QUANTIZED COORDINATION ALGORITHMS FOR RENDEZVOUS AND DEPLOYMENT*

RUGGERO CARLI† AND FRANCESCO BULLO‡

Abstract. In this paper we study motion coordination problems for groups of robots that exchange information through a rate-constrained communication network. For rendezvous and deployment problems, we propose an integrated control and communication scheme combining a logarithmic coder/decoder with linear coordination algorithms. We show that the closed-loop performance is comparable to the one achievable in the quantization-free model: the time complexity is unchanged and the exponential convergence factor degrades smoothly as the quantization accuracy becomes coarser.

1. Introduction.

Problem description and motivation. This work focuses on robotic coordination problems among robots that communicate through reliable digital channels, i.e., robots connected by a data-rate constrained network. The problem is motivated by the rising importance of robotic network technologies and the scientific interest in combined control and communication problems. We consider two prototypical control problems, namely rendezvous and deployment, and we consider jointly the communication constraints of limited topology (specifically, the communication graph is a chain) and of limited bandwidth.

Literature review. This work combines methods from data-rate constrained control and robotic networks models. First, a rich literature is available on data-rate constrained and quantized control; e.g., see the survey in [15]. The quantized coordination problems has received instead less attention. In [8] a randomized distributed algorithm for the quantized consensus is proposed. A preliminary version of the coding/decoding strategies here analyzed is presented in [3]. Second, a survey on cooperative mobile robotics is presented in [2] and an overview of control and communication issues is contained in [9]. Additional references on rendezvous and deployment problems include [1, 10, 4, 11, 5]. In [12] the authors provide the formal definitions of robotic network, control and communication law, coordination task, and time and communication complexity. In [13] the same authors analyze a number of basic coordination algorithms running on synchronous robotic network. In particular, they provide an asymptotic characterization of the time complexity of the circumcenter law, achieving rendezvous, and of the centroid law, achieving deployment over a region of interest. Building on [12, 13], this paper analyzes these two control law with a new approach: we assume that the robots communicate throughout digital channels and hence can share only quantized information about their states. Additional references on convergence rates and time complexity of motion coordination algorithms include [16, 18]

Statement of contributions. The main contributions of this article are as follows. First, we formulate a novel coordination problems with quantized information (we

[1]Submitted to the SIAM Journal on Control and Optimization on December 2, 2007. This material is based upon work supported in part by ONR Award N00014-07-1-0721 and by ARO MURI Award W911NF-05-1-0219.

[2]Department of Information Engineering, Università di Padova, Via Gradenigo 6/a, 35131 Padova, Italy, calirug@dei.unipd.it

[3]Center for Control, Dynamical Systems and Computation, University of California, Santa Barbara, CA 93106, United States, bullo@engineering.ucsb.edu, http://motion.mee.ucsb.edu

assume however that the links are reliable, i.e., no transmitted packets are lost). Second, we adapt coding/decoding strategies, that were proposed for centralized control and communication problems, to the setting of multi-agent networked systems. In particular, we present two coding/decoding strategies, one based on the exchange of logarithmically quantized information, the other on a zoom in - zoom out strategy. Third, we show that the novel quantized coordination schemes (circumcenter law and the centroid law with quantized information) achieve the same rendezvous and deployment tasks as the corresponding previously-known schemes achieved with exact information (no quantization). Fourth, we show that these coordination tasks are achieved with unchanged time complexity, i.e., the asymptotic convergence factor is not affected by the introduction of a coder/decoder pair. Additionally, we show that the convergence factors depend smoothly on the accuracy parameter of the quantized and that, remarkably, that the critical quantizer accuracy sufficient to guarantee convergence is independent from the network dimension. Fifth and finally, although our mathematical analysis establishes convergence and time complexity only for coordination algorithms with logarithmic coders, through simulations we illustrate that zoom in - zoom out uniform coders achieve the same closed-loop convergence properties.

Organization. Section 2 introduces a model of robotic networks. In Section 3 we present two strategies of coding/decoding of the data throughout reliable digital channels: one based on logarithmic quantizers, the other on uniform quantizers. We analyze a combined logarithmic coder and coordination algorithm model in Section 4; all proofs are provided in Section 5. We provide simulations results in Section 6 and our conclusions in Section 7. In the Appendix A, we review some results on augmented tridiagonal matrices. In Appendix B we generalize a well-known algebraic result and a linear parameter-varying stability result.

Notation. We let $\{\texttt{true}, \texttt{false}\}$ be the set $\{\texttt{true}, \texttt{false}\}$. We let $\prod_{i\in\{1,\dots,N\}} S_i$ denote the Cartesian product of sets $S_1, \dots, S_N$. We let $\mathbb{R}_{>0}$ denote the set of strictly positive real numbers. The set of positive natural numbers is denoted by $\mathbb{N}$ and $\mathbb{Z}_{\geq 0}$ denote the set of non-negative integers. For $x \in \mathbb{R}^d$, we denote by $\|x\|$ and $\|x\|_\infty$ the Euclidean and the ∞-norm of x, respectively (recall that $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{d}\|x\|_\infty$ for all $x \in \mathbb{R}^d$). We define the vectors $\mathbf{0} = (0, \dots, 0)^T$ and $\mathbf{1} = (1, \dots, 1)^T$ in $\mathbb{R}^d$. For $f, g\colon \mathbb{N} \to \mathbb{R}$, we say that $f \in O(g)$ if there exist $N_0 \in \mathbb{N}$ and $k \in \mathbb{R}_{>0}$ such that $|f(N)| \leq k|g(N)|$ for all $N \geq N_0$. For $f, g\colon \mathbb{N} \to \mathbb{R}$, we say that $f \in o(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. For $m = [m_1, \dots, m_N]^T \in \mathbb{R}^N$, let $\operatorname{diag}\{m\}$ or $\operatorname{diag}\{m_1, \dots, m_N\}$ denote a diagonal matrix having the components of m as diagonal elements.

2. Models and problem statement. In this section we review from [12] a simple model of robotic network, the notions of control and communication law, and the notions of coordination tasks and time complexity. We then review two known algorithms for rendezvous and deployment, and state the quantized coordination problem of interest in this paper.

2.1. A robotic network model. The paper [12] proposes a formal model for robotic networks, and defines the notions of control and communication laws, tasks, and time and communication complexity. We present here simplified versions of these notions with a discrete-time communication, discrete-time motion model: each robot evolves in the physical domain, exchanges information with other robots in discrete time and executes a state machine, which we shall refer to as a processor.

A *robotic network* $\mathcal{S}$ is a group of N identical agents endowed with the following capabilities. The agents are at positions $x_i \in X$, $i \in \{1, \dots, N\}$, where the state space

X is a subset of $\mathbb{R}^d$, for $d \in \mathbb{N}$. In $\mathbb{R}^d$, we describe the agents' motion by

$$x_i(t+1) = x_i(t) + u_i(t), \tag{2.1}$$

where u_i is the ith control input taking value in some compact set U. More compactly we can write

$$x(t+1) = x(t) + u(t), \tag{2.2}$$

where $x(t) = [x_1(t), \dots, x_N(t)]^T$ and $u(t) = [u_1(t), \dots, u_N(t)]^T$. The agents exchange messages among themselves along communication links. The collection of communication links among robots is a set of un-ordered pairs of identifiers, i.e., communication is assumed to be bidirectional and the set of communication links is a set of undirected edges E. We assume that each agent can measure its own position. In summary, a robotic network is described by the set of identifiers $\{1, \dots, N\}$, the state space X and control set U for each robot, and the collection of communication links E.

A *control and communication law* $\mathcal{CC}$ for a robotic network $\mathcal{S}$ consists of the sets:

(i) $\mathcal{A}$, a set containing the `null` element, called the *communication alphabet*; elements of $\mathcal{A}$ are called *messages*;

(ii) W called the *processor state set*;

and of the maps:

(i) msg: $X \times W \times \{1, \dots, N\} \to \mathcal{A}$, called the *message-generation function*;

(ii) stf: $X \times W \times \mathcal{A}^N \to W$, called the *(processor) state-transition function*;

(iii) ctl: $X \times W \times \mathcal{A}^N \to U$, called the *(motion) control function*.

The rationale behind these definition is the following: the state of robot i includes both the physical state $x_i \in X$ and the *processor state* $w_i \in W$ of the state machine that robot i implements. At each time $t \in \mathbb{Z}_{\geq 0}$, robot i sends to each of its neighbors j in the communication graph $\mathcal{G} = (\{1, \dots, N\}, E)$ a message computed by applying the message-generation function to the current values of its physical state x_i, processor state w_i and to the identity j. Subsequently, but still at time $t \in \mathbb{Z}_{\geq 0}$, robot i updates the value of its processor state w_i by applying the state-transition function to the current value of its physical state x_i, processor state w_i and to the messages it receives from its neighbors. Finally, the motion of the ith robot is determined by applying the control function to x_i, w_i, and the messages received at time t. These ideas are formalized as follows. The *evolution* of $(\mathcal{S}, \mathcal{CC})$ from initial conditions $x_0 \in X^N$ and $w_0 \in W^N$ is the collection of curves $x_i\colon \mathbb{Z}_{\geq 0} \to X$ and $w_i\colon \mathbb{Z}_{\geq 0} \to W$, $i \in \{1, \dots, N\}$, satisfying

$$\begin{aligned} x_i(t+1) &= x_i(t) + \mathrm{ctl}(x_i(t), w_i(t), y_i(t)), \\ w_i(t) &= \mathrm{stf}(x_i(t), w_i(t-1), y_i(t)), \end{aligned}$$

with $x_i(0) = (x_0)_i$ and $w_i(-1) = (w_0)_i$, for $i \in \{1, \dots, N\}$. In the previous equations, $y_i\colon \mathbb{Z}_{\geq 0} \to \mathcal{A}^N$ (describing the messages received by processor i) has components $y_{i,j}(t)$, for $j \in \{1, \dots, N\}$, given by

$$y_{i,j}(t) = \begin{cases} \mathrm{msg}(x_j(t), w_j(t-1), i), & \text{if } (i,j) \in E, \\ \texttt{null}, & \text{otherwise.} \end{cases}$$

For convenience, we shall also write $x(t) = (x_1(t), \dots, x_n(t)) \in X^N$ and $w(t) = (w_1(t), \dots, w_n(t)) \in W^N$.

A *coordination task* for $\mathcal{S}$ is a map $\mathcal{T}\colon X^N \to \{\texttt{true}, \texttt{false}\}$. The control and communication law $\mathcal{CC}$ *achieves* the task $\mathcal{T}$ if, for all initial conditions $x_0 \in X^N$ and $w_0 \in W^N$, the network evolution $t \mapsto (x(t), w(t))$ has the property that there exists $T \in \mathbb{N}$ such that $\mathcal{T}(x(t)) = \texttt{true}$ for all $t \geq T$.

Next, we give a notion of performance for a control and communication law. The *time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ from* $(x_0, w_0) \in X^N \times W^N$ is

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) = \inf\{t \in \mathbb{Z}_{\geq 0} \mid \mathcal{T}(x(\tau)) = \texttt{true}\,,\ \text{for all } \tau \geq t\},$$

where $t \mapsto (x(t), w(t))$ is the evolution of $(\mathcal{S}, \mathcal{CC})$ from x_0, w_0. The *time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$* is

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}) = \sup\{\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) \mid x_0 \in X^N \text{ and } w_0 \in W^N\}.$$

Remark 2.1 (Static laws). *A control and communication law is* static *if W is a singleton. In this case, the state-transition function is trivial, and the law is determined only by the communication alphabet, the message-generation function and the control function.* •

In recent years several algorithms achieving different tasks, like deployment, rendezvous, flocking, cyclic pursuit have been designed for robotic networks models akin to the one here presented. In what follows, we review some of these algorithms adopting the treatment in [13]: we present a unified "distributed linear systems" framework for simple motion coordination tasks in 1-dimensional environments and the corresponding known convergence rates. Specifically, in the following two subsections we consider the problems of (i) rendezvous on the line and (ii) deployment on a segment.

2.2. Rendezvous on the line. We start by considering the rendezvous problem on the line. We consider a network of N agents moving in $X = \mathbb{R}$ and with communication described by the *chain graph* with edges:

$$E = \{\{i, i+1\} \mid i \in \{1, \dots, N-1\}\}. \tag{2.3}$$

For $\varepsilon > 0$, the *ε-rendezvous task* $\mathcal{T}_{\varepsilon\text{-rndzvs}}\colon \mathbb{R}^N \to \{\texttt{true}, \texttt{false}\}$ is defined by $\mathcal{T}_{\varepsilon\text{-rndzvs}}(x) = \texttt{true}$ if and only if

$$|x_i - \mathrm{avrg}(\{x_1, \dots, x_N\})| < \varepsilon, \quad \text{for all } i \in \{1, \dots, N\},$$

where we use the shorthand $\mathrm{avrg}(\{y_1, \dots, y_h\}) = (y_1 + \dots + y_h)/h$. In other words, $\mathcal{T}_{\varepsilon\text{-rndzvs}}$ is $\texttt{true}$ at $x \in \mathbb{R}^N$ if, for all $i \in \{1, \dots, N\}$, x_i is at distance less than ε from the average of its own position with the position of all agents.

