\lambda'}T_\lambda \overset{L}{\otimes} N \right) \cong \Psi_w^\zeta({}_{\lambda'}T_\lambda) \overset{L}{\otimes} \Psi_w^\zeta(N).$$

Here, by $\Psi_w^\zeta({}_{\lambda'}T_\lambda)$, we mean that we twist both the left *and* the right module structures on ${}_{\lambda'}T_\lambda$ by any lift $\bar{w}$ of w to $N(T) \subset G$. To prove the lemma, it suffices to show that $\Psi_w^\zeta({}_{\lambda'}T_\lambda) \cong {}_{\lambda'}T_\lambda$.

By the same argument that we used at the beginning of the proof of Lemma 8.19, we may assume that localization holds at λ'. This implies that ${}_{\lambda'}T_\lambda \cong \Gamma_{\mathbb{S}}({}_{\lambda'}\mathcal{T}_\lambda')$. Thus $\Psi_w^\zeta({}_{\lambda'}T_\lambda)$ is the $\mathbb{S}$-invariant sections of the pullback sheaf $\bar{w}^* {}_{\lambda'}\mathcal{T}_\lambda'$; however, the pullback of ${}_{\lambda'}\mathcal{T}_\lambda'$ by any group element is again a quantization of the same line bundle, and thus isomorphic to ${}_{\lambda'}\mathcal{T}_\lambda'$. This completes the proof. $\qquad\square$

The four preceding lemmas combine to give us the following theorem.

Theorem 8.23 *Twisting functors commute with shuffling functors.*

We end the section with a pair of conjectures, motivated by our study of twisting and shuffling functors on hypertoric varieties. Suppose that we have a notion of integral periods (Section 2.4). Fixing an integral parameter $\lambda \in \Pi$ and a generic cocharacter ζ, we consider the **long twist** $\Phi^{\lambda,-\lambda} \circ \Phi^{-\lambda,\lambda}$ and the **long shuffle** $\Psi^{\zeta,-\zeta} \circ \Psi^{-\zeta,\zeta}$. The first is a pure twisting functor, the second a pure shuffling functor; in particular, they are both endofunctors of a single category $D_{\mathcal{O}_a}^b(A\text{-mod})$.

Conjecture 8.24 *Up to a shift, the long twist is isomorphic to to the right Serre functor on $D_{\mathcal{O}_a}^b(A\text{-mod})$ and the long shuffle is isomorphic to the left Serre functor.*

Remark 8.25 This conjecture is known to hold for BGG category $\mathcal{O}$ by [MS08, 4.1] and for hypertoric category $\mathcal{O}$ by [BLPW10, 6.11].

Remark 8.26 Conjecture 8.24 has recently been proven by Losev [Los, 7.4 & 7.7].

9 Examples

The purpose of this section is to summarize the structures that we have defined so far for all known classes of examples of conical symplectic resolutions. Specifically, for each class, we will address the following (the parenthetical section number indicates the point in this paper at which each of these topics was first discussed):

(i) the group G of Hamiltonian symplectomorphisms that commute with $\mathbb{S}$, along with its Weyl group $\mathbb{W}$ (Section 8.2)

(ii) the vector space $H^2(\mathfrak{M}; \mathbb{C})$ (or the full cohomology ring) along with the action of the Namikawa Weyl group W (Section 2.2)

(iii) the algebra A of $\mathbb{S}$-invariant global sections of a quantization (Section 2.3)

(iv) the periods at which localization is known to hold (Section 2.6)

(v) Koszulity of $\mathcal{O}_{\mathrm{a}}$ and $\mathcal{O}_{\mathrm{g}}$ (Section 4.1)

(vi) the map $H^*(\mathfrak{M}; \mathbb{C}) \to Z(E)$ to the center of the Yoneda algebra of $\mathcal{O}_{\mathrm{g}}$ (Section 5.4)

(vii) the poset $\mathscr{S}$ of symplectic leaves and the subposet $\mathscr{S}^{\mathrm{sp}}$ of special leaves (Section 6.3)

(viii) the twisting and shuffling functors (Section 8).

9.1 Cotangent bundles of partial flag varieties

Let G be a semi-simple complex Lie group and $P \subset G$ a parabolic subgroup. Let $\mathfrak{M} := T^*(G/P)$, equipped with the inverse scaling action on the fibers. Up to modification of the $\mathbb{S}$-action, these are the only known examples of conical symplectic resolutions that are cotangent bundles.

The G-moment map $\mathfrak{M} \to \mathfrak{g}^* \cong \mathfrak{g}$ has as its image the closure of a nilpotent orbit O_P; the orbits that arise in this way are called **Richardson**. The induced map from $\mathfrak{M}_0$ to the closure of the Richardson orbit is generically finite. If it is generically one to one, then $\mathfrak{M}_0$ is isomorphic to the normalization of the orbit closure. If $G = \mathrm{SL}_r$, then every nilpotent orbit is Richardson, every nilpotent orbit closure is normal, and the map from $\mathfrak{M}_0$ to the orbit closure is always an isomorphism.

(i) If $P \neq G$, the group of Hamiltonian symplectomorphisms of $\mathfrak{M}$ commuting with $\mathbb{S}$ is the adjoint group $G/Z(G)$, and $\mathbb{W}$ is its Weyl group.

(ii) The cohomology ring is

$$H^*(\mathfrak{M};\mathbb{C}) \cong \mathbb{C}[\mathfrak{t}^*]^{\mathbb{W}_P} \Big/ \mathbb{C}[\mathfrak{t}^*]^{\mathbb{W}_P} \cdot \mathbb{C}[\mathfrak{t}^*]_+^{\mathbb{W}},$$

where $\mathbb{W}_P$ is the Weyl group of $P/[P,P]$ and $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. In particular, $H^2(\mathfrak{M};\mathbb{C}) \cong (\mathfrak{t}^*)^{\mathbb{W}_P}$.

We describe the Namikawa Weyl group only in the special case where $G = \mathrm{SL}_r$. Let μ be a composition of r. This means that μ is a function $i \mapsto \mu_i$ from $\mathbb{Z}$ to $\mathbb{N}$ such that $\sum_i \mu_i = r$. Consider the parabolic subgroup $P = P_\mu \subset \mathrm{SL}_r$ of block-upper-triangular matrices with blocks of size $(\ldots, \mu_{-2}, \mu_{-1}, \mu_0, \mu_1, \mu_2, \ldots)$, in that order. Let $\bar{\mu}$ denote the partition with the same parts as the composition μ, sorted into nonincreasing order, and let $\bar{\mu}^t$ be its transpose; in other words, $\bar{\mu}_j^t$ is the number of parts of $\bar{\mu}$ or μ that are greater than or equal to j. Then $\mathfrak{M}_0$ is isomorphic to the closure of the nilpotent orbit in $\mathfrak{g}$ with Jordan type μ. The Namikawa Weyl group W permutes parts of the composition of the same size; more precisely, we have

$$W \cong S_{\bar{\mu}_1^t - \bar{\mu}_2^t} \times \cdots \times S_{\bar{\mu}_{r-1}^t - \bar{\mu}_r^t} \times S_{\bar{\mu}_r^t}.$$

In particular, if $\bar{\mu}^t = (r)$ (in which case P is a Borel subgroup), then $W = S_r$. At the other extreme, if $\bar{\mu}^t = (1, \ldots, 1)$ (in which case $P = G$), then W is trivial.

(iii) Let $\lambda \in H^2(\mathfrak{M};\mathbb{C}) \subset \mathfrak{t}^*$, and let A_λ be the invariant section ring of the quantization with period λ. Then A_λ is isomorphic to a quotient of $U(\mathfrak{g})$ by a primitive ideal; if $P = B$, then this ideal is generated by elements of the center. By [BPW, 4.4], A_λ is also isomorphic to the ring of global D-modules on G/P, twisted by $\lambda + \rho$.

(iv) By the work of Beilinson and Bernstein [BB81], localization holds if and only if the inner product $\langle \lambda, \alpha \rangle$ is not a non-positive integer for any positive root $\alpha \in \mathfrak{t}^*$.

(v) If the period λ is regular, then the category $\mathcal{O}_\mathfrak{g}$ is equivalent by Soergel's functor to a regular infinitesimal block of parabolic BGG category $\mathcal{O}$ [BGS96, 3.5.1] (see also [Web11, Proposition 2]). In particular, $\mathcal{O}_\mathfrak{g}$ is standard Koszul.

(vi) If the period λ is regular and integral, then the center of E is isomorphic to the center of the Koszul dual category $\mathcal{O}_\mathfrak{g}^!$, which is a singular integral block of ordinary (not parabolic) BGG category $\mathcal{O}$ [Bac99, 1.1]. The fact that the center of such a block is isomorphic to the cohomology ring of $\mathfrak{M}$ is a consequence of Soergel's Stuktursatz and

Endomorphismensatz from [Soe90]. A slightly stronger statement is that Conjecture 5.23 holds in this case.

Proposition 9.1 *The natural map from $H^*(\mathfrak{M})$ to the center $Z(E)$ is an isomorphism.*

Proof: By [BPW, 4.5], $\mathcal{D}$-mod is equivalent to the category of D-modules twisted by a line bundle L. In particular, L itself may be regarded as an object of $\mathcal{D}$-mod. The Ext-algebra $\operatorname{Ext}^\bullet(L, L)$ is isomorphic to the de Rham cohomology of $H^*(\mathfrak{M})$, and the map $H^*(\mathfrak{M}) \to \operatorname{Ext}^\bullet(L, L)$ from Section 5.4 realizes this isomorphism.

This implies that the map $H^*(\mathfrak{M}) \to Z(E)$ is injective. By [Bru08a, 5.11], the dimension of $Z(E)$ is the same as the number of simple objects in $\mathcal{O}_\mathrm{g}$, which is equal to $|\mathfrak{M}^{\mathbb{T}}| = \dim H^*(\mathfrak{M}^{\mathbb{T}}; \mathbb{C}) = \dim H^*(\mathfrak{M}; \mathbb{C})$. Thus, this map must be an isomorphism. $\square$

(vii) Let $O_P \subset \mathfrak{g}$ be the Richardson nilpotent orbit with the property that $\mathfrak{M}_0$ is finite over $\overline{O}_P$. The symplectic leaves of $\mathfrak{M}_0$ are the preimages of the G-orbits in $\overline{O}_P$. For an integral parameter, the special leaves (equivalently, the special orbits) correspond to those double cells which contain a shortest right coset representative for the Weyl group of the parabolic. These are described in Carter's book [Car93, §13]. If $\mathfrak{g} = \mathfrak{sl}_r$, then all leaves are special. For non-integral parameters, the question of which orbits are special is more complicated; we do not address it here.

(viii) In the case where P is a Borel subgroup, our twisting functors agree with those defined by Arkhipov (Remark 8.4) under the equivalence between $\mathcal{O}_\mathrm{a}$ and BGG category $\mathcal{O}$. Similarly, our shuffling functors agree with those defined by Irving (see Proposition 8.14). Thus, when P is the Borel we obtain two commuting actions of the generalized braid group B_W on $\mathcal{O}_\mathrm{a}$; at the level of the Grothendieck group, these descend to the left and right actions of W on $\mathbb{C}[W]$ (Remark 8.18).

When P is an arbitrary parabolic, Soergel's functor can be used to identify $\mathcal{O}_\mathrm{g}$ (and therefore $\mathcal{O}_\mathrm{a}$ if localization holds) with an infinitesimal block of parabolic BGG category $\mathcal{O}$. Irving's shuffling functors all still make sense in the setting of parabolic BGG category $\mathcal{O}$, and they coincide with our shuffling functors. On the other hand, not all of Arkhipov's twisting functors preserve parabolic BGG category $\mathcal{O}$; our twisting functors are just those Arkhipov functors that preserve parabolic $\mathcal{O}$. Conjecture 8.24 holds in this case.

9.2 S3-varieties

Let G be a simple complex algebraic group and let $e \in \mathfrak{g}$ be a nilpotent element; let h, f be elements which satisfy the Chevalley relations of $\mathfrak{sl}_2$ together with e and let $\mathfrak{g} = \oplus \mathfrak{g}_k$ be

the decomposition of $\mathfrak{g}$ into eigenspaces for h. Choose a Cartan $\mathfrak{t} \subset \mathfrak{g}_0$ and let T be the corresponding connected subgroup. The space $\mathfrak{g}_{-1}$ has a symplectic form defined by $\langle [-, -], e \rangle$, where $\langle -, - \rangle$ denotes the Killing form. We let $\mathfrak{l} \subset \mathfrak{g}_{-1}$ be a Lagrangian subspace with respect to this form. Let

$$\mathfrak{m} := \mathfrak{l} \oplus \bigoplus_{k \leq -2} \mathfrak{g}_k$$

and let $M \subset G$ be the associated connected algebraic subgroup. We have a natural character $\chi := \langle e, - \rangle \colon \mathfrak{m} \to \mathbb{C}$.

Let P be a parabolic subgroup of G. Consider the moment map $\mu : T^*(G/P) \to \mathfrak{g}^*$, and let $\mu_{\mathfrak{m}} : T^*(G/P) \to \mathfrak{m}^*$ be the moment map obtained by projecting onto $\mathfrak{m}^*$. As explained by Ginzburg and Gan [GG02, §3.2], the group M acts freely on $\mu_{\mathfrak{m}}^{-1}(\chi)$, so the quotient $\mathfrak{X}_P^e := \mu_{\mathfrak{m}}^{-1}(\chi)/M$ is smooth; it is a symplectic resolution of the affine quotient $\mathfrak{X}_{P,0}^e := \operatorname{Spec} \mathbb{C}[T^*(G/P)]^M$.

If the Richardson orbit $O_P \subset \mathfrak{g}$ is simply connected or if $G = \mathrm{SL}_r$, then $\mathfrak{X}_{P,0}^e$ is isomorphic to a transverse slice to the orbit $G \cdot e$ inside of $\overline{O}_P = G \cdot \mathfrak{p}^\perp \subset \mathfrak{g}^*$. More generally, it admits a finite map to such a slice. If $e = 0$, then M is trivial, and $\mathfrak{X}_P^e = T^*(G/P)$, thus these spaces generalize those considered in the previous section. There seems to be no fixed name for $\mathfrak{X}_P^e$ and $\mathfrak{X}_{P,0}^e$ in the literature; we have adopted the term **S3-varieties**, as they have been studied (independently) by Slodowy, Spaltenstein, and Springer.

Remark 9.2 A subtlety in the above construction is that M is not reductive, and one usually only considers quotients by reductive groups. In this particular case, the fact that everything works as expected must be checked carefully; this is shown in [GG02] using the freeness of the action of M on the preimage of χ.

The usual inverse scaling action of $\mathbb{S}$ on the fibers of $T^*(G/P)$ does not descend to $\mathfrak{X}_P^e$, since $\mu_{\mathfrak{m}}$ is $\mathbb{S}$-equivariant and $\chi \in \mathfrak{m}^*$ is not $\mathbb{S}$-invariant. However, we can choose a new $\mathbb{S}$-action ($\rho^\#$ in the notation of [GG02, §4]) on $T^*(G/P)$ (no longer conical and not commuting with G) that *does* descend to a conical action on $\mathfrak{X}_P^e$. The grading induced on $\mathbb{C}[\mathfrak{X}_{P,0}^e]$ by the action of $\mathbb{S}$ is called the **Kazhdan grading**; see [GG02, §4] for more details.

(i) To avoid confusion with the group G, let G_{Ham} denote the group of Hamiltonian symplectomorphisms of $\mathfrak{X}_P^e$ that commute with $\mathbb{S}$. Then G_{Ham} is a quotient of the simultaneous centralizer group $C_G(e, h, f)$ of the $\mathfrak{sl}_2 \subset \mathfrak{g}$ spanned by e, h and f. If $P = B$ then G_{Ham} is just $C_G(e, h, f)/Z(G)$, and $\mathbb{W}$ is its Weyl group. However, it can be smaller in general; for instance if $e \in O_P$, then $\mathfrak{X}_P^e = \mathfrak{X}_{P,0}^e$ is a point, and G_{Ham} is trivial.

Let us describe the group G_{Ham} explicitly when $G = \mathrm{GL}_r$. (We use GL_r instead of SL_r for convenience here, but nothing substantial changes.) The centralizer $C_G(e, h, f)$ is the product $\mathrm{GL}_{\gamma_1} \times \cdots \times \mathrm{GL}_{\gamma_r} \subset \mathrm{GL}_r$, where γ_i is the number of Jordan blocks of e

of size i and the factor GL_{γ_i} is the endomorphism group of the sum of these blocks as $\mathfrak{sl}_2$-representations. Suppose that the parabolic P is described by a composition μ, and the Jordan blocks of e are given by a partition $\nu = (\nu_1, \nu_2, \dots)$ of r. Note that $\mathfrak{X}_P^e$ is empty unless $\nu \le \bar{\mu}^t$ in the dominance order on partitions, so we will assume from now on that this is the case.

If the partitions ν and $\bar{\mu}^t$ have a different number of parts, pad the shorter one with zeros so they both have the same length ℓ. Then let J be the set of integers $1 \le j \le \ell$ for which

$$\sum_{i=1}^{j} \nu_i = \sum_{i=1}^{j} (\bar{\mu}^t)_i.$$

(Note that we always have $\ell \in J$.) Each γ_k indexes a maximal block $\nu_{j_k} = \nu_{j_k+1} = \cdots = \nu_{j_{k+1}-1}$ of equal parts of ν. Using the condition $\nu \le \bar{\mu}^t$ it is not hard to see that there are three possibilities for $J \cap [j_k, j_{k+1} - 1]$: either (1) it is empty, or (2) it consists of $j_{k+1} - 1$ only, or (3) it contains all integers $j_k \le j < j_{k+1}$. Then G_{Ham} is the quotient of $C_G(e, h, f) = \mathrm{GL}_{\gamma_1} \times \cdots \times \mathrm{GL}_{\gamma_r} \subset \mathrm{GL}_r$ by the group generated by all factors GL_{γ_k} for which (3) holds for k, together with all diagonal matrices of the form

$$\mathrm{diag}(\lambda I_{\gamma_1}, \lambda I_{\gamma_2}, \dots, \lambda I_{\gamma_k}, I_{\gamma_{k+1}}, \dots)$$

where I_{γ_i} is the identity matrix, $\lambda \in \mathbb{C} \setminus \{0\}$, and k labels a block of type (2) or (3).

Thus, the Weyl group $\mathbb{W}$ in the $\mathrm{SL}_r / \mathrm{GL}_r$ case is the product of S_{γ_k} for all k not of type (3).

For example, if $\nu = (4, 4, 2, 2)$ then $C_G(e, h, f) \cong \mathrm{GL}_2 \times \mathrm{GL}_2$. If $\bar{\mu}^t = (5, 4, 3)$ then we have $G_{\mathrm{Ham}} \cong (\mathrm{GL}_2 \times \mathrm{GL}_2)/\mathbb{C}^*$. If $\bar{\mu}^t = (5, 3, 3, 1)$ then $G_{\mathrm{Ham}} \cong \mathrm{PGL}_2 \times \mathrm{PGL}_2$, and if $\bar{\mu}^t = (5, 3, 2, 2)$ then $G_{\mathrm{Ham}} \cong \mathrm{PGL}_2$.

