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Quantized Knizhnik–Zamolodchikov Equations, Quantum Yang–Baxter Equation, and Difference Equations for q-Hypergeometric Functions

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Abstract: The $s\ell_2$ quantized Knizhnik–Zamolodchikov equations are solved in q-hypergeometric functions. New difference equations are derived for general q-hypergeometric functions. The equations are given in terms of quantum Yang–Baxter matrices and have the form similar to quantum Knizhnik–Zamolodchikov equations for quantum affine algebras introduced by Frenkel and Reshetikhin.

Introduction

The Knizhnik–Zamolodchikov (KZ) differential equation is the fundamental differential equation of the Conformal Field Theory with very rich mathematical structures. The KZ equation connects representation theories of Lie algebras and quantum groups [KZ, D, K, KL, SV, V]. Quantization of the KZ equation is of great importance. It is expected that the quantized KZ equation also will connect two representation theories. The first is presumably the theory of representations of quantum groups and the second is the theory of representations of a yet undefined structure that may be called "a double quantum group" or "an elliptic quantum group," see [FR].

The KZ equation coincides with the Gauss–Manin differential equation for general hypergeometric functions [SV]. General hypergeometric functions are integrals of special hypergeometric forms over suitable cycles depending on parameters. The special hypergeometric forms are naturally identified with objects of the representation theory of Lie algebras, the cycles are naturally identified with objects of the representation theory of quantum groups, the integration of hypergeometric forms over cycles gives a natural correspondence between representation theories of Lie algebras and quantum groups [FW, V].

There are two ways to quantize the KZ equation: through representation theory and through geometry. The quantization through representation theory

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was suggested by I. Frenkel and N. Reshetikhin [FR]. The KZ equation in the Conformal Field Theory is the differential equation for the matrix coefficients of the product of intertwining operators for an affine Lie algebra $\hat{\mathfrak{g}}$. The KZ differential equation takes values in the tensor product of representation of the corresponding simple Lie algebra $\mathfrak{g}$. I. Frenkel and N. Reshetikhin quantize the Knizhnik–Zamolodchikov differential equation deriving difference equations for the matrix coefficients of the product of intertwining operators for the quantum affine group $U_q(\hat{\mathfrak{g}})$. The quantized KZ equation takes values in the tensor product of representations and is written in terms of suitable solutions for the quantum Yang–Baxter equation.

The geometric way to quantize the KZ equation is to quantize the differential equation for general hypergeometric functions, namely, to replace hypergeometric forms, cycles, hypergeometric integrals, the differential equation for hypergeometric integrals by their difference discrete analogs: difference forms, difference cycles, Jackson integrals, a difference equation for Jackson integrals, respectively. The study of general Jackson integrals has been started recently by K. Aomoto, Y. Kato, K. Mimachi, and A. Matsuo [A, AK, AKM, M, Mi]. In this work we derive new difference equations for Jackson's integrals. The difference equations are written in terms of solutions for the quantum Yang–Baxter equations as in the I. Frenkel and N. Reshetikhin quantization. The open problems are to compare the two quantizations and to give an interpretation for the discrete geometry of Jackson integrals in terms of representation theory.

In recent very interesting works [M], A. Matsuo states formulas for solutions of the Frenkel–Reshetikhin difference equations corresponding to the quantum affine group $U_q(\widehat{s\ell_2})$. The solutions are given as suitable q-hypergeometric functions. A. Matsuo partially proves these formulas for some important cases. In Sect. 3 we extend the Matsuo results and prove the formulas for solutions to the Frenkel–Reshetikhin difference equations for $U_q(\widehat{s\ell_2})$.

In Sect. 1, we define new solutions for the quantum Yang–Baxter equation. The solutions take values in suitable spaces of forests. In Sect. 2, we derive new difference equations for Jackson integrals. The equations are written in terms of the solutions to the Yang–Baxter equation defined in Sect. 1. Section 3 is devoted to integral solutions to the Frenkel–Reshetikhin equations for $U_q(\widehat{s\ell_2})$.

