Quantized H_∞ Control for Nonlinear Stochastic Time-delay Systems with Missing Measurements

Zidong Wang, Bo Shen, Huisheng Shu and Guoliang Wei

***Abstract*—In this paper, the quantized H_∞ control problem is investigated for a class of nonlinear stochastic time-delay network-based systems with probabilistic data missing. A nonlinear stochastic system with state delays is employed to model the networked control systems where the measured output and the input signals are quantized by two logarithmic quantizers, respectively. Moreover, the data missing phenomena are modeled by introducing a diagonal matrix composed of Bernoulli distributed stochastic variables taking values of 1 and 0, which describes that the data from different sensors may be lost with different missing probabilities. Subsequently, a sufficient condition is first derived in virtue of the method of sector-bounded uncertainties, which guarantees that the closed-loop system is stochastically stable and the controlled output satisfies H_∞ performance constraint for all nonzero exogenous disturbances under the zero-initial condition. Then, the sufficient condition is decoupled into some inequalities for the convenience of practical verification. Based on that, quantized H_∞ controllers are designed successfully for some special classes of nonlinear stochastic time-delay systems by using Matlab linear matrix inequality toolbox. Finally, a numerical simulation example is exploited to show the effectiveness and applicability of the results derived.**

***Index Terms*—Nonlinear systems; stochastic systems; discrete time-delay systems; networked control systems; H_∞ control; quantized control; data Missing.**

I. Introduction

In recent years, the study of networked control systems (NCSs) has gradually become an active area of research due to their advantages in many aspects such as low cost, reduced weight and power requirements, simple installation and maintenance, as well as high reliability [15], [30]. It is well known that the devices in networks are mutually connected via communication cables which are of limited capacity. Therefore, some new challenging issues have inevitably emerged, for example, network-induced time delay, data missing (also called packet dropout or missing measurement), quantization effect, which should all be taken into account in order to achieve the required performance of the NCSs. Consequently, it is not surprising that, in the past few years, the control and filtering problems of networked systems with communication delays and/or missing measurements have been extensively considered by many researchers. For example, the H_∞ control problem has been studied in [11], [12], [28] for networked systems with random communication delays, and the same problem has been considered in [24], [27] for networked systems with random packet losses. With respect to the filtering problem, we refer the reader to [10], [16] for the case of communication delays, and [19]–[21] for the case of missing measurements.

The signal quantization is considered as another source that has significant impact on the achievable performance of the networked systems and, therefore, it is necessary to conduct analysis on the quantizers and understand how much effect the quantization makes on the overall networked systems. In fact, the problem of quantized control for non-networked system has been reported as early as in 1990 [4]. So far, a great number of results have been available in the literature, see e.g. [2], [4], [5], [8], [9], [14]. In [2], the feedback stabilization problems have been considered for linear time-invariant control systems with saturating quantized measurements. In [14], some general types of quantizers have been developed to solve the problem of feedback stabilization for general nonlinear systems. Recently, a new type of quantizer (called logarithmic quantizer) has attracted considerable research interest. Such quantizer has proven to be the coarsest one in the problem of quadratic stabilization for discrete-time single-input-single-output linear time-invariant systems using quantized feedback under the assumption that the quantizer is static and time-invariant [8]. Base on that, a number of quantized feedback design problems have been studied in [9] for linear system, where the major contribution of [9] lies that many quantized feedback design problems have been found to be equivalent to the well-known robust control problems with sector-bounded uncertainties. Later, the elegant results obtained in [8] have been generalized to the multiple-input-multiple-output systems and to control design with performance constraints.

Inspiringly, in recent years, there have appeared some new results on NCSs with the consideration of signal quantization effects. In [29], the network-based guaranteed cost has been dealt with for linear systems with state and input quantization by using the method of sector bound uncertainties. Moreover, in [18], the problem of quantized state feedback H_∞ stabilization has been addressed for linear time-invariant systems over data networks with limited network quality-of-service. Following that, the problem of output feedback control for NCSs with limited communication capacity has been investigated in

This work was supported in part by the Engineering and Physical Sciences Research Council (EPSRC) of the U.K. under Grant GR/S27658/01, the Leverhulme Trust of the U.K., the Royal Society of the U.K., the National Natural Science Foundation of China under Grants 61028008, 61134009, 61104125, 60974030 and 61074016, and the Alexander von Humboldt Foundation of Germany.

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[23], where the packet losses and quantization effect are taken into account simultaneously. It should be noticed that all the literature mentioned above has been concerned with *linear* NCSs within a *deterministic* framework, and the corresponding results for *nonlinear stochastic* case are relatively few, due primarily to the difficulty in nonlinear analysis and stochastic analysis. However, it is well known that nonlinearity and stochasticity are arguably two of the main causes in reality that have resulted in considerable system complexity, and it seems more reasonable to model the NCSs by taking into account both the nonlinearity and the stochasticity [6], [22]. Unfortunately, to the best of our knowledge, quantized H_∞ control problem for *general nonlinear stochastic time-delay network-based systems with missing measurements* has not been fully investigated despite its potential in practical applications, and the purpose of this paper is therefore to shorten such a gap by providing a rather general framework.

The main contributions of this paper can be summarized as follows. 1) A new quantized H_∞ control problem is introduced for a class of nonlinear stochastic time-delay network-based systems, where the data from different sensors may be missing with different probabilities. 2) Sufficient conditions are established under which the closed-loop system is stochastically stable and the controlled output satisfies the H_∞ performance constraint for all nonzero exogenous disturbances under zero-initial condition, where the nonlinear parameters are very general since there are no assumptions posed on them. 3) The sufficient conditions are applied to some special cases (e.g. systems with Lipschitz-type nonlinearities and systems with sector-bounded nonlinearities) so that the simplified inequalities can be numerically checked more easily. Finally, a numerical simulation example is used to demonstrate the effectiveness and applicability of the results obtained.

Notation The notation used here is fairly standard except where otherwise stated. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n dimensional Euclidean space and the set of all $n\times m$ real matrices. $\|A\|$ refers to the norm of a matrix A defined by $\|A\| = \sqrt{\mathrm{trace}(A^TA)}$. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are real symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). M^T represents the transpose of the matrix M. I denotes the identity matrix of compatible dimension. diag$\{\cdots\}$ stands for a block-diagonal matrix and the notation $\mathrm{diag}_n\{*\}$ is employed to stand for $\mathrm{diag}\{\overbrace{*,\cdots,*}^{n}\}$. Moreover, let $(\Omega,\mathscr{F},\mathrm{Prob})$ be a complete probability space where, Prob, the probability measure, has total mass 1. $\mathbb{E}\{x\}$ stands for the expectation of the stochastic variable x with respect to the given probability measure Prob. The set of all nonnegative integers is denoted by $\mathbb{I}^+$ and the set of all nonnegative real numbers is represented by $\mathbb{R}^+$. CK denotes the class of all continuous nondecreasing convex functions $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(0) = 0$ and $\mu(r) > 0$ for $r > 0$. $C^m(\mathbb{R}^n)$ denotes the class of functions $V(x)$ that is m times continuously differentiable with respect to $x \in \mathbb{R}^n$. For a function $V(x) \in C^2(\mathbb{R}^n)$, we let $V_x(x) = \left(\frac{\partial V(x)}{\partial x_1}, \frac{\partial V(x)}{\partial x_2}, \cdots, \frac{\partial V(x)}{\partial x_n}\right)^T$ and $V_{xx}(x) = \left(\frac{\partial^2 V(x)}{\partial x_i \partial x_j}\right)_{n\times n}$. The asterisk $*$ in a matrix is used to denote term that is induced by symmetry. Matrices, if they are not explicitly specified, are assumed to have compatible dimensions.

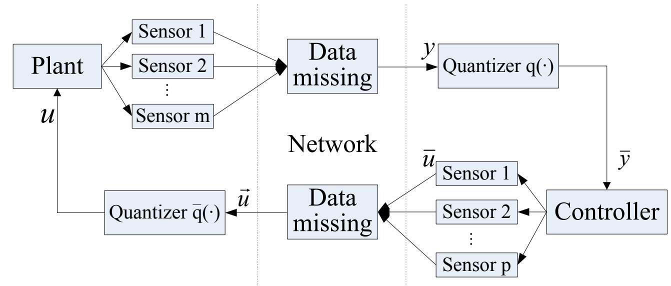

Fig. 1. Structure of a networked control system with two quantizers

II. Problem Formulation And Preliminaries

Consider the networked nonlinear stochastic control system with two quantizers shown in Fig. 1. The plant under consideration is assumed to be of the following form

$$\begin{cases} x_{k+1} = f_1(x_k, x_{k-d}) + h_1(x_k)v_k + g_1(x_k)u_k \\ \qquad\quad + f_w(x_k, x_{k-d})w_k, \\ z_k = f_2(x_k, x_{k-d}) + h_2(x_k)v_k + g_2(x_k)u_k, \\ x_k = \varphi_k, \quad k = -d, -d+1, \cdots, 0 \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state vector, $u_k \in \mathbb{R}^p$ is the control input, $z_k \in \mathbb{R}^l$ is the controlled output and w_k is a one-dimensional, zero-mean Gaussian white noise sequence on a probability space $(\Omega,\mathscr{F},\mathrm{Prob})$ with $\mathbb{E}w_k^2 = \theta$.

Let $(\Omega,\ \mathscr{F},\ \{\mathscr{F}_k\}_{k\in\mathbb{I}^+},\ \mathrm{Prob})$ be a filtered probability space where $\{\mathscr{F}_k\}_{k\in\mathbb{I}^+}$ is the family of sub σ-algebras of $\mathscr{F}$ generated by $\{w_k\}_{k\in\mathbb{I}^+}$ and assume that $\mathscr{F}_0$ is a set of some given sub σ-algebras of $\mathscr{F}$, which is independent of $\mathscr{F}_k$ for all $k > 0$. For the exogenous disturbance input $v_k \in \mathbb{R}^q$, it is assumed that $\{v_k\}_{k\in\mathbb{I}^+} \in l_2([0,\infty),\mathbb{R}^q)$, where $l_2([0,\infty),\mathbb{R}^q)$ is the space of nonanticipatory square-summable stochastic process $\{v_k\}_{k\in\mathbb{I}^+}$ with respect to $(\mathscr{F}_k)_{k\in\mathbb{I}^+}$. The nonlinear functions $f_1 : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, $f_2 : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^l$, $f_w : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}^n$, $h_1 : \mathbb{R}^n \to \mathbb{R}^{n\times q}$, $h_2 : \mathbb{R}^n \to \mathbb{R}^{l\times q}$, $g_1 : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ and $g_2 : \mathbb{R}^n \to \mathbb{R}^{l\times p}$ are smooth matrix-valued functions with $f_1(0,0) = 0$, $f_2(0,0) = 0$ and $f_w(0,0) = 0$. φ_k is a real-valued initial function on $[-d, 0]$.

The measurement with probabilistic sensor data missing is described as

$$y_k = \Gamma_k l(x_k) + k(x_k)v_k \tag{2}$$

where $y_k \in \mathbb{R}^m$ is the measurement received at the node quantizer $q(\cdot)$. The nonlinear functions $l : \mathbb{R}^n \to \mathbb{R}^m$ and $k : \mathbb{R}^n \to \mathbb{R}^{m\times q}$ are also smooth matrix-valued functions with $l(0) = 0$. $\Gamma_k = \mathrm{diag}\{\gamma_k^1, \cdots, \gamma_k^m\}$ is a diagonal matrix that accounts for the different missing rate of the individual channel. For any $1 \leq r \leq m$, γ_k^r is a Bernoulli distributed stochastic variable taking values of 1 and 0 with

$$\begin{aligned} &\mathrm{Prob}\{\gamma_k^r = 1\} = \bar{\gamma}^r, \\ &\mathrm{Prob}\{\gamma_k^r = 0\} = 1 - \bar{\gamma}^r \end{aligned} \tag{3}$$

where $\bar{\gamma}^r \in [0, 1]$ is a known constant.

As shown in Fig. 1, before entering into the controller, the signal $y_k \in \mathbb{R}^m$ is quantized by quantizer $q(\cdot)$ which is defined as

$$\bar{y}_k = q(y_k) = \begin{bmatrix} q_1(y_k^{(1)}) & q_2(y_k^{(2)}) & \cdots & q_m(y_k^{(m)}) \end{bmatrix}^T,$$

where $\bar{y}_k \in \mathbb{R}^m$ is the signal transmitted into the controller after the quantization. In this paper, the quantizer $q(\cdot)$ is assumed to be of the logarithmic type. That is, for each $q_j(\cdot)$ $(1 \le j \le m)$, the set of quantization levels is described by

$$\mathcal{U}_j = \left\{\pm\chi_i^{(j)}, \chi_i^{(j)} = \rho_j^i \chi_0^{(j)}, i = 0, \ \pm 1, \ \pm 2, \ \cdots \right\} \cup \{0\},$$
$$0 < \rho_j < 1, \quad \chi_0^{(j)} > 0.$$

Each of the quantization level corresponds to a segment such that the quantizer maps the whole segment to this quantization level. The logarithmic quantizer $q_j(\cdot)$ is defined as

$$q_j(y_k^{(j)}) = \begin{cases} \chi_i^{(j)}, & \frac{1}{1+\delta_j}\chi_i^{(j)} < y_k^{(j)} \le \frac{1}{1-\delta_j}\chi_i^{(j)} \\ 0, & y_k^{(j)} = 0 \\ -q_j(-y_k^{(j)}), & y_k^{(j)} < 0 \end{cases}$$

with $\delta_j = (1-\rho_j)/(1+\rho_j)$.

By the results derived in [9], it follows that $q_j(y_k^{(j)}) = (1+\Delta_k^{(j)})y_k^{(j)}$ such that $|\Delta_k^{(j)}| \le \delta_j$. Defining $\Delta_k = \mathrm{diag}\{\Delta_k^{(1)}, \cdots, \Delta_k^{(m)}\}$, the measurements after quantization can be expressed as

$$\bar{y}_k = (I + \Delta_k) y_k. \tag{4}$$

Therefore, the quantizing effects have been transformed into sector bound uncertainties described above.