A maximal torus in G_{Ham} is somewhat easier to describe. For all roots $\alpha \in \mathfrak{g}^* \cong \mathfrak{g}$, let e_α be the orthogonal projection of e onto the weight space $\mathfrak{g}_\alpha$. Then the Lie algebra of a maximal torus of $C_G(e, h, f)$ is

$$\mathfrak{t}^e := \mathfrak{t} \cap \mathfrak{c}_{\mathfrak{g}}(e, h, f) = \{t \in \mathfrak{t} \mid \alpha(t) = 0 \text{ for all } \alpha \text{ such that } e_\alpha \ne 0\}.$$

If $G = \mathrm{GL}_n$, then the Lie algebra of a maximal torus in G_{Ham} is the quotient of $\mathfrak{t}^e$ by the span of all $\mathrm{diag}(I_m, 0)$ where $m = \sum_{i=1}^{j} \nu_i$ for some $j \in J$.

(ii) Since $\mathfrak{m}$ is nilpotent, M is contractible, thus the M-equivariant cohomology of $T^*(G/P)$ coincides with the ordinary cohomology. This allows us to consider the Kirwan map

$$(\mathfrak{t}^*)^{\mathbb{W}_P} \cong H^2(T^*(G/P); \mathbb{C}) \cong H_M^2(T^*(G/P); \mathbb{C}) \to H^2(\mathfrak{X}_P^e; \mathbb{C}), \tag{8}$$

which is always injective if $\mathfrak{X}_P^e$ is non-empty and positive dimensional. In type A, the Kirwan map is also surjective by [BO11, 1.1] (Brundan and Ostrik also give a presentation of the full cohomology ring of $\mathfrak{X}_P^e$ in type A). The kernel of the map (8) is invariant under the action of the Namikawa Weyl group of $T^*(G/P)$ (see part (ii) of Section 9.1), so it induces an action on $H^2(\mathfrak{X}_P^e; \mathbb{C})$, and the Namikawa Weyl group of $\mathfrak{X}_P^e$ is the quotient by the elements which act trivially there.

(iii) Let A_λ be the $\mathbb{S}$-invariant section algebra for the quantization of $\mathfrak{X}_P^e$ whose period is equal to the image of $\lambda \in H^2(G/P; \mathbb{C})$. This algebra has been studied in [Web11], and it is isomorphic to a quotient of the usual W-algebra for the element e by an explicit ideal. When λ is in the image of the map $H^2(G/P; \mathbb{C}) \to H^2(\mathfrak{X}_P^e; \mathbb{C})$, the quantization of $\mathfrak{X}_P^e$ can be obtained from the quantization of $T^*(G/P)$ by quantum Hamiltonian reduction. However, except in type A, this map may not be not surjective, and the quantizations which don't arise this way are more difficult to understand.

(iv) The question of when localization holds has a simple answer for quantizations obtained by Hamiltonian reduction from $T^*(G/P)$. Choose a Borel subgroup B such that $T \subset B \subset P \subset G$. Consider an element $\lambda \in (\mathfrak{t}^*)^{\mathbb{W}_P}$, which includes into $H^2(\mathfrak{X}_P^e; \mathbb{C})$ via Equation 8. Let $\Delta_+(\mathfrak{p}) \subset \mathfrak{t}^*$ be the set of positive roots α such that $\mathfrak{g}_{-\alpha} \not\subset \mathfrak{p}$. The argument of [Gin09, 5.1.2] is easily generalized to show that localization holds at λ whenever $\langle \lambda, \alpha \rangle \notin \mathbb{Z}_{\leq 0}$ for all $\alpha \in \Delta_+(\mathfrak{p})$.

(v) Let $L \subset G$ be a Levi subgroup such that e is regular in $\mathfrak{l} = \mathrm{Lie}(L)$, and let ζ be a cocharacter of T commuting with L. If we choose ζ generically, the sum of its nonnegative weight spaces will be the Lie algebra of a parabolic R with Levi L. For example, if $G = \mathrm{SL}_r$, then the parabolic R is the subgroup of block diagonal matrices for some composition ν of r with the same Jordan type as e. Since the action of $\mathbb{T}$ by ζ fixes e, it descends to $\mathfrak{X}_P^e$.

If $\lambda \in (\mathfrak{t}^*)^{\mathbb{W}_P}$ is dominant and integral, then it is shown in [Web11] that $\mathcal{O}_\mathrm{g} \simeq \mathcal{O}_\mathrm{a}$ is equivalent to the infinitesimal block of parabolic BGG category $\mathcal{O}$ with parabolic $\mathfrak{p}$ and central character ξ, where ξ is a central character of $U(\mathfrak{g})$ corresponding to an integral highest weight whose stabilizer for the ρ-shifted action of W on $\mathfrak{t}^*$ is W_L. It follows that $\mathcal{O}_\mathrm{g}$ is standard Koszul.

(vi) The center of the Yoneda algebra of $\mathcal{O}_\mathrm{g}$ is isomorphic to the center of the Koszul dual of $\mathcal{O}_\mathrm{g}$. Let $\rho_L \in \mathfrak{t}^*$ be the half the sum of the positive roots not in $\mathfrak{l} \subset \mathfrak{g} \cong \mathfrak{g}^*$. If $\lambda + \rho_L$ is integral, then the parabolic-singular duality of Beilinson-Ginzburg-Soergel [BGS96, Bac99] tells us that the Koszul dual of $\mathcal{O}_\mathrm{g}$ is also a singular block of parabolic BGG category $\mathcal{O}$. (In this duality, the roles of the parabolic and the central character

are exchanged. The larger the parabolic on one side, the more singular the character on the other side.)

In type A, the centers of these blocks were first computed by Brundan [Bru08b]; they were shown to be isomorphic to the cohomology of $\mathfrak{X}_P^e$ independently in [BLP$^+$11, 9.9] and [BO11, 1.1]. In [Webb, 3.5], it is shown that the specific map $\gamma \colon H^*(\mathfrak{X}_P^e; \mathbb{C}) \to Z(E)$ of Conjecture 5.23 is an isomorphism.

(vii) The variety $\mathfrak{X}_{P,0}^e$ admits a finite map to the Slodowy slice $\mathcal{S}$ to e in $\mathfrak{g}$. The symplectic leaves of $\mathfrak{X}_{P,0}^e$ are the preimages of the symplectic leaves of $\mathcal{S} \cap \mathrm{nil}(\mathfrak{g})$, which are in turn the intersections of $\mathcal{S}$ with the symplectic leaves of $\mathrm{nil}(\mathfrak{g})$. We conjecture that the same statement is true of special symplectic leaves. The fact that the special symplectic leaves of $\mathcal{S} \cap \mathrm{nil}(\mathfrak{g})$ are the intersections of $\mathcal{S}$ with the special symplectic leaves of $\mathrm{nil}(\mathfrak{g})$ is proven by Losev [Los10, 1.2.2].

If $\mathfrak{g} = \mathfrak{sl}_r$ and the period is integral, then the above conjecture says that all leaves of $\mathfrak{X}_{P,0}^e$ are special. This is true; it is known for $\overline{O}_P$ (see part (vii) of the previous section), and we may obtain the result for $\mathfrak{X}_{P,0}^e$ by applying Losev's operation $(\cdot)_\dagger$ to the relevant primitive ideals in $U(\mathfrak{g})$.

(viii) As in [KR08, 2.8], one can construct a Hamiltonian reduction functor sending modules over a quantization of $T^*(G/P)$ to modules over the corresponding quantization of $\mathfrak{X}_P^e$, and we show that all twisting functors between quantizations of $T^*(G/P)$ descend to twisting functors between the reduced quantizations of $\mathfrak{X}_P^e$. However, if the map $H^2(G/P; \mathbb{C}) \to H^2(\mathfrak{X}_P^e; \mathbb{C})$ is not surjective, not every quantization of $\mathfrak{X}_P^e$ is a Hamiltonian reduction, so we cannot understand all twisting functors for $\mathfrak{X}_P^e$ in terms of those on $T^*(G/P)$. If this map is surjective, then we have

$$E_{\mathrm{tw}} \cong (\mathfrak{t}^*)^{\mathbb{W}_P} \setminus \{\lambda \mid \langle \alpha^\vee, \lambda \rangle = 0 \text{ for some coroot } \alpha \text{ not orthogonal to } (\mathfrak{t}^*)^{\mathbb{W}_P}\}$$

and every twisting functor is obtained by reduction from $T^*(G/P)$. In particular, this holds if $G = \mathrm{SL}_r$.

The parameter space E_{sh} for shuffling functors is the complement of a hyperplane arrangement in the Lie algebra $\mathfrak{t}_{\mathrm{Ham}}$ of a maximal torus in G_{Ham}. As explained in item (i) above, this is a quotient of $\mathfrak{t}^e$. If $G = \mathrm{SL}_r$, the dimension of this quotient is $m - |J|$, where m is the number of Jordan blocks of e. The root hyperplanes $\langle \alpha, t \rangle = 0$ restrict to $\mathfrak{t}^e$, and the hyperplanes in $\mathfrak{t}_{\mathrm{Ham}}$ are exactly the projections of the ones which contain the kernel of the projection $\mathfrak{t}^e \to \mathfrak{t}_{\mathrm{Ham}}$. These are given by the usual equations $\{a_i = a_j\}$, and such a hyperplane will appear if and only if i and j belong to the same "J-block".

The arrangement is thus a product of type A hyperplane arrangements. However, as we have seen, the Weyl group $\mathbb{W}$ which acts on it may not the full permutation group

associated to this arrangement, but instead is the subgroup of elements that permute Jordan blocks of the same size, except that a group of blocks of type (3) is not permuted. If $P = B$, there are no type (3) blocks, and so we have

$$\mathbb{W} \cong S_{\bar{\mu}_1^t - \bar{\mu}_2^t} \times S_{\bar{\mu}_2^t - \bar{\mu}_3^t} \times \cdots \times S_{\bar{\mu}_{\ell-1}^t - \bar{\mu}_\ell^t} \times S_{\bar{\mu}_\ell^t}.$$

Note that this coincides with the Namikawa Weyl group for $T^*(\mathrm{SL}_r / P_\mu)$.

Note that S3-spaces for SL_r are isomorphic to type A quiver varieties. We give a description of shuffling functors for all type A quiver varieties in Section 9.5 below.

9.3 Hypertoric varieties

Let V be a symplectic vector space equipped with a linear symplectic action of a torus K. The Hamiltonian reduction $\mathfrak{M}$ of V by K is called a **hypertoric variety**. If we choose a generic character for K as our GIT parameter, the reduction is an orbifold; it will be smooth if and only if the matrix determined by the inclusion of K into a maximal torus of $\mathrm{Sp}(V)$ is unimodular [BD00, 3.2 & 3.3]. When smooth, it is a conical symplectic resolution of the affine quotient $\mathfrak{M}_0$, where the action of $\mathbb{S}$ is induced by the inverse scalar action on V.

Let

$$V = \bigoplus_{\chi \in \mathfrak{k}_\mathbb{Z}^*} V_\chi$$

be the decomposition of V into weight spaces for K. For simplicity, we assume that $V_0 = 0$. (This assumption is harmless; the variety $\mathfrak{M}$ is isomorphic to $V_0 \times \mathfrak{M}'$, where $\mathfrak{M}'$ is built using the K action on V/V_0.) Choose an element $\xi \in \mathfrak{k}$ that is nonzero on every χ such that $V_\chi \neq 0$, and let

$$\Delta^+ := \{\chi \mid V_\chi \neq 0 \text{ and } (\chi, \xi) > 0\}.$$

We make the additional assumption that, if $\dim V_\chi = 1$, then χ is in the span of $\Delta^+ \smallsetminus \{\chi\}$. This has the effect of ruling out certain redundancies; for example, it rules out the case where $\dim V = 2$ and $K = \mathrm{Sp}(V)$, in which case $\mathfrak{M}$ would be a point. In particular, every hypertoric variety can be constructed using a V and a K that satisfy this condition. It will be a convenient assumption to have for part (ii) below, as well as for our discussion of symplectic duality in the next section.

(i) The group G of Hamiltonian symplectomorphisms of $\mathfrak{M}$ commuting with $\mathbb{S}$ is isomorphic to

$$\left(\prod_{\chi \in \Delta^+} \mathrm{GL}(V_\chi) \right) \Big/ K.$$

77

Its Weyl group is a product of symmetric groups:

$$\mathbb{W} \cong \prod_{\chi \in \Delta^+} S_{\dim V_\chi}.$$

If we refine the decomposition $\oplus_{\chi \in \Delta^+} V_\chi$ to a decomposition into lines, then the group $\tilde{T} \cong (\mathbb{C}^\times)^{\frac{1}{2}\dim V}$ of automorphisms of this decomposition descends to a maximal torus $T := \tilde{T}/K$ of G. The natural basis for the cocharacter lattice of $\tilde{T}$ descends to a finite multiset of cocharacters of T, which in turn define a weighted rational central multiarrangement $\mathcal{A}$ of hyperplanes in $\mathfrak{t}^*$. This hyperplane arrangement together with a character of K gives the more usual combinatorial input data for constructing $\mathfrak{M}$.

(ii) The cohomology ring of $\mathfrak{M}$ was computed independently in [Kon00] and [HS02]. In degree 2, the Kirwan map

$$\mathfrak{k}^* \cong H_K^2(V; \mathbb{C}) \to H^2(\mathfrak{M}; \mathbb{C})$$

is an isomorphism. (Surjectivity was proven by Konno, and injectivity is equivalent to our second assumption above.)

The Namikawa Weyl group is also isomorphic to a product of symmetric groups:

$$W \cong \prod_F S_{|F|},$$

where the product ranges over all rank 1 flats of $\mathcal{A}$. (One may regard the set of rank 1 flats as the set underlying the multiset $\mathcal{A}$; for an element F of this set, $|F|$ is its multiplicity in $\mathcal{A}$. Thus, if $\mathcal{A}$ contains r copies of the same hyperplane, we get a factor of S_r in W.) This group acts naturally on $\tilde{\mathfrak{t}}^* \cong \prod_F \mathbb{C}^F$ by permuting each summand. It fixes $\mathfrak{t}^*$, and thus descends to an action on $\mathfrak{k}^*$ via the exact sequence

$$0 \to \mathfrak{t}^* \to \tilde{\mathfrak{t}}^* \to \mathfrak{k}^* \to 0.$$

(iii) Let $\mathbb{D}$ be the Weyl algebra of the symplectic vector space V. The invariant algebra $\mathbb{D}^K$ is called the **hypertoric enveloping algebra** in [BLPW12]; it was originally studied by Musson and Van den Bergh [MV98]. Its center is isomorphic to $\mathrm{Sym}\,\mathfrak{k} \cong \mathbb{C}[\mathfrak{k}^*]$, and the $\mathbb{S}$-invariant section ring of the quantization of $\mathfrak{M}$ with period $\lambda \in H^2(\mathfrak{M}; \mathbb{C}) \cong \mathfrak{k}^*$ is isomorphic to the corresponding central quotient of $\mathbb{D}^K$ [BLPW12, 5.9].

(iv) A sufficient condition for localization to hold is given by Bellamy and Kuwabara [BK12, 5.8]. Using their results, we give a different combinatorial condition in [BLPW12, 6.1] that amounts to checking that certain rational polyhedra contain lattice points.

(v) It is shown in [BLPW12] that the arrangement $\mathcal{H}_{\mathrm{tw}}$ is the discriminantal arrangement of $\mathcal{A}$. For any period that is sufficiently far away from the walls of $\mathcal{H}_{\mathrm{tw}}$ (the word for this in [BLPW12] is **regular**), the category $\mathcal{O}_{\mathrm{a}}$ is equivalent to the module categories over a finite dimensional algebra introduced in [BLPW10]. In particular, it is standard Koszul [BLPW10, 5.24].

There is a unique notion of integrality for periods which satisfies the conditions of Section 2.4; it is the same as the definition of integrality in [BLPW10]. The precise recipe given there to associate an algebra to a particular quantization is straightforward if the period λ is integral, but in general it is somewhat tricky. Details are given in [BLPW12, 4.9]. The category $\mathcal{O}_{\mathrm{g}}$ is always standard Koszul, because we can always twist by a line bundle to get to a regular period where localization holds.

(vi) When λ is regular and integral, the fact that $H^*(\mathfrak{M}; \mathbb{C})$ is isomorphic to $Z(E)$ is proven in [BLPW12, 5.3], but we still need to show that the homomorphism of Conjecture 5.23 is an isomorphism.

For every $\alpha \in \mathcal{I}$, the simple module Λ_α in $\mathcal{O}_{\mathrm{g}}$ is a quantization of the structure sheaf of the relative core component $X_\alpha \subset \mathfrak{M}^+$. Computing the Ext-algebra of Λ_α using a Čech spectral sequence, we see that there is an isomorphism $H^*(X_\alpha; \mathbb{C}) \cong \mathrm{Ext}^\bullet(\Lambda_\alpha, \Lambda_\alpha)$ making the diagram

$$
\begin{array}{ccc}
H^*(\mathfrak{M}) & \longrightarrow & H^*(X_\alpha; \mathbb{C}) \\
\downarrow & & \downarrow \\
HH^*(\mathcal{D}) & \longrightarrow & \mathrm{Ext}^\bullet(\Lambda_\alpha, \Lambda_\alpha)
\end{array}
$$

commute. Thus, the kernel of the map $H^*(\mathfrak{M}; \mathbb{C}) \to Z(E)$ is contained in the intersection of the kernels of the maps $H^*(\mathfrak{M}; \mathbb{C}) \to H^*(X_\alpha; \mathbb{C})$ for all α. It is shown in [HS02, (34)] that this intersection is trivial, so the kernel of the $H^*(\mathfrak{M}; \mathbb{C}) \to Z(E)$ is trivial. Since the target and source have the same dimension, it must be an isomorphism.

(vii) The poset $\mathscr{S}$ of symplectic leaves of $\mathfrak{M}_0$ is isomorphic to the poset of coloop-free flats of $\mathcal{A}$ [PW07, 2.3]. If the period of the quantization is integral, then all leaves are special. This follows from the reformulation [BLPW12, 7.4] of work of Musson and Van den Bergh [MV98]. For non-integral weights, only some leaves remain special; which ones remain can be deduced the description of primitive ideals in A given in [MV98].

(viii) Twisting and shuffling functors for hypertoric varieties were studied in detail in [BLPW12, §8]. In particular, Conjecture 8.24 is true [BLPW12, 8.19].