1. Tensor Coordinates and the Yang–Baxter Equation

(1.1) Tensor Coordinates. Let $V_1, \dots, V_n$ be $\mathbb{C}$-vector spaces. Let $W = W(z_1, \dots, z_n)$ be a $\mathbb{C}$-vector space depending on parameters $z_1, \dots, z_n$, where $z_1, \dots, z_n$ are pair-wise different complex numbers. Assume that for every element σ of the permutation group S_n a linear isomorphism

$$L_\sigma(z_1, \dots, z_n): V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)} \to W(z_1, \dots, z_n) \tag{1.1.1}$$

is given.

The set of these linear maps $\{L_\sigma\}$ will be called *tensor coordinates* on $W(z_1, \dots, z_n)$. For arbitrary $\sigma, \nu \in S_n$ the map

$$L_{\sigma,\nu} = L_\nu^{-1} L_\sigma: V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)} \to V_{\nu(1)} \otimes \cdots \otimes V_{\nu(n)} \tag{1.1.2}$$

will be called a *transition function.* It is a function of $z_1, \dots, z_n$.

Assume that for every i and j a linear map

$$R(i,j,z_1,z_2)\colon V_i\otimes V_j\to V_j\otimes V_i \tag{1.1.3}$$

is given, where z_1, z_2 are different complex numbers. Let $\sigma_1,\dots,\sigma_{n-1}$ be the standard generators of S_n, where σ_i permutes i and $i+1$.

(1.1.4) Tensor coordinates will be called *local* with respect to $\{R(i,j,z_1,z_2)\}$ if for any $\sigma\in S_n$ and any $i\in\{1,\dots,n-1\}$ the transition function

$$L_{\sigma,\sigma\sigma_i}\colon V_{\sigma(1)}\otimes\cdots\otimes V_{\sigma(n)}\to V_{\sigma(1)}\otimes\dots V_{\sigma(i+1)}\otimes V_{\sigma(i)}\cdots\otimes V_{\sigma(n)}$$

is the operator acting as $R(\sigma(i),\sigma(i+1),z_{\sigma(i)},z_{\sigma(i+1)})$ on $V_{\sigma(i)}\otimes V_{\sigma(i+1)}$ and as the identity on other factors:

$$L_{\sigma,\sigma\sigma_i}=R_{i,i+1}(\sigma(i),\sigma(i+1),z_{\sigma(i)},z_{\sigma(i+1)})\,.$$

(*1.1.5*) *Example.* Every tensor coordinates are local if $n=2$.

(1.1.6) Lemma. *For local tensor coordinates, the operators* $\{R_{ij}\}$ *form a solution to the Yang–Baxter equation*:

(i) $$R_{23}(\sigma(2),\sigma(3),z_2,z_3)R_{12}(\sigma(1),\sigma(3),z_1,z_3)R_{23}(\sigma(1),\sigma(2),z_1,z_2)$$
$$=R_{12}(\sigma(1),\sigma(2),z_1,z_2),R_{23}(\sigma(1),\sigma(3),z_1,z_3)R_{12}(\sigma(2),\sigma(3),z_2,z_3)$$

for all pair-wise different $\sigma(1)$, $\sigma(2)$, $\sigma(3)\in\{1,\dots,n\}$. *Moreover, this solution is unitary*:

(ii) $$R(i,j,z_1,z_2)\cdot R(j,i,z_2,z_1)=1$$

for all i and j.

(1.1.7) Tensor coordinates will be called *homogeneous* if

$$L_{\sigma,\nu}(sz_1,\dots,sz_n)=L_{\sigma,\nu}(z_1,\dots,z_n)$$

for all $s\neq 0$ and all $\sigma,\nu\in S_n$.

(*1.2*) *The Weight Semigroup.* Let $\mathbb{N}$ (resp. $\mathbb{N}_+$) be the set of all non-negative (resp. positive) integers. An *admissible sequence* is an infinite sequence $\lambda=(\lambda_1,\lambda_2,\dots)$ of non-negative integers such that all of them but a finite number are equal to zero. Set

$$|\lambda|=\lambda_1+\lambda_2+\cdots. \tag{1.2.1}$$

For admissible λ,ν, we say that $\lambda\leqq\mu$ if $\lambda_j\leqq\mu_j$ for all j. Let M be the set of all admissible sequences. M forms a semigroup:

$$(\lambda_1,\lambda_2,\dots)+(\lambda_1',\lambda_2',\dots):=(\lambda_1+\lambda_1',\lambda_2+\lambda_2',\dots)\,.$$