To achieve this motion coordination task, we introduce the static *circumcenter* control and communication law. We start by recalling that the circumcenter of a point set on the line is the center of the smallest interval that encloses the set. Loosely speaking, the law is described as follows. At each time instant each agent transmits its position and receives its neighbors' positions; it computes the circumcenter of the point set comprised of its neighbors and of itself; it moves toward this circumcenter. Formally, we describe the law as follows. Given a parameter $k \in \mathbb{R}$, we denote the *circumcenter* law by $\mathcal{CC}^{(k)}_{\mathrm{crcmcntr}}$ and define its control function by

$$\begin{aligned} u_1(t+1) &= -kx_1(t) + kx_2(t), \\ u_i(t+1) &= kx_{i-1}(t) - 2kx_i(t) + kx_{i+1}(t), \qquad i \in \{2, \dots, N-1\}, \\ u_N(t+1) &= kx_{N-1}(t) - kx_N(t). \end{aligned}$$

With this control law and adopting the definitions in Appendix A, the closed loop system reads

$$x(t+1) = \mathrm{ATrid}_N^+(k, 1-2k)x(t). \tag{2.4}$$

It is easy to see that the above law preserves at each time instant the average of the positions of the agents, namely $\frac{1}{N}\mathbf{1}^T x(0) = \frac{1}{N}\mathbf{1}^T x(t)$ for all t. In other words $\mathcal{CC}_{\mathrm{crcmcntr}}^{(k)}$ is a state-sum preserving law. This implies that, if $\mathcal{CC}_{\mathrm{crcmcntr}}^{(k)}$ reaches asymptotically the rendezvous then $\lim_{t\to+\infty} x(t) = x^*\mathbf{1}$ where $x^* = \frac{1}{N}\mathbf{1}^T x(0)$ is the average of the initial condition $x(0)$. We define now $\bar{x}(t) = x(t) - x^*\mathbf{1}$. Since $\mathrm{ATrid}_N^+(k, 1-2k)x^*\mathbf{1} = x^*\mathbf{1}$, it follows immediately that $\bar{x}$ satisfies the same recursive equation of x, i.e.

$$\bar{x}(t+1) = \mathrm{ATrid}_N^+(k, 1-2k)\bar{x}(t). \tag{2.5}$$

Next, we introduce two definitions. First, the *quadratic form* associated with a symmetric matrix $B \in \mathbb{R}^{N\times N}$ is the function $\|\cdot\|_B \colon \mathbb{R}^N \to \mathbb{R}$ defined by $\|x\|_B^2 = x^T Bx$. Second, a function $f \colon \mathbb{Z}_{\geq 0} \to \mathbb{R}$ converges to 0 *exponentially fast* if there exist a time $t_0 \in \mathbb{Z}_{\geq 0}$ and a positive constant $\lambda \in [0, 1[$ such that $|f(t)| \leq \lambda^t |f(0)|$, for all $t > t_0$; we say that λ is the *exponential convergence factor* of f.

The following theorem on the above family of algorithms summarizes the known results about the asymptotic properties and the complexity of $\mathcal{CC}_{\mathrm{crcmcntr}}^{(k)}$.

Theorem 2.2 (Time complexity of circumcenter law on the line). *Consider the network $\mathcal{S}$ with edge set in* (2.3) *on the line $X = \mathbb{R}$. Take $\varepsilon \in \mathbb{R}_{>0}$ and $k \in\,]0, \frac{1}{2}]$. Then the following statements hold:*

(i) the law $\mathcal{CC}_{\mathrm{crcmcntr}}^{(k)}$ achieves the ε-rendezvous task $\mathcal{T}_{\varepsilon\text{-rndzvs}}$;

(ii) $\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}^{(k)}) \in O(N^2 \log(N\varepsilon^{-1}))$ as $\varepsilon \to 0^+$ and $N \to +\infty$; and

(iii) along any network evolution $t \mapsto x(t)$, the function $t \mapsto \|\bar{x}(t)\|_P^2$, for $P = \mathrm{ATrid}_N^+(-1, 2)$, has exponential convergence factor $1 - 2k\pi^2/N^2 + o(1/N^2)$.

The proof is a slight generalization of the treatment in [12]; we do not report it here in the interest of brevity.

2.3. Deployment on a segment. We consider a network $\mathcal{S}$ of N agents moving inside a segment $X = [-q, q]$, for $q \in \mathbb{R}_{>0}$, with communication given by the chain graph in equation (2.3). We define the desired agent placement to be the vector $x^* \in [-q, q]^N$ with components

$$x_i^* = -q + \frac{q}{N}(1 + 2(i-1)), \qquad i \in \{1, \dots, N\}.$$

This uniform placement is optimal with respect to a cost function given in [13] and discussed below. For small $\varepsilon \in \mathbb{R}_{>0}$, we define the *$\varepsilon$-deployment task* $\mathcal{T}_{\varepsilon\text{-deplmnt}} \colon [-q, q]^N \to \{\mathtt{true}, \mathtt{false}\}$ by

$$\mathcal{T}_{\varepsilon\text{-deplmnt}}(x) = \begin{cases} \mathtt{true}, & \text{if } |x_i - x_i^*| \leq \varepsilon, \text{ for all } i \in \{1, \dots, N\}, \\ \mathtt{false}, & \text{otherwise.} \end{cases}$$

To achieve this motion coordination task, we introduce the static *centroid* control and communication law. Loosely speaking, the law is described as follows. At each time instant each agent transmits its position and receives its neighbors' positions; it computes the centroid of an appropriate region, i.e., the agent's Voronoi cell inside the segment; and it takes a step toward this centroid. The Voronoi cell of agent i in the segment $[-q, q]$ is the set of points in $[-q, q]$ that are closer to agent i than to any other agent $j \neq i$. Formally, we describe the law as follows. Given a parameter $k \in \mathbb{R}$, we denote the centroid law by $\mathcal{CC}_{\text{centrd}}^{(k)}$ and define its control function by

$$\begin{aligned}
u_1(t) &= -3kx_1(t) + kx_2(t) + 2k(-q), \\
u_i(t) &= kx_{i-1}(t) - 2kx_i(t) + kx_{i+1}, \qquad i \in \{2, \dots, N-1\}, \\
u_N(t) &= kx_{N-1}(t) - 3kx_N(t) + 2kq.
\end{aligned}$$

Adopting the definitions in Appendix A, the control can be rewritten as $u(t) = \text{ATrid}_N^{-1}(k, -2k)x(t) + [-2kq, 0, \cdots, 0, 2kq]^T$ and the closed loop reads

$$x(t+1) = \text{ATrid}_N^{-}(k, 1-2k)x(t) + \begin{bmatrix} -2kq & 0 & \cdots & 0 & 2kq \end{bmatrix}^T. \tag{2.6}$$

It is easy to see that x^* is a fixed point for this closed loop system. It is convenient to define $\bar{x} = x - x^*$, so that

$$\bar{x}(t+1) = \text{ATrid}_N^{-}(k, 1-2k)\bar{x}(t). \tag{2.7}$$

The following theorem on the above family of algorithms summarizes the known results about the asymptotic properties and the complexity of $\mathcal{CC}_{\text{centrd}}^{(k)}$.

THEOREM 2.3 (Time complexity of centroid law on a segment). *Consider the network $\mathcal{S}$ with edge set in* (2.3) *in the segment $X = [-q, q]$. Take $\varepsilon \in \mathbb{R}_{>0}$ and $k \in]0, \frac{1}{2}]$. Then the following statements hold:*

(i) the law $\mathcal{CC}_{\text{centrd}}^{(k)}$ achieves the ε-deployment task $\mathcal{T}_{\varepsilon\text{-deplmnt}}$;

(ii) $\text{TC}(\mathcal{T}_{\varepsilon\text{-deplmnt}}, \mathcal{CC}_{\text{centrd}}^{(k)}) \in O(N^2 \log(N\varepsilon^{-1}))$ *as $\varepsilon \to 0^+$ and $N \to +\infty$; and*

(iii) along any network evolution $t \mapsto x(t)$, the function $t \mapsto \|\bar{x}(t)\|_P^2$, for $P = \text{ATrid}_N^{-}(-1, 2)$, has exponential convergence factor $1 - 2k\pi^2/N^2 + o(1/N^2)$.

The proof is a slight generalization of the treatment in [12]; we do not report it here in the interest of brevity.

2.4. Quantized coordination problems. The two algorithms we presented and numerous others in the literature rely upon a crucial assumption: each agent transmits to its neighboring agents the precise value of its state, i.e., the message generating function is $\text{msg}(x, w, j) = x$. This implies the exchange of perfect information through the communication network.

In what follows, we consider a more realistic case, i.e., we assume that the communication network is constituted only of rate-constrained digital links. Accordingly, the main objectives of this paper are to understand (1) how the previous algorithms may be modified to overcome the forced quantization effects due to the digital channel, and (2) how much does their performance degrade. We refer to such problems as to "quantized coordination" problems.

We note that the presence of a rate constraint prevents the agents from having a precise knowledge about the state of the other agents. In fact, through a digital channel, the i-th agent can only send to the j-th agent symbolic data in a finite or countable alphabet; using only this data, the j-th agent can build at most an estimate of the i-th agent's state. To tackle this problem we take a two step approach. First, we introduce two coding/decoding schemes; each agent uses one of these schemes to estimate the positions of its neighbors. Second, we consider the same control laws presented above for deployment and rendezvous tasks, where, in place of the exact knowledge of the states of the systems, we substitute estimates calculated according to the proposed coding/decoding schemes.

3. Coder/decoder pairs for digital channels. In this section we discuss a general and two specific coder/decoder models for reliable digital channels; we follow the treatment in the survey [15]. We will later adopt this coder/decoder structure to define communication protocols in the robotic network.

Suppose a source wants to communicate to a receiver some time-varying data $x\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}$ via repeated transmissions at time instants in $\mathbb{Z}_{\geq 0}$. Each transmission takes place through a digital channel, i.e., messages can only be symbols in a finite or countable set (to be designed). The channel is assumed to be reliable, that is, each transmitted symbol is received without error. A coder/decoder pair for a digital channel is defined by the sets:

(i) a set Ξ, serving as *state space* for the coder/decoder; a fixed $\xi_0 \in \Xi$ is the *initial coder/decoder state*;

(ii) a finite or countable set $\mathcal{A}$, serving as *transmission alphabet*; elements $\alpha \in \mathcal{A}$ are called message;

and by the maps:

(i) a map $F\colon \Xi \times \mathcal{A} \to \Xi$, called the *coder/decoder dynamics*;

(ii) a map $Q\colon \Xi \times \mathbb{R} \to \mathcal{A}$, being the *quantizer function*;

(iii) a map $H\colon \Xi \times \mathcal{A} \to \mathbb{R}$, called the *decoder function*.

The coder computes the symbols to be transmitted according to, for $t \in \mathbb{Z}_{\geq 0}$,

$$\xi(t+1) = F(\xi(t), \alpha(t)), \quad \alpha(t) = Q(\xi(t), x(t)).$$

Correspondingly, the decoder implements, for $t \in \mathbb{Z}_{\geq 0}$,

$$\xi(t+1) = F(\xi(t), \alpha(t)), \quad \widehat{x}(t) = H(\xi(t), \alpha(t)).$$

Coder and decoder are jointly initialized at $\xi(0) = \xi_0$. Note that an equivalent representation for the coder is $\xi(t+1) = F(\xi(t), Q(\xi(t), x(t)))$, and $\alpha(t) = Q(\xi(t), x(t))$. In summary, the coder/decoder dynamics is given by

$$\begin{aligned} \xi(t+1) &= F(\xi(t), \alpha(t)), \\ \alpha(t) &= Q(\xi(t), x(t)), \\ \widehat{x}(t) &= H(\xi(t), \alpha(t)). \end{aligned} \tag{3.1}$$

In what follows we present two interesting coder/decoder pairs: the logarithmic quantizer strategy and the "zoom in - zoom out" uniform quantizer strategy.

3.1. Logarithmic coder. This strategy is presented for example in [6]. Given an *accuracy parameter* $\delta \in \,]0,1[\,$, define the *logarithmic set of quantization levels*

$$S_\delta = \Big\{ \Big(\frac{1+\delta}{1-\delta}\Big)^\ell \Big\}_{\ell \in \mathbb{Z}} \cup \{0\} \cup \Big\{ -\Big(\frac{1+\delta}{1-\delta}\Big)^\ell \Big\}_{\ell \in \mathbb{Z}}, \tag{3.2}$$

and the corresponding *logarithmic quantizer* (see Figure 3.1 (left panel)) $\mathrm{lgq}_\delta\colon \mathbb{R} \to S_\delta$ by

$$\mathrm{lgq}_\delta(x) = \begin{cases} \Big(\frac{1+\delta}{1-\delta}\Big)^\ell, & \text{if } \ell \in \mathbb{Z} \text{ satisfies } \dfrac{(1+\delta)^{\ell-1}}{(1-\delta)^\ell} \leq x \leq \dfrac{(1+\delta)^\ell}{(1-\delta)^{\ell+1}}, \\ 0, & \text{if } x = 0, \\ -\mathrm{lgq}_\delta(-x), & \text{if } x < 0. \end{cases}$$

Smaller values of the parameter δ correspond to more accurate logarithmic quantizers lgq_δ. For $\delta \in]0,1[$, the *logarithmic coder/decoder* is defined by the state space $\Xi = \mathbb{R}$, initial state $\xi_0 = 0$, the alphabet $\mathcal{A} = S_\delta$, and by the maps

$$\begin{aligned} \xi(t+1) &= \xi(t) + \alpha(t), \\ \alpha(t) &= \mathrm{lgq}_\delta(x(t) - \xi(t)), \\ \widehat{x}(t) &= \xi(t) + \alpha(t). \end{aligned} \tag{3.3}$$

The coder/decoder pair is analyzed as follows. One can observe that $\xi(t+1) = \widehat{x}(t)$ for $t \in \mathbb{Z}_{\geq 0}$, that is, the coder/decoder state contains the estimate of the data x. The transmitted messages contain a quantized version of the estimate error $x - \xi$. The estimate $\widehat{x} \colon \mathbb{Z}_{\geq 0} \to \mathbb{R}$ satisfies the recursive relation

$$\widehat{x}(t+1) = \widehat{x}(t) + \mathrm{lgq}_\delta\left(x(t+1) - \widehat{x}(t)\right),$$

with initial condition $\widehat{x}(0) = \mathrm{lgq}_\delta\left(x(0)\right)$ determined by $\xi(0) = 0$. Finally, define the function $r \colon \mathbb{R} \to \mathbb{R}$ by $r(y) = \dfrac{\mathrm{lgq}_\delta(y) - y}{y}$ for $y \neq 0$ and $r(0) = 0$. Some elementary calculations show that $|r(y)| \leq \delta$ for all $y \in \mathbb{R}$. Accordingly, if we define the trajectory $\omega \colon \mathbb{Z}_{\geq 0} \to [-\delta, +\delta]$ by $\omega(t) = r(x(t+1) - \widehat{x}(t))$, then we obtain that

$$\widehat{x}(t+1) = \widehat{x}(t) + (1 + \omega(t))\big(x(t+1) - \widehat{x}(t)\big). \tag{3.4}$$

3.4 will be useful later when we will analyze from a theoretical point of view the "quantized coordination".