9.4 Hilbert schemes on ALE spaces

For any finite subgroup $\Gamma \subset \mathrm{SL}_2$, consider the associated DuVal singularity $\mathbb{C}^2/\Gamma$, along with its unique crepant resolution $\widetilde{\mathbb{C}^2/\Gamma}$. This is a conical symplectic resolution with respect to the $\mathbb{S}$-action induced by inverse scalar multiplication on $\mathbb{C}^2$. More generally, for any $r \in \mathbb{N}$, the Hilbert scheme $\mathsf{Hilb}_r(\widetilde{\mathbb{C}^2/\Gamma})$ is a conical symplectic resolution of $\mathrm{Sym}^r(\mathbb{C}^2/\Gamma) \cong \mathbb{C}^{2r}/(\Gamma \wr S_r)$ [Wan00, Cor. 4].

(i) The group G of Hamiltonian symplectomorphisms of $\mathsf{Hilb}_r(\widetilde{\mathbb{C}^2/\Gamma})$ commuting with $\mathbb{S}$ is simply the group of linear symplectomorphisms of $\mathbb{C}^{2r}$ commuting with the action of $\Gamma \wr S_r$. If $\Gamma = \mathbb{Z}/\ell\mathbb{Z}$, then $G \cong \mathbb{C}^*$, and if Γ is of type D or E, then G is trivial. In particular, we can find a Hamiltonian $\mathbb{T}$-action with isolated fixed points if and only if $\Gamma = \mathbb{Z}/\ell\mathbb{Z}$. The Weyl group $\mathbb{W}$ is always trivial.

(ii) The Namikawa Weyl group W is $W_{\mathfrak{G}} \times \mathbb{Z}/2\mathbb{Z}$, where $W_{\mathfrak{G}}$ is the Weyl group of $\mathfrak{G}$ if $r > 1$. The codimension two stratum corresponding to the factor $W_{\mathfrak{G}}$ is the set of points where 0 lies in the support of the ideal, and the stratum corresponding to the factor $\mathbb{Z}/2\mathbb{Z}$ is given by points where the ideal has a point of multiplicity two. It follows from work of Nakajima [Nak99] that for $r > 1$, we have $H^2(\mathsf{Hilb}_r(\widetilde{\mathbb{C}^2/\Gamma}); \mathbb{Z}) \cong H^2(\widetilde{\mathbb{C}^2/\Gamma}; \mathbb{Z}) \oplus \mathbb{Z}\delta$. The action of the Namikawa Weyl group on this space is via the action of $W_{\mathfrak{G}}$ on $H^2(\widetilde{\mathbb{C}^2/\Gamma}; \mathbb{Z})$ and $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}\delta$ by negation. Thus, $H^2(\widetilde{\mathbb{C}^2/\Gamma}; \mathbb{Z})$ is isomorphic as a $\mathbb{Z}W$-module to the root lattice of the finite dimensional simple Lie algebra $\mathfrak{G}$ associated to Γ via the McKay correspondence, and $H^2(\mathsf{Hilb}_r(\widetilde{\mathbb{C}^2/\Gamma}); \mathbb{Z})$ to the root lattice of its affinization.

(iii) The algebra A is isomorphic to a spherical symplectic reflection algebra for the representation of $\Gamma \wr S_r$ on $\mathbb{C}^{2r}$ [EGGO07, Gor06].

(iv) Which periods localization holds for is still not completely understood. For the case $\Gamma = \{1\}$, the answer is quite simple: localization holds at all parameters not of the form $-1/2 - m/k$ for $m \leq 0$, $1 < k \leq r$ and $(m, k) = 1$. For general Γ, this is a much more complex question, though some progress has been made in work of McGerty-Nevins [MN] and Jenkins [Jen13, §6-7].

We only have a category $\mathcal{O}$ when $\Gamma = \mathbb{Z}/\ell\mathbb{Z}$, since in the other cases, there is only the trivial $\mathbb{T}$-action. From now on, we will only consider this case.

(v) The category $\mathcal{O}_a$ is closely related to the category $\mathcal{O}$ defined by [GGOR03] for the Cherednik algebra of the complex reflection group $\mathbb{Z}/\ell\mathbb{Z} \wr S_r$. The category $\mathcal{O}_a$ is the image of the GGOR category $\mathcal{O}$ under the functor $M \mapsto eM$, where e is the spherical idempotent in the full symplectic reflection algebra. If the period lies in the set $\mathfrak{U}$ (these periods are called **spherical**), then this functor is an equivalence.

The Koszulity of $\mathcal{O}_{\mathrm{a}}$ at spherical integral parameters and thus of $\mathcal{O}_{\mathrm{g}}$ for arbitrary integral parameters is proven by Chuang and Miyachi [CM]. This was extended to all other choices of spherical parameters by Rouquier, Shan, Vasserot, and Varagnolo [RSVV, SVV14].

(vi) It is shown in [Webb, 3.5] that the map $H^*(\mathfrak{M}; \mathbb{C}) \to Z(E)$ is an isomorphism.

(vii) The special leaves for $\Gamma = \{1\}$ are described by Losev [Los12a, 5.8.1]. For $\ell > 1$, the affinization of $\mathsf{Hilb}^r(\widetilde{\mathbb{C}^2/\Gamma})$ is $\mathrm{Sym}^r(\mathbb{C}^2/\Gamma)$. The leaves of this variety are in bijection with partitions ν of integers $r' \leq r$, where the parts of the partition are the multiplicities of the points in $(\mathbb{C}^2 \setminus \{0\})/\Gamma$ that occur (and thus $\{0\}$ necessarily has multiplicity $r - r'$). It follows from work of Shan and Vasserot that the special leaves in the integral case are those where the partition is $\nu = (1^{r'})$; more generally, when $k = {}^m/e$ with $(m, e) = 1$, it follows that the special leaves are those where all parts of ν are e or 1 and $e \mid (r - r')$. Note that this is neither a subset nor a superset of the special leaves in the integral case.

(viii) Since $G = \mathbb{C}^\times$, we have $E_{\mathrm{sh}} \cong \mathbb{C}^\times$. Thus shuffling functors are expected to give an action of $\mathbb{Z} \cong \pi_1(E_{\mathrm{sh}})$, which we expect agrees with the powers of the Serre functor. Twisting functors are more interesting; even though the space $H^2(\mathsf{Hilb}^r(\widetilde{\mathbb{C}^2/\Gamma}); \mathbb{R})$ is independent of r, the hyperplane arrangement $\mathcal{H}_{\mathrm{tw}}$ is not. The hyperplanes are described by Gordon [Gor08, §4.3]; the extra hyperplanes for $r > 1$ reflect the fact that $\mathsf{Hilb}^r(\widetilde{\mathbb{C}^2/\Gamma})$ is not the only conical symplectic resolution of $\mathrm{Sym}^r(\mathbb{C}^2/\Gamma)$. If $\ell = 1$, then the twisting functors simply consist of the action of $\mathbb{Z}$ by powers of the Serre functor. If $\ell > 1$, we defer to the next section on quiver varieties.

9.5 Quiver varieties

Let Q be a finite quiver without oriented cycles. Let V be the set of vertices of Q, and let $\mathbf{w}, \mathbf{v} \in \mathbb{N}^V$ be dimension vectors. Whenever we have two weights $\mu \leq \nu$ for the Kac-Moody algebra $\mathfrak{g}_Q$ associated to Q with ν dominant and $\mu \leq \nu$ in the usual root order, there are associated dimension vectors $\mathbf{w}$ and $\mathbf{v}$ given by $\mathbf{w}_i := \alpha_i^\vee(\nu)$ and $\sum \mathbf{v}_i \alpha_i := \nu - \mu$. Our assumptions assure that these numbers are in $\mathbb{N}$.

The quiver variety $\tilde{\mathfrak{Q}}_\mu^\nu$ is a smooth open subvariety of the cotangent bundle to the moduli stack of framed representations of Q, where $\mathbf{w}$ is the dimension of the framing and $\mathbf{v}$ is the dimension of the representation [Nak94, Nak98]; its affinization is denoted $\mathfrak{Q}_\mu^\nu$. Like hypertoric varieties, a quiver variety may be described as a Hamiltonian reduction of a symplectic vector space by the group

$$\mathrm{GL}_{\mathbf{v}} \quad := \quad \prod_{i \in V} \mathrm{GL}_{\mathbf{v}_i}.$$

The smooth variety $\tilde{\mathfrak{Q}}_\mu^\nu$ is obtained by using a nontrivial GIT parameter specified by Nakajima, while the affine variety $\mathfrak{Q}_\mu^\nu$ is obtained as the affine quotient.

There are various actions of $\mathbb{S}$ that we could choose with respect to which $\tilde{\mathfrak{Q}}_\mu^\nu$ is a conical projective resolution of $\mathfrak{Q}_\mu^\nu$. For example, we could mimic the choice that we made for hypertoric varieties and take the $\mathbb{S}$-action induced by the inverse scaling action on the vector space; this has $n = 2$. Alternatively, the orientation of Q determines a Lagrangian subspace of our symplectic vector space, and the $\mathbb{S}$-action such that this subspace has weight -1 and its complement has weight 0 induces an $\mathbb{S}$-action on $\tilde{\mathfrak{Q}}_\mu^\nu$ with $n = 1$. The fact that $\tilde{\mathfrak{Q}}_\mu^\nu$ is conical with respect to this action follows from the assumption that Q has no oriented cycles. This is the $\mathbb{S}$-action that we use below.

Remark 9.3 The class of type A S3-varieties coincides with the class of quiver varieties for which Q is a type A Dynkin diagram with some choice of orientation [Maf05]. However, the $\mathbb{S}$-action that we used in Section 9.2 had $n = 2$, whereas here we are using an action with $n = 1$.

Remark 9.4 If Q is an affine Dynkin graph with some choice of orientation and ν is the highest weight of the basic representation of $\mathfrak{g}_Q$, then affine quiver variety $\mathfrak{Q}_\mu^\nu$ is isomorphic to $\mathrm{Sym}^r(\mathbb{C}^2/\Gamma)$, where $\Gamma \subset \mathrm{SL}_2$ is the finite subgroup corresponding to Q under the McKay correspondence. Thus, the class of varieties discussed in Section 9.4 is a subset of the class of affine type quiver varieties. Once again, we used an $\mathbb{S}$-action with $n = 2$ in that section, whereas here we are using one with $n = 1$.

(i) Consider the group

$$G_\mathbf{w} \quad := \quad \prod_{i \in V} \mathrm{GL}_{\mathbf{w}_i} \quad \times \quad \prod_{(i,j) \in V \times V} \mathrm{GL}_{n_{ij}},$$

where n_{ij} is the number of edges from i to j. In the Hamiltonian reduction construction described above, this is precisely the group of those automorphisms of the orientation-determined Lagrangian subspace that commute with the action of $\mathrm{GL}_\mathbf{v}$. The group G of Hamiltonian symplectomorphisms of $\tilde{\mathfrak{Q}}_\mu^\nu$ commuting with $\mathbb{S}$ is isomorphic to the quotient of $G_\mathbf{w}$ by its center:

$$G \cong G_\mathbf{w}/Z(G_\mathbf{w}) \cong G_\mathbf{w}/(\mathbb{C}^\times)^V.$$

The group $(\mathbb{C}^\times)^V$ embeds into $G_\mathbf{w}$ using the coboundary formula, with $(z_i)_{i \in V}$ landing on z_i times the identity matrix in the factor $\mathrm{GL}_{\mathbf{w}_i}$ and $z_i z_j^{-1}$ times the identity matrix in the factor $\mathrm{GL}_{n_{ij}}$.

Note that in the special case where Q is a tree with some choice of orientation, we have

$$G \cong \mathrm{PGL}_\mathbf{w} := \left(\prod \mathrm{GL}_{\mathbf{w}_i} \right) \Big/ \mathbb{C}^\times.$$

Another special case that will be of interest to us is where Q is an r-cycle with some

82

choice of orientation, in which case

$$G \cong \left(\mathrm{PGL}_{\mathbf{w}} \times \mathbb{C}^{\times} \right) / (\mathbb{Z}/r\mathbb{Z}).$$

In both of these two special cases, the Weyl group is isomorphic to the same product of symmetric groups:

$$\mathbb{W} \cong \prod_{i \in V} S_{\mathbf{w}_i}.$$

(ii) The cohomology ring of a quiver variety is poorly understood; in particular, surjectivity of the Kirwan map

$$H^*_{\mathrm{GL}_{\mathbf{v}}}(pt; \mathbb{C}) \to H^*(\tilde{\mathfrak{Q}}^{\nu}_{\mu}; \mathbb{C})$$

is an important and long-standing conjecture. The rank of the kernel of the Kirwan map in degree 2 is equal to the codimension in $\mathbb{R}^V$ of the affine span of the face of the weight polytope of the representation V_ν which contains μ. In particular, for μ in the interior of the weight polytope, the Kirwan map is injective in degree 2.

For a fixed ν, μ, consider the set S of simple roots α_i with $\mu + \alpha_i \leq \nu$ and $\alpha_i^{\vee}(\mu) = 0$. In finite type, the Namikawa Weyl group W for $\mathfrak{Q}^{\nu}_{\mu}$ is the subgroup of the Weyl group of the Kac-Moody algebra $\mathfrak{g}_Q$ generated by α_i in S. We can show this by noting that in finite type, [Nak98, 3.27] shows that codimension 2 strata of $\mathfrak{Q}^{\nu}_{\mu}$ are in bijection with the connected components of the Dynkin subdiagram with vertices given by S: the strata correspond to the weights $\mu + \alpha_D$ for α_D the highest root of a connected component. Each such stratum contributes a copy of the Weyl group of the subdiagram.

In infinite type, this group sits inside the Namikawa Weyl group, but it may be a proper subgroup, as the case of $\mathrm{Sym}^r(\mathbb{C}^2/\Gamma)$ shows.

(iii) Except in special cases in which quiver varieties coincide with other known classes of varieties, such as S3-varieties, hypertoric varieties, or Hilbert schemes on ALE spaces, the $\mathbb{S}$-invariant section algebra A has not been studied.

(iv) While the question of when localization holds is interesting, we know of no progress outside the cases of finite and affine type A quivers (discussed elsewhere in this paper), other than the general results of [BPW, MN14, MN].

(v) The categories $\mathcal{O}_{\mathrm{a}}$ and $\mathcal{O}_{\mathrm{g}}$ are studied by the fourth author in [Webd], with relatively explicit descriptions in the finite and affine cases using steadied quotients of weighted KLR algebras. In finite or affine type A, the resulting category $\mathcal{O}_{\mathrm{g}}$ is Koszul at integral parameters. In the finite case, this follows from coincidence with blocks of parabolic category $\mathcal{O}$ for $\mathfrak{sl}_m$; in the affine case, this is shown in [Webd] based on the Koszul duality results in [SVV14].

(vi) It is not clear whether the map $H^*(\tilde{\mathfrak{Q}}^\nu_\mu; \mathbb{C}) \to Z(E)$ is injective or surjective even for integral periods; resolving this question is closely tied to the question of Kirwan surjectivity for quiver varieties.

(vii) If Q is a finite type ADE Dynkin diagram with some choice of orientation, then the symplectic leaves of $\mathfrak{Q}^\nu_\mu$ are in bijection with dominant weights ν' such that $\nu' \leq \nu$ and $\nu' \geq w \cdot \mu$ for all w in the Weyl group of $\mathfrak{g}_Q$. In the integral case, all leaves are special [Webd, 5.4]. In affine type, the poset of leaves becomes more complicated and there are non-special leaves; the poset of special leaves for an integral period in affine type A is described in [Webd, 5.10].

(viii) Assume that Kirwan surjectivity holds in degree 2. The twisting functors for a quiver variety give an action of a subgroup of the Artin braid group of the corresponding root system; we can obtain an action of the whole braid group if we allow functors between different quantizations. These functors can also be constructed from the categorified quantum group which acts on these categories by the main theorem of [Weba]: the twisting action is given by the Chuang-Rouquier braid complexes, as shown by Bezrukavnikov and Losev [BL].

In the finite and affine cases, shuffling functors also have an algebraic description, given in [Webd]. In cases other than affine type A, they correspond to braiding functors from [Webc], while in affine type A they correspond to change-of-charge functors from [Webe].

9.6   Affine Grassmannian slices

Let G be a semi-simple algebraic group G over $\mathbb{C}$, and let $G((t))$ be the group of $\mathbb{C}((t))$-points of G. This has a "complementary" pair of subgroups, $G[[t]]$ and $G_1[t^{-1}]$, where $G_1[t^{-1}]$ is equal to the kernel of the evaluation map $G[t^{-1}] \to G$. There is a natural Poisson structure on the affine Grassmannian $\mathrm{Gr} := G((t))/G[[t]]$ whose Poisson leaves are the intersections of the orbits of these two subgroups.

For any cocharacter $\lambda \colon \mathbb{G}_m \to G$, we obtain a point $t^\lambda \in G((t))$, which descends to an element $[t^\lambda] \in \mathrm{Gr}$. For any pair of dominant coweights λ and μ, we can consider the intersection[19]

$$\mathrm{Gr}^{\bar\lambda}_\mu \; := \; \overline{G[[t]] \cdot [t^\lambda]} \cap G_1[t^{-1}] \cdot [t^\mu].$$

This is a transverse slice to the orbit $G[[t]] \cdot [t^\mu]$ in the closure $\overline{G[[t]] \cdot [t^\lambda]}$. It is a conical symplectic singularity with respect to the $\mathbb{S}$-action by loop rotation [KWWY14, 2.7]. It may or may not admit a conical symplectic resolution; a necessary and sufficient criterion is given in [KWWY14, 2.9]. In type A, such a resolution always exists.

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[19]Following the notational convention in [KWWY14], the $\bar\lambda$ on the left-hand side reflects the fact that we have taken the closure of $G[[t]] \cdot [t^\lambda]$ on the right-hand side.

(i) The group of Hamiltonian symplectomorphisms commuting with $\mathbb{S}$ is the simultaneous centralizer of t^λ and t^μ in G, which is typically a torus.

(iii) The question of how to quantize this variety has been considered by the fourth author jointly with Kamnitzer, Weekes and Yacobi; there is a conjectural identification of the quantizations of this symplectic variety with a quotient of a shifted Yangian [KWWY14, 4.8].

(iv-viii) At the moment, these questions have not been addressed. No serious study of the categories $\mathcal{O}_a$ and $\mathcal{O}_g$ has been done, aside from Brundan and Kleshchev's work in type A [BK06]. In the type A case, the varieties and their resolutions coincide with type A quiver varieties or type A S3-varieties, by work of Maffei [Maf05] and Mirković-Vybornov [MV], so the results of previous sections can be applied.

10   Symplectic duality

In this section we describe a close relationship between the categories associated to certain pairs of symplectic varieties. In a number of special cases, we expect this relationship to provide connections between previously studied geometric and categorical constructions, including two superficially different sets of link invariants [MS08, SS06a] (see Section 10.7).

Our relationship is defined at the categorical level, but it has two very concrete cohomological consequences. The first (Section 10.5) arises by passing to Grothendieck groups; we obtain a duality of vector spaces that explains previously known numerical identities in the combinatorics of matroids and illuminates the phenomena of Schur-Weyl duality and level-rank duality in representation theory. The second (Section 10.6) arises by considering the centers of the universal deformations of the Yoneda algebras of our categories. The relationship that we see was originally observed in certain special cases by Goresky and MacPherson [GM10]; by regarding this relationship as a shadow of symplectic duality, we generate new classes of examples and provide an explanation for the examples observed in [GM10].