We call M the *weight semigroup.* Introduce the *subset of primitive weights*

$$\Lambda=\{(\lambda_1,\lambda_2,\dots)\in M\mid|\lambda_i|\leqq 1\text{ for all }i\}\,. \tag{1.2.2}$$

Elements of Λ are in one-to-one correspondence with finite subsets of the set of positive integers. For any finite subset $J\subset\mathbb{N}_+$, define $\lambda(J)\subset\Lambda$ by

$$\lambda_j=1\text{ if }j\in J,\quad\lambda_j=0\text{ otherwise}. \tag{1.2.3}$$

For any $\lambda \in \Lambda$, define the subset $J(\lambda) \subset \mathbb{N}$ by

$$j \in J(\lambda) \text{ iff } \lambda_j > 0 . \tag{1.2.4}$$

(1.3) Tensor Coordinates on a Weight Component. Assume that all spaces $V_1, \ldots, V_n$, $W(z_1, \ldots, z_n)$ of Sect. (1.1) are graded by elements of the weight semigroup M:

$$V_j = \bigoplus_{\lambda \in M} V_{j,\lambda}, \quad W = \bigoplus_{\lambda \in M} W_\lambda .$$

These weight decompositions induce the weight decomposition of the tensor product:

$$(V_1 \otimes \cdots \otimes V_n)_\lambda := \bigoplus_{\substack{\lambda_1, \ldots, \lambda_n \\ \lambda_1 + \cdots + \lambda_n = \lambda}} V_{1,\lambda_1} \otimes \cdots \otimes V_{n,\lambda_n} .$$

For a graded space, set

$$V_{\leqq \lambda} := \bigoplus_{\nu \leqq \lambda} V_\nu .$$

We will be interested in the tensor coordinates on one of the weight components.

Fix $\lambda \in \Lambda$. Assume that for every $\sigma \in S_n$ an isomorphism

$$L_\sigma(z_1, \ldots, z_n): (V_{\sigma(1)} \otimes \cdots \otimes V_{\sigma(n)})_\lambda \to W(z_1, \ldots, z_n)_\lambda$$

is given. The set of these maps will be called *tensor coordinates* on W_λ. Define *transition functions* by the same formula: $L_{\sigma,\nu} := L_\nu^{-1} L_\sigma$.

Assume that for every i and j a linear map

$$R(i, j, z_1, z_2): (V_i \otimes V_j)_{\leqq \lambda} \to (V_j \otimes V_i)_{\leqq \lambda} \tag{1.3.1}$$

is given. Assume that the map preserves the weight decomposition.

(1.3.2) Tensor coordinates will be called *local* if for any $\sigma \in S_n$ and any $i \in \{1, \ldots, n-1\}$ we have

$$L_{\sigma,\sigma\sigma_i} = R_{i,i+1}(\sigma(i), \sigma(i+1), z_{\sigma(i)}, z_{\sigma(i+1)}) .$$

cf. (1.1.4).

(1.3.3) For local tensor coordinates the operators $\{R_{ij}\}$ form a unitary solution to the Yang–Baxter equation, see (1.1.6).

(1.4) Trees and Forests. Let T be a tree. Denote by $v(T)$ the set of its vertices and by $e(T)$ the set of its edges. For a finite set $J \subset \mathbb{N}_+$ let $\lambda(J) \in \Lambda$ be the corresponding primitive weight. A *weighted tree of weight* $\lambda(J)$ is a tree T with $|J|+1$ vertices numbered by $J \cup \{0\}$. A vertex with number j is denoted by (j). Vertex (0) is called the *root* of the tree.

Let $J_1, \ldots, J_n \subset \mathbb{N}_+$ be finite subsets. A *weighted forest of length n and multi-weight* $(\lambda(J_1), \ldots, \lambda(J_n))$ is an ordered collection of trees $F = (T_1, \ldots, T_n)$, where T_i is a tree of weight $\lambda(J_i)$ for all i. The weight $\lambda = \lambda(J_1) + \cdots + \lambda(J_n)$ is called the *weight of the forest*.

(1.4.1) A *weight form* is a collection of non-zero complex numbers $b = (b_{ij})$, where $i, j \in \mathbb{N}_+$, such that $b_{ij} = b_{ji}$ for all i, j.

(1.4.2) A *highest weight* is a sequence of non-zero complex numbers $c = (c_j)$, where $j \in \mathbb{N}$.