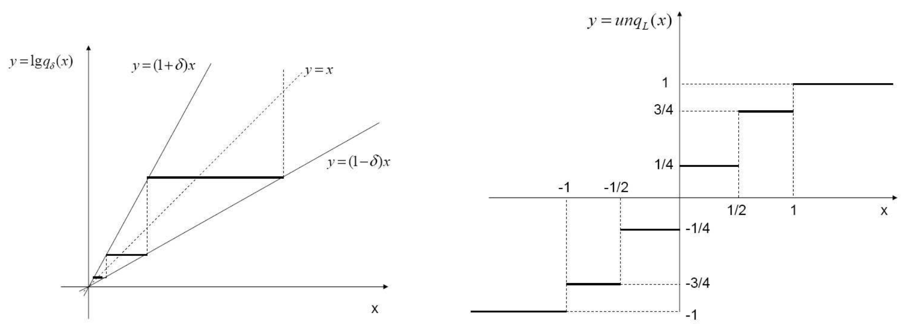

FIG. 3.1. *The logarithmic quantizer* (left) *and the uniform quantizer, for* $m = 6$ (right).

3.2. Zoom in - zoom out uniform coder. In this strategy the information transmitted from source to receiver is quantized by a scalar uniform quantizer which can be described as follows. For $L \in \mathbb{N}$, define the *uniform set of quantization levels*

$$S_L = \Big\{ -1 + \frac{2\ell - 1}{L} \mid \ell \in \{1, \ldots, L\} \Big\} \cup \{-1\} \cup \{1\}$$

and the corresponding *uniform quantizer* (see Figure 3.1 (right panel)) $\mathrm{unq}_L \colon \mathbb{R} \to S_L$ by

$$\mathrm{unq}_L(x) = \begin{cases} -1 + \dfrac{2\ell - 1}{L}, & \text{if } \ell \in \{1, \ldots, L\} \text{ satisfies } \ -1 + \dfrac{2(\ell-1)}{L} \leq x \leq -1 + \dfrac{2\ell}{L}, \\ 1, & \text{if } \ x > 1, \\ -1, & \text{if } \ x < -1. \end{cases}$$

Larger values of the parameter L correspond to more accurate uniform quantizers unq_L. Let m denote the number of quantization levels; then $m = L+2$. For $L \in \mathbb{N}$, $k_{\mathrm{in}} \in]0,1[$, and $k_{\mathrm{out}} \in]1,+\infty[$, the *zoom in - zoom out uniform coder/decoder* has the state space $\Xi = \mathbb{R} \times \mathbb{R}_{>0}$, the initial state $\xi_0 = (0,1)$, and the alphabet $\mathcal{A} = S_L$. The coder/decoder state is written as $\xi = (\widehat{x}_{-1}, f)$ and the coder/decoder dynamics are

$$\begin{aligned}\widehat{x}_{-1}(t+1) &= \widehat{x}_{-1}(t) + f(t)\alpha(t),\\ f(t+1) &= \begin{cases} k_{\mathrm{in}}\, f(t), & \text{if } |\alpha(t)| < 1,\\ k_{\mathrm{out}}\, f(t), & \text{if } |\alpha(t)| = 1.\end{cases}\end{aligned}$$

The quantizer and decoder functions are, respectively,

$$\begin{aligned}\alpha(t) &= \mathrm{unq}_L\Big(\frac{x(t)-\widehat{x}_{-1}(t)}{f(t)}\Big),\\ \widehat{x}(t) &= \widehat{x}_{-1}(t) + f(t)\alpha(t).\end{aligned}$$

The coder/decoder pair is analyzed as follows. One can observe that $\widehat{x}_{-1}(t+1) = \widehat{x}(t)$ for $t \in \mathbb{Z}_{\geq 0}$, that is, the first component of the coder/decoder state contains the estimate of the data x. The transmitted messages contain a quantized version of the estimate error $x - \widehat{x}_{-1}$ scaled by factor f. Accordingly, the second component of the coder/decoder state f is referred to as the *scaling factor*: it grows when $|x-\widehat{x}_{-1}| \geq f$ ("zoom out step") and it decreases when $|x-\widehat{x}_{-1}| < f$ ("zoom in step").

4. Coordination algorithms with exchange of quantized information. We consider now the same algorithms previously illustrated with the assumption that the agents can communicate only through digital channels. Specifically in this section we envision that each agent transmits to all its neighbors, the information regarding its position, adopting the logarithmic coder/decoder scheme (3.1) described in Subsection 3.1. We will analyze the zoom in - zoom out strategy by simulations in Section 6.

We begin by discussing how each agent communicates with all its neighbors and how the informations exchanged are used by each agent in order to build the estimates of the positions of its neighbors. Consider the i-th agent is a neighbor of the j-th agent at the time instant t. Assume that agent i and agent j maintain an estimate $\widehat{x}_{ij}(t) \in \mathbb{R}$ of the position of the j-th robot at the time instant t through a logarithmic coder/decoder with accuracy $\delta_{ij} \in]0,1[$. The estimate $\widehat{x}_{ij}$ is updated as follows. At the time instant $t+1$ the j-th agent measures its position $x_j(t+1)$. By the quantizer $\mathrm{lgq}_{\delta_{ij}}$, it quantizes logarithmically the quantity $x_j(t+1) - \widehat{x}_{ij}(t)$ obtaining the symbol $\alpha_{ij}(t)$ belonging to the logarithmic set of quantization levels as described in (3.2). It sends this symbol to the i-th agent that uses it to update $\widehat{x}_{ij}$ according to the rule (3.4), namely

$$\widehat{x}_{ij}(t+1) = \widehat{x}_{ij}(t) + (1+\omega_{ij}(t))\,(x_j(t+1) - \widehat{x}_{ij}(t))\,,$$

where we let $\omega_{ij}(t) = r\,(x_j(t+1) - \widehat{x}_{ij}(t))$.

In general we may have different encoders at the agent j, according to the intended receiver agent i. For the sake of the notational convenience, we assume that the agent j uses the same encoder for all the data transmissions. Thus, the agent j will send the same symbol $\alpha_j(t) := \alpha_{ij}(t)$ to all the other receiving agents i. In this case all the agents i will obtain the same estimate of x_j, namely we can define a single state estimate $\widehat{x}_j := \widehat{x}_{ij}$ which is common to all the agents i that receive messages from agent j. Moreover, we assume also that the logarithmic coder/decoders used by the agents have the same accuracy, namely the same value of the parameter $\delta \in]0,1[$. This implies that $\omega_{ij}(t) := \omega_j(t)$ for all i such that the i-th agent is a neighbor of the j-th agent. The above assumptions allow us to define the N-dimensional vector $\widehat{x} = [\widehat{x}_1, \dots, \widehat{x}_N]^T$. In the communication and control laws illustrated in Section 2 the state x will be now replaced by its estimate $\widehat{x}$. In the next subsections we will consider the quantized version of each algorithm in a detailed way. From now on, we will denote the communication and control laws by the symbols $\mathcal{CC}^{(k)}_{\text{q-centrd}}$, $\mathcal{CC}^{(k)}_{\text{q-midpoint}}$ and $\mathcal{CC}^{(k)}_{\text{q-crcmcntr}}$ meaning the fact that they are based on exchanges of quantized information.

4.1. Rendezvous on the line. It is worth providing an example formal definition of a control and communication law for a network with digital links. We do it here for the rendezvous problem on the line and are less formal in the other example below. Recall that the case without quantization is discussed in Section 2.2. Complying with the formal definition in Section 2.1, we define the control and communication law $\mathcal{CC}^{(k)}_{\text{q-crcmcntr}}$ with the following processor state set and message-generation, state-transition and control functions:

Processor state set: Given agent i at position x_i, the logic state w_i of agent i contains an estimate of its own position $\widehat{x}_i$ and of the position of its neighbors, namely $\widehat{x}_{i+1}$ and $\widehat{x}_{i-1}$ if $1 < i < N$, $\widehat{x}_2$ if $i = 1$, and $\widehat{x}_{N-1}$ if $i = N$. In other words, we select the processor state set to be $\mathbb{R}^2$ if $i = 1$ or $i = N$, and $\mathbb{R}^3$ if $1 < i < N$; in components, we set $w_1 = (\widehat{x}_1, \widehat{x}_2) \in \mathbb{R}^2$, $w_N = (\widehat{x}_{N-1}, \widehat{x}_N) \in \mathbb{R}^2$, and $w_i = (\widehat{x}_i, \widehat{x}_{i+1}, \widehat{x}_{i-1}) \in \mathbb{R}^3$ if $1 < i < N$;

Message-generation function: Given an accuracy $\delta \in]0,1[$, for $1 < i < N$, we have that agent i implements the message-generation function

$$\operatorname{msg}(x_i, (\widehat{x}_i, \widehat{x}_{i+1}, \widehat{x}_{i-1}), j) = \operatorname{lgq}_\delta(x_i - \widehat{x}_i), \quad j \in \{i-1, i+1\},$$

where we define the alphabet to be the logarithmic set of quantization levels S_δ, see Section 3.1; if $i = 1$ and $i = N$, then the above message-generation function modifies suitably to the communication graph, namely

$$\begin{aligned} \operatorname{msg}(x_1, (\widehat{x}_1, \widehat{x}_2), 2) &= \operatorname{lgq}_\delta(x_1 - \widehat{x}_1), \quad \text{and} \\ \operatorname{msg}(x_N, (\widehat{x}_N, \widehat{x}_{N-1}), N-1) &= \operatorname{lgq}_\delta(x_N - \widehat{x}_N). \end{aligned}$$

State-transition function: Consider first the case $1 < i < N$. We design the state-transition function component-wise as

$$\operatorname{stf}_j(x_i, (\widehat{x}_i, \widehat{x}_{i+1}, \widehat{x}_{i-1}), y_i) = \begin{cases} \widehat{x}_i + \operatorname{lgq}_\delta(x_i - \widehat{x}_i), & \text{if } j = i, \\ \widehat{x}_j + \alpha_j, & \text{if } j \in \{i-1, i+1\}, \end{cases}$$

where y_i, the array of received messages, has components $y_{i,j} = \alpha_j$, for $j \in \{i-1, i+1\}$; if $i = 1$ and $i = N$ we have respectively that

$$\operatorname{stf}_j(x_1, (\widehat{x}_1, \widehat{x}_2), y_1) = \begin{cases} \widehat{x}_1 + \operatorname{lgq}_\delta(x_1 - \widehat{x}_1), & \text{if } j = 1, \\ \widehat{x}_2 + \alpha_2, & \text{if } j = 2, \end{cases}$$

where $y_{1,2} = \alpha_2$, and

$$\mathrm{stf}_j(x_N, (\widehat{x}_N, \widehat{x}_{N-1}), y_1) = \begin{cases} \widehat{x}_N + \mathrm{lgq}_\delta(x_N - \widehat{x}_N), & \text{if } j = N, \\ \widehat{x}_{N-1} + \alpha_{N-1}, & \text{if } j = N-1, \end{cases}$$

where $y_{N,N-1} = \alpha_{N-1}$,

Control function: We define the control function by

$$\begin{aligned} u_1(t+1) &= -k\widehat{x}_1(t) + k\widehat{x}_2(t), \\ u_i(t+1) &= k\widehat{x}_{i-1}(t) - 2k\widehat{x}_i(t) + k\widehat{x}_{i+1}(t), \qquad i \in \{2, \dots, N-1\}, \\ u_N(t+1) &= k\widehat{x}_{N-1}(t) - k\widehat{x}_N(t). \end{aligned}$$

It is easy to see that, in matrix notation the above control function is written as $u(t) = \mathrm{ATrid}_N^+(k, -2k)\widehat{x}(t)$.