Throughout this section, we will assume that every conical symplectic resolution comes equipped with a set of "integral periods" in $H^2(\mathfrak{M}; \mathbb{C})$, consistent with the three conditions in Section 2.4. We will always work with a quantization for which localization holds, so that we need not distinguish between $\mathcal{O}_a$ and $\mathcal{O}_g$. We will assume that Conjecture 5.18 holds, so that our category $\mathcal{O}$ is standard Koszul (and therefore Koszul by Theorem 4.14). In particular, this means that $\mathcal{O}$ comes equipped with a graded lift $\tilde{\mathcal{O}}$, that is, a mixed category whose degrading is $\mathcal{O}$ (see Section 4.1). Finally, we will assume that the twisting and shuffling actions lift naturally to $D^b(\tilde{\mathcal{O}})$; this is the case in all of the examples from Section 9 where a Koszul grading is known to exist.

10.1 The definition

Consider a conical symplectic resolution $\mathfrak{M}$, equipped with a Hamiltonian action of $\mathbb{T}$, commuting with $\mathbb{S}$, such that $\mathfrak{M}^{\mathbb{T}}$ is finite. We denote by $\mathcal{O}$ the category $\mathcal{O}_{\mathrm{a}} \simeq \mathcal{O}_{\mathrm{g}}$ for an integral period at which localization holds. The fact that we do not need to specify the period of the quantization follows from Lemma 3.17, which says that the categories $\mathcal{O}_{\mathrm{g}}$ associated to any two integral parameters are canonically equivalent. We will write $\pi_1(E_{\mathrm{tw}}/W)$ to denote $\pi_1(E_{\mathrm{tw}}/W, [\lambda])$ for any integral λ sufficiently deep in the ample cone of $\mathfrak{M}$.

Let G be the group of Hamiltonian symplectomorphisms of $\mathfrak{M}$ that commute with $\mathbb{S}$, and let $T \subset G$ be a maximal torus containing the image of $\mathbb{T}$. By Lemma 8.6, this is unique if G is reductive, and more generally unique up to conjugation by the unipotent radical of $C_G(\mathbb{T})$. Let $\zeta \in \mathfrak{t}_{\mathbb{R}}$ be the cocharacter of T by which $\mathbb{T}$ acts, and let C be the chamber of $\mathcal{H}_{\mathrm{tw}}$ containing ζ. By Lemma 8.11, we could replace ζ (and with it the action of $\mathbb{T}$) by any other element of C without changing $\mathcal{O}$. We will write $\pi_1(E_{\mathrm{sh}}/\mathbb{W})$ to denote $\pi_1(E_{\mathrm{sh}}/\mathbb{W}, [\zeta])$ for any $\zeta \in C$.

Let $\mathfrak{M}^!$ be another conical symplectic resolution on which $\mathbb{T}$ acts with isolated fixed points, commuting with $\mathbb{S}$. We denote all of the corresponding structures related to $\mathfrak{M}^!$ with an upper shriek; for example, the fixed points of $\mathfrak{M}^!$ will be indexed by the set $\mathcal{I}^!$, the group $\pi_1(E_{\mathrm{sh}}^!/\mathbb{W}^!)$ will act on $D^b(\tilde{\mathcal{O}}^!)$, and so on.

Definition 10.1 A **symplectic duality** from $\mathfrak{M}$ to $\mathfrak{M}^!$ consists of

- a bijection $\alpha \mapsto \alpha^!$ from $\mathcal{I}$ to $\mathcal{I}^!$ which is order-reversing for the geometric order $\hookleftarrow$ defined in Section 6.2,

- a bijection $S \mapsto S^!$ from $\mathscr{S}_{\mathfrak{M}}^{\mathrm{sp}}$ to $\mathscr{S}_{\mathfrak{M}^!}^{\mathrm{sp}}$ which is order-reversing for the closure order,

- group isomorphisms $W \cong \mathbb{W}^!$ and $\mathbb{W} \cong W^!$,

- a pair of linear isomorphisms $\mathfrak{t}_{\mathbb{R}} \cong H^2(\mathfrak{M}^!; \mathbb{R})$ and $H^2(\mathfrak{M}; \mathbb{R}) \cong \mathfrak{t}_{\mathbb{R}}^!$, which identify the lattice of cocharacters with the lattice of integer homology classes, and

- a Koszul duality from $\tilde{\mathcal{O}}$ to $\tilde{\mathcal{O}}^!$ (Definition 4.8).

These structures are required to satisfy the following conditions:

- The bijection of fixed points is compatible with the bijection of special leaves via the operation that associates a special leaf $\mathring{\mathfrak{M}}_{\alpha,0} \in \mathscr{S}^{\mathrm{sp}}$ to an element $\alpha \in \mathcal{I}$ (see Corollary 6.12 and the preceding discussion). That is, for any $\alpha \in \mathcal{I}$, we require that

$$(\mathring{\mathfrak{M}}_{\alpha,0})^! = \mathring{\mathfrak{M}}_{\alpha^!,0}^! .$$

- The isomorphism $\mathfrak{t}_{\mathbb{R}} \cong H^2(\mathfrak{M}^!; \mathbb{R})$, intertwines the action of $\mathbb{W}$ with that of $W^!$, takes the arrangement $\mathcal{H}_{\mathrm{sh}}$ to $\mathcal{H}_{\mathrm{tw}}^!$, and takes the chamber $C \subset \mathfrak{t}_{\mathbb{R}}$ to the ample cone in $H^2(\mathfrak{M}^!; \mathbb{R})$.

Furthermore, all of the analogous statements hold for the isomorphism $H^2(\mathfrak{M}; \mathbb{R}) \cong \mathfrak{t}_{\mathbb{R}}^!$. In particular, this means that we have canonical isomorphisms

$$\pi_1(E_{\mathrm{sh}}/\mathbb{W}) \cong \pi_1(E_{\mathrm{tw}}^!/W^!) \qquad \text{and} \qquad \pi_1(E_{\mathrm{tw}}/W) \cong \pi_1(E_{\mathrm{sh}}^!/\mathbb{W}^!).$$

- The Koszul duality from $\tilde{\mathcal{O}}$ to $\tilde{\mathcal{O}}^!$ exchanges twisting functor $\Phi^{*,*}$ (as defined in Section 8.1) and shuffling functor $\Xi^{*,*}$ (as defined in Section 8.2) and similarly with $\tilde{\mathcal{O}}$ and $\tilde{\mathcal{O}}^!$ reversed. That is, the equivalence $D^b(\tilde{\mathcal{O}}) \to D^b(\tilde{\mathcal{O}}^!)$ takes the shuffling action of $\pi_1(E_{\mathrm{sh}}/\mathbb{W})$ on $D^b(\tilde{\mathcal{O}})$ to the twisting action of $\pi_1(E_{\mathrm{tw}}^!/W^!)$ on $D^b(\tilde{\mathcal{O}}^!)$, and vice versa.

Remark 10.2 Symplectic duality is symmetric; that is, if there is a symplectic duality from $\mathfrak{M}$ to $\mathfrak{M}^!$, then there is a symplectic duality from $\mathfrak{M}^!$ to $\mathfrak{M}$. To see this, we invoke Proposition 4.9 and Remark 4.10, which say that if $\Psi : D^b(\tilde{\mathcal{O}}) \to D^b(\tilde{\mathcal{O}}^!)$ is a Koszul duality from $\tilde{\mathcal{O}}$ to $\tilde{\mathcal{O}}^!$, then the composition of Ψ^{-1} with the derived Nakayama functor $\mathbb{R}\mathcal{N}$ on $D^b(\tilde{\mathcal{O}})$ is a Koszul duality from $\mathcal{O}^!$ to $\mathcal{O}$.

We still need to check that $\Psi^{-1} \circ \mathbb{R}\mathcal{N}$ exchanges twisting and shuffling functors. Since $\tilde{\mathcal{O}}$ has finite global dimension, $\mathbb{R}\mathcal{N}$ is a right Serre functor [MS08]; by the uniqueness of Serre functors, it commutes with any equivalence of derived categories, in particular with any twisting or shuffling functor. Thus, since Ψ exchanges twisting and shuffling functors, so does $\Psi^{-1} \circ \mathbb{R}\mathcal{N}$.

10.2 Examples of symplectic dualities

In this section we describe all of the examples of pairs of conical symplectic resolutions that we know to be dual, along with some conjectural generalizations of these examples.

10.2.1 Cotangent bundles of flag varieties

Theorem 10.3 *Let G be a reductive algebraic group with Langlands dual ^LG, and let $B \subset G$ and $^LB \subset {}^LG$ be Borel subgroups. Then $T^*(G/B)$ is symplectic dual to $T^*({}^LG/{}^LB)$.*

Proof: For a generic cocharacter ζ of G, the fixed points of $T^*(G/B)$ are indexed by the Weyl group $W \cong W^!$, and the order-reversing bijection of W is given by sending w to $w^{-1}w_0$. The fact that this bijection induces an order-reversing bijection of special nilpotent orbits is proven in [KL79, 3.3]. The W-equivariant linear isomorphisms are part of the package of Langlands duality. The Koszul duality is proven in [BGS96, 1.1.3][20], and the fact that twisting and shuffling are exchanged is proven in [MOS09, Theorem 39]. □

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[20]This paper actually proves that a regular integral block of BGG category $\mathcal{O}$ is self-dual, but those categories are isomorphic for Langlands dual groups, since they can be computed in terms of the Weyl group [Soe90].

10.2.2 S3-varieties

Next, we consider S3-varieties associated to SL_r, as described in Section 9.2. For a composition μ of r, we define a new composition μ^o by $\mu_i^o := \mu_{-i}$. Also, recall that $\bar{\mu}$ denotes the partition of r obtained by sorting the positive entries of μ, and $\bar{\mu}^t$ denotes the transposed partition. Note that $\bar{\mu} = \overline{\mu^o}$.

Fix a pair of compositions μ and ν of r. Let $e \colon \mathbb{C}^r \to \mathbb{C}^r$ be a nilpotent element in Jordan normal form with block sizes given by ν in order. Let $\mathfrak{X}_\mu^\nu$ be the S3-variety $\mathfrak{X}_{P_\mu}^e$ that was introduced in Section 9.2; it is nonempty if and only if $\bar{\nu} \leq \bar{\mu}^t$ in the dominance order.

Let T_ν^μ be a maximal torus of the group G_{Ham} for the variety $\mathfrak{X}_\mu^\nu$, as described in item (i) of Section 9.2. The description of the cohomology of these varieties in [BO11] gives a natural isomorphism $\operatorname{Lie} T_\nu^\mu \cong H^2(\mathfrak{X}_{\mu^o}^\nu; \mathbb{C})$. Let C_ν^μ be the unique chamber of the arrangement $\mathcal{H}_{\mathrm{sh}}$ for $\mathfrak{X}_\nu^\mu$ which lies on the positive side of every root hyperplane which appears. The following theorem appears in [Webd, 5.32].

Theorem 10.4 *The variety $\mathfrak{X}_\nu^\mu$ is symplectic dual to $\mathfrak{X}_\mu^\nu$, where the action of $\mathbb{T}$ on $\mathfrak{X}_\nu^\mu$ is given by a cocharacter in C_ν^μ and the action of $\mathbb{T}$ on $\mathfrak{X}_\mu^\nu$ is given by a cocharacter in $-C_\mu^\nu$.*

Remark 10.5 Theorem 10.4 does not appear to be fully symmetric; of course, by negating the isomorphisms $\mathfrak{t}_\mathbb{R} \cong H^2(\mathfrak{M}^!; \mathbb{R})$ and *vice versa*, we can switch the sign of the chambers appearing, and thus the role of μ and ν.

Alternatively, we could take $\mathfrak{X}_{\mu^o}^\nu$ with the chamber $C_{\mu^o}^\nu$. In this case, the symmetry depends on the G-equivariant isomorphism $\mathfrak{X}_{\nu^o}^{\mu^o} \cong \mathfrak{X}_\nu^\mu$, using the automorphism of $\mathfrak{sl}_r$ given by the adjoint action of any representative of w_0. See [BLP$^+$11, 9.3] for the analogous statement about algebras.

Remark 10.6 If we take ν to be a composition with r parts each of size 1, Theorem 10.4 specializes to the statement that $T^*(G/P_\mu)$ is symplectic dual to the Slodowy slice to the nilpotent orbit of Jordan type $\bar{\mu}^t$ inside of the full nilpotent cone. If we further specialize to the case where $\mu = \nu$, we obtain Theorem 10.3 for $\mathfrak{sl}_r$.

Remark 10.7 More generally, we expect that a quiver variety whose quiver is a finite ADE Dynkin diagram with some choice of orientation (Section 9.5) will be dual to a slice in the affine Grassmannian for the Langlands dual group (Section 9.6). Since quiver varieties exist for every integral highest weight, we should consider them as associated to the simply connected group for that Dynkin diagram, and thus consider the affine Grassmannian of the adjoint form. In type A, both of these varieties are type A S3-varieties, and the precise statement that we want is given in Theorem 10.4.

We note that this conjectural duality provides a connection between two well-known constructions of weight spaces of irreducible representations of simply laced simple Lie algebras.

One, due to Nakajima, realizes these weight spaces as top homology groups of quiver varieties [Nak98, 10.2]. The other, using the geometric Satake correspondence of Ginzburg [Gin, 3.11 & 5.2] and Mirković-Vilonen [MV07], realizes them as top-degree $\mathbb{T}$-equivariant intersection cohomology groups of slices in the affine Grassmannian. See Example 10.27 for an explanation of how symplectic duality (conjecturally) allows us to identify these two vector spaces.

10.2.3 Hypertoric varieties

Next, we consider symplectic duality for hypertoric varieties. Let X and $X^!$ be a pair of unimodular, Gale dual polarized arrangements [BLPW12, 2.17]. These data can be used to construct hypertoric varieties $\mathfrak{M}$ and $\mathfrak{M}^!$ with specified actions of $\mathbb{T}$ [BLPW12, §5.1].

Theorem 10.8 *The hypertoric varieties $\mathfrak{M}$ and $\mathfrak{M}^!$ are symplectic dual.*

Proof: The order-reversing bijection on fixed points is given in [BLPW10, 2.10]. Symplectic leaves of $\mathfrak{M}_0$ and $\mathfrak{M}_0^!$ (all of which are special) are indexed by coloop-free flats of the hyperplane arrangements $\mathcal{A}$ and $\mathcal{A}^!$ associated to X and $X^!$ [PW07, 2.3], and it is well-known that such flats are in order-reversing bijection for Gale dual arrangements. The compatibility of the bijections follows from [BLPW12, 7.16].

The group isomorphisms $W \cong \mathbb{W}^!$ and $\mathbb{W} \cong W^!$ are described in [BLPW12, §8.1], and the equivariant isomorphisms of vector spaces with hyperplane arrangements are straightforward from the combinatorics of Gale duality. The Koszul duality between $\mathcal{O}$ and $\mathcal{O}^!$ is proven in [BLPW12, 4.7 & 4.10], and the fact that twisting and shuffling are exchanged is [BLPW12, 8.24 & 8.26]. $\square$

10.2.4 Affine type A quiver varieties

A fourth example of symplectic duality is given by quiver varieties for affine type A quivers. We leave most of the combinatorics to the papers [Webd, Webe] which treat this case in more detail, and only give a rough outline below.

Fix a positive integer e, and consider quiver varieties for $\widehat{\mathfrak{sl}}_e$. That is, we take a quiver whose underlying graph is an e-cycle, which we will identify with the Cayley graph of $\mathbb{Z}/e\mathbb{Z}$ for the generators $\{\pm 1\}$. Fix a highest weight $\nu = \sum \nu_i \omega_i$ for $\widehat{\mathfrak{sl}}_e$ as in Section 9.5; let $\ell := \sum \nu_i$ be the **level** of this highest weight. Pick a basis of the framing vector spaces, which have total dimension ℓ. The Lie algebra of the torus $\mathfrak{t}$ is spanned by the cocharacters

- ε_j which acts with weight 1 on the jth basis vector in the framing space, and

- γ which acts with weight 1 on every clockwise oriented edge of the cycle (and thus weight -1 on counterclockwise oriented edges).

There are certain distinguished choices of ζ which contain a representative of each chamber of $\mathcal{H}_{\mathrm{sh}}$. We call these **Uglov** actions, since they naturally correspond to the choice of charges for a higher level Fock space *à la* Uglov [Ugl00]. Let $\mathbf{s} = (s_1, \ldots, s_\ell)$ be a collection of integers such that there are precisely ν_j of the entries of this sequence such that $s_i \equiv j \pmod{e}$. We let $\zeta_{\mathbf{s}}^+$ be the cocharacter whose derivative is $\ell\gamma + (s_i\ell + ie)\varepsilon_j$, and let $C_{\mathbf{s}}^{\pm}$ be the chamber of $\mathcal{H}_{\mathrm{sh}}$ containing it. The action of $\zeta_{\mathbf{s}}^+$ always has isolated fixed points, and every chamber contains one of these cocharacters, as shown in [Webd, 5.16]. The fixed points of an Uglov action are in canonical bijection with ℓ-multipartitions. We wish to visualize these partitions as abaci as in, for example, [Tin08, §2.1].

We have ℓ runners on our abacus, numbered from bottom to top, each of which we visualize as a copy of the real line with slots at each integer which can hold a bead. To each multipartition

$$\xi = (\xi_1^{(1)} \geq \xi_2^{(1)} \geq \cdots ; \xi_1^{(2)} \geq \xi_2^{(2)} \geq \cdots ; \ldots ; \xi_1^{(\ell)} \geq \xi_2^{(\ell)} \geq \cdots),$$

we associate the abacus where on the kth runner, we fill the beads at

$$\xi_1^{(k)} + s_k, \xi_2^{(k)} + s_k - 1, \ldots, \xi_j^{(k)} + s_k - j + 1, \ldots,$$

and no others. Note this means that every position is filled at sufficiently negative integers, and open at sufficiently positive.