Let T be a weighted tree. A weight form and a highest weight define a color of every edge of the tree. The *color of the edge* connecting vertices (i) and (j) for $i, j \in \mathbb{N}_+$ is the number b_{ij}. The *color of the edge* connecting vertices (0) and (j) is the number c_j.

We say that a weighted tree is *admissible* with respect to a weight form and a highest weight if the colors of all its edges are different from one. We say that a weighted forest $(T_1, \ldots, T_n)$ is *admissible* with respect to a weight form b and highest weights $c^1, \ldots, c^n$ if the tree T_i is admissible with respect to b and c^i for all i.

The *space of admissible trees with a weight form b and a highest weight c* is the $\mathbb{C}$-linear space $V = V(b, c)$ with the basis $\{[T]\}$, where T runs through all weighted trees admissible with respect to b and c.

The space of trees has the *weight decomposition*:

$$V(b, c) = \bigoplus_{\lambda \in \Lambda} V(b, c)_\lambda , \tag{1.4.3}$$

where the space $V(b, c)_\lambda$ is the subspace with the basis $\{[T]\}$, T runs through all weighted admissible trees with weight λ.

(1.4.4) The *space of admissible forests of length n* with a weight form b and highest weights $c^1, \ldots, c^n$ is the $\mathbb{C}$-linear space

$$V(b, c^1, \ldots, c^n) = V(b, c^1) \otimes \cdots \otimes V(b, c^n) .$$

The space of forests has the basis $\{[T_1] \otimes \cdots \otimes [T_n]\}$ numerated by admissible forests $(T_1, \ldots, T_n)$. The space of forests has the weight decomposition:

$$V = \bigoplus_{\lambda \in M} V_\lambda ,$$

$$V_\lambda = \bigoplus_{\lambda_1 + \cdots + \lambda_n = \lambda} V(b, c^1)_{\lambda_1} \otimes \cdots \otimes V(b, c^n)_{\lambda_n} , \tag{1.4.5}$$

where M is the semigroup of weights.

(1.5) Realization of a Primitive Weight Component of the Space of Forests as a Space of Functions. Fix a weight form b and highest weights $c^\ell = (c_1^\ell, c_2^\ell, \ldots)$ for $\ell = 1, \ldots, n$. Fix a primitive weight $\lambda = (\lambda_1, \lambda_2, \ldots) \in \Lambda$. We realize the space

$$(V(b, c^1) \otimes \cdots \otimes V(b, c^n))_\lambda$$

as a space of suitable rational functions.

Note that this will be done only for a primitive weight.

Let $J = J(\lambda)$ be the finite subset of $\mathbb{N}_+$ corresponding to λ. Consider the space $\mathbb{C}^{|\lambda|}$ with coordinates $\{t_j\}_{j \in J}$. Fix pair-wise different complex numbers $z_1, \ldots, z_n$. For any $\ell \in \{1, \ldots, n\}$ and any $j \in J$ such that $c_j^\ell \neq 1$ define the hyperplane

$$H_j^\ell : t_j - c_j^\ell z_\ell = 0 .$$

For every $i, j \in J$ such that $i < j$ and $b_{ij} \neq 1$ define the hyperplane

$$H_{ij} : t_i - b_{ij} t_j = 0 .$$

The collection of all these hyperplanes will be called the *configuration of hyperplanes associated with* $\lambda, b, c^1, \ldots, c^n, z_1, \ldots, z_n$. Notations:

$$\mathscr{C} = \mathscr{C}(\lambda, b, c^1, \ldots, c^n, z_1, \ldots, z_n) .$$

An *edge* of a configuration is a non-empty intersection of some of its hyperplanes. A *vertex* of a configuration is a zero-dimensional edge. Let $F=(T_1,\dots,T_n)$ be a forest of weight λ admissible with respect to $b, c^1,\dots,c^n$.

(1.5.1) For any tree T_ℓ of F and for any of its edges e define the *edge function* f_e on $\mathbb{C}^{|\lambda|}$: if the edge connects vertices (i) and (j) for $i,j\in J$ and $i<j$, we set

$$f_e=z_\ell/(b_{ij}t_j-t_i)\,,$$

if the edge connects vertices (0) and (j), we set

$$f_e=z_\ell/(c_j^\ell z_\ell-t_j)\,.$$

The edge function is a rational function and its poles form a hyperplane of $\mathscr{C}$.