The dynamic observer-based control scheme for the plant (1) is described by

$$\begin{cases} \hat{x}_{k+1} = f_c(\hat{x}_k) + g_c(\hat{x}_k)\bar{y}_k, \\ \bar{u}_k = u_c(\hat{x}_k), \quad f_c(0) = 0, \quad u_c(0) = 0, \\ \hat{x}_k = 0, \quad k = -d, -d+1, \cdots, 0 \end{cases} \tag{5}$$

where $\hat{x}_k \in \mathbb{R}^n$ is the state estimate of the plant (1), $\bar{u}_k \in \mathbb{R}^p$ is the control input without transmission missing, and the matrix-valued nonlinear functions $f_c : \mathbb{R}^n \to \mathbb{R}^n$, $g_c : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $u_c : \mathbb{R}^n \to \mathbb{R}^p$ are controller parameters to be determined.

When the control signal $\bar{u}_k$ is transmitted on the network from the controller to the quantizer $\bar{q}(\cdot)$, the data missing phenomenon will probably occur again owing to the limited bandwidth of the communication channel. Therefore, the data missing model is applied to $\bar{u}_k$ again. Here, we introduce another diagonal matrix $\Xi_k = \mathrm{diag}\{\xi_k^1, \cdots, \xi_k^p\}$ where ξ_k^r is also assumed to be a Bernoulli distributed stochastic variable satisfying

$$\begin{aligned} &\mathrm{Prob}\{\xi_k^r = 1\} = \bar{\xi}^r, \\ &\mathrm{Prob}\{\xi_k^r = 0\} = 1 - \bar{\xi}^r. \end{aligned} \tag{6}$$

Then, the control input with data missing $\vec{u}_k \in \mathbb{R}^p$ can be described as

$$\vec{u}_k = \Xi_k \bar{u}_k. \tag{7}$$

Similar to the signal $y_k \in \mathbb{R}^m$, the control signal $\vec{u}_k \in \mathbb{R}^p$ is also quantized by the quantizer $\bar{q}(\cdot)$ before entering the plant (1). Here, the quantizer $\bar{q}(\cdot)$ is also assumed to be of the logarithmic type and has the same form as the quantizer $q(\cdot)$. Specifically, the quantizer $\bar{q}(\cdot)$ is defined as

$$u_k = \bar{q}(\vec{u}_k) = \begin{bmatrix} \bar{q}_1(\vec{u}_k^{(1)}) & \bar{q}_2(\vec{u}_k^{(2)}) & \cdots & \bar{q}_p(\vec{u}_k^{(p)}) \end{bmatrix}^T$$

where $u_k \in \mathbb{R}^p$ is the control input actually entering the plant (1). For each $\bar{q}_j(\cdot)$ $(1 \le j \le p)$, the set of quantization levels is described by

$$\bar{\mathcal{U}}_j = \left\{\pm\bar{\chi}_i^{(j)}, \bar{\chi}_i^{(j)} = \bar{\rho}_j^i \bar{\chi}_0^{(j)}, i = 0, \ \pm 1, \ \pm 2, \ \cdots \right\} \cup \{0\},$$
$$0 < \bar{\rho}_j < 1, \quad \bar{\chi}_0^{(j)} > 0,$$

and the quantizer $\bar{q}_j(\cdot)$ is defined as

$$\bar{q}_j(\vec{u}_k^{(j)}) = \begin{cases} \bar{\chi}_i^{(j)}, & \frac{1}{1+\bar{\delta}_j}\bar{\chi}_i^{(j)} < \vec{u}_k^{(j)} \le \frac{1}{1-\bar{\delta}_j}\bar{\chi}_i^{(j)} \\ 0, & \vec{u}_k^{(j)} = 0 \\ -\bar{q}_j(-\vec{u}_k^{(j)}), & \vec{u}_k^{(j)} < 0 \end{cases}$$

with $\bar{\delta}_j = (1-\bar{\rho}_j)/(1+\bar{\rho}_j)$. To the end, the control input u_k can be expressed as

$$u_k = (I + \bar{\Delta}_k)\vec{u}_k \tag{8}$$

where $\bar{\Delta}_k = \mathrm{diag}\{\bar{\Delta}_k^{(1)}, \cdots, \bar{\Delta}_k^{(p)}\}$ and $\bar{\Delta}_k^{(j)}$ satisfies $|\bar{\Delta}_k^{(j)}| \le \bar{\delta}_j$ for each $1 \le j \le p$.

For the sake of easy manipulation, we introduce two matrices $C_p^r := \mathrm{diag}\{\underbrace{\overbrace{0, \cdots, 0}^{r-1}, 1, 0, \cdots, 0}_{p}\}$ and $C_m^r := \mathrm{diag}\{\underbrace{\overbrace{0, \cdots, 0}^{r-1}, 1, 0, \cdots, 0}_{m}\}$, and then rewrite the signals $\bar{y}_k \in \mathbb{R}^m$ and $u_k \in \mathbb{R}^p$ as

$$\bar{y}_k = (I + \Delta_k)\sum_{r=1}^{m} \gamma_k^r C_m^r l(x_k) + (I + \Delta_k)k(x_k)v_k \tag{9}$$

and

$$u_k = (I + \bar{\Delta}_k)\sum_{r=1}^{p} \xi_k^r C_p^r u_c(\hat{x}_k), \tag{10}$$

respectively.

By setting $\eta_k = \begin{bmatrix} x_k^T & \hat{x}_k^T \end{bmatrix}^T$, $\eta_{k-d} = \begin{bmatrix} x_{k-d}^T & \hat{x}_{k-d}^T \end{bmatrix}^T$ and substituting (9)-(10) into (1) and (5), we obtain the following closed-loop system:

$$\begin{cases} \eta_{k+1} = \mathcal{F}_1(\eta_k, \eta_{k-d}) + \mathcal{H}_1(\eta_k)v_k + \mathcal{F}_w(\eta_k, \eta_{k-d})w_k \\ \qquad + \sum_{r=1}^{p}(\xi_k^r - \bar{\xi}^r)\mathcal{G}_1^r(\eta_k) + \sum_{r=1}^{m}(\gamma_k^r - \bar{\gamma}^r)\mathcal{G}_2^r(\eta_k), \\ z_k = \mathcal{F}_2(\eta_k, \eta_{k-d}) + \mathcal{H}_2(\eta_k)v_k + \sum_{r=1}^{p}(\xi_k^r - \bar{\xi}^r)\mathcal{G}_3^r(\eta_k) \end{cases} \tag{11}$$

where

$$\mathcal{F}_1(\eta_k,\eta_{k-d}) = \begin{bmatrix} f_1(x_k,x_{k-d}) + g_1(x_k)(I+\bar{\Delta}_k)\Xi u_c(\hat{x}_k) \\ f_c(\hat{x}_k) + g_c(\hat{x}_k)(I+\Delta_k)\bar{\Gamma} l(x_k) \end{bmatrix},$$
$$\mathcal{H}_1(\eta_k) = \begin{bmatrix} h_1(x_k) \\ g_c(\hat{x}_k)(I+\Delta_k)k(x_k) \end{bmatrix}, \quad \mathcal{H}_2(\eta_k) = h_2(x_k),$$
$$\mathcal{G}_1^r(\eta_k) = \begin{bmatrix} g_1(x_k)(I+\bar{\Delta}_k)C_p^r u_c(\hat{x}_k) \\ 0 \end{bmatrix},$$
$$\mathcal{G}_2^r(\eta_k) = \begin{bmatrix} 0 \\ g_c(\hat{x}_k)(I+\Delta_k)C_m^r l(x_k) \end{bmatrix},$$
$$\mathcal{F}_w(\eta_k,\eta_{k-d}) = \begin{bmatrix} f_w(x_k,x_{k-d}) \\ 0 \end{bmatrix},$$
$$\mathcal{F}_2(\eta_k,\eta_{k-d}) = f_2(x_k,x_{k-d}) + g_2(x_k)(I+\bar{\Delta}_k)\Xi u_c(\hat{x}_k),$$
$$\mathcal{G}_3^r(\eta_k) = g_2(x_k)(I+\bar{\Delta}_k)C_p^r u_c(\hat{x}_k),$$
$$\bar{\Gamma} = \mathrm{diag}\{\bar{\gamma}^1,\cdots,\bar{\gamma}^m\}, \quad \Xi = \mathrm{diag}\{\bar{\xi}^1,\cdots,\bar{\xi}^p\}. \tag{12}$$

Throughout this paper, we assume that all the stochastic variables v_k, w_k, ξ_k^i $(i=1,\cdots,p)$ and γ_k^j $(i=1,\cdots,m)$ are uncorrelated each other.

Definition 1: The zero-solution of the closed-loop system (11) with $v_k=0$ is said to be stochastically stable if, for any $\varepsilon>0$, there exists a $\delta>0$ such that

$$\mathbb{E}\{\|\eta_k\|\}<\varepsilon,$$

whenever $k\in\mathbb{I}^+$ and $\max_{k\in\{-d,-d+1,\cdots,0\}}\{\|\bar{\varphi}_k\|\}<\delta$ where $\bar{\varphi}_k = \begin{bmatrix}\varphi_k^T & 0\end{bmatrix}^T$ for $k=-d,-d+1,\cdots,0$.

In Definition 1, the notion of stochastic stability is proposed for the stochastic discrete time-delayed system (11). Other definitions of stability for different kinds of stochastic systems can be found in [13], [17].

The purpose of the problem addressed in this paper is to design the parameters $f_c(\hat{x}_k)$, $g_c(\hat{x}_k)$ and $u_c(\hat{x}_k)$ of the nonlinear controller such that the following requirements are satisfied simultaneously for the given system (1) as well as the quantizers $q(\cdot)$ and $\bar{q}(\cdot)$:

a) The zero-solution of the closed-loop system (11) with $v_k=0$ is stochastically stable.
b) Under the zero-initial condition, the controlled output z_k satisfies

$$\sum_{k=0}^{\infty}\mathbb{E}\{\|z_k\|^2\} \le \gamma^2\sum_{k=0}^{\infty}\mathbb{E}\{\|v_k\|^2\} \tag{13}$$

for all nonzero v_k, where $\gamma>0$ is a given disturbance attenuation level.

III. Main Results

To state our main results, we need the following lemma.

Lemma 1: If there exist a Lyapunov function $V(\zeta)\in C^1(\mathbb{R}^{2(d+1)n})$ and a function $a(r)\in CK$ satisfying the following conditions:

$$V(0)=0, \tag{14a}$$
$$a(\|\zeta\|)\le V(\zeta), \tag{14b}$$
$$\mathbb{E}\{V(\zeta_{k+1})\}\le\mathbb{E}\{V(\zeta_k)\}, \quad k\in\mathbb{I}^+, \tag{14c}$$

where $\zeta_k = \begin{bmatrix}\eta_k^T & \eta_{k-1}^T & \cdots & \eta_{k-d}^T\end{bmatrix}^T$, then the zero-solution of closed-loop system (11) with $v_k=0$ is stochastically stable.

Proof: First of all, it is noted that $V(0)=0$ and $V(\zeta)$ is continuous. Therefore, for any $\varepsilon>0$, there exists a scalar $\delta>0$ such that $V(\zeta_0)<a(\varepsilon)$ when $\|\zeta_0\|<\delta$. We aim to prove that $\mathbb{E}\{\|\eta_k\|\}<\varepsilon$ whenever $k\in\mathbb{I}^+$ and $\max_{k\in\{-d,-d+1,\cdots,0\}}\{\|\bar{\varphi}_k\|\}<\delta$. By considering $\|\bar{\varphi}_k\|\le\|\zeta_0\|$ for all $k=-d,-d+1,\cdots,0$, we only need to prove that every solution η_k with $\|\zeta_0\|<\delta$ implies $\mathbb{E}\{\|\eta_k\|\}<\varepsilon$ for all $k\in\mathbb{I}^+$. Let us now prove the latter by contradiction. Suppose that, for a solution η_k satisfying $\|\zeta_0\|<\delta$, there exists a $k_1\in\mathbb{I}^+$ such that $\mathbb{E}\{\|\eta_{k_1}\|\}\ge\varepsilon$. Noting the fact of $\|\eta_k\|\le\|\zeta_k\|$, one has $\mathbb{E}\{\|\eta_{k_1}\|\}\le\mathbb{E}\{\|\zeta_{k_1}\|\}$. In addition, by using the Jensen inequality and considering the property of function $a(r)$, it follows from (14b) and (14c) that $a(\varepsilon)\le a(\mathbb{E}\{\|\eta_{k_1}\|\})\le a(\mathbb{E}\{\|\zeta_{k_1}\|\})\le\mathbb{E}\{a(\|\zeta_{k_1}\|)\}\le\mathbb{E}\{V(\zeta_{k_1})\}\le\mathbb{E}\{V(\zeta_0)\}<a(\varepsilon)$, which is a contradiction. Therefore, it follows easily from Definition 1 that the zero-solution of the augmented system (11) with $v_k=0$ is stochastically stable. The proof is complete. ■

The following theorem provides a sufficient condition under which the closed-loop system (11) is stochastically stable and the controlled output z_k satisfies the H_∞ criterion (13) under zero-initial condition for the given quantizers $q(\cdot)$ and $\bar{q}(\cdot)$.