This completes the formal description of the law $\mathcal{CC}^{(k)}_{\text{q-crcmcntr}}$. To analyze the asymptotic properties of this law, it is convenient to write the close-loop system in terms of the quantities $\bar{x} = [\bar{x}_1, \dots, \bar{x}_N]^T$ and $e = [e_1, \dots, e_N]^T$, where $\bar{x}_i(t) = x_i(t) - \frac{1}{N}\mathbf{1}^T x(0)$ is the distance of the i-th agent from the average of the initial condition, and $e_i(t) = \widehat{x}_i(t) - x_i(t)$ is the estimate error. From now on, let $x^* = (1/N)\mathbf{1}^T x(0)$. We can write

$$\begin{aligned} x(t+1) &= x(t) + \mathrm{ATrid}_N^+(k, -2k)\widehat{x}(t) \\ &= (\mathrm{ATrid}_N^+(k, -2k) + I_N)x(t) + \mathrm{ATrid}_N^+(k, -2k)e(t) \\ &= \mathrm{ATrid}_N^+(k, 1-2k)x(t) + \mathrm{ATrid}_N^+(k, -2k)e(t). \end{aligned}$$

By recalling that $\mathrm{ATrid}_N^+(k, 1-2k)\mathbf{1} = \mathbf{1}$, it follows that

$$\bar{x}(t+1) = \mathrm{ATrid}_N^+(k, 1-2k)\bar{x}(t) + \mathrm{ATrid}_N^+(k, -2k)e(t).$$

Consider now the recursive equation for $\widehat{x}$:

$$\widehat{x}(t+1) = \widehat{x}(t) + (I_N + \Omega(t))\,(x(t+1) - \widehat{x}(t)),$$

where $\Omega(t) := \mathrm{diag}\{\omega_1(t), \dots, \omega_N(t)\}$. We rewrite this equation in terms of $\bar{x}$ and e:

$$\begin{aligned} \widehat{x}(t+1) - x(t+1) &= \Omega(t)\,(x(t+1) - \widehat{x}(t)) = \Omega(t)\,(x(t+1) - x(t) - e(t)) \\ &= \Omega(t)\,(x(t+1) - x^*\mathbf{1} - (x(t) - x^*\mathbf{1}) - e(t)) \\ &= \Omega(t)\left(\mathrm{ATrid}_N^+(k, -2k)\bar{x}(t) + \mathrm{ATrid}_N^+(k, -1-2k)e(t)\right). \end{aligned}$$

In summary, we can write the closed loop system in matrix form as

$$\begin{bmatrix} \bar{x}(t+1) \\ e(t+1) \end{bmatrix} = \begin{bmatrix} I_N & 0 \\ 0 & \Omega(t) \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ e(t) \end{bmatrix}, \tag{4.1}$$

where $A = \mathrm{ATrid}_N^+(k, 1-2k)$, $B = A - I_N$, and $C = A - 2I_N$. The following result illustrates the asymptotic properties and the time complexity of $\mathcal{CC}^{(k)}_{\text{q-crcmcntr}}$

THEOREM 4.1 (Time complexity of quantized circumcenter law). *Let $\varepsilon \in \mathbb{R}_{>0}$, let $k \in \mathbb{R}$ and $\delta \in \mathbb{R}$ such that $k \in \,]0, \frac{1}{2}]$ and $\delta \in \,]0, (1-2k)/(1+2k)[$. Then the following statements hold:*

(i) the law $\mathcal{CC}^{(k)}_{\text{q-crcmcntr}}$ achieves the ε-rendezvous task $\mathcal{T}_{\varepsilon\text{-rndzvs}}$;

(ii) $\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-rndzvs}}, \mathcal{CC}^{(k)}_{\text{crcmcntr}}) \in O(N^2 \log(N\varepsilon^{-1}))$ *as* $\epsilon \to 0$ *and* $N \to +\infty$*; and*

(iii) let $z(t) = [\bar{x}(t) \;\; e(t)]^T$*. Then, along any network evolution* $t \to z(t)$*, the function* $t \to \|z(t)\|^2_{\bar{P}}$ *has exponential convergence factor* $1 - \big(2 - \gamma(k,\delta)\big)\, k\pi^2/N^2 + o(1/N^2)$*, where*

$$\bar{P} = \begin{bmatrix} \mathrm{ATrid}^+_N(k, 1-2k) & 0 \\ 0 & k\delta^2\gamma(k,\delta) I_N \end{bmatrix},$$

for $\gamma(k,\delta) = 1 - 2k - (1+2k)\delta^2 - \sqrt{(1 - 2k - (1+2k)\delta^2)^2 - 16k^2\delta^2} \in \,]0, 1[\,.$

Proof. The proof follows immediately from Theorem 5.2 and Theorem 5.3 and Corollary 5.4 below. □

Remark 4.2.

(i) For a given fixed k, we have that $\lim_{\delta \to 0^+} \gamma(k,\delta) = 0^+$ *and hence, in the limit as* $\delta \to 0^+$*, the exponential convergence factors for the quantized circumcenter law and for the circumcenter law become equal.*

(ii) The control law described above is state-sum preserving, namely $\mathbf{1}^T x(t) = \mathbf{1}^T x(0)$ *for all t. This implies that, if* $\mathcal{CC}^{(k)}_{\text{q-crcmcntr}}$ *asymptotically reaches rendezvous, then* $\lim_{t \to +\infty} x(t) = x^* \mathbf{1}$ *where* $x^* = \frac{1}{N}\mathbf{1}^T x(0)$ *is the average of the initial condition* $x(0)$*. This same property holds also in the setting without quantization.*

(iii) It is possible to define a control and communication law in which each agent i uses the exact value x_i *instead of the estimate* $\widehat{x}_i$ *to compute the control law* u_i*. For such a law, time complexity and convergence factor are very similar to the ones obtained in Theorem 4.1.* •

4.2. Deployment on a segment. The quantized version of the control input $\mathcal{CC}^{(k)}_{\text{centrd}}$, denoted by $\mathcal{CC}^{(k)}_{\text{q-centrd}}$, is

$$\begin{aligned} u_1(t) &= -3k\widehat{x}_1(t) + k\widehat{x}_2(t) + 2k(-q), \\ u_i(t) &= k\widehat{x}_{i-1}(t) - 2k\widehat{x}_i(t) + k\widehat{x}_{i+1}, \qquad i \in \{2, \dots, N-1\}, \\ u_N(t) &= k\widehat{x}_{N-1}(t) - 3k\widehat{x}_N(t) + 2kq. \end{aligned}$$

In matrix notation, the control law is $u = \mathrm{ATrid}^-_N(k, -2k)\widehat{x} + [-2kq \; 0 \; \cdots \; 0 \; 2kq]^T$, and the evolution of x is

$$x(t+1) = x(t) + \mathrm{ATrid}^-_N(k, -2k)\widehat{x}(t) + \begin{bmatrix} -2kq & 0 & \cdots & 0 & 2kq \end{bmatrix}^T.$$

To analyze the asymptotic properties of this law, we introduce the estimation error $e(t) = \widehat{x}(t) - x(t)$ to write:

$$x(t+1) = \mathrm{ATrid}^-_N(k, 1-2k)x(t) + \mathrm{ATrid}^-_N(k, -2k)e(t) + \begin{bmatrix} -2kq & 0 & \cdots & 0 & 2kq \end{bmatrix}^T.$$

From Section 2.3, we recall $\bar{x}(t) = x(t) - x^*$, with $x^* = \mathrm{ATrid}^-_N(k, 1-2k)x^* + [-2kq, 0, \cdots, 0, 2kq]^T$, to write:

$$\bar{x}(t+1) = \mathrm{ATrid}^-_N(k, 1-2k)\bar{x}(t) + \mathrm{ATrid}^-_N(k, -2k)e(t).$$

Again we rewrite the recursive equation for $\widehat{x}$

$$\widehat{x}(t+1) = \widehat{x}(t) + (I_N + \Omega(t))\,(x(t+1) - \widehat{x}(t)),$$

in terms of $\bar{x}$ and e. Straightforward calculations similar to the ones of the previous subsection lead to the following

$$e(t+1) = \Omega(t)\left(\mathrm{ATrid}_N^-(k,-2k)\bar{x}(t) + \mathrm{ATrid}_N^-(k,-1-2k)e(t)\right).$$

In summary, we can write the closed loop system in matrix form as

$$\begin{bmatrix}\bar{x}(t+1)\\ e(t+1)\end{bmatrix} = \begin{bmatrix}I_N & 0\\ 0 & \Omega(t)\end{bmatrix}\begin{bmatrix}A & B\\ B & C\end{bmatrix}\begin{bmatrix}\bar{x}(t)\\ e(t)\end{bmatrix}, \tag{4.2}$$

where $A = \mathrm{ATrid}_N^-(k, 1-2k)$, $B = A - I_N$, and $C = A - 2I_N$. The following result illustrates the asymptotic properties and the time complexity of $\mathcal{CC}^{(k)}_{\text{q-centrd}}$

THEOREM 4.3 (Time complexity of quantized centroid law). *Let $\varepsilon \in \mathbb{R}_{>0}$, $k \in \mathbb{R}$, $\delta \in \mathbb{R}$ such that $k \in]0, \frac{1}{2}[$ and $\delta \in]0, (1-2k)/(1+2k)[$. Define $\gamma(k,\delta)$ as in Theorem 4.1. Then the following statements hold:*

(i) the law $\mathcal{CC}^{(k)}_{\text{q-centrd}}$ achieves the ε-deployment task $\mathcal{T}_{\varepsilon\text{-deplmnt}}$;

(ii) $\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-deplmnt}}, \mathcal{CC}^{(k)}_{\text{q-centrd}}) \in O(N^2\log(N\varepsilon^{-1}))$ as $\epsilon \to 0^+$ and $N \to +\infty$; and

(iii) let $z(t) = [\bar{x}(t)\ \ e(t)]^T$. Then, along any network evolution $t \to z(t)$, the function $t \to \|z(t)\|^2_{\bar{P}}$, has exponential convergence factor $1-\big(2-\gamma(k,\delta)\big)k\pi^2/N^2 + o(1/N^2)$, where

$$\bar{P} = \begin{bmatrix}\mathrm{ATrid}_N^-(-1,2) & 0\\ 0 & k\delta^2\gamma(k,\delta)I_N\end{bmatrix}.$$

Proof. The proof follows immediately from Theorem 5.6 and Theorem 5.7 and Corollary 5.8 below. ∎

5. Convergence analysis. This section is devoted to the study, based on a worst-case analysis, of the convergence properties of the systems (4.1) and (4.2) previously introduced. In this section we provide a detailed analysis only for (4.1). Since (4.2) can be treated in a similar way, for this case we will state only the statements of theorems and some remarks emphasizing the more relevant aspects.

5.1. Rendezvous. Consider the system (4.1) and let us define

$$\mathcal{A}(t) = \begin{bmatrix}I_N & 0\\ 0 & \Omega(t)\end{bmatrix}\begin{bmatrix}A & B\\ B & C\end{bmatrix}. \tag{5.1}$$

We recall that $A = \mathrm{ATrid}_N^+(k, 1-2k)$, $B = A - I_N$ and $C = A - 2I_N$. We start our analysis by rewriting (5.1) in a more suitable way. Let

$$\mathcal{E} = \left\{E \in \mathbb{R}^{N\times N} \mid E = \mathrm{diag}\{e_1,\dots,e_N\},\ \ e_i \in \{-1,+1\}, i \in \{1,\dots,N\}\right\}.$$

Notice that $\mathcal{E}$ contains 2^N elements. Hence, we can write $\mathcal{E} = \{E_1,\dots,E_{2^N}\}$, where we are assuming that some suitable way to enumerate the matrices inside $\mathcal{E}$ has been used. We assume that $E_1 = I_N$. By means of the above definitions we can introduce an another set of matrices

$$\mathcal{R} = \left\{R_i = \begin{bmatrix}I_N & 0\\ 0 & \delta E_i\end{bmatrix}\begin{bmatrix}A & B\\ B & C\end{bmatrix} \,\Big|\, E_i \in \mathcal{E}\right\}.$$

The set $\mathcal{R}$ is useful because the matrix $\mathcal{A}(t)$ belongs to $Co\,\{\mathcal{R}\}$ for all $t \geq 0$, where $Co\,\{\mathcal{R}\}$ denotes the convex hull of the set $\mathcal{R}$. In other words, for all $t \geq 0$, there exist nonnegative real numbers $\lambda_1(t), \ldots, \lambda_{2^N}(t)$ such that $\sum_{i=1}^{2^N} \lambda_i(t) = 1$ and

$$\mathcal{A}(t) = \sum_{i=1}^{2^N} \lambda_i(t) R_i.$$

Before proceeding with the analysis of (4.1), for $N \in \mathbb{N}$ and $k \in]0, 1/2[$, we define

$$\delta_{\text{q-crcmcntr}}(N) = \frac{1 - k + k\cos\frac{(N-1)\pi}{N}}{1 + k - k\cos\frac{(N-1)\pi}{N}}, \tag{5.2}$$

with following provable properties

(i) $\delta_{\text{q-midpoint}}(N) \geq (1-2k)/(1+2k)$, for all $N \in \mathbb{N}$, and the equality holds precisely when N is even;

(ii) $\lim_{N\to+\infty} \delta_{\text{q-midpoint}}(N) = (1-2k)/(1+2k)$.

We are able now to state the following result that will permit us to analyze the system (4.1) by means of Theorem B.2 (see Appendix B).

LEMMA 5.1. *For $v = \left[\mathbf{1}^T \mathbf{0}^T\right]^T$, we have*

$$R_i v = v, \quad \textit{and} \quad v^T R_i = v^T, \quad \textit{for all } i \in \{1, \ldots, 2^N\}.$$

Moreover, for $\delta_{\text{q-crcmcntr}}$ as in equation (5.2), the following facts are equivalent:

(i) 1 is the only eigenvalue of unit magnitude of the matrix $R_1 = R_1(\delta)$, and all its other eigenvalues are strictly inside the unit disc;

(ii) $\delta < \delta_{\text{q-crcmcntr}}$.

Proof. The first part of the lemma is easily proved by observing that

$$\begin{bmatrix} I_N & 0 \\ 0 & \delta E_i \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} I_N & 0 \\ 0 & \delta E_i \end{bmatrix} \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}. \tag{5.3}$$

Consider now R_1; to compute its eigenvalues we calculate

$$\det(\lambda I_N - R_1) = \det \begin{bmatrix} \lambda I_N - A & -B \\ -\delta B & \lambda I_N - \delta C \end{bmatrix}. \tag{5.4}$$

Since all the blocks of the above matrices are of the form ATrid_N^+, they commute and, therefore, we have from [17] that

$$\begin{aligned} \det(\lambda I_N - R_1) &= \det\left[\lambda^2 I_N - (A + \delta C)\lambda + \delta(AC - B^2)\right] \\ &= \det\left[\text{ATrid}_N^+\left(-k(1+\delta)\lambda, \lambda^2 - (1 - 2k + (-1-2k)\delta)\lambda - \delta\right)\right] \\ &= \left(\lambda^2 - (1-\delta)\lambda - \delta\right) \prod_{i=1}^{N-1} \left(\lambda^2 - b_i\lambda - \delta\right). \end{aligned}$$

where we let $b_i = 1 - 2k + (-1-2k)\,\delta + 2k(1+\delta)\cos\frac{\pi i}{N}$. Hence, the eigenvalues of R_1 are given by the solution of the N second order equations

$$\lambda^2 - (1-\delta)\lambda - \delta = 0 \tag{5.5}$$

$$\lambda^2 - b_i\lambda - \delta = 0, \qquad i \in \{1, \ldots, N-1\}. \tag{5.6}$$

The solutions of equation (5.5) are 1 and δ. For $i \in \{1, \ldots, N-1\}$, the two solutions of equation (5.6) are

$$\lambda_1^{(i)} = \frac{b_i - \sqrt{b_i^2 + 4\delta}}{2}, \qquad \text{and} \qquad \lambda_2^{(i)} = \frac{b_i + \sqrt{b_i^2 + 4\delta}}{2}.$$

By straightforward algebraic calculations, one can see that the conditions $|\lambda_1^{(i)}| < 1$ and $|\lambda_2^{(i)}| < 1$, for all $i \in \{1, \ldots, N-1\}$, are satisfied precisely when $\delta < \delta_{\text{q-crcmcntr}}$. □

The following two theorems characterize the asymptotic properties of system (4.1).