The combinatorics of the duality is encapsulated in the map between fixed points. This is given by cutting the abacus into $e \times \ell$ rectangles; that is rectangles consisting of the $me, me+1, \ldots, me+e-1$ positions of each runner as m ranges over $\mathbb{Z}$. Then we flip the rectangle, so that the first runner becomes the beads at the points $m\ell$ for $m \in \mathbb{Z}$, the second runner becomes the beads the points $m\ell + 1$, etc. as in the picture below.
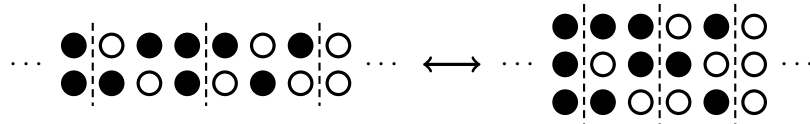
The lefthand picture above corresponds to

$$e = 3, \quad \ell = 2, \quad \mathbf{s} = (0,1), \quad \xi^{(1)} = (2,1), \xi^{(2)} = (2,1,1,1),$$

while the dual righthand picture corresponds to

$$e = 2, \quad \ell = 3, \quad \mathbf{t} = (0,0,1), \quad \xi^{(1)} = (2), \xi^{(2)} = (1,1), \xi^{(3)} = (1).$$

If we fix the triple $(\nu, \mu, \mathbf{s})$, and perform the duality above on the abacus for a multipartitions

with this weight and charge, the resulting weights and charge $(\mu^!, \nu^!, \mathbf{t})$ are combinatorially determined, as discussed in [Webd, §5.3]. We can also associate to this the triple $(\tilde{\mathfrak{Q}}^\nu_\mu, T, C^+_\mathbf{s})$. When the combinatorial data is switched by rank-level duality, we obtain a symplectic duality [Webd, 5.25].

Theorem 10.9 *The variety $\tilde{\mathfrak{Q}}^\nu_\mu$ is symplectic dual to $\tilde{\mathfrak{Q}}^{\mu^!}_{\nu^!}$, where the action of $\mathbb{T}$ on $\tilde{\mathfrak{Q}}^\nu_\mu$ is given by a cocharacter in $C^+_\mathbf{s}$ and the action of $\mathbb{T}$ on $\tilde{\mathfrak{Q}}^{\mu^!}_{\nu^!}$ is given by a cocharacter in $-C^+_\mathbf{t}$.*

Remark 10.10 The proof of Theorem 10.9 ultimately relies on Koszul duality results for certain categories of affine representations and category $\mathcal{O}$ for Cherednik algebras based on work of Rouquier, Shan, Varagnolo, and Vasserot [RSVV, SVV14].

Theorem 10.9 has the following special case. Let $\mathcal{H}(k, r)$ be the Hilbert scheme of r points on a crepant resolution of $\mathbb{C}^2/\Gamma$, where $\Gamma := \mathbb{Z}/k\mathbb{Z}$ acts effectively and symplectically on $\mathbb{C}^2$. Let $\mathcal{M}(k, r)$ be the moduli space of torsion-free sheaves E on $\mathbb{P}^2$ with $\operatorname{rk} E = k$ and $c_2(E) = r$, along with a framing $\Phi : E|_{\mathbb{P}^1} \xrightarrow{\sim} \mathcal{O}^{\oplus k}_{\mathbb{P}^1}$. On $\mathcal{H}(k, r)$, the torus T is 1-dimensional, induced by the symplectic action on $\mathbb{C}^2$ commuting with Γ. On $\mathcal{M}(k, r)$, the torus $T^!$ is naturally identified with $\mathbb{C}^\times$ times the projective diagonal matrices in PGL_k; let ϑ_i denote the weights of ζ in PGL_k (thus only well-defined up to simultaneous translation) and h the weight in $\mathbb{C}^\times$. The hyperplanes in $\mathcal{H}^!_{\mathrm{sh}}$ are the points of $\mathfrak{t}^!$ for which $\vartheta_i - \vartheta_j = mh$ for $m \in [-k+1, k-1]$, along with the single additional hyperplane $h = 0$. Note that these are precisely the GIT walls for $\mathcal{H}(k, r)$ as described by Gordon [Gor08, §4.3] (Gordon's H_i is our $\vartheta_i - \vartheta_{i+1}$). Let C_+ be the positive chamber in $\mathfrak{t}_\mathbb{R} \cong \mathbb{R}$ and let C_- be the chamber in $\mathfrak{t}^!_\mathbb{R}$ where $\vartheta_i \ll \vartheta_{i+1}$ and $h < 0$.

Corollary 10.11 *The variety $\mathcal{H}(k, r)$ is symplectic dual to $\mathcal{M}(k, r)$.*

Further specializing to the case where $k = 1$, we have $\mathcal{H}(1, r) \cong \operatorname{Hilb}^r \mathbb{C}^2 \cong \mathcal{M}(1, r)$. The Hilbert scheme $\operatorname{Hilb}^r \mathbb{C}^2$ does not satisfy our assumption that the minimal leaf is a point, but we may replace it with the reduced Hilbert scheme $\operatorname{Hilb}^r_0 \mathbb{C}^2$ (in which the center of mass is required to lie at the origin) without affecting category $\mathcal{O}$ or any of its structure.

Corollary 10.12 *The reduced Hilbert scheme $\operatorname{Hilb}^r_0 \mathbb{C}^2$ is self-dual.*

Remark 10.13 More generally, we expect the moduli space of G-instantons on a crepant resolution of $\mathbb{C}^2/\Gamma$ to be dual to the moduli space of G'-instantons on a crepant resolution of $\mathbb{C}^2/\Gamma'$, where G is matched to Γ' and G' is matched to Γ via the McKay correspondence. Corollary 10.11 constitutes the special case where both G and G' are of type A.

Braverman and Finkelberg have suggested that resolutions of slices in the "double affine Grassmannian" should be isomorphic to certain spaces of instantons. Via this philosophy, our conjecture may be regarded as an affine version of the conjecture in Remark 10.7.

10.3 Duality of cones

The notion of symplectic duality is in fact more naturally defined at the level of cones, as we explain below.

Proposition 10.14 *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be resolutions of the same cone $\mathfrak{M}_0$.*

(i) *$G \cong G'$, and therefore $\mathbb{W} \cong \mathbb{W}'$. Given a maximal torus $T \subset G$, the hyperplane arrangements $\mathcal{H}_{\mathrm{sh}}$ and $\mathcal{H}'_{\mathrm{sh}}$ in $\mathfrak{t}_{\mathbb{R}}$ coincide.*

(ii) *$W \cong W'$, and we have a W-equivariant isomorphism $H^2(\mathfrak{M};\mathbb{R}) \cong H^2(\mathfrak{M}';\mathbb{R})$ taking $\mathcal{H}_{\mathrm{tw}}$ to $\mathcal{H}'_{\mathrm{tw}}$.*

(iii) *For any $\lambda \in H^2(\mathfrak{M};\mathbb{C}) \cong H^2(\mathfrak{M}';\mathbb{C})$, we have $\mathscr{S}^{\mathrm{sp}}_{\mathfrak{M}} = \mathscr{S}^{\mathrm{sp}}_{\mathfrak{M}'}$.*

(iv) *Given a period $\lambda \in H^2(\mathfrak{M};\mathbb{C})$ and a generic cocharacter ζ of T, the categories $\mathcal{O}_{\mathrm{a}}$ and $\mathcal{O}'_{\mathrm{a}}$ associated to $\mathfrak{M}$ and $\mathfrak{M}'$ are canonically equivalent.*

Proof:

(i) The Lie algebra $\mathfrak{g}$ of G is isomorphic to the Lie algebra of Hamiltonian vector fields on $\mathfrak{M}_0$ with $\mathbb{S}$-weight 1; this tells us that $\mathfrak{g} \cong \mathfrak{g}'$. Let $\tilde{G}$ be a simultaneous cover of both G and G'; then $\tilde{G}$ acts on both $\mathfrak{M}$ and $\mathfrak{M}'$. Since an automorphism of $\mathfrak{M}$ or $\mathfrak{M}'$ is trivial if and only if it induces the trivial automorphism of $\mathfrak{M}_0$, the maps $\tilde{G} \to G$ and $\tilde{G} \to G'$ have the same kernel.

Let $T \subset G$ be a maximal torus. Any cocharacter ζ of T induces an action of $\mathbb{T}$ on $\mathfrak{M}$ with isolated fixed points if and only if the induced action on $\mathfrak{M}_0$ has isolated fixed points, thus the hyperplane arrangement $\mathcal{H}_{\mathrm{sh}}$ in $\mathfrak{t}$ is independent of the choice of resolution.

(ii) The fact that the Namikawa Weyl group is determined completely by $\mathfrak{M}_0$ is immediate from Namikawa's definition. The fact that the groups $H^2(\mathfrak{M};\mathbb{R})$ and $H^2(\mathfrak{M}';\mathbb{R})$ are canonically isomorphic for two different resolutions $\mathfrak{M}$ and $\mathfrak{M}'$ is explained in [BPW, 2.18]. This isomorphism is clearly W-equivariant, and takes the arrangement $\mathcal{H}_{\mathrm{tw}}$ to $\mathcal{H}'_{\mathrm{tw}}$.

(iii) The fact that the poset $\mathscr{S}^{\mathrm{sp}}$ of special leaves does not depend on the choice of resolution follows from the fact that the algebra A does not depend on the choice of resolution; that is, the algebra of $\mathbb{S}$-invariant global sections of the quantization of $\mathfrak{M}$ with period λ is canonically isomorphic to the algebra of $\mathbb{S}$-invariant global sections of the quantization of $\mathfrak{M}'$ with period λ [BPW, 3.9].

(iv) The category $\mathcal{O}_{\mathrm{a}}$ is defined in terms of the algebra A and the subalgebra $A^+ \subset A$, and we have established that these structures do not depend on the choice of resolution. $\square$

We are now prepared to define symplectic duality of cones. Let $\mathfrak{M}_0$ be a Poisson cone that admits a conical symplectic resolution, which we do not fix. Let G be the group of Hamiltonian symplectomorphisms that commute with $\mathbb{S}$. Fix a Borel subgroup $B \subset G$, and assume that one (and thus any) maximal torus $T \subset B$ fixes only the cone point of $\mathfrak{M}_0$.

For any movable chamber C_{tw} of $\mathcal{H}_{\mathrm{tw}}$ and B-dominant chamber C_{sh} of $\mathcal{H}_{\mathrm{sh}}$, we obtain a category $\mathcal{O}(C_{\mathrm{tw}}, C_{\mathrm{sh}})$ by choosing the unique conical symplectic resolution $\mathfrak{M}$ with ample cone C_{tw}, taking a quantization with period sufficiently deep in the ample cone, and allowing $\mathbb{T}$ to act by a cocharacter in C_{sh}.

Definition 10.15 A symplectic duality between $(\mathfrak{M}_0, B)$ and $(\mathfrak{M}_0^!, B^!)$ consists of

- an order-reversing bijection $S \mapsto S^!$ from $\mathscr{S}_{\mathfrak{M}}^{\mathrm{sp}}$ to $\mathscr{S}_{\mathfrak{M}^!}^{\mathrm{sp}}$ (for any choice of $\mathfrak{M}$ and $\mathfrak{M}^!$);

- group isomorphisms $W \cong \mathbb{W}^!$ and $\mathbb{W} \cong W^!$;

- a pair of linear isomorphisms $\mathfrak{t}_{\mathbb{R}} \cong H^2(\mathfrak{M}^!; \mathbb{R})$ and $H^2(\mathfrak{M}; \mathbb{R}) \cong \mathfrak{t}_{\mathbb{R}}^!$, which are equivariant with respect to the isomorphisms of the previous item, compatible with the lattices, take $\mathcal{H}_{\mathrm{tw}}$ to $\mathcal{H}_{\mathrm{sh}}^!$ and $\mathcal{H}_{\mathrm{sh}}$ to $\mathcal{H}_{\mathrm{tw}}^!$, and take movable twisting chambers to dominant shuffling chambers;

- for any $C_{\mathrm{tw}}, C_{\mathrm{sh}}$ and $C_{\mathrm{tw}}^!, C_{\mathrm{sh}}^!$ related by the linear isomorphisms above, a Koszul duality from $\mathcal{O}(C_{\mathrm{tw}}, C_{\mathrm{sh}})$ to $\tilde{\mathcal{O}}^!(C_{\mathrm{tw}}^!, C_{\mathrm{sh}}^!)$.

These structures are required to satisfy the following conditions:

- Let L be a simple object of $\mathcal{O}(C_{\mathrm{tw}}, C_{\mathrm{sh}})$, and suppose that $\tilde{L}$ is sent by the Koszul duality functor to a graded lift of the projective cover of the simple object $L^!$ of $\mathcal{O}^!(C_{\mathrm{tw}}^!, C_{\mathrm{sh}}^!)$. Then we require that our bijection of special leaves takes the dense leaf in $\mathfrak{M}_{0,L}$ to the dense leaf in $\mathfrak{M}_{0,L^!}^!$.

- The equivalences $D^b(\tilde{\mathcal{O}}(C_{\mathrm{tw}}, C_{\mathrm{sh}})) \simeq D^b(\tilde{\mathcal{O}}^!(C_{\mathrm{tw}}^!, C_{\mathrm{sh}}^!))$ interchange twisting and shuffling functors.

Remark 10.16 Essentially, Definition 10.15 says that for any choice of $\mathfrak{M}$ resolving $\mathfrak{M}_0$ and $\mathfrak{M}^!$ resolving $\mathfrak{M}_0^!$, there are corresponding actions of $\mathbb{T}$ on both sides such that $\mathfrak{M}$ is dual to $\mathfrak{M}^!$. (It also says that the linear isomorphisms involved in all of these dualities can be chosen consistently.) It is straightforward to promote Theorems 10.3, 10.4, 10.8, and 10.9 to dualities between cones.

10.4 Duality of leaf closures and slices

Let $\mathfrak{M} \to \mathfrak{M}_0$ be a conical symplectic resolution, and $S \subset \mathfrak{M}_0$ a symplectic leaf. The closure $\bar{S} \subset \mathfrak{M}_0$ is again a Poisson cone, which may or may not admit a symplectic resolution. For

example, if $\mathfrak{M}_0$ is the nilpotent cone in $\mathfrak{sl}_r$ and S is a nilpotent orbit, then $\bar{S}$ admits a symplectic resolution of the form $T^*(\mathrm{SL}_r/P)$ for some P. For other simple Lie algebras, however, $\bar{S}$ may admit no symplectic resolution. If $\mathfrak{M}_0$ admits a Hamiltonian action of $\mathbb{T}$, commuting with $\mathbb{S}$ and fixing only the cone point, then this restricts to an action on $\bar{S}$ with the same properties.

Let $p \in S$ be any point. We say that another Poisson cone $\mathfrak{N}_0$ is a **slice** to S at p if a formal neighborhood of p in $\mathfrak{M}_0$ is isomorphic to a formal neighborhood of p in S times a formal neighborhood of the cone point in $\mathfrak{N}_0$. Assuming that such an $\mathfrak{N}_0$ exists[21], it will always admit a conical symplectic resolution; in an infinitesimal neighborhood of the cone point, this resolution is obtained by base change along the inclusion of $\mathfrak{N}_0$ into $\mathfrak{M}_0$. However, even if $\mathfrak{M}_0$ admits a Hamiltonian $\mathbb{T}$-action that commutes with $\mathbb{S}$ and fixes only the cone point, $\mathfrak{N}_0$ might not admit such an action.

Let $\mathfrak{M}_0$ and $\mathfrak{M}_0^!$ be dual in the sense of Definition 10.15. Let S be a special leaf of $\mathfrak{M}_0$, and let $S^!$ be the corresponding special leaf of $\mathfrak{M}_0^!$. Let $\mathfrak{N}_0$ be a slice to S at a point $p \in S$.

Conjecture 10.17 *Suppose that $\bar{S}^!$ admits a conical symplectic resolution and $\mathfrak{N}_0$ admits a Hamiltonian $\mathbb{T}$ action commuting with $\mathbb{S}$ and fixing only a point. Then $\bar{S}^!$ is dual to $\mathfrak{N}_0$.*

Example 10.18 Conjecture 10.17 is true for S3-varieties in type A. We showed in Theorem 10.4 that $\mathfrak{X}_\nu^\mu$ is dual to $\mathfrak{X}_{\mu^o}^\nu$. The leaf closures in the affinization of $\mathfrak{X}_\nu^\mu$ have resolutions of the form $\mathfrak{X}_\rho^\mu$ and the slices in the affinization of $\mathfrak{X}_{\mu^o}^\nu$ have resolutions of the form $\mathfrak{X}_{\mu^o}^\rho$, where $\bar{\rho}$ lies between $\bar{\nu}$ and $\bar{\mu}^t$ in the dominance order.

More generally, Conjecture 10.17 should hold for finite-type quiver varieties and slices in the affine Grassmannian (Remark 10.7), as both of these families of varieties are closed under the operations of leaf-closure and slice.

Example 10.19 Conjecture 10.17 is true for hypertoric varieties. Special leaves correspond to coloop-free flats [PW07, 2.1], leaf closures correspond to restrictions at flats [PW07, 2.1], and slices correspond to localization at flats [PW07, 2.4]. Thus Conjecture 10.17 for hypertoric varieties follows from Theorem 10.8 and the interchange of localization and restriction under Gale duality [BLPW10, 2.6].

Example 10.20 Conjecture 10.17 is true for affine type A quiver varieties. A special leaf of $\mathfrak{Q}_\mu^\nu$ is indexed by a highest weight ξ, a weight ϖ and an integer r. From the description of leaves and slices in [Nak94, §6], the closure of this leaf is isomorphic to $\mathfrak{Q}_\varpi^\nu \times \mathrm{Sym}^r\left(\mathbb{C}^2/(\mathbb{Z}/e\mathbb{Z})\right)$, and its slice is isomorphic to $\mathfrak{Q}_\mu^{\varpi - r\delta} \times \mathrm{Sym}^r\left(\mathbb{C}^2/(\mathbb{Z}/e\mathbb{Z})\right)$. These switch roles under rank-level duality by [Webd, 5.18], and the result follows.

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[21]Kaledin [Kal09, 1.6] shows that we can always find a symplectic singularity $\mathfrak{N}_0$ with this property, but he does not prove that the Poisson structure on $\mathfrak{N}_0$ is always homogeneous with respect to a conical $\mathbb{S}$ action. He does, however, conjecture that this is the case [Kal09, 1.8].

Remark 10.21 In all of the examples that we know, $\bar{S}^!$ admits a conical symplectic resolution if and only if $\mathfrak{N}_0$ admits a Hamiltonian $\mathbb{T}$-action commuting with $\mathbb{S}$ and fixing only a point. This suggests that there might be a more general notion of duality than Definition 10.15 in which both of these conditions are relaxed, and one holds on one side if and only if the other holds on the other side.

10.5 Duality of leaf filtrations

Suppose that $\mathfrak{M}$ is symplectic dual to $\mathfrak{M}^!$. For every special leaf $S \subset \mathfrak{M}$, let

$$D_S := K(\mathcal{O}_{\mathrm{g}}^S)_{\mathbb{C}} = \mathbb{C}\{[\Lambda_\alpha] \mid \mathfrak{M}_{\alpha,0} \subset \bar{S}\}$$

and

$$E_S := K(\mathcal{O}_{\mathrm{g}}^{\partial S})_{\mathbb{C}} = \mathbb{C}\{[\Lambda_\alpha] \mid \mathfrak{M}_{\alpha,0} \subsetneq \bar{S}\}.$$

This is precisely the filtration of $K(\mathcal{O}_{\mathrm{g}})_{\mathbb{C}}$ whose associated graded gives us the direct sum decomposition of Equation (5); that is, $D_S/E_S \cong K(\mathfrak{P}_S)_{\mathbb{C}}$. If $\mathfrak{M}$ and $\mathfrak{M}^!$ are both interleaved (Definition 6.14), then this filtration agrees, via the characteristic cycle isomorphism, with the BBD filtration of $H^{2d}_{\mathfrak{M}^+}(\mathfrak{M}; \mathbb{C})$ (Theorem 7.14).