Theorem 1: Let the disturbance attenuation level $\gamma>0$ be given. If there exist two real-valued functional $V_1(\eta)\in C^2(\mathbb{R}^{2n})$ and $V_2(\eta)\in C^1(\mathbb{R}^{2n})$ satisfying

$$V_1(0)=0, \quad V_2(0)=0, \tag{15}$$

$$a(\|\eta\|)\le V_1(\eta), \quad a(\|\eta\|)\le V_2(\eta), \tag{16}$$

where $a(r)\in CK$, and the following inequalities for any $\eta,\eta_\alpha,\eta_d\in\mathbb{R}^{2n}$:

$$\begin{aligned}&\mathscr{A}(\eta,\eta_\alpha)\\ &=\gamma^2 I - \frac{1}{2}\mathcal{H}_1^T(\eta)V_{1\eta\eta}(\eta_\alpha)\mathcal{H}_1(\eta) - \mathcal{H}_2^T(\eta)\mathcal{H}_2(\eta) > 0,\end{aligned} \tag{17}$$

$$\begin{aligned}&\mathscr{J}(\eta,\eta_\alpha,\eta_d)\\ &:= \mathscr{B}(\eta,\eta_\alpha,\eta_d)\mathscr{A}^{-1}(\eta,\eta_\alpha)\mathscr{B}^T(\eta,\eta_\alpha,\eta_d)\\ &\quad+\frac{1}{2}\mathcal{F}_1^T(\eta,\eta_d)V_{1\eta\eta}(\eta_\alpha)\mathcal{F}_1(\eta,\eta_d)\\ &\quad+\frac{1}{2}\theta\mathcal{F}_w^T(\eta,\eta_d)V_{1\eta\eta}(\eta_\alpha)\mathcal{F}_w(\eta,\eta_d)+\frac{1}{2}\eta^T V_{1\eta\eta}(\eta_\alpha)\eta\\ &\quad+\mathcal{F}_2^T(\eta,\eta_d)\mathcal{F}_2(\eta,\eta_d)-\mathcal{F}_1^T(\eta,\eta_d)V_{1\eta\eta}(\eta_\alpha)\eta\\ &\quad+V_{1\eta}^T(\eta)\mathcal{F}_1(\eta,\eta_d)-V_{1\eta}^T(\eta)\eta+V_2(\eta)-V_2(\eta_d)\\ &\quad+\frac{1}{2}\sum_{r=1}^{p}\alpha_r^2\mathcal{G}_1^{rT}(\eta)V_{1\eta\eta}(\eta_\alpha)\mathcal{G}_1^r(\eta)\\ &\quad+\frac{1}{2}\sum_{r=1}^{m}\beta_r^2\mathcal{G}_2^{rT}(\eta)V_{1\eta\eta}(\eta_\alpha)\mathcal{G}_2^r(\eta)\\ &\quad+\sum_{r=1}^{p}\alpha_r^2\mathcal{G}_3^{rT}(\eta)\mathcal{G}_3^r(\eta)\le 0\end{aligned} \tag{18}$$

where

$$\begin{aligned}&\mathscr{B}(\eta,\eta_\alpha,\eta_d)\\ &=\frac{1}{2}V_{1\eta}^T(\eta)\mathcal{H}_1(\eta)+\frac{1}{2}\mathcal{F}_1^T(\eta,\eta_d)V_{1\eta\eta}(\eta_\alpha)\mathcal{H}_1(\eta)\\ &\quad-\frac{1}{2}\eta^T V_{1\eta\eta}(\eta_\alpha)\mathcal{H}_1(\eta)+\mathcal{F}_2^T(\eta,\eta_d)\mathcal{H}_2(\eta)\end{aligned}$$

with $\alpha_r = \sqrt{\bar{\xi}^r(1-\bar{\xi}^r)}$ and $\beta_r = \sqrt{\bar{\gamma}^r(1-\bar{\gamma}^r)}$, then the system (11) with $v_k = 0$ is stochastically stable and the controlled output z_k satisfies the H_∞ criterion (13) for all nonzero v_k under the zero-initial condition.

Proof: Choose the Lyapunov functional $V(\zeta_k)$ as

$$V(\zeta_k) = V_1(\eta_k) + \sum_{i=k-d}^{k-1} V_2(\eta_i) \tag{19}$$

where ζ_k is defined in Lemma 1. Note that the first term in (19) corresponds to the stability conditions for the discrete-time nonlinear stochastic systems *without* delays, and the second term in (19) corresponds to delay-independent stability conditions that account for the delay effects.

Obviously, the Lyapunov functional $V(\zeta_k)$ constructed as (19) satisfies (14a) and (14b). By using Taylor's formula, there exists a scalar $\bar{\alpha}_k \in (0,1)$ such that

$$\begin{aligned} V_1(\eta_{k+1}) - V_1(\eta_k) &= V_{1\eta}^T(\eta_k)(\eta_{k+1}-\eta_k) \\ &\quad + \frac{1}{2}(\eta_{k+1}-\eta_k)^T V_{1\eta\eta}(\eta_{\alpha_k})(\eta_{k+1}-\eta_k) \end{aligned}$$

where $\eta_{\alpha_k} = \eta_k + \bar{\alpha}_k(\eta_{k+1}-\eta_k)$.

Now, we first prove the stochastic stability of the closed-loop system (11) with $v_k = 0$. By noting $\mathbb{E}w_k^2 = \theta$,

$$\mathbb{E}\{(\xi_k^i - \bar{\xi}^i)(\xi_k^j - \bar{\xi}^j)\} = \begin{cases} \bar{\xi}^i(1-\bar{\xi}^i), & i = j \\ 0, & i \neq j \end{cases}$$

for $1 \le i \le p, 1 \le j \le p$, and

$$\mathbb{E}\{(\gamma_k^i - \bar{\gamma}^i)(\gamma_k^j - \bar{\gamma}^j)\} = \begin{cases} \bar{\gamma}^i(1-\bar{\gamma}^i), & i = j \\ 0, & i \neq j \end{cases}$$

for $1 \le i \le m, 1 \le j \le m$, it can be calculated along the closed-loop system (11) with $v_k = 0$ that

$$\begin{aligned} &\mathbb{E}\{V(\zeta_{k+1})\} - \mathbb{E}\{V(\zeta_k)\} \\ &= \mathbb{E}\{V_1(\eta_{k+1}) - V_1(\eta_k) + V_2(\eta_k) - V_2(\eta_{k-d})\} \\ &= \mathbb{E}\{V_{1\eta}^T(\eta_k)(\eta_{k+1}-\eta_k) + V_2(\eta_k) - V_2(\eta_{k-d}) \\ &\quad + \frac{1}{2}(\eta_{k+1}-\eta_k)^T V_{1\eta\eta}(\eta_{\alpha_k})(\eta_{k+1}-\eta_k)\} \\ &= \mathbb{E}\Big\{\frac{1}{2}\mathcal{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{F}_1(\eta_k,\eta_{k-d}) \\ &\quad + \frac{1}{2}\theta\mathcal{F}_w^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{F}_w(\eta_k,\eta_{k-d}) \\ &\quad + \frac{1}{2}\sum_{r=1}^{p}\alpha_r^2\mathcal{G}_1^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{G}_1^r(\eta_k) \\ &\quad + \frac{1}{2}\sum_{r=1}^{m}\beta_r^2\mathcal{G}_2^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{G}_2^r(\eta_k) \\ &\quad + \frac{1}{2}\eta_k^T V_{1\eta\eta}(\eta_{\alpha_k})\eta_k - \mathcal{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\eta_k \\ &\quad + V_{1\eta}^T(\eta_k)\mathcal{F}_1(\eta_k,\eta_{k-d}) - V_{1\eta}^T(\eta_k)\eta_k \\ &\quad + V_2(\eta_k) - V_2(\eta_{k-d})\Big\} \\ &\le \mathscr{J}(\eta_k,\eta_{\alpha_k},\eta_{k-d}) \le 0 \end{aligned} \tag{20}$$

which, from Lemma 1, confirms that the system (11) with $v_k = 0$ is stochastically stable.

Next, let us show that the closed-loop system (11) satisfies the H_∞ performance constraint for all nonzero exogenous disturbances under the zero-initial condition. From (11), it follows that

$$\begin{aligned} &\mathbb{E}\left\{V(\zeta_{k+1}) - V(\zeta_k) + \|z_k\|^2 - \gamma^2\|v_k\|^2\right\} \\ &= \mathbb{E}\{V_1(\eta_{k+1}) - V_1(\eta_k) + V_2(\eta_k) - V_2(\eta_{k-d}) \\ &\quad + \|z_k\|^2 - \gamma^2\|v_k\|^2\} \\ &= \mathbb{E}\Big\{V_{1\eta}^T(\eta_k)(\eta_{k+1}-\eta_k) + V_2(\eta_k) - V_2(\eta_{k-d}) \\ &\quad + \frac{1}{2}(\eta_{k+1}-\eta_k)^T V_{1\eta\eta}(\eta_{\alpha_k})(\eta_{k+1}-\eta_k) \\ &\quad + \|z_k\|^2 - \gamma^2\|v_k\|^2\Big\} \\ &= \mathbb{E}\Bigg\{V_{1\eta}^T(\eta_k)\mathcal{F}_1(\eta_k,\eta_{k-d}) + V_{1\eta}^T(\eta_k)\mathcal{H}_1(\eta_k)v_k \\ &\quad - V_{1\eta}^T(\eta_k)\eta_k + \frac{1}{2}\mathcal{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{F}_1(\eta_k,\eta_{k-d}) \\ &\quad + \frac{1}{2}v_k^T\mathcal{H}_1^T(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{H}_1(\eta_k)v_k \\ &\quad + \frac{1}{2}\theta\mathcal{F}_w^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{F}_w(\eta_k,\eta_{k-d}) \\ &\quad + \frac{1}{2}\eta_k^T V_{1\eta\eta}(\eta_{\alpha_k})\eta_k + \mathcal{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{H}_1(\eta_k)v_k \\ &\quad - \eta_k^T V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{H}_1(\eta_k)v_k - \mathcal{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\eta_k \\ &\quad + \frac{1}{2}\sum_{r=1}^{p}\alpha_r^2\mathcal{G}_1^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{G}_1^r(\eta_k) \\ &\quad + \frac{1}{2}\sum_{r=1}^{m}\beta_r^2\mathcal{G}_2^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{G}_2^r(\eta_k) \\ &\quad + \sum_{r=1}^{p}\alpha_r^2\mathcal{G}_3^{rT}(\eta_k)\mathcal{G}_3^r(\eta_k) + \mathcal{F}_2^T(\eta_k,\eta_{k-d})\mathcal{F}_2(\eta_k,\eta_{k-d}) \\ &\quad + 2\mathcal{F}_2^T(\eta_k,\eta_{k-d})\mathcal{H}_2(\eta_k)v_k + v_k^T\mathcal{H}_2^T(\eta_k)\mathcal{H}_2(\eta_k)v_k \\ &\quad + V_2(\eta_k) - V_2(\eta_{k-d}) - \gamma^2\|v_k\|^2\Bigg\} \\ &= \mathbb{E}\Bigg\{ - v_k^T\mathscr{A}(\eta_k,\eta_{\alpha_k})v_k + 2\mathscr{B}(\eta_k,\eta_{\alpha_k},\eta_{k-d})v_k \\ &\quad + \frac{1}{2}\mathcal{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{F}_1(\eta_k,\eta_{k-d}) \\ &\quad + \frac{1}{2}\theta\mathcal{F}_w^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{F}_w(\eta_k,\eta_{k-d}) \\ &\quad + \frac{1}{2}\eta_k^T V_{1\eta\eta}(\eta_{\alpha_k})\eta_k + \mathcal{F}_2^T(\eta_k,\eta_{k-d})\mathcal{F}_2(\eta_k,\eta_{k-d}) \\ &\quad - \mathcal{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\eta_k + V_{1\eta}^T(\eta_k)\mathcal{F}_1(\eta_k,\eta_{k-d}) \\ &\quad - V_{1\eta}^T(\eta_k)\eta_k + V_2(\eta_k) - V_2(\eta_{k-d}) \\ &\quad + \frac{1}{2}\sum_{r=1}^{p}\alpha_r^2\mathcal{G}_1^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{G}_1^r(\eta_k) \\ &\quad + \frac{1}{2}\sum_{r=1}^{m}\beta_r^2\mathcal{G}_2^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{G}_2^r(\eta_k) \\ &\quad + \sum_{r=1}^{p}\alpha_r^2\mathcal{G}_3^{rT}(\eta_k)\mathcal{G}_3^r(\eta_k)\Bigg\}. \end{aligned} \tag{21}$$

Applying the "completing the square" rule, it can be easily seen that (21) is equal to

$$\begin{aligned}&\mathbb{E}\Big\{-(v_k-v_k^*)^T\mathscr{A}(\eta_k,\eta_{\alpha_k})(v_k-v_k^*)\\&+\mathscr{B}(\eta_k,\eta_{\alpha_k},\eta_{k-d})\mathscr{A}^{-1}(\eta_k,\eta_{\alpha_k})\mathscr{B}^T(\eta_k,\eta_{\alpha_k},\eta_{k-d})\\&+\frac{1}{2}\mathcal{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{F}_1(\eta_k,\eta_{k-d})\\&+\frac{1}{2}\theta\mathcal{F}_w^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{F}_w(\eta_k,\eta_{k-d})\\&+\frac{1}{2}\eta_k^TV_{1\eta\eta}(\eta_{\alpha_k})\eta_k+\mathcal{F}_2^T(\eta_k,\eta_{k-d})\mathcal{F}_2(\eta_k,\eta_{k-d})\\&-\mathcal{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\eta_k+V_{1\eta}^T(\eta_k)\mathcal{F}_1(\eta_k,\eta_{k-d})\\&-V_{1\eta}^T(\eta_k)\eta_k+V_2(\eta_k)-V_2(\eta_{k-d})\\&+\frac{1}{2}\sum_{r=1}^{p}\alpha_r^2\mathcal{G}_1^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{G}_1^r(\eta_k)\\&+\frac{1}{2}\sum_{r=1}^{m}\beta_r^2\mathcal{G}_2^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{G}_2^r(\eta_k)\\&+\sum_{r=1}^{p}\alpha_r^2\mathcal{G}_3^{rT}(\eta_k)\mathcal{G}_3^r(\eta_k)\Big\}\end{aligned}\tag{22}$$

where $v_k^*=\mathscr{A}^{-1}(\eta_k,\eta_{\alpha_k})\mathscr{B}^T(\eta_k,\eta_{\alpha_k},\eta_{k-d})$. Noticing (17), it follows from (22) that

$$\begin{aligned}&\mathbb{E}\left\{V(\zeta_{k+1})-V(\zeta_k)+\|z_k\|^2-\gamma^2\|v_k\|^2\right\}\\\le&\mathbb{E}\Big\{\mathscr{B}(\eta_k,\eta_{\alpha_k},\eta_{k-d})\mathscr{A}^{-1}(\eta_k,\eta_{\alpha_k})\mathscr{B}^T(\eta_k,\eta_{\alpha_k},\eta_{k-d})\\&+\frac{1}{2}\mathcal{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{F}_1(\eta_k,\eta_{k-d})\\&+\frac{1}{2}\theta\mathcal{F}_w^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{F}_w(\eta_k,\eta_{k-d})\\&+\frac{1}{2}\eta_k^TV_{1\eta\eta}(\eta_{\alpha_k})\eta_k+\mathcal{F}_2^T(\eta_k,\eta_{k-d})\mathcal{F}_2(\eta_k,\eta_{k-d})\\&-\mathcal{F}_1^T(\eta_k,\eta_{k-d})V_{1\eta\eta}(\eta_{\alpha_k})\eta_k+V_{1\eta}^T(\eta_k)\mathcal{F}_1(\eta_k,\eta_{k-d})\\&-V_{1\eta}^T(\eta_k)\eta_k+V_2(\eta_k)-V_2(\eta_{k-d})\\&+\frac{1}{2}\sum_{r=1}^{p}\alpha_r^2\mathcal{G}_1^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{G}_1^r(\eta_k)\\&+\frac{1}{2}\sum_{r=1}^{m}\beta_r^2\mathcal{G}_2^{rT}(\eta_k)V_{1\eta\eta}(\eta_{\alpha_k})\mathcal{G}_2^r(\eta_k)\\&+\sum_{r=1}^{p}\alpha_r^2\mathcal{G}_3^{rT}(\eta_k)\mathcal{G}_3^r(\eta_k)\Big\}\\=&\mathbb{E}\left\{\mathscr{J}(\eta_k,\eta_{\alpha_k},\eta_{k-d})\right\},\end{aligned}$$

and then it can be seen from (18) that

$$\mathbb{E}\left\{V(\zeta_{k+1})-V(\zeta_k)+\|z_k\|^2-\gamma^2\|v_k\|^2\right\}\le 0.\tag{23}$$