(1.5.2) For any tree T of the forest define its *tree function* by the formula

$$f_T=\prod_{e\subset T} f_e\,,$$

where f_e is the edge function of an edge e of T and the product is taken over all edges of the tree.

If a tree T has weight $(0,0,\dots,)$ and consists of its root, we set

$$f_T=1\,.$$

(1.5.3) Define the *untwisted forest function* of a forest $F=(T_1,\dots,T_n)$ by the formula

$$f_F=\prod_{\ell=1}^{n} f_{T_\ell}\,.$$

(1.5.4) The *space of forest functions* of weight λ admissible with respect to $b, c^1,\dots,c^n$ is the $\mathbb{C}$-linear space

$$W_\lambda(b,c^1,\dots,c^n,z_1,\dots,z_n)$$

consisting of all $\mathbb{C}$-linear combinations $\sum_F a_F f_F$, where F runs through the set of all forests of weight λ admissible for $b, c^1,\dots,c^n$, and $\{a_F\}$ are complex coefficients.

Let $V_\lambda(b,c^1,\dots,c^n)$ be the space of all forests of weight λ, see (1.4.4) and (1.4.5). There is the natural linear map

$$f(b,c^1,\dots,c^n,z_1,\dots,z_n)\colon V_\lambda(b,c^1,\dots,c^n)\to W_\lambda(b,c^1,\dots,c^n,z_1,\dots,z_n)\,, \tag{1.5.5}$$

namely, let $[T_1]\otimes\cdots\otimes[T_n]\in V_\lambda$ be the basic vector corresponding to a forest $F=(T_1,\dots,T_n)$, then we set

$$f(b,c^1,\dots,c^n,z_1,\dots,z_n)\colon [T_1]\otimes\cdots\otimes[T_n]\mapsto f_F\,.$$

(1.5.6) Lemma. *The map f is an isomorphism for non-degenerate $\lambda, b, c^1,\dots,c^n$ in the sense defined below.*

Denote by $(\mathbb{C}^*)^{|\lambda|}\subset\mathbb{C}^{|\lambda|}$ the subset of all points with non-zero coordinates. For any admissible forest F define the point

$$p(F)=\{t\in\mathbb{C}^{|\lambda|}\mid 1/f_e(t)=0 \text{ for all edges } e \text{ of } F\}\,.$$

$p(F)$ is a vertex of the configuration $\mathscr{C}$, and $p(F)\in(\mathbb{C}^*)^{|\lambda|}$. Any vertex of $\mathscr{C}$ lying in $(\mathbb{C})^{|\lambda|}$ may be written in this form.

(1.5.7) We say that $\lambda, b, c^1, \ldots, c^n$ are *non-degenerate* if the points $\{p(F)\}$ are pair-wise different for pair-wise different admissible forests.

The proof of the lemma is obvious

(*1.5.8*) *Remark.* The configuration $\mathscr{C}(\lambda, b, c^1, \ldots, c^n, z_1, \ldots, z_n)$ and forest functions homogeneously depend on $z_1, \ldots, z_n$. Namely, for $s \in \mathbb{C}^*$ the transformation $\mathbb{C}^{|\lambda|} \to \mathbb{C}^{|\lambda|}$, sending t to st, sends $\mathscr{C}(z_1, \ldots, z_n)$ to $\mathscr{C}(sz_1, \ldots, sz_n)$ and forest functions to forest functions.

(*1.5.9*) *Remark.* For any permutation $\sigma \in S_n$, the configuration $\mathscr{C}(\lambda, b, c^{\sigma(1)}, \ldots, c^{\sigma(n)}, z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ coincides with the configuration $\mathscr{C}(\lambda, b, c^1, \ldots, c^n, z_1, \ldots, z_n)$, and the space $W_\lambda(b, c^{\sigma(1)}, \ldots, c^{\sigma(n)}, z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ coincides with the space $W_\lambda(b^1, c^1, \ldots, c^n, z_1, \ldots, z_n)$. Therefore, for any $\sigma \in S_n$, we have constructed the linear isomorphism

$$f(b, c^{\sigma(1)}, \ldots, c^{\sigma(n)}, z_{\sigma(1)}, \ldots, z_{\sigma(n)}) : (V(b, c^{\sigma(1)}) \otimes \cdots \otimes V(b, c^{\sigma(n)}))_\lambda \to W_\lambda(b, c^1, \ldots, c^n, z_1, \ldots, z_n) .$$