Consider the perfect pairing between $K(\mathcal{O})_{\mathbb{C}}$ and $K(\mathcal{O}^!)_{\mathbb{C}}$ for which the simples form dual bases, under the bijection between simples provided by Koszul duality.

Proposition 10.22 *For each special leaf S, the above pairing induces a perfect pairing between D_S/E_S and $D^!_{S^!}/E^!_{S^!}$.*

Proof: For every special leaf S, we have

$$
\begin{aligned}
D_{\bar{S}}^{\perp} \cap D^!_{S^!} &= \mathbb{C}\{[\Lambda_\alpha] \mid \mathfrak{M}_{\alpha,0} \subset \bar{S}\}^{\perp} \cap \mathbb{C}\{[\Lambda^!_{\alpha^!}] \mid \mathfrak{M}^!_{\alpha^!,0} \subset \bar{S}^!\} \\
&= \mathbb{C}\{[\Lambda^!_{\alpha^!}] \mid \mathfrak{M}_{\alpha,0} \not\subset \bar{S}\} \cap \mathbb{C}\{[\Lambda^!_{\alpha^!}] \mid \mathfrak{M}^!_{\alpha^!,0} \subset \bar{S}^!\} \\
&= \mathbb{C}\{[\Lambda^!_{\alpha^!}] \mid \mathfrak{M}^!_{\alpha^!,0} \not\supset \bar{S}^!\} \cap \mathbb{C}\{[\Lambda^!_{\alpha^!}] \mid \mathfrak{M}^!_{\alpha^!,0} \subset \bar{S}^!\} \\
&= \mathbb{C}\{[\Lambda^!_{\alpha^!}] \mid \mathfrak{M}^!_{\alpha^!,0} \subsetneq \bar{S}^!\} \\
&= E^!_{S^!}.
\end{aligned}
$$

By symmetry, we also have $(D^!_{S^!})^{\perp} \cap D_S = E_S$. This completes the proof. $\qquad\square$

Example 10.23 If $\mathfrak{M} = \mathfrak{M}^! = T^*(\mathrm{SL}_r / B)$ and the period is integral, then the space $K(\mathcal{O})_{\mathbb{C}}$ can be identified with $\mathbb{C}[S_n]$, and the space D_S/E_S is the unique subquotient which is isomorphic to the isotypic component of the Springer representation for S.

Example 10.24 If $\mathfrak{M}$ and $\mathfrak{M}^!$ are hypertoric and the period is regular and integral, the dimension of the space D_S/E_S is $T_{\mathcal{A}^F}(1,0)T_{\mathcal{A}_F}(0,1)$ where $T_{\mathcal{A}^F}$ and $T_{\mathcal{A}_F}$ are the Tutte polynomials of the restriction and localization of the hyperplane arrangement at the coloop-free flat F corresponding to the leaf S. The dual variety $\mathfrak{M}^!$ is the variety associated to the Gale dual hyperplane arrangement, and Gale duality takes restrictions to localizations and vice versa, along with reversing the variables in the Tutte polynomial.

Corollary 10.25 *If $(\mathfrak{M}, \mathcal{D})$ is fat-tailed and $(\mathfrak{M}^!, \mathcal{D}^!)$ is light-headed, then the vector space $H^{\dim \mathfrak{M}}(\mathfrak{M}; \mathbb{C})$ is dual to $IH_{\mathbb{T}}^{\dim \mathfrak{M}^!}(\mathfrak{M}_0^!; \mathbb{C})$.*

Proof: Taking S to be the minimal leaf $\{o\}$, Proposition 10.22 says that $D_{\{o\}}$ is dual to $D_{\mathring{\mathfrak{M}}_0^!}^! / E_{\mathring{\mathfrak{M}}_0^!}^!$. By fat-tailedness (Theorem 7.20), we have

$$D_{\{o\}} \cong K(\mathfrak{P}_{\{o\}})_{\mathbb{C}} \cong \Sigma(\mathcal{F}_{\{o\}}) \cong H^{2d}(\mathfrak{M}; \mathbb{C}).$$

By light-headedness (Theorems 7.23), we have

$$D_{\mathring{\mathfrak{M}}_0^!}^! / E_{\mathring{\mathfrak{M}}_0^!}^! \cong K(\mathfrak{P}_{\mathring{\mathfrak{M}}_0^!}^!)_{\mathbb{C}} \cong \Sigma(\mathcal{F}_{\mathring{\mathfrak{M}}_0^!}^!) \cong IH_{\mathbb{T}}^{\dim \mathfrak{M}^!}(\mathfrak{M}_0^!; \mathbb{C}).$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 10.26 The vector space $IH_{\mathbb{T}}^{\dim \mathfrak{M}}(\mathfrak{M}; \mathbb{C})$ is naturally filtered, with k^{th} filtered piece $H_{\mathbb{T}}^{\dim \mathfrak{M} - 2k}(pt; \mathbb{C}) \cdot IH_{\mathbb{T}}^{2k}(\mathfrak{M}; \mathbb{C})$, and the associated graded is isomorphic to $IH^*(\mathfrak{M}; \mathbb{C})$. Thus Corollary 10.25 says that the dimension of the total intersection cohomology of $\mathfrak{M}_0$ is equal to the dimension of the top homology of $\mathfrak{M}^!$. If you have a pair of conical symplectic resolutions that you think might be dual, this is the first calculation that you should do.

Example 10.27 Let G be a simple algebraic group, simply laced and simply connected. Fix a pair of λ, μ of dominant weights for G, and consider the Nakajima quiver variety $\tilde{\mathfrak{Q}}_\mu^\lambda$ that we discussed in Section 9.5. The top homology group of $\tilde{\mathfrak{Q}}_\mu^\lambda$ is isomorphic to the $V(\lambda)_\mu$, the μ-weight space of the irreducible representation $V(\lambda)$ [Nak98, 10.2].

Let G^L be the Langlands dual group, and consider the slice $\mathrm{Gr}_\mu^{\bar{\lambda}}$ in the affine Grassmannian for G^L that we discussed in Section 9.6. This variety always admits a Hamiltonian $\mathbb{T}$-action fixing only the cone point, and the intersection cohomology group $IH_{\mathbb{T}}^{\dim \mathrm{Gr}_\mu^{\bar{\lambda}}}(\mathrm{Gr}_\mu^{\bar{\lambda}}; \mathbb{C})$ is also isomorphic to the $V(\lambda)_\mu$, the μ-weight space of the irreducible representation $V(\lambda)$ [Gin, 3.11 & 5.2].

If λ is a sum of minuscule weights, then there exists a symplectic resolution $\tilde{\mathrm{Gr}}_\mu^{\bar{\lambda}}$ of $\mathrm{Gr}_\mu^{\bar{\lambda}}$ and a Hamiltonian $\mathbb{T}$-action on $\tilde{\mathfrak{Q}}_\mu^\lambda$ with isolated fixed points, and we conjecture that $\tilde{\mathrm{Gr}}_\mu^{\bar{\lambda}}$ is symplectic dual to $\tilde{\mathfrak{Q}}_\mu^\lambda$ (Remark 10.7). We know that $\tilde{\mathfrak{Q}}_\mu^\lambda$ is fat-tailed (Example 7.26) and

we conjecture that $\tilde{\mathrm{Gr}}_\mu^{\bar\lambda}$ is light-headed (Conjecture 7.27). Assuming these two conjectures, Corollary 10.25 would provide an isomorphism

$$H_{\dim \tilde{\mathfrak{Q}}_\mu^\lambda}(\tilde{\mathfrak{Q}}_\mu^\lambda; \mathbb{C}) \cong H^{\dim \tilde{\mathfrak{Q}}_\mu^\lambda}(\tilde{\mathfrak{Q}}_\mu^\lambda; \mathbb{C})^* \cong IH_{\mathbb{T}}^{\dim \mathrm{Gr}_\mu^{\bar\lambda}}(\mathrm{Gr}_\mu^{\bar\lambda}; \mathbb{C}) \tag{9}$$

between the two geometric realizations of $V(\lambda)_\mu$.

Note that if G is of type A, then both $\tilde{\mathfrak{Q}}_\mu^\lambda$ and $\tilde{\mathrm{Gr}}_\mu^{\bar\lambda}$ are S3-varieties [Maf05, MV]. In this case, Conjecture 7.27 is covered by Example 7.25, and the symplectic duality statement is Theorem 10.4.

Remark 10.28 We defined a pairing between $K(\mathcal{O})$ and $K(\mathcal{O}^!)$ by making the simple bases on each side be orthonormal. However, in the proof of Proposition 10.22, we only used that they are orthogonal. There is some evidence to suggest that there is a more natural pairing that is orthogonal but not orthonormal.

To define such a pairing, we suppose that there is a function $\epsilon : \mathcal{I} \to \{\pm 1\}$ such that for all $\alpha, \alpha' \in \mathcal{I}$,

$$\epsilon(\alpha) = \epsilon(\alpha') \implies \mathrm{Ext}^1(\Lambda_\alpha, \Lambda_{\alpha'}) = 0.$$

Such a function always exists for regular blocks of BGG category $\mathcal{O}$ (it is given by the sign function on the Weyl group), as well as blocks of hypertoric category $\mathcal{O}$ (it is the same function that controls the orientations in [BLPW10, 4.3]). We know of no examples for which such a function does not exist. As long as $\mathcal{O}$ is indecomposable, any such function would be unique up to a global sign.

Given such a function, define a new pairing by putting

$$\left\langle [\Lambda_\alpha], [\Lambda_{\beta^!}^!] \right\rangle_\epsilon := \epsilon(\alpha)\delta_{\alpha\beta}.$$

In other words, we take the simples to form twisted orthonormal bases, where the twist is determined by ϵ. The reason that this pairing might be more natural is that the standards and the projectives would also form twisted orthonormal bases.[22] In contrast, the untwisted pairing that we originally defined is not well behaved with respect to projectives or standards.

We conjecture that the isomorphism in Equation (9) between the two geometric realizations of $V(\lambda)_\mu$ will only be compatible with the action of G if we use this twisted pairing.

Example 10.29 In type A, the relationship between quiver variety geometry and affine Grassmannian geometry is somewhat special. In particular, as observed by Mirković-Vybornov, Nakajima quiver varieties of type A also arise as transverse slices in the affine Grassmannian of type A [MV]. This geometric phenomenon is reflected algebraically in skew-Howe duality

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[22]We leave this statement as an exercise for the reader. The proof uses the fact that the isomorphism $K(\tilde{\mathcal{O}}) \cong K(\tilde{\mathcal{O}}^!)$ induced by Koszul duality takes q to $-q^{-1}$.

for representations of type A simple Lie algebras. More precisely, this duality asserts that there is a canonical vector space isomorphism between weight spaces in representations of $\mathfrak{gl}_m$ and multiplicity spaces in representations of $\mathfrak{gl}_n$ in the $\mathfrak{gl}_m \times \mathfrak{gl}_n$-module $\bigwedge^N(\mathbb{C}^n \otimes \mathbb{C}^m)$. These weight spaces and multiplicity spaces can be interpreted geometrically using either the Ginzburg-Nakajima construction or the geometric Satake construction. The various geometric realisations of weight and multiplicity spaces are then summarized in the following diagram:

	Geometric Satake	Ginzburg-Nakajima
Multiplicity space for $\mathfrak{gl}_m$	$H_{\dim \mathfrak{M}}(\mathfrak{M}; \mathbb{C})$	$IH_{\mathbb{T}^!}^{\dim \mathfrak{M}^!}(\mathfrak{M}_0^!; \mathbb{C})$
Weight space for $\mathfrak{gl}_n$	$IH_{\mathbb{T}^!}^{\dim \mathfrak{M}^!}(\mathfrak{M}_0^!; \mathbb{C})$	$H_{\dim \mathfrak{M}}(\mathfrak{M}; \mathbb{C})$

The observation of Mirković-Vybornov is essentially that the varieties in the diagonal of the above matrix are identical. Our additional observation, which is special to type A, is that the varieties in a given row (or column) will be symplectic duals. Moreover, the isomorphism between the vector spaces in a given row (or column) is a cohomological consequence of symplectic duality; see Remark 10.26. Thus, in type A, symplectic duality exchanges the geometric Satake realization of skew-Howe duality with the Ginzburg-Nakajima realization of skew-Howe duality.

Example 10.30 For affine type $\widehat{A}$ quiver varieties, symplectic duality recovers a duality in representation theory due to I.B. Frenkel known as level-rank duality. The situation is quite reminiscent of that of skew-Howe duality in finite type A, and is discussed in more detail in [Webd, §5.2]. In this case, the commuting actions of the affine Lie algebras $\widehat{\mathfrak{sl}}_n$ and $\widehat{\mathfrak{sl}}_m$ on the semi-infinite wedge space give rise to canonical identifications between weight spaces in a level m representation of $\widehat{\mathfrak{sl}}_n$ and multiplicity spaces in a level n representation of $\widehat{\mathfrak{sl}}_m$. (See, for example, equation A.5 in [Nak09].) The geometric Satake construction of representations in affine type is not completely understood. However, in affine type $\widehat{A}$, Braverman-Finkelberg [BF10] have an explicit proposal for a geometric Satake construction of representations. We then have the following diagram of geometric realisations of representations:

	Geometric Satake	Nakajima
Multiplicity space in a level n rep. of $\widehat{\mathfrak{sl}}_m$	$H_{\dim \mathfrak{M}}(\mathfrak{M}; \mathbb{C})$	$IH_{\mathbb{T}^!}^{\dim \mathfrak{M}^!}(\mathfrak{M}_0^!; \mathbb{C})$
Weight space in a level m rep. of $\widehat{\mathfrak{sl}}_n$	$IH_{\mathbb{T}^!}^{\dim \mathfrak{M}^!}(\mathfrak{M}_0^!; \mathbb{C})$	$H_{\dim \mathfrak{M}}(\mathfrak{M}; \mathbb{C})$

Here $\mathfrak{M}$ and $\mathfrak{M}^!$ are affine type $\widehat{A}$ quiver varieties. The Nakajima column realizes the weight and multiplicity spaces as homology groups of Nakajima quiver varieties. That the diagonal (resp. off-diagonal) entries in Geometric Satake column involve the same variety is part of the content of the Braverman-Finkelberg proposal for geometric Satake in affine type $\widehat{A}$. Our additional observation is that the varieties in a given row (or column) will be symplectic duals.

Remark 10.31 Etingof and Schedler [ES14, 1.3.1(b)] conjecture that $H_{\dim \mathfrak{M}^!}(\mathfrak{M}^!; \mathbb{C})$ has the same dimension as the zeroth Poisson homology group $HP_0(\mathfrak{M}^!_0)$, which is defined as the ring of functions modulo the linear span of all Poisson brackets. By Corollary 10.25 and Remark 10.26, this is equivalent to the conjecture that $IH^*(\mathfrak{M}; \mathbb{C})$ has the same dimension as $HP_0(\mathfrak{M}^!_0)$. In [Pro14, 3.4], we strengthen this conjecture by proposing that they should be isomorphic as graded vector spaces (the grading on Poisson homology is induced by the $\mathbb{S}$-action).

Furthermore, $IH^*(\mathfrak{M}; \mathbb{C})$ admits the natural deformation $IH^*_T(\mathfrak{M}; \mathbb{C})$ over $\mathfrak{t}$, and $HP_0(\mathfrak{M}^!_0)$ admits the natural deformation $HP_0(\mathscr{N}^!)$ over $H^2(\mathfrak{M}^!)$ (recall that $\mathscr{N}^! := \operatorname{Spec} \mathbb{C}[\mathscr{M}^!]$ is a Poisson deformation of $\mathfrak{M}^!_0$ over $H^2(\mathfrak{M}^!; \mathbb{C})$). Finally, recall that we have $\mathfrak{t} \cong H^2(\mathfrak{M}^!; \mathbb{C})$ as part of the package of symplectic duality, therefore the two deformations share the same base. The conjecture in [Pro14, 3.4] asserts that these two deformations should be isomorphic, as well; it is proven for hypertoric varieties [Pro14, 3.1].

10.6 Duality of localization algebras

In this section we extend Conjecture 5.23 in order to relate symplectic duality to a cohomological duality first studied by Goresky and MacPherson [GM10], and later by the authors [BLP+11]. Let $\mathfrak{M}$ and $\mathfrak{M}^!$ be a symplectic dual pair.

Recall that we define E to be the Yoneda algebra of $\mathcal{O}$, and we conjecture that its center is isomorphic to $H^*(\mathfrak{M}; \mathbb{C})$ (Conjecture 5.23). Let $\tilde{E}$ be the universal deformation of E, as defined in [BLP+11, 4.2]; this is a flat deformation over the base $Z(E^!)^*_2$, the dual of the degree 2 part of the center of $E^!$. Conjecture 5.23 for $\mathfrak{M}^!$ says that the center of $E^!$ is isomorphic to $H^*(\mathfrak{M}^!; \mathbb{C})$, which means that the base of the universal deformation is isomorphic to $H_2(\mathfrak{M}^!; \mathbb{C})$. As part of the package of symplectic duality, this is isomorphic to $\mathfrak{t}$.

Conjecture 10.32 *The graded ring $Z(\tilde{E})$ is isomorphic to $H^*_T(\mathfrak{M}; \mathbb{C})$.*

Remark 10.33 Conjecture 10.32 is a natural extension of Conjecture 5.23; if you believe that $Z(E)$ is isomorphic to $H^*(\mathfrak{M}; \mathbb{C})$, and they both admit canonical flat deformations over the base $\mathfrak{t}$, it is natural to guess that these deformations are the same. Unfortunately, unlike in Conjecture 5.23, we do not have a geometric definition of a map from $H^*_T(\mathfrak{M}; \mathbb{C})$ to $Z(\tilde{E})$.

Remark 10.34 Conjecture 10.32 holds for hypertoric varieties [BLP+11, 8.5] and for S3-varieties in type A [BLP+11, 9.9].

In their paper [GM10], Goresky and MacPherson observe a somewhat mysterious cohomological relationship between certain pairs of varieties with torus actions. Below we will describe this relationship, and demonstrate that it is a consequence of symplectic duality and Conjecture 10.32.