Under the zero-initial condition, summing up (23) from 0 to ∞ with respect to k and considering $\mathbb{E}\{V(\zeta_\infty)\}\ge 0$, we obtain

$$\sum_{k=0}^{\infty}\mathbb{E}\{\|z_k\|^2\}\le\gamma^2\sum_{k=0}^{\infty}\mathbb{E}\{\|v_k\|^2\}$$

which means that the desired H_∞ performance requirement is achieved and therefore the proof of Theorem 1 is complete. ■

Remark 1: In Theorem 1, a very general condition described by a second-order nonlinear inequality has been derived to guarantee the H_∞ performance as well as the stochastic stability of the closed-loop system (11). Such a nonlinear inequality, although it is difficult to solved, will play a theoretically significant role in the analysis and synthesis of H_∞ control problems. Based on Theorem 1, the corresponding H_∞ control problems for some special classes of nonlinear systems can be solved effectively. Take the polynomial nonlinear system as an example. One just needs to choose the Lyapunov function as a positive homogeneous polynomial. Then, by using the result in Theorem 1 together with the technique of complete square matricial representation (SMR) [3], the existence condition of the desired H_∞ controllers can be formulated in terms of the feasibility problem for a linear matrix inequality (LMI), which can be readily verified by the available SOS (sum of squares) solvers [3].

In order to derive more tractable sufficient conditions, in the sequel, we take the real-valued functions as $V_1(\eta)=\eta^TP\eta$ and $V_2(\eta)=\eta^TQ\eta$ where P and Q are positive definite matrices. The following corollary is obtained directly from Theorem 1.

Corollary 1: Let the disturbance attenuation level $\gamma>0$ be given. If there exist two positive definite matrices $P=P^T>0$ and $Q=Q^T>0$ satisfying the following conditions for all nonzero $\eta,\eta_d\in\mathbb{R}^{2n}$:

$$A(\eta)=\gamma^2I-\mathcal{H}_1^T(\eta)P\mathcal{H}_1(\eta)-\mathcal{H}_2^T(\eta)\mathcal{H}_2(\eta)>0,\tag{24}$$

$$\begin{aligned}&\mathscr{H}(\eta,\eta_d)\\:=&B(\eta,\eta_d)A^{-1}(\eta)B^T(\eta,\eta_d)+\mathcal{F}_1^T(\eta,\eta_d)P\mathcal{F}_1(\eta,\eta_d)\\&+\theta\mathcal{F}_w^T(\eta,\eta_d)P\mathcal{F}_w(\eta,\eta_d)+\mathcal{F}_2^T(\eta,\eta_d)\mathcal{F}_2(\eta,\eta_d)\\&+\sum_{r=1}^{p}\alpha_r^2\mathcal{G}_1^{rT}(\eta)P\mathcal{G}_1^r(\eta)+\sum_{r=1}^{m}\beta_r^2\mathcal{G}_2^{rT}(\eta)P\mathcal{G}_2^r(\eta)\\&+\sum_{r=1}^{p}\alpha_r^2\mathcal{G}_3^{rT}(\eta)\mathcal{G}_3^r(\eta)+\eta^T(Q-P)\eta-\eta_d^TQ\eta_d\\\le&0,\end{aligned}\tag{25}$$

where

$$B(\eta,\eta_d)=\mathcal{F}_1^T(\eta,\eta_d)P\mathcal{H}_1(\eta)+\mathcal{F}_2^T(\eta,\eta_d)\mathcal{H}_2(\eta)\tag{26}$$

with α_r and β_r defined in Theorem 1, then the system (11) with $v_k=0$ is stochastically stable and the controlled output z_k satisfies the H_∞ criterion (13) for all nonzero v_k under the zero-initial condition.

Remark 2: From (24)-(26), it can be observed that the inequalities of Corollary 1 are dependent on both the missing probability and the quantization effects Δ_k and $\bar{\Delta}_k$. If the quantization effects are taken as $\Delta_k=0$ and $\bar{\Delta}_k=0$, one can immediately obtain a sufficient condition to guarantee that the system without quantization effect (when $v_k=0$) is stochastically stable while achieving the H_∞ performance constraint for all admissible missing observations and nonzero exogenous disturbances under the zero-initial condition. Such an problem for *linear deterministic* system has been investigated in [24], [27], where the data missing phenomena have been modeled by one stochastic variable only. Obviously, Corollary 1 generalize the results in [24], [27].

Remark 3: If $\bar{\gamma}^i = 1$ $(1 \le i \le m)$ and $\bar{\xi}^j = 1$ $(1 \le j \le p)$, i.e., the data missing phenomena do not arise, then a sufficient condition is easily obtained from Corollary 1 to make sure that the system without data missing (when $v_k = 0$) is stochastically stable with a guaranteed H_∞ performance index for nonzero exogenous disturbances under the zero-initial condition. Similar results for *linear deterministic* system can be found in [9].

Corollary 1 provides a sufficient condition which guarantees the H_∞ performance as well as the stochastic stability of the closed-loop system (11). However, it should be pointed out that the condition in Corollary 1 is dependent on the quantization effects Δ_k and $\bar{\Delta}_k$, which results in significant difficulty in checking such a sufficient condition in practice. Fortunately, the quantization effects of the logarithmic type quantizers can be transformed into sector bound uncertainties. In fact, by defining $\bar{\Lambda} = \text{diag}\{\bar{\delta}_1, \cdots, \bar{\delta}_p\}$, $\Lambda = \text{diag}\{\delta_1, \cdots, \delta_m\}$ and $F_k = \text{diag}\{\bar{\Delta}_k\bar{\Lambda}^{-1}, \Delta_k\Lambda^{-1}\}$, we can obtain an unknown real-valued time-varying matrix F_k satisfying $F_kF_k^T = F_k^TF_k \le I$. In what follows, we are devoted to eliminating the quantization effects and establishing some conditions that can be solved effectively. For this purpose, the coefficients of the system (11) are rewritten as follows:

$$\begin{aligned}
&\mathcal{F}_1(\eta_k, \eta_{k-d}) = \mathcal{A}_1(\eta_k, \eta_{k-d}) + (\mathcal{S}_1(\eta_k) + \mathcal{S}_2(\eta_k))F_k\mathcal{T}_1(\eta_k),\\
&\mathcal{H}_1(\eta_k) = \mathcal{B}_1(\eta_k) + \mathcal{S}_2(\eta_k)F_k\mathcal{T}_2(\eta_k),\\
&\mathcal{F}_2(\eta_k, \eta_{k-d}) = \mathcal{A}_2(\eta_k, \eta_{k-d}) + \mathcal{S}_3(\eta_k)F_k\mathcal{T}_3(\eta_k),\\
&\mathcal{G}_1^r(\eta_k) = \mathcal{C}_1^r(\eta_k) + \mathcal{S}_1(\eta_k)F_k\mathcal{T}_4^r(\eta_k),\\
&\mathcal{G}_2^r(\eta_k) = \mathcal{C}_2^r(\eta_k) + \mathcal{S}_2(\eta_k)F_k\mathcal{T}_5^r(\eta_k),\\
&\mathcal{G}_3^r(\eta_k) = \mathcal{C}_3^r(\eta_k) + \mathcal{S}_3(\eta_k)F_k\mathcal{T}_4^r(\eta_k),
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{A}_1(\eta_k, \eta_{k-d}) = \begin{bmatrix} f_1(x_k, x_{k-d}) + g_1(x_k)\bar{\Xi}u_c(\hat{x}_k) \\ f_c(\hat{x}_k) + g_c(\hat{x}_k)\bar{\Gamma}l(x_k)\end{bmatrix},\\
&\mathcal{B}_1(\eta_k) = \begin{bmatrix} h_1(x_k) \\ g_c(\hat{x}_k)k(x_k)\end{bmatrix}, \quad \mathcal{T}_1(\eta_k) = \begin{bmatrix}\bar{\Lambda}\bar{\Xi}u_c(\hat{x}_k) \\ \Lambda\bar{\Gamma}l(x_k)\end{bmatrix},\\
&\mathcal{T}_2(\eta_k) = \begin{bmatrix} 0 \\ \Lambda k(x_k)\end{bmatrix}, \ \mathcal{C}_1^r(\eta_k) = \begin{bmatrix} g_1(x_k)C_p^r u_c(\hat{x}_k) \\ 0\end{bmatrix},\\
&\mathcal{T}_3(\eta_k) = \begin{bmatrix}\bar{\Lambda}\bar{\Xi}u_c(\hat{x}_k) \\ 0\end{bmatrix}, \mathcal{C}_2^r(\eta_k) = \begin{bmatrix} 0 \\ g_c(\hat{x}_k)C_m^r l(x_k)\end{bmatrix}, \qquad (27)\\
&\mathcal{T}_4^r(\eta_k) = \begin{bmatrix}\bar{\Lambda}C_p^r u_c(\hat{x}_k) \\ 0\end{bmatrix}, \quad \mathcal{T}_5^r(\eta_k) = \begin{bmatrix} 0 \\ \Lambda C_m^r l(x_k)\end{bmatrix},\\
&\mathcal{A}_2(\eta_k, \eta_{k-d}) = f_2(x_k, x_{k-d}) + g_2(x_k)\bar{\Xi}u_c(\hat{x}_k),\\
&\mathcal{C}_3^r(\eta_k) = g_2(x_k)C_p^r u_c(\hat{x}_k), \ \mathcal{S}_1(\eta_k) = \text{diag}\{g_1(x_k), 0\},\\
&\mathcal{S}_2(\eta_k) = \text{diag}\{0, g_c(\hat{x}_k)\}, \quad \mathcal{S}_3(\eta_k) = \begin{bmatrix} g_2(x_k) & 0\end{bmatrix}.
\end{aligned}$$

Before giving the next theorem, we firstly recall some well-known lemmas.

Lemma 2: *[26]:* For any matrices A, H, E and $U = U^T$ of appropriate dimensions, there exists a positive definite matrix X such that $(A + HFE)^TX(A + HFE) + U < 0$ holds for all F satisfying $F^TF \le I$ if and only if there exists a positive constant $\alpha > 0$ such that $\alpha^{-1}I - H^TXH > 0$ and $A^T(X^{-1} - \alpha HH^T)^{-1}A + \alpha^{-1}E^TE + U < 0$.

Lemma 3: *[25]:* Assume that the matrices A, H, E and F are given with $FF^T \le I$. Let X be a symmetric positive definite matrix and $\alpha > 0$ be an arbitrary constant such that $\alpha^{-1}I - EXE^T > 0$. Then, we have $(A + HFE)X(A + HFE)^T \le A(X^{-1} - \alpha E^TE)^{-1}A^T + \alpha^{-1}HH^T$.

Lemma 4: Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $\varepsilon > 0$. Then we have $2x^Ty \le \varepsilon x^Tx + \varepsilon^{-1}y^Ty$.

The following theorem provides a sufficient condition that is independent of the quantization effects Δ_k and $\bar{\Delta}_k$ but still guarantees the H_∞ performance as well as the stochastic stability of the closed-loop system (11) for the given two quantizers $q(\cdot)$ and $\bar{q}(\cdot)$.