(*1.6*) *Tensor Coordinates on a Primitive Weight Component of the Forest Space.* Fix $\lambda \in \Lambda, b, c^1, \ldots, c^n, z_1, \ldots, z_n$ as in (1.5). We define local homogeneous tensor coordinates on $W_\lambda(b, c^1, \ldots, c^n, z_1, \ldots, z_n)$. Let $J = J(\lambda)$ be the finite subset of $\mathbb{N}_+$ corresponding to the primitive weight λ as in (1.5).

(1.6.1) For every $i, j \in J$ define the *twisting function* by the formula

$$D_{ij} = (t_j - b_{ij} t_i)/(b_{ij} t_j - t_i) .$$

For every $\ell \in \{1, \ldots, n\}$ and $j \in J$ define the twisting function by the formula

$$D_j^\ell = (z_\ell - c_j^\ell t_j)/(c_j^\ell z_\ell - t_j) .$$

(1.6.2) Properties of twisting functions:

(1) Twisting functions are rational functions homogeneous with respect to transformations $(t, z) \mapsto (st, sz)$.
(2) $D_{ij} = 1$ if $b_{ij} = 1$.
$D_j^\ell = 1$ if $c_j^\ell = 1$.
(3) $D_{ij} = b_{ij} + (1 - (b_{ij})^2 t_j/(b_{ij} t_j - t_i) = (b_{ij})^{-1} + ((b_{ij})^{-1} - b_{ij}) t_i/(b_{ij} t_j - t_i)$.
$D_j^\ell = c_j^\ell + (1 - (c_j^\ell)^2) z_\ell/(c_j^\ell z_\ell - t_j)$.

Let $F = (T_1, \ldots, T_n)$ be an admissible forest of length n and weight λ. Let $\lambda_1, \ldots, \lambda_n$ be the weights of the trees $T_1, \ldots, T_n$, resp. Let $J_1, \ldots, J_n \subset \mathbb{N}_+$ be the subsets corresponding to $\lambda_1, \ldots, \lambda_n$, resp.

(1.6.3) For such a forest F and a permutation $\sigma \in S_n$, define the *twisting function* by the formula

$$D_{F,\sigma} = (\prod D_{ij})(\prod D_j^\ell) .$$

Here the first product is taken over all pairs $(i, j) \subset J$ such that

(1) $i < j$,
(2) $i \in J_{\sigma(a)}, j \in J_{\sigma(b)}$ for some $a > b$.

The second product is taken over all $j \in J$ and $\ell \in \{1, \ldots, n\}$ such that $j \in J_{\sigma(a)}$ and $\ell = \sigma(b)$ for some $a > b$.

(*1.6.4*) *Example.* For $n=2$, $J_1=\{1\}$, $J_2=\{2\}$, and the single non-trivial permutation $\sigma\in S_2$, we have

$$D_{F,\sigma}=D_{12}D_1^2\ ,$$

and for $J_1=\{2\}$, $J_2=\{1\}$ we have

$$D_{F,\sigma}=D_2^2\ .$$

(1.6.5) For any admissible forest F of length n and weight λ and for any permutation $\sigma\in S_n$, define the *twisted forest function*

$$f_{F,\sigma}=f_F D_{F,\sigma}\ ,$$

where f_F is the untwisted forest function defined in (1.5.4).

This construction defines a linear map $L_\sigma(z_1,\dots,z_n)$ of the forest space $(V(b,c^{\sigma(1)}))\otimes\cdots\otimes V(b,c^{\sigma(1)}))_\lambda$ defined in (1.4) to the space of rational functions on $\mathbb{C}^{|\lambda|}$. Namely, let a forest $F=(T_1,\dots,T_n)$ be admissible to $b,c^1,\dots,c^n$, then the forest $(T_{\sigma(1)},\dots,T_{\sigma(n)})$ is admissible to $b,c^{\sigma(1)},\dots,c^{\sigma(n)}$ and $[T_{\sigma(1)}]\otimes\cdots\otimes[T_{\sigma(n)}]$ is a basic vector in $(V(b,c^{\sigma(1)})\otimes\cdots\otimes V(b,c^{\sigma(n)}))_\lambda$. We set

$$L_\sigma(z_1,\dots,z_n)\colon [T_{\sigma(1)}]\otimes\cdots\otimes[T_{\sigma(n)}]\mapsto f_{F,\sigma}\ . \tag{1.6.6}$$