Consider the ring homomorphisms

$$\mathrm{Sym}\,\mathfrak{t}^* \hookrightarrow \mathrm{Sym}\,H_T^2(\mathfrak{M};\mathbb{C}) \;\to\; H_T^*(\mathfrak{M};\mathbb{C}) \hookrightarrow H_T^*(\mathfrak{M}^T;\mathbb{C}) \;\cong\; \bigoplus_{\alpha\in\mathcal{I}} H_T^*(p_\alpha;\mathbb{C}).$$

Dualizing, we obtain maps

$$\mathfrak{t} \;\leftarrow\; H_2^T(\mathfrak{M};\mathbb{C}) \;\leftarrow\; \mathrm{Spec}\,H_T^*(\mathfrak{M};\mathbb{C}) \;\leftarrow\; \bigsqcup_{\alpha\in\mathcal{I}} \mathfrak{t}.$$

Let $H_\alpha \subset H_2^T(\mathfrak{M};\mathbb{C})$ be the image of the copy of $\mathfrak{t}$ indexed by $\alpha\in\mathcal{I}$. This is a linear subspace that projects isomorphically to $\mathfrak{t}$ via the left-most map, and the union of these linear subspaces is equal to the image of $\mathrm{Spec}\,H_T^*(\mathfrak{M};\mathbb{C})$ in $H_2^T(\mathfrak{M};\mathbb{C})$.

Theorem 10.35 *There is a canonical perfect pairing between $H_2^T(\mathfrak{M};\mathbb{C})$ and $H_2^{T^!}(\mathfrak{M}^!;\mathbb{C})$. Assuming that Conjecture 10.32 holds, then:*

- *the kernel of the projection from $H_2^T(\mathfrak{M};\mathbb{C})$ to $\mathfrak{t}$ is the perp space to the kernel of the projection from $H_2^{T^!}(\mathfrak{M}^!;\mathbb{C})$ to $\mathfrak{t}^!$*

- *for all $\alpha\in\mathcal{I}$, $H_\alpha \subset H_2^T(\mathfrak{M};\mathbb{C})$ is the perp space to $H_{\alpha^!}^! \subset H_2^{T^!}(\mathfrak{M}^!;\mathbb{C})$.*

Proof: This result follows immediately from the Koszul duality of $\mathcal{O}$ and $\mathcal{O}^!$ using [BLP+11, 1.2]. □

Remark 10.36 The phenomenon in Theorem 10.35 was observed by Goresky and MacPherson [GM10, §8] for the varieties described in Remark 10.6 (a special case of type A S3-varieties). The connection to symplectic duality allowed us to find many new examples, such as hypertoric varieties and more general type A S3-varieties.

10.7 Knot homologies and symplectic duality

There are close relationships between the representation theory of quantum groups and low-dimensional topology. Perhaps the best known example of such a relationship is the construction of Reshetikhin-Turaev [RT90] invariants of links in S^3; these invariants are defined using the braiding on the category of $U_q(\mathfrak{g})$ modules. Much of the subsequent work categorifying the representation theory of quantum groups has been motivated by the desire to lift the polynomial invariants of Reshetikhin-Turaev to richer homological invariants. The best known such lift - at least to representation theorists - is Khovanov's $U_q(\mathfrak{sl}_2)$ link homology [Kho00], which categorifies the Jones polynomial.

On the other hand, representations of quantum groups (and their categorifications) can be constructed using the geometry of conical symplectic resolutions. As a result, one expects to

obtain constructions of knot homologies using the geometry of conical symplectic resolutions or the representation theory of Koszul algebras. Indeed, such constructions have been studied by many authors in the last ten years. The examples most closely related to the geometry of conical symplectic resolutions include:

- Sussan's algebraic construction of $U_q(\mathfrak{sl}_n)$ link homologies (for fundamental weights) using a graded version of parabolic BGG category $\mathcal{O}$ for $\mathfrak{gl}_m$ [Sus07];

- Mazorchuk-Stroppel's algebraic construction of $U_q(\mathfrak{sl}_n)$ link homologies (for fundamental weights) using a graded version of singular BGG category $\mathcal{O}$ for $\mathfrak{gl}_m$ [MS09];

- Cautis-Kamnitzer's geometric constructions of $U_q(\mathfrak{sl}_n)$ link homologies (for fundamental weights) using the affine Grassmannian [CK08b, CK08a, Kam11];

- Seidel-Smith and Manolescu's constructions of Khovanov homology using Fukaya categories of type A quiver varieties [SS06b, Man07];

- Cautis's geometric construction of of $U_q(\mathfrak{sl}_n)$ link homologies (for all weights) using derived categories of coherent sheaves and subvarieties of the affine Grassmannian [Cau15];

- Smith-Thomas and Cautis-Licata's constructions of Khovanov homology using derived categories of coherent sheaves on Hilbert schemes of points on ALE spaces [Tho10, CL12];

- the fourth author's construction of $U_q(\mathfrak{g})$ link homologies, categorifying the entire family of RT polynomial link invariants [Webd] using category $\mathcal{O}$ for quiver varieties.

The geometry underlying these categories is not the same in each case. Rather, it seems to come in two different flavors, which are related to two different ways of geometrizing a representation of $\mathfrak{g}$: via quiver varieties, or via the affine Grassmannian for the Langlands dual group $^L G$. The work of Seidel-Smith, Manolescu, and Cautis-Kamnitzer is on the affine Grassmannian side, while the work of the fourth author is on the quiver variety side. Philosophically, all of these approaches involve defining a braid group action on certain geometrically defined categories, and adding some special functors which can be used to define cups and caps. In all cases, we can interpret these data as coming from ideas that we have discussed.

On the affine Grassmannian side, the relevant braid group action comes from twisting functors, while cups and caps arise from Lagrangian correspondences; in very loose terms, one should think of them as versions of pushforward and pullback from leaves.

- Seidel-Smith and Manolescu work in the Fukaya category. Their braid group action comes from parallel transport in a space of complex structures, which we interpret as

a Fukaya version of twisting. Their cup and cap functors are the functors on quilted Fukaya categories induced by natural Lagrangian correspondences.

- Cautis-Kamnitzer work with coherent sheaves on convolution varieties for affine Grassmannians, but nothing is lost by replacing these varieties with certain open subsets which are conical symplectic resolutions (they are also $\mathfrak{sl}_\infty$ quiver varieties). After this modification, their braid group action is obtained via tensor produces with associated gradeds of twisting bimodules, and their cups and caps via tensor products with the associated gradeds of certain Harish-Chandra bimodules. Cautis has explained how their construction is a special case of a general construction from a categorical $\mathfrak{sl}_\infty$ action, and the quantum version of Cautis-Kamnitzer arises from applying this to $\mathfrak{sl}_\infty$ quiver varieties as in the paper [Weba].

On the other hand, from the quiver variety perspective, the braid group action one uses comes from shuffling functors, as suggested by Nakajima's work on tensor product quiver varieties. The cup and cap functors are harder to describe in this case, but should be thought of as some sort of restriction to slices. The two different approaches to categorified knot invariants can be summarized by the following table.

	Affine Grassmannian	Quiver varieties
braid actions	twisting	shuffling
cups and caps	push to/pull from leaves	push to/pull from slices
examples	[CK08b, CK08a, Cau15, Kam11, SS06b, Man07]	[Webc, Webd]

Strikingly, while these two contexts look very different, the basic geometric concepts involved (twisting/shuffling, leaves/slices) are interchanged by our conjectural duality. Thus, the fact that these two constructions exist and give the same knot invariants in type A serves as a powerful piece of evidence for our conjecture. In other types, there is no construction which yet exists on the affine Grassmannian side.

In type A, these knot homologies also have Koszul dual realizations, due to Sussan and Mazorchuk-Stroppel; these fit with the left and right hand columns of the table above, since the former uses twisting functors for the braid group action and the latter uses shuffling functors.

A An Ext-vanishing result (appendix by Ivan Losev)

Let P be a vector space over $\mathbb{C}$ equipped with a linear map $P \to H^2(\mathfrak{M}; \mathbb{C})$ whose image does not lie in any of the discriminant hyperplanes. Recall that $\mathscr{M}$ is the universal deformation of $\mathfrak{M}$, and $\mathscr{N}$ the affinization of this variety. Let $\mathscr{N}_P$ be the fiber product $\mathscr{N} \times_{H^2(\mathfrak{M}; \mathbb{C})} P$. The fiber product $\mathscr{M}_P = \mathscr{M} \times_{H^2(\mathfrak{M}; \mathbb{C})} P$ is a fiberwise symplectic resolution of singularities for $\mathscr{N}_P$; let ω_P be the fiberwise symplectic form on $\mathscr{M}_P$.

Let $\mathcal{D}_{P,h}$ be a $\mathbb{T} \times \mathbb{S}$-equivariant formal quantization of $\mathscr{M}_P$. This means that $\mathcal{D}_{P,h}$ is a $\mathbb{T} \times \mathbb{S}$-equivariant sheaf of $\mathbb{C}[P][[h]]$-algebras (that are flat over $\mathbb{C}[[h]]$ and are complete and separated in the h-adic topology) with a fixed isomorphism $\theta : \mathcal{D}_{P,h}/(h) \xrightarrow{\sim} \mathfrak{S}_{\mathscr{M}_P}$. As before, we choose $h\xi \in \Gamma(\mathscr{M}_P, \mathcal{D}_{P,h})$ such that $\xi := h^{-1}(h\xi)$ is a non-commutative moment map for the action of $\mathbb{T}$ on $\mathcal{D}_P := \mathcal{D}_{P,h}[h^{-1}]$.

Let $A_{P,h}$ denote the subalgebra of all $\mathbb{S}$-finite elements in $\Gamma(\mathscr{M}_P, \mathcal{D}_{P,h})$. This is an algebra over $\mathbb{C}[P][h]$ equipped with an action of $\mathbb{T} \times \mathbb{S}$ by automorphisms such that $A_{P,0} := A_{P,h}/(h)$ is identified with $\mathbb{C}[\mathscr{M}_P] = \mathbb{C}[\mathscr{N}_P]$. For $p \in P$ we set $A_{p,0} := A_{P,0}/(\ker p)$, where we view p as a homomorphism $\mathbb{C}[P] \to \mathbb{C}$; this is the space of functions on the fiber of $\mathscr{M}_P$ over the point p.

Set $A_P := A_{P,h}/(h-1) \cong A_{P,h}[h^{-1}]^{\mathbb{S}}$, $A_p := A_P/(\ker p)$. We have gradings on the algebras $A_{P,h}, A_{P,0}, A_P, A_{p,0}, A_p$ coming from the $\mathbb{T}$-action. The ith graded components will be denoted by $A_P(i), A_{P,0}(i)$, etc. We remark that on A_P, A_p this grading is inner – it comes from the inner derivation $[\xi, \cdot]$.

As in Section 3.2 (and in previous works such as [Los12c, BLPW12, GL14]), we can consider the full subcategory $\mathcal{O}_p$ in the category A_p-mod of finitely generated modules consisting of all modules where A_p^+ acts locally finitely.

Let Δ_P denote the left A_P-module $A_P/A_P A_P^{>0}$. We use the notation $\Delta_?$ for various specializations of Δ_P. We remark that Δ_p is an object in $\mathcal{O}_p$. Now consider the right A_P-module $\nabla_P^\vee := A_P/A_P^{<0} A_P$. The $\mathbb{T}$-grading on A_P induces a $\mathbb{T}$-grading on $\nabla_P^\vee$. We can consider the specializations $\nabla_?^\vee$ of $\nabla_P^\vee$. We will see below that, for any $(p, h) \in P \oplus \mathbb{C}$, all graded components $\nabla_{p,h}^\vee(i)$ are finite dimensional. Let $\nabla_{p,h}$ denote the restricted dual of $\nabla_{p,h}^\vee$, i.e., $\nabla_{p,h} = \bigoplus_i \left(\nabla_{p,h}^\vee(i) \right)^*$. We have $\nabla_{p,h}(i) = 0$ for $i > 0$. So ∇_p lies in the ind completion of $\mathcal{O}_p$. In fact, one can show that ∇_p is finitely generated and so lies in $\mathcal{O}_p$.

The purpose of this appendix is to prove that:

Theorem A.1 *There is a non-empty Zariski open subset P^0 in the hyperplane $h = 1$ such that $\mathrm{Ext}^i_{A_p}(\Delta_p, \nabla_p) = 0$ for $p \in P^0$ and $i > 0$.*

There is a Zariski open subset of P where derived localization holds. Since the algebra A_p has finite global dimension on this open subset, the vanishing of the coherent sheaves $\mathrm{Ext}^i(\Delta_P, \nabla_P)$ at p for all i is an open condition. The content of this theorem is that this set is non-empty. In fact, it contains all but finitely many points of any affine line not parallel to the discriminant locus.

One corollary of our choice of p is that the category $\mathcal{O}_p$ becomes highest weight. Its standard objects are indecomposable direct summands of Δ_p and the costandard objects are indecomposable direct summands of ∇_p. In particular, $\mathrm{Ext}^1_{A_p}(\Delta_p, \nabla_p) = \mathrm{Ext}^1_{\mathcal{O}_p}(\Delta_p, \nabla_p) = 0$. Another corollary is that the algebra A_p has finite homological dimension not exceeding $\dim \mathfrak{M}$.

We also would like to mention that our proof of Theorem A.1 is inspired by the proof of an analogous statement for Rational Cherednik algebras, see [Eti12].

A.1 The proof

We start with an easy lemma that is analogous to [GL14, 3.1.4]. The proof is precisely like that of Lemma 3.13:

Lemma A.2 *The graded components of $\Delta_P, \nabla_P^\vee$ are finitely generated $\mathbb{C}[P]$-modules.* □

Next, we will need a structural result related to symplectic $\mathbb{C}^\times$-actions. Let $\mathbb{T}$ act on a smooth affine symplectic variety X with finitely many fixed points. Let x be one of the fixed points. Then the linear action of $\mathbb{T}$ on $T_x X$ is symplectic.

Lemma A.3 *There are homogeneous elements $x_1, \ldots, x_m \in \mathbb{C}[X]^{>0}, m = \frac{1}{2}\dim X$ such that the differentials $dx_1, \ldots, dx_m$ are linearly independent at the point x. Moreover, the ideal $\mathbb{C}[X]\mathbb{C}[X]^{>0}$ is a locally complete intersection generated in a neighborhood of x by the elements $x_1, \ldots, x_m$.*

Proof: This is a standard fact that can be deduced, for example, from the Luna slice theorem. □

Now we are going to reinterpret the Ext's between Δ_p and ∇_p in terms of Tor's between Δ_p and $\nabla_p^\vee$.

Lemma A.4 *We have $\operatorname{Ext}^i(\Delta_{p,h}, \nabla_{p,h}) = (\operatorname{Tor}_i(\nabla_{p,h}^\vee, \Delta_{p,h}))^*$. Here both the Ext's and the Tor's are taken over $A_{p,h}$.*

The spaces $\operatorname{Tor}_i(\nabla_{p,h}^\vee, \Delta_{p,h})$ are graded (via the $\mathbb{T}$-action) and the graded components are finite dimensional. This is because both modules are finitely generated and their graded components are finite dimensional. The superscript * means the restricted dual.

Proof: Let $(P_\bullet, d)$ be a free $\mathbb{T}$-equivariant resolution of $\Delta_{p,h}$. Then the Ext's in interest are the cohomology of the complex $\operatorname{Hom}(P_\bullet, \nabla_{p,h}) = P_\bullet^* \otimes_{A_{p,h}} \nabla_{p,h}$ and the differential is d^*. But $P_\bullet^* \otimes_{A_{p,h}} \nabla_{p,h} = (P_\bullet \otimes_{A_{p,h}} \nabla_{p,h}^\vee)^*$ and the differential d^* is the dual of the differential on the complex $P_\bullet \otimes_{A_{p,h}} \nabla_{p,h}^\vee$. The cohomology of the latter complex are the Tor's. Since the restricted duality * is an exact functor, we are done. □

So we only need to prove the analog of Theorem A.1 for $\operatorname{Tor}_i^{A_p}(\nabla_p^\vee, \Delta_p)$. First we are going to understand the behavior of the Tor's for $h = 0$.

Let ℓ be a line in P not lying in the discriminant hyperplanes. Let $M = \bigoplus_{i \in \mathbb{Z}} M(i), N = \bigoplus_{i \in \mathbb{Z}} N(i)$ be $\mathbb{T}$-equivariant $A_{\ell,0}$-modules with the following properties.

(i) M and N are finitely generated.

(ii) $M(i) = N(-i) = 0$ for $i \gg 0$ and $M(i), N(i)$ are finitely generated $\mathbb{C}[\ell]$-modules for all i.

(iii) The specializations of M, N at $p \in \ell \setminus \{0\}$ are $\Delta_{p,0}, \nabla^{\vee}_{p,0}$, respectively.

Proposition A.5 $\dim \operatorname{Tor}_i^{A_{\ell,0}}(M, N) < \infty$ *for all* $i > 0$.

Proof: The proof is in several steps.

Step 1. Let us show that $\operatorname{Tor}_i^{A_{\ell,0}}(M, N)$ is a finitely generated $\mathbb{C}[\ell]$-module for any i.

First of all, we claim that the $A_{\ell,0}$-module $\operatorname{Tor}_i^{A_{\ell,0}}(M, N)$ is supported on the $\mathbb{T}$-fixed point set $\mathcal{M}_\ell^{\mathbb{T}}$. Indeed, the condition $M(i) = 0$ for $i \gg 0$ implies that M is supported on the repelling locus for the $\mathbb{T}$-action. Similarly, the condition $N(i) = 0$ for $i \ll 0$ implies that N is supported on the contracting locus. The intersection of the two loci is precisely the set of $\mathbb{T}$-fixed points. Let $x \in \mathcal{N}_\ell$ be a point that is not $\mathbb{T}$-fixed. Form the completion $A_{\ell,0}^{\wedge x}$ of $A_{\ell,0}$ at x, this algebra is flat over $A_{\ell,0}$. From the above remarks, $A_{\ell,0}^{\wedge x} \otimes_{A_{\ell,0}} M = 0$ or $A_{\ell,0}^{\wedge x} \otimes_{A_{\ell,0}} N = 0$. Then we have $A_{\ell,0}^{\wedge x} \otimes_{A_{\ell,0}} \operatorname{Tor}_i^{A_{\ell,0}}(M, N) = \operatorname{Tor}_i^{A_{\ell,0}^{\wedge x}}(A_{\ell,0}^{\wedge x} \otimes_{A_{\ell,0}} M, A_{\ell,0}^{\wedge x} \otimes_{A_{\ell,0}} N) = 0$. Our claim in the beginning of this paragraph follows.

Now the original claim of this step will follow if we check that the morphism $\mathcal{N}_\ell^{\mathbb{T}} \to \ell$ is finite. The latter is a consequence of the following two observations. First, the morphism $\mathcal{M}_\ell^{\mathbb{T}} \to \mathcal{N}_\ell^{\mathbb{T}}$ is proper, surjective and locally finite. The last property follows from the assumption that $\mathbb{T}$ has finitely many fixed points on each $\mathcal{M}_p$. Second, the morphism $\mathcal{M}_\ell^{\mathbb{T}} \to \mathcal{N}_\ell^{\mathbb{T}}$ is also surjective, proper and locally finite.