Theorem 2: Consider the system (1). For a given disturbance attenuation level $\gamma > 0$ and two quantizers $q(\cdot)$ and $\bar{q}(\cdot)$, if there exist two positive definite matrices $P^T = P > 0$, $Q^T = Q > 0$ and two positive scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ satisfying

$$\gamma^2 I - \varepsilon_2\mathcal{T}_2^T(\eta)\mathcal{T}_2(\eta) > 0, \qquad (28)$$

$$R^{-1} - \Omega(\eta) - \varepsilon_1\tilde{\mathcal{S}}_1(\eta)\tilde{\mathcal{S}}_1^T(\eta) > 0 \qquad (29)$$

for all $\eta \in \mathbb{R}^{2n}$, and

$$\begin{aligned}
&\widetilde{\mathscr{H}}(\eta, \eta_d)\\
&:= \tilde{\mathcal{A}}^T(\eta, \eta_d)\left(R^{-1} - \Omega(\eta) - \varepsilon_1\tilde{\mathcal{S}}_1(\eta)\tilde{\mathcal{S}}_1^T(\eta)\right)^{-1}\tilde{\mathcal{A}}(\eta, \eta_d)\\
&\quad + \mathcal{C}_{1c}^T(\eta)\left(P_p^{-1} - \varepsilon_1\mathcal{S}_{1p}(\eta)\mathcal{S}_{1p}^T(\eta)\right)^{-1}\mathcal{C}_{1c}(\eta)\\
&\quad + \mathcal{C}_{2c}^T(\eta)\left(P_m^{-1} - \varepsilon_1\mathcal{S}_{2m}(\eta)\mathcal{S}_{2m}^T(\eta)\right)^{-1}\mathcal{C}_{2c}(\eta)\\
&\quad + \mathcal{C}_{3c}^T(\eta)\left(I - \varepsilon_1\mathcal{S}_{3p}(\eta)\mathcal{S}_{3p}^T(\eta)\right)^{-1}\mathcal{C}_{3c}(\eta)\\
&\quad + \varepsilon_1^{-1}\mathcal{T}_1^T(\eta)\mathcal{T}_1(\eta) + \varepsilon_1^{-1}\mathcal{T}_3^T(\eta)\mathcal{T}_3(\eta)\\
&\quad + 2\varepsilon_1^{-1}\mathcal{T}_{1c}^T(\eta)\mathcal{T}_{1c}(\eta) + \varepsilon_1^{-1}\mathcal{T}_{2c}^T(\eta)\mathcal{T}_{2c}(\eta)\\
&\quad + \mathcal{U}(\eta, \eta_d)\\
&< 0 \qquad (30)
\end{aligned}$$

for all nonzero $\eta, \eta_d \in \mathbb{R}^{2n}$, where

$$\begin{aligned}
&\tilde{\mathcal{A}}(\eta, \eta_d) = \begin{bmatrix}\mathcal{A}_1^T(\eta, \eta_d) & \mathcal{A}_2^T(\eta, \eta_d)\end{bmatrix}^T, P_p = \text{diag}_p\{P\},\\
&\tilde{\mathcal{S}}_1(\eta) = \text{diag}\left\{\mathcal{S}_1(\eta) + \mathcal{S}_2(\eta), \mathcal{S}_3(\eta)\right\}, P_m = \text{diag}_m\{P\},\\
&\tilde{\mathcal{S}}_2(\eta) = \begin{bmatrix}\mathcal{S}_2^T(\eta) & 0\end{bmatrix}^T, \quad \tilde{\mathcal{B}}(\eta) = \begin{bmatrix}\mathcal{B}_1^T(\eta) & \mathcal{H}_2^T(\eta)\end{bmatrix}^T,\\
&\mathcal{S}_{1p}(\eta) = \text{diag}_p\{\mathcal{S}_1(\eta)\}, \quad \mathcal{S}_{2m}(\eta) = \text{diag}_m\{\mathcal{S}_2(\eta)\},\\
&\mathcal{S}_{3p}(\eta) = \text{diag}_p\{\mathcal{S}_3(\eta)\}, \quad R = \text{diag}\{P, I\},\\
&\mathcal{C}_{1c}(\eta) = \begin{bmatrix}\alpha_1\mathcal{C}_1^{1T}(\eta) & \cdots & \alpha_p\mathcal{C}_1^{pT}(\eta)\end{bmatrix}^T,\\
&\mathcal{T}_{1c}(\eta) = \begin{bmatrix}\alpha_1\mathcal{T}_4^{1T}(\eta) & \cdots & \alpha_p\mathcal{T}_4^{pT}(\eta)\end{bmatrix}^T,\\
&\mathcal{C}_{2c}(\eta) = \begin{bmatrix}\beta_1\mathcal{C}_2^{1T}(\eta) & \cdots & \beta_m\mathcal{C}_2^{mT}(\eta)\end{bmatrix}^T,\\
&\mathcal{T}_{2c}(\eta) = \begin{bmatrix}\beta_1\mathcal{T}_5^{1T}(\eta) & \cdots & \beta_m\mathcal{T}_5^{mT}(\eta)\end{bmatrix}^T,\\
&\mathcal{C}_{3c}(\eta) = \begin{bmatrix}\alpha_1\mathcal{C}_3^{1T}(\eta) & \cdots & \alpha_p\mathcal{C}_3^{pT}(\eta)\end{bmatrix}^T,\\
&\Omega(\eta) = \tilde{\mathcal{B}}(\eta)\left(\gamma^2 I - \varepsilon_2\mathcal{T}_2^T(\eta)\mathcal{T}_2(\eta)\right)^{-1}\tilde{\mathcal{B}}^T(\eta)\\
&\qquad + \varepsilon_2^{-1}\tilde{\mathcal{S}}_2(\eta)\tilde{\mathcal{S}}_2^T(\eta),\\
&\mathcal{U}(\eta, \eta_d) = \theta\mathcal{F}_w^T(\eta, \eta_d)P\mathcal{F}_w(\eta, \eta_d) + \eta^T(Q - P)\eta\\
&\qquad - \eta_d^TQ\eta_d, \qquad (31)
\end{aligned}$$

for some nonlinear parameter-functions f_c, g_c and u_c, then the quantized nonlinear stochastic H_∞ control problem for the system (1) is solved by the controller (5).

Proof: For presentation convenience, we first define

$$\tilde{\mathcal{F}}(\eta,\eta_d) = \left[\mathcal{F}_1^T(\eta,\eta_d) \quad \mathcal{F}_2^T(\eta,\eta_d)\right]^T,$$
$$\tilde{\mathcal{H}}(\eta) = \left[\mathcal{H}_1^T(\eta) \quad \mathcal{H}_2^T(\eta)\right]^T.$$

After some manipulations, we have

$$\tilde{\mathcal{F}}(\eta,\eta_d) = \tilde{\mathcal{A}}(\eta,\eta_d) + \tilde{\mathcal{S}}_1(\eta)\tilde{F}\tilde{\mathcal{T}}(\eta), \tag{32}$$
$$\tilde{\mathcal{H}}(\eta) = \tilde{\mathcal{B}}(\eta) + \tilde{\mathcal{S}}_2(\eta)F\mathcal{T}_2(\eta) \tag{33}$$

where $\tilde{\mathcal{T}}(\eta) = \left[\mathcal{T}_1^T(\eta) \quad \mathcal{T}_3^T(\eta)\right]^T$, $\tilde{F} = \text{diag}\{F,F\}$ and $\tilde{\mathcal{A}}(\eta,\eta_d)$, $\tilde{\mathcal{B}}(\eta)$, $\tilde{\mathcal{S}}_1(\eta)$, $\tilde{\mathcal{S}}_2(\eta)$ are defined in (31).

By applying Schur complement, it is known that the condition (28) is equivalent to

$$\varepsilon_2^{-1}I - \gamma^{-2}\mathcal{T}_2(\eta)\mathcal{T}_2^T(\eta) > 0.$$

Hence, it follows from Lemma 3 that

$$\gamma^{-2}\tilde{\mathcal{H}}(\eta)\tilde{\mathcal{H}}^T(\eta) \le \Omega(\eta) \tag{34}$$

where $\Omega(\eta)$ is defined in (31). In addition, it can be easily seen from (29) that

$$R^{-1} - \Omega(\eta) > 0. \tag{35}$$

Consequently, from (34)-(35), we have

$$R^{-1} - \gamma^{-2}\tilde{\mathcal{H}}(\eta)\tilde{\mathcal{H}}^T(\eta) > 0$$

which is obviously equivalent to (24) in Corollary 1.

On the other hand, we rewrite $\mathscr{H}(\eta,\eta_d)$ as the following compact form:

$$\begin{aligned}&\mathscr{H}(\eta,\eta_d)\\ =&\tilde{\mathcal{F}}^T(\eta,\eta_d)R\tilde{\mathcal{H}}(\eta)\left(\gamma^2 I - \tilde{\mathcal{H}}^T(\eta)R\tilde{\mathcal{H}}(\eta)\right)^{-1}\tilde{\mathcal{H}}^T(\eta)R\\ &\times\tilde{\mathcal{F}}(\eta,\eta_d) + \tilde{\mathcal{F}}^T(\eta,\eta_d)R\tilde{\mathcal{F}}(\eta,\eta_d) + \mathcal{G}_{1c}^T(\eta)P_p\mathcal{G}_{1c}(\eta)\\ &+\mathcal{G}_{2c}^T(\eta)P_m\mathcal{G}_{2c}(\eta) + \mathcal{G}_{3c}^T(\eta)\mathcal{G}_{3c}(\eta) + \mathcal{U}(\eta,\eta_d)\end{aligned}$$

where

$$\mathcal{G}_{1c}(\eta) = \mathcal{C}_{1c}(\eta) + \mathcal{S}_{1p}(\eta)F_p\mathcal{T}_{1c}(\eta), \tag{36}$$
$$\mathcal{G}_{2c}(\eta) = \mathcal{C}_{2c}(\eta) + \mathcal{S}_{2m}(\eta)F_m\mathcal{T}_{2c}(\eta), \tag{37}$$
$$\mathcal{G}_{3c}(\eta) = \mathcal{C}_{3c}(\eta) + \mathcal{S}_{3p}(\eta)F_p\mathcal{T}_{1c}(\eta) \tag{38}$$

with $F_p = \text{diag}_p\{F\}$, $F_m = \text{diag}_m\{F\}$ and $\mathcal{U}(\eta,\eta_d)$, $\mathcal{C}_{1c}(\eta)$, $\mathcal{C}_{2c}(\eta)$, $\mathcal{C}_{3c}(\eta)$, $\mathcal{S}_{1p}(\eta)$, $\mathcal{S}_{2m}(\eta)$, $\mathcal{S}_{3p}(\eta)$, $\mathcal{T}_{1c}(\eta)$, $\mathcal{T}_{2c}(\eta)$ are defined in (31). Then, in virtue of the Matrix Inverse Lemma, we obtain

$$\begin{aligned}&\mathscr{H}(\eta,\eta_d)\\ =&\tilde{\mathcal{F}}^T(\eta,\eta_d)\left(R^{-1} - \gamma^{-2}\tilde{\mathcal{H}}(\eta)\tilde{\mathcal{H}}^T(\eta)\right)^{-1}\tilde{\mathcal{F}}(\eta,\eta_d)\\ &+\mathcal{G}_{1c}^T(\eta)P_p\mathcal{G}_{1c}(\eta) + \mathcal{G}_{2c}^T(\eta)P_m\mathcal{G}_{2c}(\eta)\\ &+\mathcal{G}_{3c}^T(\eta)\mathcal{G}_{3c}(\eta) + \mathcal{U}(\eta,\eta_d).\end{aligned} \tag{39}$$

Noting (34) and (35), it follows from (39) that

$$\begin{aligned}&\overline{\mathscr{H}}(\eta,\eta_d)\\ :=&\tilde{\mathcal{F}}^T(\eta,\eta_d)\left(R^{-1} - \Omega(\eta)\right)^{-1}\tilde{\mathcal{F}}(\eta,\eta_d) + \mathcal{G}_{1c}^T(\eta)P_p\mathcal{G}_{1c}(\eta)\\ &+\mathcal{G}_{2c}^T(\eta)P_m\mathcal{G}_{2c}(\eta) + \mathcal{G}_{3c}^T(\eta)\mathcal{G}_{3c}(\eta) + \mathcal{U}(\eta,\eta_d)\\ \ge&\mathscr{H}(\eta,\eta_d).\end{aligned} \tag{40}$$

Next, let us "eliminate" the uncertainties in (40) by using Lemma 2. From (29), we have

$$\varepsilon_1^{-1}I - \tilde{\mathcal{S}}_1^T(\eta)\left(R^{-1} - \Omega(\eta)\right)^{-1}\tilde{\mathcal{S}}_1(\eta) > 0. \tag{41}$$

Considering $\Omega(\eta) \ge 0$, it can also be obtained from (29) that $R^{-1} - \varepsilon_1\tilde{\mathcal{S}}_1(\eta)\tilde{\mathcal{S}}_1^T(\eta) > 0$, which results in

$$I - \varepsilon_1\mathcal{S}_3(\eta)\mathcal{S}_3^T(\eta) > 0, \tag{42}$$
$$P^{-1} - \varepsilon_1\left(\mathcal{S}_1(\eta) + \mathcal{S}_2(\eta)\right)\left(\mathcal{S}_1(\eta) + \mathcal{S}_2(\eta)\right)^T > 0. \tag{43}$$

Noting $\mathcal{S}_1(\eta)\mathcal{S}_2^T(\eta) = 0$, we know that (43) implies

$$P^{-1} - \varepsilon_1\mathcal{S}_1(\eta)\mathcal{S}_1^T(\eta) > 0, \tag{44}$$
$$P^{-1} - \varepsilon_1\mathcal{S}_2(\eta)\mathcal{S}_2^T(\eta) > 0. \tag{45}$$

After using Schur complement again and conducting the augmented manipulation, it can be seen that (42), (44) and (45) are equivalent to

$$\varepsilon_1^{-1}I - \mathcal{S}_{3p}^T(\eta)\mathcal{S}_{3p}(\eta) > 0, \tag{46}$$
$$\varepsilon_1^{-1}I - \mathcal{S}_{1p}^T(\eta)P_p\mathcal{S}_{1p}(\eta) > 0, \tag{47}$$
$$\varepsilon_1^{-1}I - \mathcal{S}_{2m}^T(\eta)P_m\mathcal{S}_{2m}(\eta) > 0, \tag{48}$$

respectively. Subsequently, by Lemma 2, we know that under the conditions (41), (46)-(48) together with (30), the inequality $\overline{\mathscr{H}}(\eta,\eta_d) < 0$ is true, which implies $\mathscr{H}(\eta,\eta_d) < 0$ from (40). So far, (24) and (25) in Corollary 1 have been shown to hold. Therefore, the rest of the proof can be directly obtained from Corollary 1, which is omitted here. ■

Before giving further results, we make the following assumption on the plant (1) for the purpose of simplicity.

Assumption 1: The system matrices $h_1(x)$, $h_2(x)$ and $k(x)$ are assumed to satisfy

$$h_1(x)h_2^T(x) = 0, \tag{49}$$
$$h_1(x)k^T(x) = 0, \tag{50}$$
$$h_2(x)k^T(x) = 0. \tag{51}$$

Remark 4: Assumption 1 means that the measurement noise, the output noise and the system noise are mutually independent. Similar assumptions can be found in [1], [7].