THEOREM 5.2. *Consider the system (4.1). The following facts are equivalent:*

(a) $\delta < \delta_{\text{q-crcmcntr}}$*;*

(b) for each initial condition $[\bar{x}(0)^T \;\; e(0)^T]^T$ *and for any sequence* $\{\Omega(t)\}_{t=0}^{+\infty}$*, we have*

$$\lim_{t \to +\infty} \begin{bmatrix} \bar{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} \alpha \mathbf{1} \\ 0 \end{bmatrix}, \qquad \text{for } \alpha = \frac{1}{N} \mathbf{1}^T \bar{x}(0).$$

Proof. We start by proving that (a) implies (b). In order to do so, we will show that, for $\delta < \delta_{\text{q-crcmcntr}}$, there exists a suitable matrix $\bar{P} \in \mathbb{R}^{2N \times 2N}$ satisfying

$$\bar{P} \left[\mathbf{1}^T \;\; \mathbf{0}^T\right]^T = 0, \tag{5.7}$$

$$z^T \bar{P} z > 0, \tag{5.8}$$

$$z^T \left(\frac{1}{2} \left(R_i^T \bar{P} R_j + R_j^T \bar{P} R_i \right) - \bar{P} \right) z < 0, \quad \text{for all } R_i,\, R_j \in \mathcal{R}, \tag{5.9}$$

for each nonzero $z \notin \operatorname{span}\{\left[\mathbf{1}^T \;\; \mathbf{0}^T\right]^T\}$. This fact, together with Theorem B.2 (see Appendix B) and Lemma 5.1, ensures that fact (a) implies (b). As candidate matrix $\bar{P}$ we select

$$\bar{P} = \begin{bmatrix} P & 0 \\ 0 & \gamma I_N \end{bmatrix}, \tag{5.10}$$

where $P = \text{ATrid}_N^+(-1, 2)$ and where γ is a suitable positive scalar to be determined. Observe that the eigenvalues of P are 0 and $2 - 2\cos\frac{i\pi}{N}$ for $i \in \{1, \ldots, N-1\}$ (see Appendix A), where it is immediate to see that $2 - 2\cos\frac{i\pi}{N} > 0$ for $i \in \{1, \ldots, N-1\}$. Since the spectrum of $\bar{P}$ is the union of the spectrum of P and of γI_N, it follows that also $\bar{P}$ has an eigenvalue equal to 0 and all other eigenvalues positive. Moreover, since $\bar{P} \left[\mathbf{1}^T \;\; \mathbf{0}^T\right]^T = \left[(P\mathbf{1})^T \;\; \mathbf{0}^T\right]^T = 0$, we have that the eigenspace associated to the eigenvalue 0 is generated by the vector $[\mathbf{1}^T \;\; 0^T]^T$. Hence, $\bar{P}$ satisfies (5.7) and (5.8). Moreover, by the structure of $\bar{P}$, it is easy to check that $R_i^T \bar{P} R_j = R_j^T \bar{P} R_i$ for all $R_i, R_j \in \mathcal{R}$. Thus, verifying (5.9) is equivalent to verify

$$z^T \left(R_i^T \bar{P} R_j - \bar{P} \right) z < 0, \quad \text{for all } R_i,\, R_j \in \mathcal{R}, \tag{5.11}$$

for any nonzero $z \notin \text{span}\{\begin{bmatrix}\mathbf{1}^T & \mathbf{0}^T\end{bmatrix}^T\}$. We have that

$$\begin{aligned}
R_i^T \bar{P} R_j - \bar{P} &= \begin{bmatrix} A^2P + \gamma\delta^2 BE_iE_jB - P & ABP + \gamma\delta^2 BE_iE_jC \\ ABP + \gamma\delta^2 CE_iE_jB & B^2P + \gamma\delta^2 CE_iE_jC - \gamma I_N \end{bmatrix} \\
&= \begin{bmatrix} A^2P + \gamma\delta^2 B^2 - P & ABP + \gamma\delta^2 BC \\ ABP + \gamma\delta^2 CB & B^2P + \gamma\delta^2 C^2 - \gamma I_N \end{bmatrix} \\
&\qquad + \begin{bmatrix} \gamma\delta^2 B(E_iE_j - I_N)B & \gamma\delta^2 B(E_iE_j - I_N)C \\ \gamma\delta^2 CB + \gamma\delta^2 C(E_iE_j - I_N)B & \gamma\delta^2 C(E_iE_j - I_N)C \end{bmatrix} \\
&= R_1 \bar{P} R_1 - \bar{P} - Q,
\end{aligned}$$

where

$$R_1^T \bar{P} R_1 - \bar{P} = \begin{bmatrix} A^2P + \gamma\delta^2 B^2 - P & ABP + \gamma\delta^2 BC \\ ABP + \gamma\delta^2 BC & B^2P + \gamma\delta^2 C^2 - \gamma I_N \end{bmatrix},$$

and

$$Q = \gamma\delta^2 \begin{bmatrix} B(I_N - E_iE_j)B & B(I_N - E_iE_j)C \\ C(I_N - E_iE_j)B & C(I_N - E_iE_j)C \end{bmatrix} = \gamma\delta^2 \begin{bmatrix} BK \\ CK \end{bmatrix} \begin{bmatrix} KB & KC \end{bmatrix},$$

with K such that $K^2 = I_N - E_iE_j$. Clearly, $Q \geq 0$ and $Q\begin{bmatrix}\mathbf{1}^T & \mathbf{0}^T\end{bmatrix}^T = 0$. Hence, if $R_1^T \bar{P} R_1 - \bar{P}$ satisfies (5.11), then (5.11) holds also for any pair R_i, R_j belonging to $\mathcal{R}$. We recall that all the matrices A, B, C, P are ATrid_N^+ and hence diagonalizable by the matrix F_+ (see Appendix A). In general, given $Z = \text{ATrid}_N^+(a, b)$ we have that $F_+^{-1} Z F_+ = \text{diag}\{\lambda_0(Z), \ldots, \lambda_{N-1}(Z)\}$ where $\lambda_0(Z), \ldots, \lambda_{N-1}(Z)$ denote the eigenvalues of Z and $\lambda_0(Z) = b + 2a$ (see Appendix A). We shall now find a condition on γ by applying Lemma B.3 to $R_1^T \bar{P} R_1 - \bar{P}$. Observe preliminary that $\lambda_0(P) = 0$ and $\lambda_0(B) = \lambda_0(A - I_N) = 0$ and that $P\mathbf{1} = B\mathbf{1} = 0$. It follows that $\lambda_0\left(P - A^2P - \gamma\delta^2B^2\right) = \lambda_0(I_N - A^2)\lambda_0(P) - \gamma\delta^2\lambda_0^2(B) = 0$ and that $\left(P - A^2P - \gamma\delta^2B^2\right)\mathbf{1} = (I_N - A^2)P\mathbf{1} - \gamma\delta^2B^2\mathbf{1} = \mathbf{0}$. Clearly,

$$z_1^T\left(P - A^2P - \gamma\delta^2B^2\right)z_1 > 0, \quad \text{for all } z_1 \notin \text{span}\{\mathbf{1}\}, \tag{5.12}$$

if and only if

$$\lambda_i\left(P - A^2P - \gamma\delta^2B^2\right) > 0, \quad \text{for all } i \in \{1, \ldots, N-1\}. \tag{5.13}$$

By a simple manipulation one can show that $P = \frac{1}{k}(I_N - A)$. Hence

$$\begin{aligned}
\lambda_i(P - A^2P - \gamma\delta^2B^2) &= \frac{1}{k}\lambda_i\left(I_N - A^2\right)\lambda_i\left(I_N - A\right) - \gamma\delta^2\left(A - I_N\right)^2 \\
&= \lambda_i^2\left(I_N - A\right)\left(\frac{1}{k}\lambda_i(I_N + A) - \gamma\delta^2\right).
\end{aligned}$$

Let $\lambda_{\min}(I_N + A)$ denote the smallest eigenvalue of $I_N + A$. Since the eigenvalues of A are 1 and $1 - 2k + 2k\cos\frac{i\pi}{N}$, for $i \in \{1, \ldots, N-1\}$, we have that $\lambda_{\min}(I_N + A) = 1 + \min\{1, \min_{1 \leq i \leq N-1}\{1 - 2k + 2k\cos\frac{i\pi}{N}\}\} \geq 2 - 4k$. Hence, in order for (5.13) to be satisfied, it suffices that

$$\gamma < \frac{2 - 4k}{k\delta^2}. \tag{5.14}$$

Obviously, if this last condition is satisfied, then we have that $\ker(P - A^2P - \gamma\delta^2B^2) = \mathrm{span}\{\mathbf{1}\}$. From $P\mathbf{1} = 0$ and $BC\mathbf{1} = (A - I_N)(A - 2I_N)\mathbf{1} = (A - I_N)(-\mathbf{1}) = 0$ we know that $(ABP + \gamma\delta^2BC)\mathbf{1} = \mathbf{0}$ and, in turn, that $\ker(P - A^2P - \gamma\delta^2B^2) \subseteq \ker(ABP + \gamma\delta^2BC)$. Consider now the condition

$$\gamma I_N - (B^2P + \gamma\delta^2C^2) - (ABP + \gamma\delta^2BC)(P - (A^2P + \gamma\delta^2B^2))^\dagger(ABP + \gamma\delta^2BC) > 0.$$

$\lambda_0\left(P - \left(A^2P + \gamma\delta^2B^2\right)\right) = 0$ implies that also $\lambda_0\left(P - \left(A^2P + \gamma\delta^2B^2\right)\right)^\dagger = 0$. Then

$$\begin{aligned}\lambda_0\Big(\gamma I_N - (B^2P + \gamma\delta^2C^2) - (ABP + \gamma\delta^2BC)(P - (A^2P + \gamma\delta^2B^2))^\dagger(ABP + \gamma\delta^2BC)\Big)\\ = \lambda_0\left(\gamma I_N - (B^2P + \gamma\delta^2C^2)\right) = \gamma\left(1 - \delta^2\right) > 0.\end{aligned}$$

Let now $i \in \{1, \dots, N-1\}$. From $\lambda_i(P - (A^2P + \gamma\delta^2B^2)) > 0$ we have that $\lambda_i(P - (A^2P + \gamma\delta^2B^2))^\dagger = 1/\lambda_i(P - (A^2P + \gamma\delta^2B^2))$. Therefore,

$$\begin{aligned}&\lambda_i\big(\gamma I_N - (B^2P + \gamma\delta^2C^2)\\ &\qquad -(ABP + \gamma\delta^2BC)(P - (A^2P + \gamma\delta^2B^2))^\dagger(ABP + \gamma\delta^2BC)\big) > 0\\ \iff\quad &\lambda_i\left((P - A^2P - \gamma\delta^2B^2)(\gamma I_N - B^2P - \gamma\delta^2C^2) - (ABP + \gamma\delta^2BC)^2\right) > 0\\ \iff\quad &\lambda_i\Big(\gamma\delta^2\left(AC - B^2\right)^2P - \gamma A^2P - B^2P^2 - \gamma^2\delta^2B^2 - \gamma\delta^2C^2P + \gamma P\Big) > 0.\end{aligned}$$

Let now $Y = \gamma\delta^2\left(AC - B^2\right)^2P - \gamma A^2P - B^2P^2 - \gamma^2\delta^2B^2 - \gamma\delta^2C^2P + \gamma P$. We have

$$\begin{aligned}Y &= \frac{\gamma\delta^2}{k}\left(A(A - 2I_N) - (A - I_N)^2\right)^2(I_N - A) - \frac{\gamma}{k}A^2(I_N - A) - \frac{1}{k^2}(A - I_N)^2(I_N - A)^2\\ &\quad - \gamma^2\delta^2(A - I_N)^2 - \frac{\gamma\delta^2}{k}(A - 2I_N)^2(I_N - A) + \frac{\gamma}{k}(I_N - A)\\ &= (I_N - A)\left[\frac{\gamma\delta^2}{k}I_N - \frac{\gamma}{k}A^2 - \frac{1}{k^2}(I_N - A)^3 - \gamma^2\delta^2(I_N - A) - \frac{\gamma\delta^2}{k}(A - 2I_N)^2 + \frac{\gamma}{k}I_N\right]\\ &= (I_N - A)\left[\frac{\gamma\delta^2}{k}(A - 3I_N)(I_N - A) + \frac{\gamma}{k}(I_N + A)(I_N - A) - \frac{1}{k^2}(I_N - A)^3 - \gamma^2\delta^2(I_N - A)\right]\\ &= (I_N - A)^2\left[\frac{\gamma\delta^2}{k}(A - 3I_N) + \frac{\gamma}{k}(I_N + A) - \frac{1}{k^2}(I_N - A)^2 - \gamma^2\delta^2I_N\right].\end{aligned}$$

From this last expression, straightforward calculations show that that the eigenvalues of Y are $\lambda_i(Y) = 4k\left(a_i\gamma^2 + b_i\gamma + c_i\right)\left(1 - \cos\frac{i\pi}{N}\right)^2$, for $i \in \{1, \dots, N-1\}$, where

$$a_i = -k\delta^2, b_i = 2\left[1 - k + k\cos\frac{i\pi}{N} - \delta^2\left(1 + k - k\cos\frac{i\pi}{N}\right)\right], c_i = -4k\left(1 - \cos\frac{i\pi}{N}\right)^2.$$