Step 2. Let us show that $\operatorname{Tor}_i^{A_{p,0}}(\Delta_{p,0}, \nabla^{\vee}_{p,0}) = 0$ for $i > 0$. Similarly to Step 1, the $A_{p,0}$-module $\operatorname{Tor}_i^{A_{p,0}}(\Delta_{p,0}, \nabla^{\vee}_{p,0})$ is supported at the $\mathbb{T}$-fixed points. Pick such a point x. It is enough to show that the localization $\operatorname{Tor}_i^{A_{p,0}}(\Delta_{p,0}, \nabla^{\vee}_{p,0})_x$ is zero. Let $x_1, \ldots, x_m$ be as in Lemma A.3 and let $y_1, \ldots, y_m$ be similar elements but with negative eigen-characters. Then $d_x x_1, \ldots, d_x x_m, d_x y_1, \ldots, d_x y_m$ form a basis in $T_x X$. Locally, $\Delta_{p,0}$ is $A_{p,0}/(x_1, \ldots, x_m)$, while $\nabla^{\vee}_{p,0}$ is $A_{p,0}/(y_1, \ldots, y_m)$. Then the Tor vanishing is a standard fact.

Step 3. Let $\mathfrak{m}_p$ denote the maximal ideal of $p \in \ell$ in $\mathbb{C}[\ell]$. The previous step implies $\operatorname{Tor}_i^{A_{p,0}}(M/\mathfrak{m}_p M, N/\mathfrak{m}_p N) = 0$ for $i > 0$. But M, N are flat over $\mathbb{C}[\ell \setminus \{0\}]$ by our assumptions. It follows that the completion of $\operatorname{Tor}_i^{A_{\ell,0}}(M, N)$ at p is zero. Together with Step 1, this implies that $\operatorname{Tor}_i^{A_{\ell,0}}(M, N)$ is finite dimensional for $i > 0$. $\qquad\qquad\square$

Proposition A.6 *Let* $q \in P$. *Then* $\dim \operatorname{Tor}_i^{A_{q+\ell}}(\nabla^{\vee}_{q+\ell}, \Delta_{q+\ell}) < \infty$ *for all* $i > 0$.

Proof: The proof is again in several steps.

Step 1. Our goal is to deduce Proposition A.6 from Proposition A.5.

Let us notice that $\operatorname{gr} A_{q+\ell} = A_{\ell,0}$. The space $\operatorname{Tor}_i^{A_{q+\ell}}(\Delta_{q+\ell}, \nabla^{\vee}_{q+\ell})$ inherits a filtration from $A_{q+\ell}$. We can lift a $\mathbb{T} \times \mathbb{S}$-graded free resolution for $\operatorname{gr} \Delta_{q+\ell}$ to a free $\mathbb{T}$-graded resolution of $\Delta_{q+\ell}$. This easily implies that $\operatorname{gr} \operatorname{Tor}_i^{A_{q+\ell}}(\Delta_{q+\ell}, \nabla^{\vee}_{q+\ell})$ is a subquotient of $\operatorname{Tor}_i^{A_{\ell,0}}(\operatorname{gr} \Delta_{q+\ell}, \operatorname{gr} \nabla^{\vee}_{q+\ell})$,

where the filtrations on $\Delta_{q+\ell}, \nabla^\vee_{q+\ell}$ are induced from $A_{q+\ell}$. So it is enough to show that $\mathrm{Tor}_i^{A_{\ell,0}}(\mathrm{gr}\,\Delta_{q+\ell}, \mathrm{gr}\,\nabla^\vee_{q+\ell})$ is finite dimensional for all $i > 0$.

We are going to check the last claim using Proposition A.5. For this we only need to check that $M := \mathrm{gr}\,\Delta_{q+\ell}, N := \mathrm{gr}\,\nabla_{q+\ell}$ satisfy the conditions (i)-(iii) above. (i) and (ii) are clear (both $\Delta_{q+\ell}, \nabla_{q+\ell}$ are cyclic modules and so are $\mathrm{gr}\,\Delta_{q+\ell}, \mathrm{gr}\,\nabla^\vee_{q+\ell}$). We only need to check (iii), i.e., that the fibers of $\mathrm{gr}\,\Delta_{q+\ell}, \mathrm{gr}\,\nabla^\vee_{q+\ell}$ at $p \neq 0$ are $\Delta_{p,0}, \nabla^\vee_{p,0}$, respectively. We will give a proof for Δ, for ∇ it is analogous.

Step 2. In this step we will reduce the proof of the equality between the fiber of $\mathrm{gr}\,\Delta_{q+\ell}$ at p and $\Delta_{p,0}$ to the claim that a certain completion vanishes.

Consider the quotient $A_{\ell+hq,h}$ of $A_{P,h}$ by the ideal of the plane $(\ell + hq, h)$ in $\mathbb{C}[P, h]$. Consider the left ideal $I_{\ell+hq,h} := A_{\ell+hq,h} A^{>0}_{\ell+hq,h}$. Of course, $\Delta_{\ell+hq,h} = A_{\ell+hq,h}/I_{\ell+hq,h}$. Let $\tilde{I}_{\ell+hq,h}$ denote the h-saturation of $I_{h,\ell}$, i.e., $\tilde{I}_{\ell+hq,h}$ consists of all elements $a \in A_{\ell+hq,h}$ such that $h^k a \in I_{\ell+hq,h}$ for some $k > 0$. Let $\tilde{I}_{\ell,0}$ be the specialization of $\tilde{I}_{\ell+hq,h}$ at $h = 0$. Then, more or less by definition, $\mathrm{gr}\,\Delta_{q+\ell} = A_{\ell,0}/\tilde{I}_{\ell,0}$. So we only need to show that the $A_{\ell,0}$-module $M_\ell := \tilde{I}_{\ell,0}/I_{\ell,0}$ (that is the kernel of a natural epimorphism $\Delta_{\ell,0} \to \mathrm{gr}\,\Delta_{q+\ell}$) is supported on $\mathrm{Spec}(A_{0,0})$.

Since $M_\ell \subset \Delta_{\ell,0}$, it follows that $M_\ell(i) = 0$ for $i > 0$. So M_ℓ is supported on the repelling locus of the $\mathbb{T}$-action. Therefore it is enough to show that any $\mathbb{T}$-fixed point $x \in \mathscr{N}_p$ with $p \in \ell \setminus \{0\}$ does not lie in the support of M_p, the specialization of M_ℓ at p. Let $y \in \mathscr{N}_\ell//\mathbb{T}$ be the image of x. It will suffice to check that $M_p^{\wedge y} := \mathbb{C}[\mathscr{N}_p//\mathbb{T}]^{\wedge y} \otimes_{\mathbb{C}[\mathscr{N}_p//\mathbb{T}]} M_p$ is zero.

Step 3. Here we will reduce the proof of $M_p^{\wedge y} = 0$ to the claim that a certain ideal in a non-commutative completion of $A_{p+hq,h}$ is h-saturated.

By definition, M_p is the specialization of $\tilde{I}_{\ell+hq,h}/I_{\ell+hq,h}$ at $(p, 0)$. Let $\tilde{I}_{p+hq,h}$ be the h-saturation of $I_{p+hq,h}$. Clearly, the specialization of $\tilde{I}_{\ell+hq,h}|_p$ of $\tilde{I}_{\ell+hq,h}$ at p is contained in $\tilde{I}_{p+hq,h}$. The module $M_p^{\wedge y}$ is the specialization of the quotient

$$\left((A^{\mathbb{T}}_{h,p})^{\wedge y} \otimes_{A^{\mathbb{T}}_{h,p}} \tilde{I}_{\ell+hq,h}|_p\right) / \left((A^{\mathbb{T}}_{h,p})^{\wedge y} \otimes_{A^{\mathbb{T}}_{h,p}} I_{p+hq,h}\right)$$

at $h = 0$. Here $(A^{\mathbb{T}}_{h,p})^{\wedge y}$ is the completion of $A^{\mathbb{T}}_{h,p}$ at the maximal ideal that is the preimage under the epimorphism $A^{\mathbb{T}}_{h,p} \twoheadrightarrow \mathbb{C}[\mathscr{N}_p//\mathbb{T}]$ of the maximal ideal of y. Similarly, to [BPW], Step 3 of the proof of Proposition 5.4.4, we see that $(A^{\mathbb{T}}_{h,p})^{\wedge y}$ is a flat right module over $A^{\mathbb{T}}_{h,p}$. So

$$(A^{\mathbb{T}}_{h,p})^{\wedge y} \otimes_{A^{\mathbb{T}}_{h,p}} \tilde{I}_{\ell+hq,h}|_p \hookrightarrow (A^{\mathbb{T}}_{h,p})^{\wedge y} \otimes_{A^{\mathbb{T}}_{h,p}} \tilde{I}_{p+hq,h}.$$

So to check that $M_p^{\wedge y}$ is zero it is enough to show that the quotient

$$\left((A^{\mathbb{T}}_{h,p})^{\wedge y} \otimes_{A^{\mathbb{T}}_{h,p}} \tilde{I}_{p+hq,h}\right) / \left((A^{\mathbb{T}}_{h,p})^{\wedge y} \otimes_{A^{\mathbb{T}}_{h,p}} I_{p+hq,h}\right)$$

has no h-torsion.

In fact, we will need a few more reductions. First, let $\widehat{\otimes}$ denote the tensor product followed by the h-adic completion. The claim on the absence of the h-torsion is the same as

$$(A_{h,p}^{\mathbb{T}})^{\wedge_y} \widehat{\otimes}_{A_{h,p}^{\mathbb{T}}} \tilde{I}_{p+hq,h} = (A_{h,p}^{\mathbb{T}})^{\wedge_y} \widehat{\otimes}_{A_{h,p}^{\mathbb{T}}} I_{p+hq,h}.$$

Also let us notice that the left hand side lies in the h-saturation of the right hand side. So it is enough to show that the left ideal

$$(A_{h,p}^{\mathbb{T}})^{\wedge_y} \widehat{\otimes}_{A_{h,p}^{\mathbb{T}}} I_{p+hq,h} \subset A_{h,p}^{\wedge_{y,h}} := (A_{h,p}^{\mathbb{T}})^{\wedge_y} \widehat{\otimes}_{A_{h,p}^{\mathbb{T}}} A_{p+hq,h}$$

is h-saturated.

Step 4. Here we will investigate some properties of $(A_{h,p}^{\mathbb{T}})^{\wedge_y} \widehat{\otimes}_{A_{h,p}^{\mathbb{T}}} I_{p+hq,h}$.

Let us show that the left ideal of interest is closed in the h-adic topology. The algebra $A_{p,0}^{\wedge_y}$ is the algebra of $\mathbb{T}$-finite vectors in the completion $A_{p,0}^{\wedge_x}$. The latter is Noetherian. From here it is easy to deduce that $A_{p,0}^{\wedge_y}$ is Noetherian, compare with [Gin09], the proof of Lemma 2.4.2. The usual Hilbert argument (for power series) can be used now to show that any left ideal in $A_{p+hq,h}^{\wedge_{y,h}}$ is closed in the h-adic topology.

Modulo h, the left ideal $(A_{p+hq,h}^{\mathbb{T}})^{\wedge_y} \widehat{\otimes}_{A_{p+hq,h}^{\mathbb{T}}} I_{p+hq,h}$ is a complete intersection generated by some $\mathbb{T}$-equivariant elements $x_1, \ldots, x_m \in A_{p,0}^{>0}$, see Lemma A.3. Let us lift $x_1, \ldots, x_m$ to $\mathbb{T}$-semiinvariant elements $\tilde{x}_1, \ldots, \tilde{x}_m \in A_{p+hq,h}^{>0}$. We claim that $\tilde{x}_1, \ldots, \tilde{x}_m$ generate the ideal $(A_{p+hq,h}^{\mathbb{T}})^{\wedge_y} \widehat{\otimes}_{A_{p+hq,h}^{\mathbb{T}}} I_{p+hq,h}$. To establish this it is enough to check that any element of $A_{p+hq,h}^{\wedge_{y,h},>0}$ lies in the left ideal generated by $\tilde{x}_1, \ldots, \tilde{x}_m$. This easily follow from the observations that

- $A_{p+hq,h}^{\wedge_{y,h}}(i)$ is a closed and h-saturated subspace of $A_{p+hq,h}^{\wedge_{y,h}}$ for every i

- and that $A_{p+hq,h}^{\wedge_{y,h},>0}$ modulo h lies in the ideal generated by $x_1, \ldots, x_m$.

Step 5. This step will complete the proof of the claim that the left ideal

$$(A_{p+hq,h}^{\mathbb{T}})^{\wedge_y} \widehat{\otimes}_{A_{p+hq,h}^{\mathbb{T}}} I_{p+hq,h} \subset A_{p+hq,h}^{\wedge_{y,h}}$$

is h-saturated and hence the proof of the proposition, as well.

Thanks to the previous step, it is enough to show that the left ideal generated by $\tilde{x}_1, \ldots, \tilde{x}_m$ is h-saturated. This a corollary of a more general statement: that the coisotropic complete intersection always admits a quantization, but we are going to provide a proof here since we do not know a reference for that fact.

Assume the converse, let $a \in A_{p+hq,h}^{\wedge_{y,h}}$ be such that $ha \in A_{p+hq,h}^{\wedge_{y,h}}(\tilde{x}_1, \ldots, \tilde{x}_m)$ but $a \notin A_{p+hq,h}^{\wedge_{y,h}}(\tilde{x}_1, \ldots, \tilde{x}_m)$. Let $ha = \sum_{i=1}^m \tilde{b}_i \tilde{x}_i$ for some elements $\tilde{b}_1, \ldots, \tilde{b}_m \in A_{p+hq,h}^{\wedge_{y,h}}$. Let $b_1, \ldots, b_m$ be the classes of $\tilde{b}_1, \ldots, \tilde{b}_m$ modulo h. Then not all of $b_1, \ldots, b_m$ are 0 and we have $\sum_{i=1}^m b_i x_i = 0$. From the exactness of the Koszul complex, we deduce that there are elements $b_{ij} \in A_{p,0}^{\wedge_y}$ with $b_{ij} = -b_{ji}$ and $b_i = \sum_j b_{ij} x_j$. Let us choose liftings $\tilde{b}_{ij}$ of b_{ij} to

$A_{p+hq,h}^{\wedge h,y}$ so that $\tilde{b}_{ij} = -\tilde{b}_{ji}$. Set $c_i = h^{-1}(\tilde{b}_i - \sum_{j=1}^{m} \tilde{b}_{ij}\tilde{x}_j)$. We get

$$ha = \sum_{i=1}^{m} \tilde{b}_i \tilde{x}_i = \sum_{i,j=1}^{m} \tilde{b}_{ij}\tilde{x}_j\tilde{x}_i + hc_{ij}\tilde{x}_i = h(\sum_{i<j} \tilde{b}_{ij}\frac{1}{h}[\tilde{x}_j, \tilde{x}_i] + \sum_{i,j} c_{ij}\tilde{x}_i).$$

To complete the proof it remains to check that $\frac{1}{h}[\tilde{x}_i, \tilde{x}_j]$ lies in the left ideal generated by $\tilde{x}_i$. This follows from the observation that $\frac{1}{h}[\tilde{x}_i, \tilde{x}_j] \in A_{p+hq,h}^{>0}$. □

Corollary A.7 *Let $q \in P^0$. For each $n > 0$ there is an open subset $\ell^0 \subset \ell$ such that $\mathrm{Tor}_i^{A_{q+p}}(\nabla_{q+p}^{\vee}, \Delta_{q+p}) = 0$ for $0 < i < n$ and all $p \in \ell^0$.*

Proof: The $\mathbb{C}[\ell]$-module $\mathrm{Tor}_i^{A_{q+\ell}}(\nabla_{q+\ell}^{\vee}, \Delta_{q+\ell})$ is finite dimensional by Proposition A.6 and so is supported in finitely many points of ℓ. Let ℓ^0 be the complement of the supports of $\mathrm{Tor}_i^{A_{q+\ell}}(\nabla_{q+\ell}^{\vee}, \Delta_{q+\ell})$ for $0 < i < n$ in the intersection $\ell \cap P^0$. We claim that $\mathrm{Tor}_i^{A_{q+p}}(\nabla_{q+p}^{\vee}, \Delta_{q+p}) = 0$ for every $p \in \ell^0$. Indeed, let z be a coordinate on ℓ near p so that $\mathbb{C}[\ell] = \mathbb{C}[z]$ and the maximal ideal of p is generated by z. Then $\mathrm{Tor}_i^{A_{q+p}}(\nabla_{q+p}^{\vee}, \Delta_{q+p}) = \mathrm{Tor}_i^{A_{q+\ell}}(\nabla_{q+\ell}^{\vee}, \Delta_{q+p})$. Also we have the short exact sequence

$$0 \to \Delta_{q+\ell} \xrightarrow{z} \Delta_{q+\ell} \to \Delta_{q+p} \to 0$$

which yields the long exact sequence

$$\mathrm{Tor}_i^{A_{q+\ell}}(\nabla_{q+\ell}^{\vee}, \Delta_{q+\ell}) \xrightarrow{z} \mathrm{Tor}_i^{A_{q+\ell}}(\nabla_{q+\ell}, \Delta_{q+\ell}) \to \mathrm{Tor}_i^{A_{q+\ell}}(\nabla_{q+\ell}^{\vee}, \Delta_{q+p}) \to$$
$$\to \mathrm{Tor}_{i-1}^{A_{q+\ell}}(\nabla_{q+\ell}^{\vee}, \Delta_{q+\ell}) \xrightarrow{z} \mathrm{Tor}_{i-1}^{A_{q+\ell}}(\nabla_{q+\ell}^{\vee}, \Delta_{q+\ell}).$$

The first arrow is bijective, thanks to our choice of p, for any i with $0 < i < n$, while the last arrow is bijective for any i with $1 < i \leqslant n$. So we see that $\mathrm{Tor}_i^{A_{q+p}}(\nabla_{q+p}^{\vee}, \Delta_{q+p}) = 0$ for all i with $1 < i < n$. But $\mathrm{Tor}_1^{A_{q+p}}(\nabla_{q+p}^{\vee}, \Delta_{q+p}) = \mathrm{Ext}_{A_{q+p}}^1(\Delta_{q+p}, \nabla_{q+p})^*$. The former is zero by our initial assumptions on p. □

Proof of Theorem A.1: Let ℓ that passes through p which is not parallel to any discriminant hyperplane. Apply Corollary A.7 to $n = \dim X + 1$ and the line ℓ. We will get $\mathrm{Ext}_{A_{p'}}^i(\Delta_{p'}, \nabla_{p'}) = (\mathrm{Tor}_i^{A_{p'}}(\nabla_{p'}^{\vee}, \Delta_{p'}))^*$ for any $i \in \{1, \ldots, \dim \mathfrak{M}\}$ provided p' avoids the finitely number of bad points. Hence we can choose p' in the form $p + n\eta$ for $n \gg 0$; for n sufficiently large, the algebra $A_{p'}$ has finite global dimension equal to the dimension of $\mathfrak{M}$ since localization holds by Corollary 5.17. The desired conclusion follows. □

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