Theorem 3: Let the disturbance attenuation level $\gamma > 0$, the two quantizers $q(\cdot)$, $\bar{q}(\cdot)$ and the controller parameter-functions f_c, g_c, u_c be given. The quantized nonlinear stochastic H_∞ control problem for the system (1) is solved by the controller (5), if there exist positive definite matrices $P_1^T = P_1 > 0$, $P_2^T = P_2 > 0$, $Q_1^T = Q_1 > 0$, $Q_2^T = Q_2 > 0$ and positive scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\lambda > 0$ satisfying the following inequalities:

$$\gamma^2 I - \varepsilon_2 k^T(x)\Lambda^2 k(x) \geq \lambda I, \tag{52}$$

$$\Phi_1(x) := P_1^{-1} - \lambda^{-1} h_1(x) h_1^T(x) - \varepsilon_1 g_1(x) g_1^T(x) > 0, \tag{53}$$

$$\begin{aligned}\Phi_2(x,\hat{x}) := P_2^{-1} - \lambda^{-1} g_c(\hat{x}) k(x) k^T(x) g_c^T(\hat{x}) \\ -(\varepsilon_1 + \varepsilon_2^{-1}) g_c(\hat{x}) g_c^T(\hat{x}) > 0,\end{aligned} \tag{54}$$

$$\Phi_3(x) := I - \lambda^{-1} h_2(x) h_2^T(x) - \varepsilon_1 g_2(x) g_2^T(x) > 0 \tag{55}$$

for all $x, \hat{x} \in \mathbb{R}^n$, and

$$\begin{aligned}&\widehat{\mathscr{H}}(x, x_d, \hat{x}, \hat{x}_d) \\ :=&\mathcal{W}_1(x, x_d, \hat{x}) + \mathcal{W}_2(x, \hat{x}) + 2\mathcal{W}_3(x, x_d, \hat{x}) + \mathcal{U}(\eta, \eta_d) \\ <&0\end{aligned}$$

for all nonzero $x, \hat{x}, x_d, \hat{x}_d \in \mathbb{R}^n$, where

$$\begin{aligned}
&\mathcal{W}_1(x, x_d, \hat{x}) \\
=& f_1^T(x, x_d)\Phi_1^{-1}(x) f_1(x, x_d) + f_c^T(\hat{x})\Phi_2^{-1}(x,\hat{x}) f_c(\hat{x}) \\
&+ f_2^T(x, x_d)\Phi_3^{-1}(x) f_2(x, x_d) \\
&+ u_c^T(\hat{x})\bar{\Xi} g_1^T(x)\Phi_1^{-1}(x) g_1(x)\bar{\Xi} u_c(\hat{x}) \\
&+ l^T(x)\bar{\Gamma} g_c^T(\hat{x})\Phi_2^{-1}(x,\hat{x}) g_c(\hat{x})\bar{\Gamma} l(x) \\
&+ u_c^T(\hat{x})\bar{\Xi} g_2^T(x)\Phi_3^{-1}(x) g_2(x)\bar{\Xi} u_c(\hat{x}) \\
&+ 2\varepsilon_1^{-1}\|\bar{\Lambda}\bar{\Xi} u_c(\hat{x})\|^2 + \varepsilon_1^{-1}\|\Lambda\bar{\Gamma} l(x)\|^2, \\
&\mathcal{W}_2(x, \hat{x}) \\
=& \sum_{r=1}^{p} \alpha_r^2 u_c^T(\hat{x}) C_p^r g_1^T(x)\Psi_1^{-1}(x) g_1(x) C_p^r u_c(\hat{x}) \\
&+ \sum_{r=1}^{m} \beta_r^2 l^T(x) C_m^r g_c^T(\hat{x})\Psi_2^{-1}(\hat{x}) g_c(\hat{x}) C_m^r l(x) \\
&+ \sum_{r=1}^{p} \alpha_r^2 u_c^T(\hat{x}) C_p^r g_2^T(x)\Psi_3^{-1}(x) g_2(x) C_p^r u_c(\hat{x}) \\
&+ 2\varepsilon_1^{-1}\sum_{r=1}^{p} \alpha_r^2 \|\bar{\Lambda} C_p^r u_c(\hat{x})\|^2 + \varepsilon_1^{-1}\sum_{r=1}^{m} \beta_r^2 \|\Lambda C_m^r l(x)\|^2, \\
&\mathcal{W}_3(x, x_d, \hat{x}) \\
=& f_1^T(x, x_d)\Phi_1^{-1}(x) g_1(x)\bar{\Xi} u_c(\hat{x}) \\
&+ f_c^T(\hat{x})\Phi_2^{-1}(x,\hat{x}) g_c(\hat{x})\bar{\Gamma} l(x) \\
&+ f_2^T(x, x_d)\Phi_3^{-1}(x) g_2(x)\bar{\Xi} u_c(\hat{x}), \\
&\mathcal{U}(\eta, \eta_d) \\
=& \theta f_w^T(x, x_d) P_1 f_w(x, x_d) + x^T(Q_1 - P_1)x \\
&+ \hat{x}^T(Q_2 - P_2)\hat{x} - x_d^T Q_1 x_d - \hat{x}_d^T Q_2 \hat{x}_d,
\end{aligned} \tag{56}$$

with

$$\begin{aligned}
\Psi_1(x) &= P_1^{-1} - \varepsilon_1 g_1(x) g_1^T(x), \\
\Psi_2(\hat{x}) &= P_2^{-1} - \varepsilon_1 g_c(\hat{x}) g_c^T(\hat{x}), \\
\Psi_3(x) &= I - \varepsilon_1 g_2(x) g_2^T(x).
\end{aligned}$$

Proof: Let $P = \mathrm{diag}\{P_1, P_2\}$ and $Q = \mathrm{diag}\{Q_1, Q_2\}$. It follows from (27) that (52) is equivalent to

$$\gamma^2 I - \varepsilon_2 \mathcal{T}_2^T(\eta)\mathcal{T}_2(\eta) \geq \lambda I$$

which means (28) is guaranteed by (52).

Under Assumption 1 and by a series of computations, it can be obtained from (52) that

$$\begin{aligned}&R^{-1} - \Omega(\eta) - \varepsilon_1 \tilde{\mathcal{S}}_1(\eta)\tilde{\mathcal{S}}_1^T(\eta) \\ &\quad \geq \begin{bmatrix} \Phi_1(x) & 0 & 0 \\ 0 & \Phi_2(x,\hat{x}) & 0 \\ 0 & 0 & \Phi_3(x) \end{bmatrix}.\end{aligned} \tag{57}$$

Hence, (29) is obtained from (53)-(55).

Now, it remains to show that $\widehat{\mathscr{H}}(\eta, \eta_d) < 0$. Considering (27) and (31), it follows from (57) that

$$\begin{aligned}
&\tilde{\mathcal{A}}^T(\eta, \eta_d)\left(R^{-1} - \Omega(\eta) - \varepsilon_1 \tilde{\mathcal{S}}_1(\eta)\tilde{\mathcal{S}}_1^T(\eta)\right)^{-1}\tilde{\mathcal{A}}(\eta, \eta_d) \\
\leq& f_1^T(x, x_d)\Phi_1^{-1}(x) f_1(x, x_d) + f_c^T(\hat{x})\Phi_2^{-1}(x,\hat{x}) f_c(\hat{x}) \\
&+ f_2^T(x, x_d)\Phi_3^{-1}(x) f_2(x, x_d) + u_c^T(\hat{x})\bar{\Xi} g_1^T(x)\Phi_1^{-1}(x) \\
&\times g_1(x)\bar{\Xi} u_c(\hat{x}) + l^T(x)\bar{\Gamma} g_c^T(\hat{x})\Phi_2^{-1}(x,\hat{x}) g_c(\hat{x})\bar{\Gamma} l(x) \\
&+ u_c^T(\hat{x})\bar{\Xi} g_2^T(x)\Phi_3^{-1}(x) g_2(x)\bar{\Xi} u_c(\hat{x}) + 2\Big(f_1^T(x, x_d) \\
&\times \Phi_1^{-1}(x) g_1(x)\bar{\Xi} u_c(\hat{x}) + f_c^T(\hat{x})\Phi_2^{-1}(x,\hat{x}) g_c(\hat{x})\bar{\Gamma} l(x) \\
&+ f_2^T(x, x_d)\Phi_3^{-1}(x) g_2(x)\bar{\Xi} u_c(\hat{x})\Big).
\end{aligned} \tag{58}$$

By some straightforward manipulations and noting that $\Psi_1(x) > 0$, $\Psi_2(\hat{x}) > 0$ and $\Psi_3(x) > 0$ from (53)-(55), one can obtain

$$\begin{aligned}
&\mathcal{T}_1^T(\eta)\mathcal{T}_1(\eta) = \|\bar{\Lambda}\bar{\Xi} u_c(\hat{x})\|^2 + \|\Lambda\bar{\Gamma} l(x)\|^2, \\
&\mathcal{T}_3^T(\eta)\mathcal{T}_3(\eta) = \|\bar{\Lambda}\bar{\Xi} u_c(\hat{x})\|^2, \\
&\mathcal{T}_{1c}^T(\eta)\mathcal{T}_{1c}(\eta) = \sum_{r=1}^{p} \alpha_r^2 \|\bar{\Lambda} C_p^r u_c(\hat{x})\|^2, \\
&\mathcal{T}_{2c}^T(\eta)\mathcal{T}_{2c}(\eta) = \sum_{r=1}^{m} \beta_r^2 \|\Lambda C_m^r l(x)\|^2, \\
&\mathcal{C}_{1c}^T(\eta)\left(P_p^{-1} - \varepsilon_1 \mathcal{S}_{1p}(\eta)\mathcal{S}_{1p}^T(\eta)\right)^{-1}\mathcal{C}_{1c}(\eta) \\
=& \sum_{r=1}^{p} \alpha_r^2 u_c^T(\hat{x}) C_p^r g_1^T(x)\Psi_1^{-1}(x) g_1(x) C_p^r u_c(\hat{x}), \\
&\mathcal{C}_{2c}^T(\eta)\left(P_m^{-1} - \varepsilon_1 \mathcal{S}_{2m}(\eta)\mathcal{S}_{2m}^T(\eta)\right)^{-1}\mathcal{C}_{2c}(\eta) \\
=& \sum_{r=1}^{m} \beta_r^2 l^T(x) C_m^r g_c^T(\hat{x})\Psi_2^{-1}(\hat{x}) g_c(\hat{x}) C_m^r l(x), \\
&\mathcal{C}_{3c}^T(\eta)\left(I - \varepsilon_1 \mathcal{S}_{3p}(\eta)\mathcal{S}_{3p}^T(\eta)\right)^{-1}\mathcal{C}_{3c}(\eta) \\
=& \sum_{r=1}^{p} \alpha_r^2 u_c^T(\hat{x}) C_p^r g_2^T(x)\Psi_3^{-1}(x) g_2(x) C_p^r u_c(\hat{x}), \\
&\mathcal{U}(\eta, \eta_d) \\
=& \theta f_w^T(x, x_d) P_1 f_w(x, x_d) + x^T(Q_1 - P_1)x \\
&+ \hat{x}^T(Q_2 - P_2)\hat{x} - x_d^T Q_1 x_d - \hat{x}_d^T Q_2 \hat{x}_d.
\end{aligned} \tag{59}$$

It can be obtained from (58) and (59) that $\widetilde{\mathscr{H}}(\eta, \eta_d) \leq \widehat{\mathscr{H}}(x, x_d, \hat{x}, \hat{x}_d) < 0$. Therefore, the proof of this theorem follows immediately from that of Theorem 2. ∎

In practice, the matrix functions $h_1(x)$, $h_2(x)$, $g_1(x)$, $g_2(x)$ and $k(x)$ are usually taken as constant matrices as follows

$$\begin{aligned}&h_1(x) = H_1, \quad h_2(x) = H_2, \quad g_1(x) = G_1, \\ &G_2(x) = G_2, \quad k(x) = K,\end{aligned}$$

and it is assumed that

$$H_1 H_2^T = 0, \quad H_1 K^T = 0, \quad H_2 K^T = 0. \tag{60}$$

Furthermore, considering the issue of easy implementation, *linear time-invariant controller* is often designed in practical engineering. In view of this, we are going to show that the main results obtained so far can be directly specialized to the system with linear controller. We adopt the following linear observer-based controller

$$\begin{cases} \hat{x}_{k+1} = F_c \hat{x}_k + G_c \bar{y}_k, \\ \bar{u}_k = U_c \hat{x}_k, \quad \hat{x}_0 = 0 \end{cases} \tag{61}$$

where F_c, G_c and U_c are the parameter-matrices to be determined.

Corollary 2: Let the disturbance attenuation level $\gamma > 0$, two quantizers $q(\cdot)$, $\bar{q}(\cdot)$ and the controller parameter-matrices F_c, G_c, U_c be given. If there exist positive definite matrices $P_1^T = P_1 > 0$, $P_2^T = P_2 > 0$, $Q_1^T = Q_1 > 0$, $Q_2^T = Q_2 > 0$ and positive scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\lambda > 0$, $\mu > 0$ satisfying the following inequalities:

$$\gamma^2 I - \varepsilon_2 K^T \Lambda^2 K \geq \lambda I, \tag{62}$$

$$\Phi_1 := P_1^{-1} - \lambda^{-1} H_1 H_1^T - \varepsilon_1 G_1 G_1^T \geq \mu I, \tag{63}$$

$$\Phi_2 := P_2^{-1} - \lambda^{-1} G_c K K^T G_c^T - (\varepsilon_1 + \varepsilon_2^{-1}) G_c G_c^T \geq \mu I, \tag{64}$$

$$\Phi_3 := I - \lambda^{-1} H_2 H_2^T - \varepsilon_1 G_2 G_2^T \geq \mu I, \tag{65}$$

$$\begin{aligned} \mathscr{H}_1 := & \mu^{-1}(1+\varepsilon_3^{-1}) U_c^T \bar{\Xi} (G_1^T G_1 + G_2^T G_2) \bar{\Xi} U_c \\ & + \mu^{-1} \sum_{r=1}^{p} \alpha_r^2 U_c^T C_p^r (G_1^T G_1 + G_2^T G_2) C_p^r U_c \\ & + \mu^{-1}(1+\varepsilon_3) F_c^T F_c + 2\varepsilon_1^{-1} \sum_{r=1}^{p} \alpha_r^2 U_c^T C_p^r \bar{\Lambda}^2 C_p^r U_c \\ & + 2\varepsilon_1^{-1} U_c^T \bar{\Xi} \bar{\Lambda}^2 \bar{\Xi} U_c + Q_2 - P_2 < 0, \end{aligned} \tag{66}$$

and

$$\begin{aligned} & \mathscr{H}_2(x, x_d) \\ := & \mu^{-1}(1+\varepsilon_3)\left(\|f_1(x,x_d)\|^2 + \|f_2(x,x_d)\|^2\right) \\ & + \mu^{-1}(1+\varepsilon_3^{-1}) \|G_c \bar{\Gamma} l(x)\|^2 + \varepsilon_1^{-1} \|\Lambda \bar{\Gamma} l(x)\|^2 \\ & + \mu^{-1} \sum_{r=1}^{m} \beta_r^2 \|G_c C_m^r l(x)\|^2 + \varepsilon_1^{-1} \sum_{r=1}^{m} \beta_r^2 \|\Lambda C_m^r l(x)\|^2 \\ & + \theta f_w^T(x,x_d) P_1 f_w(x,x_d) + x^T(Q_1 - P_1)x - x_d^T Q_1 x_d \\ < & \; 0 \end{aligned} \tag{67}$$

for all nonzero $x, x_d \in \mathbb{R}^n$, then the quantized nonlinear stochastic H_∞ control problem for the system (1) is solved by the controller (61).