Denote the solutions of $a_i\gamma^2 + b_i\gamma + c_i = 0$ by

$$\gamma_i^{(1)} = \frac{-b_i + \sqrt{b_i^2 - 4a_ic_i}}{2a_i}, \qquad \gamma_i^{(2)} = \frac{-b_i - \sqrt{b_i^2 - 4a_ic_i}}{2a_i}.$$

We define now three functions $a : [-1, 1] \to \mathbb{R}$, $b : [-1, 1] \to \mathbb{R}$ and $c : [-1, 1] \to \mathbb{R}$ by

$$a(x) = -k\delta^2,\ b(x) = 2\left[1 - k + kx - \delta^2\left(1 + k - kx\right)\right], c(x) = -4k\left(1 - x\right)^2.$$

Using basic calculus ideas, one can show that the assumption $\delta < \delta_{\text{q-crcmcntr}}$ implies $b(x) > 0$ and $b(x)^2 - 4a(x)c(x) > 0$ for $x \in [-1, 1]$. These facts, together with the observations that $a(x) < 0$ and $c(x) < 0$, imply, by the Cartesian rule, that all the roots $\gamma_i^{(1)}, \gamma_i^{(2)}$, for $i \in \{1, \ldots, N-1\}$, are real and positive. Clearly, $\lambda_i(Y) > 0$, for $i \in \{1, \ldots, N-1\}$, if and only if $\gamma_i^{(1)} < \gamma < \gamma_i^{(2)}$. Let us define the functions $\gamma^{(1)} : [-1, 1] \to \mathbb{R}$ and $\gamma^{(2)} : [-1, 1] \to \mathbb{R}$ by

$$\gamma^{(1)}(x) = \frac{-b(x) + \sqrt{b(x) - 4a(x)c(x)}}{2a(x)}, \qquad \gamma^{(2)}(x) = \frac{-b(x) - \sqrt{b(x) - 4a(x)c(x)}}{2a(x)}$$

Now, using again basic calculus ideas, one can see that, if $\delta < \delta_{\text{q-crcmcntr}}$, then $\gamma^{(1)}(x)$ is monotonically decreasing for $x \in [-1, 1]$ whereas $\gamma^{(2)}(x)$ is monotonically increasing for $x \in [-1, 1]$. Moreover, $\gamma^{(1)}(-1) < \gamma^{(2)}(-1)$. Therefore, we can argue that

$$\max_{i \in \{1, \ldots, N-1\}} \gamma_i^{(1)} < \min_{i \in \{1, \ldots, N-1\}}, \gamma_i^{(2)}$$

and

$$\max_{i \in \{1, \ldots, N-1\}} \gamma_i^{(1)} = \gamma_{N-1}^{(1)}, \qquad \min_{i \in \{1, \ldots, N-1\}} \gamma_i^{(2)} = \gamma_{N-1}^{(2)}.$$

Moreover,

$$\gamma_{N-1}^{(1)} < \frac{1 - 2k - (1+2k)\delta^2 - \sqrt{\left(1 - 2k - (1+2k)\delta^2\right)^2 - 16k^2\delta^2}}{k\delta^2} = \bar{\gamma}_{N-1}^{(1)},$$

and

$$\gamma_{N-1}^{(2)} > \frac{1 - 2k - (1+2k)\delta^2 + \sqrt{\left(1 - 2k - (1+2k)\delta^2\right)^2 - 16k^2\delta^2}}{k\delta^2} = \bar{\gamma}_{N-1}^{(2)}.$$

Hence, in order to have $\lambda_i(Y) > 0$, for all $i \in \{1, \ldots, N-1\}$, it suffices that

$$\bar{\gamma}_{N-1}^{(1)} \leq \gamma \leq \bar{\gamma}_{N-1}^{(2)}, \tag{5.15}$$

Furthermore, observe that

$$\bar{\gamma}_{N-1}^{(2)} < \frac{2\left(1 - 2k - (1+2k)\delta^2\right)}{k\delta^2} < \frac{2\,(1-2k)}{k\delta^2},$$

thus implying that if $\gamma \leq \bar{\gamma}_{N-1}^{(2)}$, then also equation (5.14) is satisfied. Hence, if (5.15) holds, then the matrix $\bar{P}$ introduced in (5.10) satisfies (5.7), (5.8) and (5.9).

Finally, in order to prove that (b) implies (a) we consider the sequence $\mathcal{A}(0) = \mathcal{A}(1) = \mathcal{A}(2) = \ldots = R_1$. If $\delta \geq \delta_{\text{q-crcmcntr}}$, then at least two eigenvalues of R_1 are not strictly inside the unit disc and thus $\prod_{i=0}^{+\infty} R_1$ is a matrix which has at least two eigenvalues different from 0. □

Theorem 5.3. *Let $\bar{P}$ as in the previous theorem with γ as in (5.15). There exists $\beta(N) > 0$ such that*

$$z^T R_1^T \bar{P} R_1 z - z^T \bar{P} z < -\beta(N) z^T \bar{P} z, \qquad \textit{for all } z \notin \operatorname{span}\{\begin{bmatrix}\mathbf{1}^T & \mathbf{0}^T\end{bmatrix}^T\}, \tag{5.16}$$

and, as $N \to +\infty$,

$$\beta(N) = \left(2k - \gamma\delta^2 k^2\right) \frac{\pi^2}{N^2} + o\left(\frac{1}{N^2}\right).$$

Proof. We have that

$$R_1^T \bar{P} R_1 - (1-\beta)\bar{P} = \begin{bmatrix} A^2P + \gamma\delta^2 B^2 - P + \beta P & ABP + \gamma\delta^2 BC \\ ABP + \gamma\delta^2 BC & B^2P + \gamma\delta^2 C^2 - \gamma I_N + \gamma\beta I_N \end{bmatrix}.$$

Similarly to the previous theorem we shall apply Lemma B.3 to find a condition on β satisfying $z^T\left(R_1^T P R_1 - (1-\beta)P\right)z < 0$, for all $z \notin \operatorname{span}\{\left[\mathbf{1}^T \;\; \mathbf{0}^T\right]^T\}$. Observe that $\lambda_0\left(A^2P + \gamma\delta^2 B^2 - P + \beta P\right) = 0$ and that $\left(A^2P + \gamma\delta^2B^2 - P + \beta P\right)\mathbf{1} = 0$. We have that

$$\begin{aligned} P - A^2P - \gamma\delta^2B^2 - \beta P &= \frac{1}{k}(I_N - A) - \frac{1}{k}A^2(I_N - A) - \gamma\delta^2(I_N - A)^2 - \frac{\beta}{k}(I_N - A) \\ &= \frac{1}{k}(I_N - A)\left(I_N - A^2 - k\gamma\delta^2(I_N - A) - \beta I_N\right). \end{aligned}$$

Hence, it turns out that the condition $\lambda_i\left(P - A^2P - \gamma\delta^2B^2 - \beta P\right) > 0$ is satisfied if and only if

$$\begin{aligned} \beta &< \min_{i\in\{1,\dots,N-1\}} \Big(1 - \lambda_i(A)^2 - k\gamma\delta^2\left(1 - \lambda_i(A)\right)\Big) \\ &= \min_{i\in\{1,\dots,N-1\}} \Big(-4k^2 - 4k^2\cos^2\frac{i\pi}{N} + 4k - (4k - 8k^2)\cos\frac{i\pi}{N} - k\gamma\delta^2(2k - 2k\cos\frac{i\pi}{N})\Big). \end{aligned}$$

Define now the function $f : [-1,1] \to \mathbb{R}$ by

$$f(x) = 4k - 4k^2 + (-4k + 8k^2)x - 4k^2x^2 - k\gamma\delta^2\left(2k - 2kx\right)$$

Using basic calculus ideas one can show that, for $\delta < \delta_{\text{q-crcmcntr}}$, f is monotonically decreasing for $x \in [-1,1]$. Moreover, $f(x) > 0$ for $x \in [-1,1)$ and $f(1) = 0$. Therefore,

$$\begin{aligned} \min_{i\in\{1,\dots,N-1\}} &\Big(1 - \lambda_i(A)^2 - k\gamma\delta^2\left(1 - \lambda_i(A)\right)\Big) = 1 - \lambda_1(A)^2 - k\gamma\delta^2\left(1 - \lambda_1(A)\right) \\ &= 4k - 4k^2 + (-4k + 8k^2)\cos\frac{\pi}{N} - 4k^2\cos^2\frac{\pi}{N} - k\gamma\delta^2\left(2k - 2k\cos\frac{\pi}{N}\right) > 0. \end{aligned}$$

Clearly, if $\lambda_i(P - A^2P - \gamma\delta^2B^2 - \beta P) > 0$ for $i \in \{1,\dots,N-1\}$ then

$$\ker\left(P - A^2P - \gamma\delta^2B^2 - \beta P\right) = \operatorname{span}\{\mathbf{1}\}.$$

Since $\left(ABP + \gamma\delta^2BC\right)\mathbf{1} = 0$, it follows that

$$\ker\left(P - A^2P - \gamma\delta^2B^2 - \beta P\right) \subseteq \ker\left(ABP + \gamma\delta^2BC\right).$$

Consider now the condition

$$\begin{aligned} &\gamma I_N - \gamma\beta I_N - B^2P - \gamma\delta^2C^2 \\ &\quad - \left(ABP + \gamma\delta^2BC\right)\left(P - A^2P - \gamma\delta^2B^2 - \beta P\right)^{\dagger}\left(ABP + \gamma\delta^2BC\right) > 0. \end{aligned}$$

Let $i = 0$. Since $\lambda_0(P - (\beta P + A^2P + \gamma\delta^2 B^2))^\dagger = 0$, we have that

$$\begin{aligned}\lambda_0\Big(&\gamma I_N - (\gamma\beta I_N + B^2P + \gamma\delta^2C^2)\\ &- (ABP + \gamma\delta^2BC)(P - (\beta P + A^2P + \gamma\delta^2B^2))^\dagger(ABP + \gamma\delta^2BC)\Big)\\ &= \lambda_0\left(\gamma I_N - (\gamma\beta I_N + B^2P + \gamma\delta^2C^2)\right) = \gamma - \gamma\beta - \gamma\delta^2.\end{aligned}$$

Hence, it follows that the following condition must be satisfied

$$\beta < 1 - \delta^2.$$

Let now $i \in \{1, \dots, N-1\}$. Reasoning similarly to the proof of the previous theorem, it is straightforward to verify that

$$\begin{aligned}&\lambda_i\left(\gamma I_N - (\gamma\beta I_N + B^2P + \gamma\delta^2C^2)\right.\\ &\quad\left.-(ABP + \gamma\delta^2BC)(P - (\beta P + A^2P + \gamma\delta^2B^2))^\dagger(ABP + \gamma\delta^2BC)\right) > 0\\ \iff\quad &\lambda_i\left((P - (\beta P + A^2P + \gamma\delta^2B^2))(\gamma I_N - (\gamma\beta I_N + B^2P + \gamma\delta^2C^2))\right.\\ &\quad\left.-(ABP + \gamma\delta^2BC)^2\right) > 0\\ \iff\quad &\lambda_i\left(\beta^2\gamma P - \left(\gamma P - B^2P^2 - \gamma\delta^2C^2P + \gamma\left(P - A^2P - \gamma\delta^2B^2\right)\right)\beta\right.\\ &\quad\left.+\left(P - A^2P - \gamma\delta^2B^2\right)\left(\gamma I_N - B^2P - \gamma\delta^2C^2\right) - \left(ABP + \gamma\delta^2BC\right)^2\right) > 0\\ \iff\quad &\lambda_i\left(\beta^2\gamma P - \left(PY_1 + \gamma Y_2\right)\beta + Y_1Y_2 - Y_3^2\right) > 0,\end{aligned}$$

where $Y_1 = \gamma I_N - B^2P - \gamma\delta^2C^2$, $Y_2 = P - A^2P - \gamma\delta^2B^2$ and $Y_3 = ABP + \gamma\delta^2BC$. We know from the previous theorem that $\lambda_i\left(PY_1 + \gamma Y_2\right) > 0$ and $\lambda_i\left(Y_1Y_2 - Y_3^2\right) > 0$ for $i \in \{1, \dots, N-1\}$. From the Cartesian rule, it follows that the equation

$$\gamma\lambda_i(P)\beta^2 - \lambda_i\left(PY_1 + \gamma Y_2\right)\beta + \lambda_i\left(Y_1Y_2 - Y_3^2\right) = 0$$

has two solutions

$$\beta_1^{(i)} = \frac{\lambda_i\left(PY_1 + \gamma Y_2\right) - \sqrt{\lambda_i\left(PY_1 - \gamma Y_2\right)^2 + 4\gamma\lambda_i\left(PY_3^2\right)}}{2\gamma\lambda_i(P)}$$

and

$$\beta_2^{(i)} = \frac{\lambda_i\left(PY_1 + \gamma Y_2\right) + \sqrt{\lambda_i\left(PY_1 - \gamma Y_2\right)^2 + 4\gamma\lambda_i\left(PY_3^2\right)}}{2\gamma\lambda_i(P)},$$

which are both positive with $\beta_1^{(i)} \le \beta_2^{(i)}$. Therefore, we argue that there exists $\beta > 0$ such that (5.16) is satisfied. Indeed, it suffices to take β such that

$$\beta < \min\left\{1 - \lambda_1(A)^2 - k\gamma\delta^2\left(1 - \lambda_1(A)\right), \min_{i\in\{1,\dots,N-1\}}\beta_1^{(i)}, 1 - \delta^2\right\}.$$

Assume now that $N \to \infty$. By expanding $\cos\frac{\pi}{N}$ in its Taylor series we obtain that

$$\min_{i\in\{1,\dots,N-1\}}\Big(1 - \lambda_i(A)^2 - k\gamma\delta^2\left(1 - \lambda_i(A)\right)\Big) = \left(2k - \gamma\delta^2k^2\right)\frac{\pi^2}{N^2} + o\left(\frac{1}{N^2}\right).$$