(1.6.7) Theorem. *Let λ, b be path non-degenerate in the sense defined below. Then the previous construction defines a map*

$$L_\sigma(z_1,\dots,z_n)\colon (V(b,c^{\sigma(1)})\otimes\cdots\otimes V(b,c^{\sigma(n)}))_\lambda\to W_\lambda(b,c^1,\dots,c^n,z_1,\dots,z_n)\ .$$

In other words, any twisted function is a linear combination of untwisted forest functions.

Define the notion of path non-degeneracy.

A *path* in J is a sequence $j_1,\dots,j_\ell\in J$ of pair-wise different elements, where $\ell>2$. Define the *graph of the path* as the graph with ℓ vertices $(j_1),\dots,(j_n)$ and ℓ edges $e_1=(j_1,j_2)$, $e_2=(j_2,j_3),\dots,e_\ell=(j_\ell,j_1)$. Consider the system of equations

$$f_{e_1}^{-1}=\cdots=f_{e_1}^{-1}=0\ ,$$

where f_e is the edge function of the edge e. This is a system of ℓ linear equations on ℓ variables $t_{j_1},\dots,t_{j_\ell}$. We say that the path is *non-degenerate* if the space of solutions to the system has codimension ℓ.

The pair λ, b will be called *path non-degenerate* if all paths in J are non-degenerate.

Theorem (1.6.7) is proved in (1.8).

Assume now that $\lambda, b, c^1,\dots,c^n$, are non-degenerate in the sense of (1.5.7) and λ, b are path non-degenerate. Then the map (1.5.5),

$$f(b,c^1,\dots,c^n,z_1,\dots,z_n)\colon (V(b,c^1)\otimes\cdots\otimes V(b,c^n))_\lambda$$

$$\to W_\lambda(b,c^1,\dots,c^n,z_1,\dots,z_n)$$

is an isomorphism, and the map (1.6.7)

$$L_\sigma(z_1,\dots,z_n)\colon (V(b,c^{\sigma(1)})\otimes\cdots\otimes V(b,c^{\sigma(n)}))_\lambda$$

$$\to W_\lambda(b,c^1,\dots,c^n,z_1,\dots,z_n)$$

is well defined for all $\sigma\in S_n$.

The space W_λ has the basis $\{f_F\}$ of untwisted forest functions. This basis is parametrized by forests $F=(T_1,\dots,T_n)$ admissible to $b, c^1,\dots,c^n$. The space $(V(b,c^{\sigma(1)})\otimes\cdots\otimes V(b,c^{\sigma(n)}))_\lambda$ has the basis $\{[T_{\sigma(1)}]\otimes\cdots\otimes[T_{\sigma(n)}]\}$ parametrized by forests $F=(T_1,\dots,T_n)$ admissible to $b, c^1,\dots,c^n$. For these bases, the map $L_\sigma(z_1,\dots,z_n)$ has a matrix $(L_\sigma(z_1,\dots,z_n)_{F,F'})$. The entries of this matrix are rational functions of $\{b_{ij}, c_j^\ell, z_1,\dots,z_n\}$.

(1.6.8) Lemma. *The determinant of the matrix* $(L_\sigma(z_1,\dots,z_n)_{F,F'})$ *is a non-trivial rational function.*

(1.6.9) Corollary. *For generic values of* $\{b_{ij}, c_j^\ell, z_1,\dots,z_n\}$ *the maps* $\{L_\sigma(z_1,\dots,z_n)\}$, $\sigma\in S_n$, *are isomorphisms, and therefore form tensor coordinates on* $W_\lambda(b,c^1,\dots,c^n,z_1,\dots,z_n)$.

Proof of the lemma. Let all parameters $\{b_{ij}, c_j^\ell\}$ tend to 1. Then all twisted forest functions tend to the corresponding untwisted functions, see (1.6.2). Hence the matrix of $L_\sigma(z_1,\dots,z_n)$ tends to the unit matrix. The lemma is proved.