Proof: Under the assumption (60), the inequalities (52)-(55) follow from (62)-(65) by replacing H_1, H_2, G_1, G_2, K and G_c with $h_1(x)$, $h_2(x)$, $g_1(x)$, $g_2(x)$, $k(x)$ and $g_c(\hat{x})$, respectively. Also, it follows from (63)-(65) that

$$\begin{aligned} & \mathcal{W}_1(x, x_d, \hat{x}) \\ \leq & \mu^{-1}\left(\|f_1(x,x_d)\|^2 + \|F_c \hat{x}\|^2 + \|f_2(x,x_d)\|^2 \right. \\ & \left. + \|G_1 \bar{\Xi} U_c \hat{x}\|^2 + \|G_c \bar{\Gamma} l(x)\|^2 + \|G_2 \bar{\Xi} U_c \hat{x}\|^2\right) \\ & + 2\varepsilon_1^{-1} \|\bar{\Lambda} \bar{\Xi} U_c \hat{x}\|^2 + \varepsilon_1^{-1} \|\Lambda \bar{\Gamma} l(x)\|^2. \end{aligned} \tag{68}$$

Noting that (63)-(65) imply $\Psi_1(x) \geq \mu I$, $\Psi_2(\hat{x}) \geq \mu I$ and $\Psi_3(x) \geq \mu I$, respectively, one has

$$\begin{aligned} & \mathcal{W}_2(x, \hat{x}) \\ \leq & \mu^{-1}\Big(\sum_{r=1}^{p} \alpha_r^2 \|G_1 C_p^r U_c \hat{x}\|^2 + \sum_{r=1}^{m} \beta_r^2 \|G_c C_m^r l(x)\|^2 \\ & + \sum_{r=1}^{p} \alpha_r^2 \|G_2 C_p^r U_c \hat{x}\|^2\Big) + 2\varepsilon_1^{-1} \sum_{r=1}^{p} \alpha_r^2 \|\bar{\Lambda} C_p^r U_c \hat{x}\|^2 \\ & + \varepsilon_1^{-1} \sum_{r=1}^{m} \beta_r^2 \|\Lambda C_m^r l(x)\|^2. \end{aligned} \tag{69}$$

By Lemma 4, it follows from (63)-(65) that

$$\begin{aligned} & \mathcal{W}_3(x, x_d, \hat{x}) \\ \leq & \frac{1}{2}\mu^{-1}\Big(\varepsilon_3(\|f_1(x,x_d)\|^2 + \|F_c \hat{x}\|^2 + \|f_2(x,x_d)\|^2) \\ & + \varepsilon_3^{-1}(\|G_1 \bar{\Xi} U_c \hat{x}\|^2 + \|G_c \bar{\Gamma} l(x)\|^2 + \|G_2 \bar{\Xi} U_c \hat{x}\|^2)\Big). \end{aligned} \tag{70}$$

Consequently, it can be obtained from (68)-(70) together with (56) that

$$\widehat{\mathscr{H}}(x, x_d, \hat{x}, \hat{x}_d) \leq \hat{x}^T \mathscr{H}_1 \hat{x} + \mathscr{H}_2(x, x_d) - \hat{x}_d^T Q_2 \hat{x}_d.$$

In view of (66)-(67) and noticing $Q_2 > 0$, we have $\widehat{\mathscr{H}}(x, x_d, \hat{x}, \hat{x}_d) < 0$ for all nonzero $x, \hat{x}, x_d, \hat{x}_d \in \mathbb{R}^n$. Therefore, the rest of the proof follows from that of Theorem 3 immediately. ■

Remark 5: Note that Theorem 3 is proved mainly by the "completing the square" technique which results in very little conservatism.

IV. Some special cases

In this section, we aim to show that Theorem 3 can be specialized to the following two kinds of stochastic systems that have been extensively studied in the literature: 1) systems with Lipschitz-type nonlinearities; and 2) systems with sector-bounded nonlinearities.

Case 1. We first consider a special class of nonlinear stochastic systems with nonlinearities described by Lipschitz condition. For this purpose, we assume that

$$f_1(x, x_d) = A_1 x + A_{1d} x_d + E\psi(x) + E_d \psi_d(x_d), \tag{71}$$

$$f_2(x, x_d) = A_2 x + A_{2d} x_d, \quad l(x) = Lx, \tag{72}$$

$$f_w(x, x_d) = A_w x + A_{wd} x_d, \tag{73}$$

where A_i, A_{id} $(i = 1, 2)$, E, E_d, A_w, A_{wd} and L are known real matrices. The nonlinear terms $\psi(x)$ and $\psi_d(x_d)$ satisfy the Lipschitz conditions $\|\psi(x)\| \leq \|Mx\|$ and $\|\psi_d(x_d)\| \leq \|M_d x_d\|$, where M and M_d are given real matrices.

Corollary 3: Let the disturbance attenuation level $\gamma > 0$ be given. The quantized nonlinear stochastic H_∞ control problem for the system (1) with the nonlinearities bounded by Lipschitz conditions $\|\psi(x)\| \leq \|Mx\|$ and $\|\psi_d(x_d)\| \leq \|M_d x_d\|$ is solved by the linear observer-based controller (61) if there exist positive definite matrices $P_1^T = P_1 > 0$, $R_2^T = R_2 > 0$, $Q_1^T = Q_1 > 0$, $\tilde{Q}_2^T = \tilde{Q}_2 > 0$, real matrices X, G_c, Y, and positive scalars $\kappa_1 > 0$, $\kappa_2 > 0$, $\varepsilon_2 > 0$, $\lambda > 0$ such that the following LMIs hold for given positive scalars $\varepsilon_1 > 0$, $\varepsilon_3 > 0$ and $\mu > 0$:

$$\gamma^2 I - \varepsilon_2 K^T \Lambda^2 K \geq \lambda I, \tag{74}$$

$$\begin{bmatrix} -P_1 & P_1 H_1 & P_1 G_1 & P_1 \\ * & -\lambda I & 0 & 0 \\ * & * & -\varepsilon_1^{-1} I & 0 \\ * & * & * & -\mu^{-1} I \end{bmatrix} < 0, \tag{75}$$

$$\begin{bmatrix} -R_2 & G_c K & G_c & G_c & I \\ * & -\lambda I & 0 & 0 & 0 \\ * & * & -\varepsilon_1^{-1} I & 0 & 0 \\ * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & -\mu^{-1} I \end{bmatrix} < 0, \tag{76}$$

$$\begin{bmatrix} -I & H_2 & G_2 & I \\ * & -\lambda I & 0 & 0 \\ * & * & -\varepsilon_1^{-1} I & 0 \\ * & * & * & -\mu^{-1} I \end{bmatrix} < 0, \tag{77}$$

$$\begin{bmatrix} \tilde{Q}_2 - R_2 & \Theta_{12} \\ * & \Theta_{22} \end{bmatrix} < 0, \tag{78}$$

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & \theta A_w^T P_1 & \Pi_{15} \\ * & \Pi_{22} & \Pi_{23} & \theta A_{wd}^T P_1 & 0 \\ * & * & \Pi_{33} & 0 & 0 \\ * & * & * & -\theta P_1 & 0 \\ * & * & * & * & \Pi_{55} \end{bmatrix} < 0, \tag{79}$$

where

$$\begin{aligned}
\Theta_{12} &= \big[X^T \Xi G_1^T \quad X^T \Xi G_2^T \quad X^T C_{pc}^T G_{1p}^T \\
&\qquad X^T C_{pc}^T G_{2p}^T \quad Y^T \quad X^T C_{pc}^T \bar{\Lambda}_p \quad X^T \Xi \bar{\Lambda}\big], \\
\Theta_{22} &= \text{diag}\Big\{-\frac{\mu I}{1+\varepsilon_3^{-1}}, -\frac{\mu I}{1+\varepsilon_3^{-1}}, -\mu I, \\
&\qquad -\mu I, -\frac{\mu I}{1+\varepsilon_3}, -\frac{\varepsilon_1}{2}, -\frac{\varepsilon_1}{2}\Big\}, \\
\Pi_{11} &= \mu^{-1}(1+\varepsilon_3)\big(A_1^T A_1 + A_2^T A_2\big) + Q_1 - P_1 \\
&\quad + \varepsilon_1^{-1} L^T \bar{\Gamma} \Lambda^2 \bar{\Gamma} L + \varepsilon_1^{-1} L^T C_{mc}^T \Lambda_m^2 C_{mc} L \\
&\quad + \kappa_1 M^T M, \\
\Pi_{12} &= \mu^{-1}(1+\varepsilon_3)\big(A_1^T A_{1d} + A_2^T A_{2d}\big), \\
\Pi_{13} &= \mu^{-1}(1+\varepsilon_3)\begin{bmatrix} A_1^T E & A_1^T E_d \end{bmatrix}, \\
\Pi_{15} &= \begin{bmatrix} L^T \bar{\Gamma} G_c^T & L^T C_{mc}^T G_{cm}^T \end{bmatrix}, \\
\Pi_{22} &= \mu^{-1}(1+\varepsilon_3)\big(A_{1d}^T A_{1d} + A_{2d}^T A_{2d}\big) - Q_1 \\
&\quad + \kappa_2 M_d^T M_d, \\
\Pi_{23} &= \mu^{-1}(1+\varepsilon_3)\begin{bmatrix} A_{1d}^T E & A_{1d}^T E_d \end{bmatrix}, \\
\Pi_{33} &= \begin{bmatrix} \mu^{-1}(1+\varepsilon_3) E^T E - \kappa_1 I & \mu^{-1}(1+\varepsilon_3) E^T E_d \\ * & \mu^{-1}(1+\varepsilon_3) E_d^T E_d - \kappa_2 I \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\Pi_{55} &= \text{diag}\{-\frac{\mu I}{1+\varepsilon_3^{-1}}, -\mu I\}, \quad G_{1p} = \text{diag}_p\{G_1\}, \\
G_{2p} &= \text{diag}_p\{G_2\}, \quad G_{cm} = \text{diag}_m\{G_c\}, \\
\bar{\Lambda}_p &= \text{diag}_p\{\bar{\Lambda}\}, \quad \Lambda_m = \text{diag}_m\{\Lambda\}, \\
C_{pc} &= \begin{bmatrix} \alpha_1 C_p^1 & \alpha_2 C_p^2 & \cdots & \alpha_p C_p^p \end{bmatrix}^T, \\
C_{mc} &= \begin{bmatrix} \beta_1 C_m^1 & \beta_2 C_m^2 & \cdots & \beta_m C_m^m \end{bmatrix}^T.
\end{aligned}$$

Moreover, if the LMIs (74)-(79) are feasible, the desired controller parameters are given by $F_c = Y R_2^{-1}$, G_c and $U_c = X R_2^{-1}$.

Proof: Setting $R_2 = P_2^{-1}$, $\tilde{Q}_2 = P_2^{-1} Q_2 P_2^{-1}$, $X = U_c R_2$, $Y = F_c R_2$ and applying Schur complement together with some algebraic manipulations, (63)-(66) follow directly from (75)-(78), respectively.

Letting

$$\vartheta = \begin{bmatrix} x & x_d & \psi(x) & \psi_d(x_d) \end{bmatrix}^T$$

and noting (71)-(73), (67) can be rewritten as

$$\mathscr{H}_2(x, x_d) = \vartheta^T \Upsilon_1 \vartheta$$

where

$$\Upsilon_1 = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \mu^{-1}(1+\varepsilon_3) A_1^T E & \mu^{-1}(1+\varepsilon_3) A_1^T E_d \\ * & \Sigma_{22} & \mu^{-1}(1+\varepsilon_3) A_{1d}^T E & \mu^{-1}(1+\varepsilon_3) A_{1d}^T E_d \\ * & * & \mu^{-1}(1+\varepsilon_3) E^T E & \mu^{-1}(1+\varepsilon_3) E^T E_d \\ * & * & * & \mu^{-1}(1+\varepsilon_3) E_d^T E_d \end{bmatrix},$$

$$\begin{aligned}
\Sigma_{11} =& \mu^{-1}(1+\varepsilon_3)\big(A_1^T A_1 + A_2^T A_2\big) + \theta A_w^T P_1 A_w \\
&+ Q_1 - P_1 + \mu^{-1}(1+\varepsilon_3^{-1}) L^T \bar{\Gamma} G_c^T G_c \bar{\Gamma} L \\
&+ \varepsilon_1^{-1} L^T \bar{\Gamma} \Lambda^2 \bar{\Gamma} L + \varepsilon_1^{-1} L^T C_{mc}^T \Lambda_m^2 C_{mc} L \\
&+ \mu^{-1} L^T C_{mc}^T G_{cm}^T G_{cm} C_{mc} L, \\
\Sigma_{12} =& \mu^{-1}(1+\varepsilon_3)\big(A_1^T A_{1d} + A_2^T A_{2d}\big) + \theta A_w^T P_1 A_{wd}, \\
\Sigma_{22} =& \mu^{-1}(1+\varepsilon_3)\big(A_{1d}^T A_{1d} + A_{2d}^T A_{2d}\big) \\
&+ \theta A_{wd}^T P_1 A_{wd} - Q_1.
\end{aligned}$$