We consider now $\min_{i\in\{1,\dots,N-1\}} \beta_1^{(i)}$. To analyze this quantity, we define the functions $y_1 : [-1,1] \to \mathbb{R}$, $y_2 : [-1,1] \to \mathbb{R}$ and $y_3 : [-1,1] \to \mathbb{R}$ by

$$\begin{aligned} y_1(x) &= \gamma - 8k^2(1-x)^3 - \gamma\delta^2(1+2k-2kx)^2 \\ y_2(x) &= 4k(1-x)^2\left[2-2k+2kx-k\gamma\delta^2\right] \\ y_3(x) &= 2k(-1+x)\left[2(1-2k+2kx)+\gamma\delta^2(-1-2k+2kx)\right]. \end{aligned}$$

Observe that $\lambda_i(Y_1) = y_1\left(\cos\frac{i\pi}{N}\right)$, $\lambda_i(Y_2) = y_2\left(\cos\frac{i\pi}{N}\right)$ and $\lambda_i(Y_3) = y_3\left(\cos\frac{i\pi}{N}\right)$. Finally, let $\bar\beta : [-1,1) \to \mathbb{R}$ such that

$$\bar\beta(x) = \frac{2(1-x)y_1(x)+\gamma y_2(x) - \sqrt{\left(2(1-x)y_1(x)-\gamma y_2(x)\right)^2 + 8\gamma(1-x)y_3^2(x)}}{4\gamma(1-x)}.$$

Using basic ideas calculus it is possible to see that $\bar\beta$ is concave in $[-1,1)$. Moreover, $\bar\beta(-1) > 0$ and $\lim_{x\to1}\bar\beta(x) = 0$. It follows that there exists $\bar N$ such that

$$\min_{i\in\{1,\dots,N-1\}} \beta_1^{(i)} = \beta_1^{(1)}, \quad \text{for all } N > \bar N.$$

Straightforward calculations show that

$$\lim_{x\to1} \frac{8\gamma(1-x)y_3^2(x)}{\left(2(1-x)y_1(x)-\gamma y_2(x)\right)^2} = 0$$

Thus, we have also that

$$\lim_{N\to+\infty} \frac{\lambda_1\left(PY_3^2\right)}{\lambda_1\left(PY_1-\gamma Y_2\right)^2} = 0.$$

Hence, for $N\to\infty$, we have that

$$\begin{aligned} \beta_1^{(1)} &\approx \frac{2\gamma\lambda_1(Y_2)}{2\gamma\lambda_1(P)} = \frac{2\gamma\lambda_1\left(P-A^2P-\gamma\delta^2B^2\right)}{2\gamma\lambda_1(P)} \\ &= \frac{\frac{1}{k}\lambda_1\left(I_N-A\right)\lambda_1\left(I_N-A^2\right) - \gamma\delta^2\lambda_1\left(\left(I_N-A\right)^2\right)}{\frac{1}{k}\lambda_1\left(I_N-A\right)} \\ &= \lambda_1\left(I_N-A^2\right) - k\gamma\delta^2\lambda_1\left(I_N-A\right) = 2k\frac{\pi^2}{N^2} - \gamma k^2\delta^2\frac{\pi^2}{N^2} + o\left(\frac{1}{N^2}\right), \end{aligned}$$

where in the last equality we have used the fact that $\lambda_1(A) = 1 - 2k\frac{\pi^2}{N^2} + o\left(\frac{1}{N^2}\right)$. In summary, for $N\to\infty$, we can find β satisfying (5.16) such that

$$\beta = 2k\frac{\pi^2}{N^2} - \gamma k^2\delta^2\frac{\pi^2}{N^2} + o\left(\frac{1}{N^2}\right).$$

□

The following corollary characterizes the time complexity of the quantized version of the rendezvous algorithm.

Corollary 5.4. *Consider the system (4.1). Then*

$$\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-rndzvs}}, \mathcal{CC}^{(k)}_{\text{q-crcmcntr}}) \in O(N^2\log(N\varepsilon^{-1})). \tag{5.17}$$

Proof. Recall that if Z is a positive semidefinite matrix, then

$$\lambda_{\min}(Z)\|v\|^2 \leq \|v\|_Z^2 \leq \lambda_{\max}(Z)\|v\|^2,$$

where $\lambda_{\min}$, $\lambda_{\max}$ denote respectively the smallest and the largest eigenvalues of Z. We can extend the above inequality in the following way. Let $v \in \{\ker Z\}^{\perp}$ then

$$\bar{\lambda}_{\min}(Z)\|v\|^2 \leq \|v\|_Z^2 \leq \lambda_{\max}(Z)\|v\|^2,$$

where $\bar{\lambda}_{\min}(Z)$ denotes the smallest eigenvalue of Z different from 0.

Consider now the matrix $\bar{P}$ introduced in Theorem 5.2 and the system (4.1). Let $z(t) = [\bar{x}(t)^T \;\; e(t)^T]^T$. We have that

$$\|z(t)\|_{\bar{P}} \leq (1-\beta)^t \|z(0)\|_{\bar{P}}.$$

Next, let us convert the above contraction inequality into an appropriate inequality on ∞-norm. First, we compute $\bar{\lambda}_{\min}(\bar{P})$. Since γ is independent on N, it follows that, for $N \to \infty$, $\bar{\lambda}_{min}(\bar{P}) = \bar{\lambda}_{min}(P) = 2 - 2\cos\frac{\pi}{N} = \frac{\pi^2}{N^2} + o\left(1/N^2\right)$. Recall now the basic inequality $\|w\|_\infty \leq \|w\| \leq \sqrt{N}\|w\|_\infty$. Since $z(t) \in (\ker Z)^{\perp}$, we have that

$$\|z(t)\|_\infty \leq \frac{\|z(t)\|_{\bar{P}}}{\sqrt{\bar{\lambda}_{\min}(\bar{P})}} \leq \frac{(1-\beta)^t\|z(0)\|_{\bar{P}}}{\sqrt{\bar{\lambda}_{\min}(\bar{P})}}.$$

In order to have $\|z(t)\|_\infty < \epsilon$, it suffices that

$$t > \frac{\log \frac{\epsilon\sqrt{\bar{\lambda}_{\min}(\bar{P})}}{\|z(0)\|_{\bar{P}}}}{\log(1-\beta)} = \frac{\log \frac{\epsilon^{-1}\|z(0)\|_{\bar{P}}}{\sqrt{\bar{\lambda}_{\min}(\bar{P})}}}{-\log(1-\beta)},$$

from which, recalling the expression of β and that $\log(1+x) = x + o(x)$ and performing some manipulations, it follows that

$$t \in O\left(N^2 \log\left(\epsilon^{-1}N\right)\right).$$

□

5.2. Deployment on a segment. Consider the system (4.2). Let $\mathcal{A}(t)$ and $\mathcal{R}$ be defined as in the previous subsection, with the only difference that now in all the definitions $A = \mathrm{ATrid}_N^-(k, 1-2k)$. Note that the matrix $\mathrm{ATrid}_N^-(k, 1-2k)$, differently from the matrix $\mathrm{ATrid}_N^+(k, 1-2k)$, has all the eigenvalues strictly inside the unit circle. This leads to the following result that will permit us to analyze equation (4.2) by means of Theorem B.1 (see Appendix B).

Lemma 5.5. *Let* $\delta_{\text{q-centrd}} := \delta_{\text{q-crcmcntr}}$ *as in equation* (5.2). *The following facts are equivalent:*

(i) all eigenvalues of $R_1 = R_1(\delta)$ *are strictly inside the unit disc; and*

(ii) $\delta < \delta_{\text{q-centrd}}$.

Proof. The proof follows the lines of Lemma 5.1. □

The following two theorems characterize the asymptotic properties of (4.2).

Theorem 5.6. *Consider the system (4.2). The following facts are equivalent:*

(a) $\delta < \delta_{\text{q-centrd}}$*;*

(b) *For each initial condition* $[\bar{x}(0)^T \;\; e(0)^T]^T$ *and for any sequence* $\{\Omega(t)\}_{t=0}^{+\infty}$, *we have*

$$\lim_{t\to+\infty} \begin{bmatrix} \bar{x}(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}.$$

Proof. The proof is based on Theorem B.1 and is along the lines of the proof of Theorem 5.2. We remark only that, in this case the matrix $\bar{P} \in \mathbb{R}^{2N\times 2N}$ verifying the conditions of Theorem B.1 is

$$\bar{P} = \begin{bmatrix} P & 0 \\ 0 & \gamma I_N \end{bmatrix}, \tag{5.18}$$

where $P = \mathrm{ATrid}_N^-(-1,2)$ and where γ is as in Theorem 5.2. □

THEOREM 5.7. *Let* $\bar{P}$ *be as in (5.18). There exists* $\beta(N) > 0$ *such that*

$$R_1^T \bar{P} R_1 - \bar{P} < -\beta(N)\bar{P},$$

and, as $N \to +\infty$,

$$\beta(N) = \left(2k - \gamma\delta^2 k^2\right)\frac{\pi^2}{N^2} + o\left(\frac{1}{N^2}\right).$$

Proof. The proof follows the lines of the proof of Theorem 5.3 using Theorem 7.7.6 in [7] in place of Lemma B.3. □

Finally, the following corollary characterizes the time complexity of the Deployment on a segment.

COROLLARY 5.8. *Consider the system (4.2). Then*

$$\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-deplmnt}}, \mathcal{CC}^{(k)}_{\text{q-centrd}}) \in O(N^2 \log(N\varepsilon^{-1})). \tag{5.19}$$

Proof. The proof is similar to the proof of Corollary 5.4. □

6. Numerical simulations. In this section we provide some numerical results illustrating the performance of the zoom in - zoom out strategy. In these simulations we consider a network of $N = 100$ agents. Both for the rendezvous on the line and for deployment on a segment the initial conditions has been generated randomly inside the interval $[-100, 100]$. For all the experiments, we set the parameters k_{in} and k_{out} to the values 1/2 and 2 and initialized the scaling factor f of each agent to the value 50. Moreover, we run simulations for two different values of m, $m = 5$ and $m = 12$. The result obtained are reported in Figure 6.1. The variable plotted is the square of the covariance of the vector $\bar{x}(t)$ both for the Rendezvous on the line and for the Deployment on a segment, that is,

$$s(t) = \frac{1}{N}\sum_{i=1}^{N} \bar{x}_i^2(t)$$

Note that, as depicted in Figure 6.1, also the zoom in- zoom out uniform coder-decoder strategy seems to be very efficient in achieving the rendezvous and the deployment tasks. In particular it is remarkable that this strategy works well even if the uniform quantizer has a low number of quantization levels ($m = 5$). Finally, it is worth observing that, as theoretically proved in the logarithmic coder-decoder strategy, also in this case the performance degrades smoothly as the quantization becomes coarser.

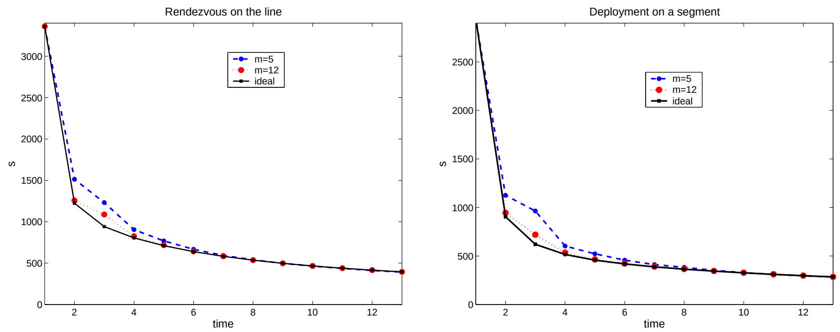

FIG. 6.1. *Zoom in - zoom out strategy for the scalar case: rendezvous an the line* (left)*; deployment on a segment* (right).

TABLE 7.1
Summary and comparison of the results

Control law	Parameters	Time Complexity	Convergence Factor
$\mathcal{CC}^{(k)}_{\text{crcmcntr}}$ and $\mathcal{CC}^{(k)}_{\text{centrd}}$	$k \in \left]0, \frac{1}{2}\right[$	$O(N^2 \log(N\varepsilon^{-1}))$	$1 - 2\dfrac{k\pi^2}{N^2} + o(\dfrac{1}{N^2})$
$\mathcal{CC}^{(k)}_{\text{q-crcmcntr}}$ and $\mathcal{CC}^{(k)}_{\text{q-centrd}}$	$k \in \left]0, \frac{1}{2}\right[$ $\delta \in \left[0, \frac{1-2k}{1+2k}\right]$	$O(N^2 \log(N\varepsilon^{-1}))$	$1 - \big(2 - \gamma(k,\delta)\big)\dfrac{k\pi^2}{N^2} + o(\dfrac{1}{N^2})$

7. Conclusion. This paper's main results in Theorems 4.1 and 4.3 are summarized in Table 7: the time complexity of the quantized rendezvous on a line and of the quantized deployment on a segment is unchanged when compared to quantization-free results in [13]. Moreover, for fixed control gain k, the exponential convergence factor is a decreasing and asymptotically-vanishing function of the quantizer accuracy δ, i.e., the convergence rate degrades smoothly as the quantization becomes coarser. Remarkably, while the algorithms' time complexity is a function of N, the interval of admissible values for the quantizer accuracy δ is independent of it.

REFERENCES

[1] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita, *Distributed memoryless point convergence algorithm for mobile robots with limited visibility*, IEEE Transactions on Robotics and Automation, 15 (1999), pp. 818–828.

[2] Y. U. Cao, A. S. Fukunaga, and A. Kahng, *Cooperative mobile robotics: Antecedents and directions*, Autonomous Robots, 4 (1997), pp. 7–27.

[3] R. Carli, F. Fagnani, and S. Zampieri, *On the state agreement with quantized information*, in Mathematical Theory of Networks and Systems, Kyoto, Japan, July 2006. Electronic Proceedings.

[4] J. Cortés, S. Martínez, and F. Bullo, *Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions*, IEEE Transactions on Automatic Control, 51 (2006), pp. 1289–1298.