From the conditions $\|\psi(x)\| \leq \|Mx\|$ and $\|\psi_d(x_d)\| \leq \|M_d x_d\|$, it can be easily seen that

$$\begin{aligned}
\mathscr{H}_2(x, x_d) \leq& \vartheta^T \Upsilon_1 \vartheta + \kappa_1 \begin{bmatrix} x \\ \psi(x) \end{bmatrix}^T \begin{bmatrix} M^T M & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x \\ \psi(x) \end{bmatrix} \\
&+ \kappa_2 \begin{bmatrix} x_d \\ \psi_d(x_d) \end{bmatrix}^T \begin{bmatrix} M_d^T M_d & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} x_d \\ \psi_d(x_d) \end{bmatrix} \\
=& \vartheta^T \Upsilon_2 \vartheta
\end{aligned}$$

where

$$\Upsilon_2 = \begin{bmatrix} \Sigma_{11} + \kappa_1 M^T M & \Sigma_{12} & \mu^{-1}(1+\varepsilon_3) A_1^T E & \mu^{-1}(1+\varepsilon_3) A_1^T E_d \\ * & \Sigma_{22} + \kappa_2 M_d^T M_d & \mu^{-1}(1+\varepsilon_3) A_{1d}^T E & \mu^{-1}(1+\varepsilon_3) A_{1d}^T E_d \\ * & * & \mu^{-1}(1+\varepsilon_3) E^T E - \kappa_1 I & \mu^{-1}(1+\varepsilon_3) E^T E_d \\ * & * & * & \mu^{-1}(1+\varepsilon_3) E_d^T E_d - \kappa_2 I \end{bmatrix}.$$

By Schur complement, (79) is equivalent to $\Upsilon_2 < 0$, which implies $\mathscr{H}_2(x, x_d) < 0$. Therefore, the proof of this corollary is accomplished in virtue of Corollary 2. ■

Case 2. Let us now deal with the nonlinear terms $\psi(x)$ and $\psi_d(x_d)$ described by the following sector-bounded conditions that are more general than the Lipschitz-like ones:

$$(\psi(x) - Ux)^T(\psi(x) - Vx) \leq 0, \tag{80}$$

$$(\psi_d(x_d) - U_d x_d)^T(\psi_d(x_d) - V_d x_d) \leq 0, \tag{81}$$

where U, V, U_d, V_d are known real constant matrices, and $U - V$, $U_d - V_d$ are symmetric positive definite matrices.

In order to obtain the corresponding results for Case 2, we decompose the sector-bounded nonlinear term $\psi(x)$ and $\psi_d(x_d)$ into a linear part and a nonlinear part as follows:

$$\psi(x) = \frac{1}{2}(U+V)x + \tilde{\psi}(x),$$
$$\psi_d(x_d) = \frac{1}{2}(U_d+V_d)x_d + \tilde{\psi}_d(x_d),$$

where $\|\tilde{\psi}(x)\| \leq \|\frac{1}{2}(U-V)x\|$ and $\|\tilde{\psi}_d(x_d)\| \leq \|\frac{1}{2}(U_d - V_d)x_d\|$.

Letting

$$\begin{aligned}
&\tilde{A}_1 = A_1 + \frac{1}{2}E(U+V), \quad \tilde{M} = \frac{1}{2}(U-V), \\
&\tilde{A}_{1d} = A_{1d} + \frac{1}{2}E_d(U_d+V_d), \quad \tilde{M}_d = \frac{1}{2}(U_d - V_d),
\end{aligned} \tag{82}$$

the nonlinear functions $f_1(x, x_d)$ can be rewritten as

$$f_1(x, x_d) = \tilde{A}_1 x + \tilde{A}_{1d} x_d + E\tilde{\psi}(x) + E_d\tilde{\psi}_d(x_d)$$

where $\|\tilde{\psi}(x)\| \leq \|\tilde{M}x\|$ and $\|\tilde{\psi}_d(x_d)\| \leq \|\tilde{M}_d x_d\|$. Subsequently, by replacing A_1, A_{1d}, M and M_d with $\tilde{A}_1$, $\tilde{A}_{1d}$, $\tilde{M}$ and $\tilde{M}_d$, respectively, the following corollary can be obtained immediately from Corollary 3.

Corollary 4: Let the disturbance attenuation level $\gamma > 0$ be given. The quantized nonlinear stochastic H_∞ control problem for the system (1) with the nonlinearities bounded by sector-bounded conditions (80) and (81) is solved by linear observer-based controller (61) if there exist positive definite matrices $P_1^T = P_1 > 0$, $R_2^T = R_2 > 0$, $Q_1^T = Q_1 > 0$, $\tilde{Q}_2^T = \tilde{Q}_2 > 0$, real matrices X, G_c, Y, and positive scalars $\kappa_1 > 0$, $\kappa_2 > 0$, $\varepsilon_2 > 0$, $\lambda > 0$ satisfying the LMIs (74)-(79) with

$$\begin{aligned}
\Pi_{11} &= \mu^{-1}\big(1+\varepsilon_3\big)\big(\tilde{A}_1^T\tilde{A}_1 + A_2^T A_2\big) + \varepsilon_1^{-1}L^T\bar{\Gamma}\Lambda^2\bar{\Gamma}L \\
&\quad + \varepsilon_1^{-1}L^T C_{mc}^T\Lambda_m^2 C_{mc}L + \kappa_1\tilde{M}^T\tilde{M} + Q_1 - P_1, \\
\Pi_{12} &= \mu^{-1}\big(1+\varepsilon_3\big)\big(\tilde{A}_1^T\tilde{A}_{1d} + A_2^T A_{2d}\big), \\
\Pi_{13} &= \mu^{-1}\big(1+\varepsilon_3\big)\left[\tilde{A}_1^T E \quad \tilde{A}_1^T E_d\right], \\
\Pi_{22} &= \mu^{-1}\big(1+\varepsilon_3\big)\big(\tilde{A}_{1d}^T\tilde{A}_{1d} + A_{2d}^T A_{2d}\big) - Q_1 \\
&\quad + \kappa_2\tilde{M}_d^T\tilde{M}_d, \\
\Pi_{23} &= \mu^{-1}\big(1+\varepsilon_3\big)\left[\tilde{A}_{1d}^T E \quad \tilde{A}_{1d}^T E_d\right],
\end{aligned} \tag{83}$$

for given positive scalars $\varepsilon_1 > 0$, $\varepsilon_3 > 0$ and $\mu > 0$, where Θ_{12}, Θ_{22}, Π_{15}, Π_{33}, Π_{55}, $\bar{\Lambda}_p$, Λ_m, G_{1p}, G_{2p}, G_{cm}, C_{pc} and C_{mc} are defined in Corollary 3, and $\tilde{A}_1$, $\tilde{A}_{1d}$, $\tilde{M}$, $\tilde{M}_d$ are defined in (82). Moreover, if the LMIs (74)-(79) with (83) are feasible, the desired controller parameters are given by $F_c = YR_2^{-1}$, G_c and $U_c = XR_2^{-1}$.

Remark 6: In this paper, we first consider a very general stochastic system (1) where *all the system parameters and controller parameters* are nonlinear functions or functionals. In this case, sufficient conditions are given in Theorem 1 which make sure that the system (11) is stochastically stable and H_∞ criterion in (13) is satisfied. Note that, at this stage, the nonlinear parameters are very general since there are no assumptions posed on them. Therefore, as expected, the sufficient conditions established in Theorem 1 serve as a theoretical basis for *general* nonlinear stochastic systems. It is shown in subsequent analysis that the fundamental results given in Theorem 1 can be specialized to numerically tractable ones in practical cases when the nonlinear parameters take certain commonly used forms. Based on Theorem 1, the aim of Theorem 2 is to provide a particular condition that eliminates the quantization effects Δ_k and $\bar{\Delta}_k$ but still guarantees the H_∞ performance as well as the stochastic stability. Next, we take some practically justifiable forms, in a gradual way, for the nonlinear parameters with hope to obtain easy-to-verify conditions for the addressed design problem. Under the assumption that the measurement noise, the output noise and the system noise are mutually independent, Theorem 3 offers a more specific condition that ensures both the stability and the H_∞ performance, and such a condition is further simplified in Corollary 2.

V. An Illustrative Example

Consider the following nonlinear discrete-time stochastic system

$$\begin{cases}
x_{k+1} = \frac{1}{3}x_k + \frac{1}{6}x_{k-1}\sin x_k + \frac{1}{4}v_k + \frac{1}{3}u_k \\
\qquad + \frac{1}{50}x_k\cos x_{k-1}w_k, \\
z_k = \frac{1}{3}x_k\sin x_k - \frac{1}{6}x_{k-1} + \frac{1}{\sqrt{2}}u_k
\end{cases} \tag{84}$$

with the initial conditions $\varphi_{-1} = \varphi_0 = 0$. The measurement with sensors data missing is described as $y_k = \frac{1}{3}\gamma_k x_k\cos x_k$.

We choose the dynamic observer-based controller parameters as $F_c = \frac{2}{7}$, $G_c = 1$, $U_c = \frac{1}{5}$, and obtain the following dynamic observer-based controller:

$$\begin{cases}
\hat{x}_{k+1} = \frac{2}{7}\hat{x}_k + \bar{y}_k, \\
\vec{u}_k = \frac{1}{5}\xi_k\hat{x}_k.
\end{cases} \tag{85}$$

Let the probability $\bar{\gamma} = \bar{\xi} = 0.8$, the variance $\theta = 0.25$, the disturbance attenuation level $\gamma = 0.85$, and the disturbance input $v_k = \exp(-k/35) \times n_k$ where n_k is uniformly distributed over $[0, 0.1]$. The parameters of the two logarithmic quantizers $q(\cdot)$ and $\bar{q}(\cdot)$ are set as $\chi_0 = \bar{\chi}_0 = 0.003$ and $\rho = \bar{\rho} = 0.9$. According to Corollary 2, it can be seen that the controller of form (85) is a desired controller for system (84) with parameters $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, $\lambda = 0.7155$, $\mu = 0.5$, $P_1 = 1.4317$, $P_2 = 0.4$, $Q_1 = 0.2223$ and $Q_2 = 0.002$.

Simulation results are shown in Figs. 2-4. Specifically, the control input after quantization by quantizers $\bar{q}(\cdot)$ is given in

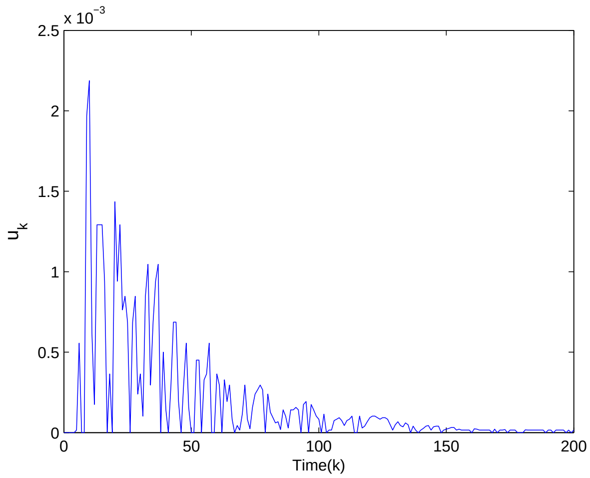

Fig. 2. The control input with quantization by $\bar{q}(\cdot)$

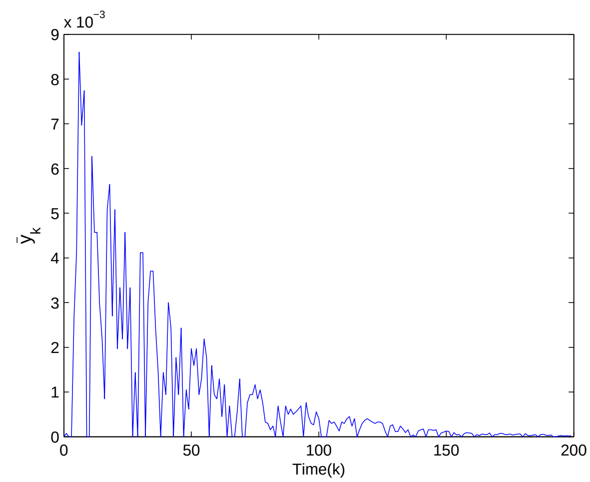

Fig. 3. The measurement with quantization by $q(\cdot)$

Fig. 2 and the measurement after quantization by quantizers $q(\cdot)$ is shown in Fig. 3, which correspond to the controlled system and the dynamic controller, respectively. Fig. 4 depicts the simulation result of the state response of the closed-loop system. It can be calculated that the H_∞ performance constraint is $0.0469 < \gamma = 0.85$. Therefore, this example has verified the theories obtained in this paper.

VI. Conclusions

In this paper, the quantized H_∞ control problem has been addressed for a class of nonlinear stochastic time-delay network-based systems with data missing. Two logarithmic quantizers have been employed to quantize both the measured output and the input signals in the NCSs and one diagonal matrix whose leading diagonal elements are Bernoulli distributed stochastic variables has been used to model the data missing phenomena. Then, we have derived a sufficient condition under which the closed-loop system is stochastically stable and the controlled output satisfies H_∞ performance constraint for all

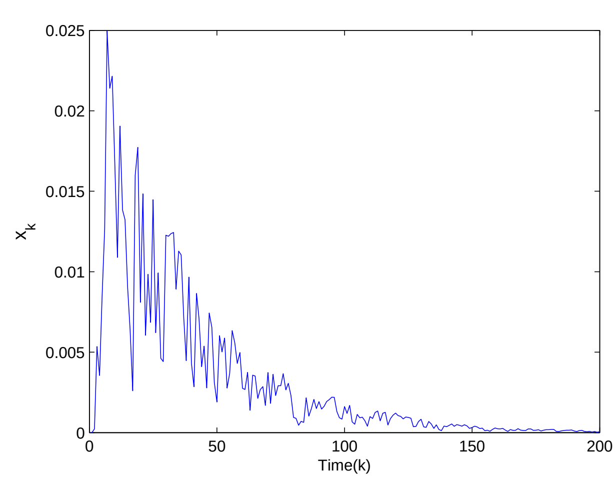

Fig. 4. The state response of the closed-loop system

nonzero exogenous disturbances under the zero-initial condition by applying the method of sector bound uncertainties. For the purpose of easy checking, the sufficient condition has been decoupled into some inequalities. Based on that, quantized H_∞ controllers have been designed successfully for some special classes of nonlinear stochastic time-delay systems.

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