[5] J. Cortés, S. Martínez, T. Karatas, and F. Bullo, *Coverage control for mobile sensing networks*, IEEE Transactions on Robotics and Automation, 20 (2004), pp. 243–255.

[6] N. Elia and S. K. Mitter, *Stabilization of linear systems with limited information*, IEEE Transactions on Automatic Control, 46 (2001), pp. 1384–1400.

[7] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1985.

[8] A. Kashyap, T. Başar, and R. Srikant, *Quantized consensus*, Automatica, 43 (2007), pp. 1192–1203.

[9] E. Klavins and R. M. Murray, *Distributed algorithms for cooperative control*, IEEE Pervasive Computing, 3 (2004), pp. 56–65.

[10] J. Lin, A. S. Morse, and B. D. O. Anderson, *The multi-agent rendezvous problem - the asynchronous case*, in IEEE Conf. on Decision and Control, Paradise Island, Bahamas, Dec. 2004, pp. 1926–1931.

[11] Z. Lin, M. Broucke, and B. Francis, *Local control strategies for groups of mobile autonomous agents*, IEEE Transactions on Automatic Control, 49 (2004), pp. 622–629.

[12] S. Martínez, F. Bullo, J. Cortés, and E. Frazzoli, *On synchronous robotic networks – Part I: Models, tasks and complexity*, IEEE Transactions on Automatic Control, 52 (2007). To appear.

[13] ———, *On synchronous robotic networks – Part II: Time complexity of rendezvous and deployment algorithms*, IEEE Transactions on Automatic Control, 52 (2007). To appear.

[14] C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, Philadelphia, PA, 2001.

[15] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, *Feedback control under data rate constraints: An overview*, IEEE Proceedings, 95 (2007), pp. 108–137.

[16] V. Sharma, M. Savchenko, E. Frazzoli, and P. Voulgaris, *Transfer time complexity of conflict-free vehicle routing with no communications*, International Journal of Robotics Research, 26 (2007), pp. 255–272.

[17] J. R. Silvester, *Determinants of block matrices*, The Mathematical Gazette, 84 (2000), pp. 460–467.

[18] S. L. Smith, M. E. Broucke, and B. A. Francis, *A hierarchical cyclic pursuit scheme for vehicle networks*, Automatica, 41 (2005), pp. 1045–1053.

Appendix A. Tridiagonal Toeplitz and circulant dynamical systems. For $n \geq 2$ and a, $b \in \mathbb{R}$ with $a \neq 0$, define the $n \times n$ symmetric tridiagonal Toeplitz matrix $\mathrm{Trid}_n(a,b)$ by

$$\mathrm{Trid}_n(a,b) = \begin{bmatrix} b & a & 0 & \dots & 0 \\ a & b & a & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & a & b & a \\ 0 & \dots & 0 & a & b \end{bmatrix}.$$

From Example 7.2.20 in [14], we know that the eigenvalues and eigenvectors of $\mathrm{Trid}_n(a,b)$ are, for $i \in \{1,\dots,n\}$,

$$b + 2a\cos\left(\frac{i\pi}{n+1}\right), \quad v_i := \begin{bmatrix}\sin(\frac{i\pi}{n+1}) & \sin(\frac{2i\pi}{n+1}) & \cdots & \sin(\frac{ni\pi}{n+1})\end{bmatrix}^T.$$

Note that the eigenvectors are independent of a and b. If we define the matrix $T_n = [v_1 \cdots v_n]$, then all matrices of the form $\mathrm{Trid}_n(a,b)$ for arbitrary a, b, are diagonalizable by T_n.

Next, for $n \geq 2$ and $a, b \in \mathbb{R}$, we define the $n \times n$ *augmented tridiagonal matrices* $\mathrm{ATrid}_n^+(a,b)$ and $\mathrm{ATrid}_n^-(a,b)$ by

$$\mathrm{ATrid}_n^\pm(a,b) = \mathrm{Trid}_n(a,b) \pm \begin{bmatrix} a & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 0 \\ 0 & \dots & \dots & 0 & a \end{bmatrix}.$$

If we define

$$P_+ = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & -1 & 1 & 0 & \dots & 0 \\ 1 & 0 & -1 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 1 & 0 & \dots & 0 & -1 & 1 \\ 1 & 0 & \dots & 0 & 0 & -1 \end{bmatrix}, \qquad P_- = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ (-1)^{n-2} & 0 & \dots & 0 & 1 & 1 \\ (-1)^{n-1} & 0 & \dots & 0 & 0 & 1 \end{bmatrix},$$

then the following similarity transforms are satisfied:

$$\operatorname{ATrid}_n^{\pm}(a,b) = P_{\pm} \begin{bmatrix} b \pm 2a & 0 \\ 0 & \operatorname{Trid}_{n-1}(a,b) \end{bmatrix} P_{\pm}^{-1}. \tag{A.1}$$

Consider now the matrix

$$\bar{T}_n = \begin{bmatrix} 1 & 0 \\ 0 & T_{n-1} \end{bmatrix}.$$

It follows that all the matrices of the form $\operatorname{ATrid}_n^+(a,b)$ (respectively, $\operatorname{ATrid}_n^-(a,b)$), for arbitrary a, b, are diagonalizable by the matrix $P_+\bar{T}_n$ (respectively, $P_-\bar{T}_n$). Accordingly, we define F_+ and F_- by

$$F_{\pm} = P_{\pm}\bar{T}_n,$$

and, given $Z_+ = \operatorname{ATrid}_n^+(a,b)$ and $Z_- = \operatorname{ATrid}_n^-(a,b)$, we know

$$F_+^{-1} Z_+ F_+ = \operatorname{diag}\{\lambda_0(Z_+), \dots, \lambda_{n-1}(Z_+)\}, \text{ and } F_+^{-1} Z_- F_- = \operatorname{diag}\{\lambda_0(Z), \dots, \lambda_{n-1}(Z)\}),$$

where $\lambda_0(Z_\pm), \dots, \lambda_{n-1}(Z_\pm)$ denote the eigenvalues of $Z_\pm$. Clearly, $\lambda_0(Z_\pm) = b \pm 2a$. We conclude by noting that augmented tridiagonal matrices commute, because they are all diagonalized by the same similarity transformation.

Appendix B. General facts.

Given $A_1, \dots, A_k \in \mathbb{R}^{n \times n}$, we let $\{A(t)\}_{t=0}^{+\infty} \subset \operatorname{Co}\{A_1, \dots, A_k\}$ denote a sequence of matrices taking values in the convex hull of $\{A_1, \dots, A_k\}$. We consider the dynamical system

$$x(t+1) = A(t)x(t). \tag{B.1}$$

Theorem B.1 (Common Lyapunov function). *For $A_1, \dots, A_k \in \mathbb{R}^{n \times n}$, if there exists a symmetric matrix $P \in \mathbb{R}^{n \times n}$ such that*

$$\frac{A_i^T P A_j + A_j^T P A_i}{2} - P < 0, \quad \textit{for all } i, j \in \{1, \dots, k\},$$

then, for all initial conditions $x(0) \in \mathbb{R}^n$ and sequences $\{A(t)\}_{t=0}^{+\infty} \subset \operatorname{Co}\{A_1, \dots, A_k\}$, the solution to (B.1) *satisfies*

$$\lim_{t \to +\infty} x(t) = 0.$$

The objective of this appendix is to generalize this classic result as follows.

Theorem B.2 (Common Lyapunov function for convergence to eigenspace). *Assume that 1 is a simple eigenvalue with left and right eigenvector $v \in \mathbb{R}^n$ for each matrix $A_1, \dots, A_k \in \mathbb{R}^{n\times n}$. If there exists a symmetric matrix $P \in \mathbb{R}^{n\times n}$ satisfying, for all nonzero $z \notin \operatorname{span}\{v\}$ and for all $i, j \in \{1, \dots, k\}$,*

$$Pv = 0, \quad z^T P z > 0, \quad \text{and} \quad z^T \left(\frac{A_i^T P A_j + A_j^T P A_i}{2} - P \right) z < 0, \tag{B.2}$$

then, for all initial conditions $x(0) \in \mathbb{R}^n$ and sequences $\{A(t)\}_{t=0}^{+\infty} \subset \operatorname{Co}\{A_1, \dots, A_k\}$, the solution to (B.1) *satisfies*

$$\lim_{t\to+\infty} x(t) = \alpha v, \quad \alpha = \frac{1}{n} v^T x(0).$$

Proof. Because v is a left and right eigenvector with eigenvalue 1, we have

$$A_i v = v, \quad \text{and} \quad v^T A_i = v^T, \quad \text{for } i \in \{1, \dots, k\}. \tag{B.3}$$

Consider the following decomposition

$$x(t) = x_{\text{ave}}(t)\, v + x_{\perp}(t),$$

where $x_\perp \perp v$ and where $x_{\text{ave}}(t) = \frac{1}{n} v^T x(t) \in \mathbb{R}$. Straightforward calculations show that x_{ave} satisfies the recursive relation

$$x_{\text{ave}}(t+1) = x_{\text{ave}}(t) + \frac{1}{\|v\|^2}\, v^T A(t) x_\perp(t) = x_{\text{ave}}(t),$$

where in the last equality we have used the facts that $v^T A(t) = v^T$ and $v^T x_\perp = 0$. Hence, $x_{\text{ave}}(t) = x_{\text{ave}}(0) = (1/n)\mathbf{1}^T x(0)$, for all t. Now, let $v_1 = v$ and consider a basis $\{v_1, v_2, \dots, v_n\}$ of $\mathbb{R}^n$ with the orthogonality property $v_1 \perp v_i$ for all $i \in \{2, \dots, n\}$. Let $T = [v_1, \dots, v_n] \in \mathbb{R}^{n\times n}$. Let $\bar{x} = T^{-1}x = (\tilde{x}_1, \dots, \tilde{x}_n)$ and let $\tilde{x} = (\tilde{x}_2, \dots, \tilde{x}_n) \in \mathbb{R}^{n-1}$. By the assumption (B.3) we have that

$$T^{-1} P T = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P} \end{bmatrix}, \quad T^{-1} A_i T = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_i \end{bmatrix}, \quad T^{-1} A_j T = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_j \end{bmatrix},$$

where $\tilde{P}$, $\tilde{A}_i$, $\tilde{A}_j \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\tilde{P} > 0$. It follows that

$$\frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_i^T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_j \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_j^T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tilde{A}_i \end{bmatrix} \right) - \begin{bmatrix} 0 & 0 \\ 0 & \tilde{P} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}\left(\tilde{A}_i^T \tilde{P} \tilde{A}_j + \tilde{A}_j^T \tilde{P} \tilde{A}_i \right) - \tilde{P} \end{bmatrix},$$

where $\frac{1}{2}\left(\tilde{A}_i^T \tilde{P} \tilde{A}_j + \tilde{A}_j^T \tilde{P} \tilde{A}_i \right) - \tilde{P} < 0$, for all $i, j \in \{1, \dots, k\}$. Hence, Theorem B.1 implies $\lim_{t\to+\infty} \tilde{x}(t) = 0$ and, in turn, $\lim_{t\to+\infty} x_{\text{ave}}(t) = (1/n)\mathbf{1}^T x(0)$. □

We conclude this section with the following useful extension of Theorem 7.7.6 page 472 in [7].

Lemma B.3. *Suppose that a symmetric matrix X is partitioned as*

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix},$$

where X_1 and X_3 are square matrices of dimensions $n_1 \times n_1$ and $n_2 \times n_2$. Let $\mathbf{1}$ be an n_1-dimensional vector of ones and let $\mathbf{0}$ be an n_2-dimensional vector of zeros. The following statements are equivalent:

(i) $z^T X z > 0$, *for all* $z \notin \operatorname{span}\{[\mathbf{1}^T\ \mathbf{0}^T]^T\}$*;*

(ii) the following three facts are true:

(a) $z_1^T X_1 z_1 > 0$, *for all* $z_1 \notin \operatorname{span}\{\mathbf{1}\}$,

(b) $X_3 - X_2^T X_1^\dagger X_2 > 0$, *and*

(c) $\ker X_1 \subseteq \ker X_2^T$.

Proof. First, we establish that (ii) implies (i). Since $\ker X_1 \subseteq \ker X_2^T$, we have that $\operatorname{Im} X_2 \subseteq \operatorname{Im} X_1$ and hence $X_2 = X_1 K$ for a suitable matrix K. Now, we let $Y = -X_1^\dagger X_2$ and, through some simple bookkeeping, we calculate

$$\begin{bmatrix} I_{n_1} & 0 \\ Y^T & I_{n_2} \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \begin{bmatrix} I_{n_1} & Y \\ 0 & I_{n_2} \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_3 - X_2^T X_1^\dagger X_2 \end{bmatrix}.$$

Since the right-hand side is semidefinite positive, the positive semidefiniteness of X follows from the exhibited congruence.

Next, we show that (i) implies (ii). By choosing $z = [z_1^T\ \mathbf{0}^T]^T$, with $z_1 \notin \operatorname{span}\{\mathbf{1}\}$, it is immediate to show that $z_1^T X_1 z_1 > 0$, for all $z_1 \notin \operatorname{span}\{\mathbf{1}\}$. Suppose now that there exists z_1 such that $v = X_2^T z_1 \neq 0$ and $X_1 z_1 = 0$. Let z_2 satisfy $z_2^T v = \gamma \neq 0$. Then

$$\begin{bmatrix} \alpha z_1^T & z_2^T \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} \begin{bmatrix} \alpha z_1 \\ z_2 \end{bmatrix} = \alpha z_2^T X_2^T z_1 + \alpha z_1^T X_2 z_2 + z_2^T X_3 z_2 = 2\alpha\gamma + z_2^T X_3 z_2.$$

If we choose $\alpha = -\gamma$ with γ sufficiently large, then the above quantity is negative contradicting the assumption. Hence, $\ker X_1 \subseteq \ker X_2^T$. The necessity of $X_3 - X_2^T X_1^\dagger X_2$ follows from the congruence exhibited previously. □