(1.6.10) Lemma. *The tensor coordinates* $\{L_\sigma(z_1,\dots,z_n)\}$ *are homogeneous with respect to* $z_1,\dots,z_n$, *see* (1.1.7).

The lemma is obvious, see (1.5.9) and (1.6.2.1).

We have constructed tensor coordinates simultaneously for all primitive weights $\lambda\in\Lambda$ and all n. It turns out that these tensor coordinates are simultaneously local with respect to the same operators $R(i,j,z_1,z_2)$.

Namely, set

$$(V(b,c^i)\otimes V(b,c^j))_\Lambda=\bigoplus_{\lambda\in\Lambda}(V(b,c^i)\otimes V(b,c^j))_\lambda\,.$$

This is the sum of components with primitive weights of the space of admissible forests with two trees, see (1.4).

Set

$$W_\Lambda(b,c^i,c^j,z_1,z_2)=\bigoplus_{\lambda\in\Lambda}(W_\lambda(b,c^i,c^j,z_1,z_2))_\lambda\,.$$

This is the space of forest functions with primitive weights and two trees.

Let $\sigma, \nu\in S_2$ be the trivial and non-trivial elements, resp. The tensor coordinates

$$L_\sigma(z_1,z_2)\colon (V(b,c^i)\otimes V(b,c^j))_\Lambda\to W_\Lambda(b,c^i,c^j,z_1,z_2)\,,$$

$$L_\nu(z_1,z_2)\colon (V(b,c^i)\otimes V(b,c^i))_\Lambda\to W_\Lambda(b,c^i,c^j,z_1,z_2) \tag{1.6.11}$$

defined above give the transition function

$$L_{\sigma,\nu}(z_1,z_2)\colon (V(b,c^i)\otimes V(b,c^j))_\Lambda\to(V(b,c^j)\otimes V(b,c^i))_\Lambda\,. \tag{1.6.12}$$

Denote the transition function $L_{\sigma,\nu}(z_1,z_2)$ by $R(i,j,z_1,z_2)$.

(1.6.13) Lemma. *The tensor coordinates* $\{L_\sigma(z_1,\dots,z_n)\}$ *defined in* (1.6.7) *and* (1.6.9) *are local with respect to the operators* $\{R(i,j,z_1,z_2)\}$ *defined in* (1.6.12).

See the definition of local coordinates in (1.1.4) and (1.3.2).

The lemma is obvious because each factor of a twisting function $D_{F,\sigma}$ is defined by two of n sets $J_1, \ldots, J_n$, see (1.6.1) and (1.6.3).

(*1.7*) *Example*, cf. [M1]. Let $n=2$ and $\lambda=(0,0,\ldots)$. Fix b, c^1, c^2. Denote by T^0 the tree consisting of its root (0). Denote the vector $[T^0]\in V(b,c^1)$ by v_1, the vector $[T^0]\in V(b,c^2)$ by v_2. The spaces $(V(b,c^1)\otimes V(b,c^2))_\lambda$ and $(V(b,c^2)\otimes V(b,c^1))_\lambda$ are 1 dimensional with the basic vectors $v_1\otimes v_2$ and $v_2\otimes v_1$, resp. The space $W_\lambda(b,c^1,c^2,z_1,z_2)$ is 1 dimensional, it consists of constants. It has the basic vector $f_F=1$, where $F=(T^0,T^0)$ is the single forest of weight λ. The tensor coordinates are trivial:

$$L_\sigma(z_1,z_2)\colon v_1\otimes v_2\mapsto 1, L_\nu(z_1,z_2)\colon v_2\otimes v_1\mapsto 1$$

where $\sigma, \nu\in S_2$ are the trivial and non-trivial elements respectively. The transition function

$$L_{\sigma,\nu}(z_1,z_2)\colon (V(b,c^1)\otimes V(b,c^2))_\lambda\to(V(b,c^2)\otimes V(b,c^1))_\lambda$$

is the transposition of factors.

Let $\lambda=(1,0,0,\ldots)$. Assume that c^1 and c^2 are such that $c_1^1\neq 1$ and $c_1^2\neq 1$. Set $c_1=c_1^1$, $c_2=c_1^2$. Denote by T_1 the tree consisting of the edge connecting the root (0) with the vertex (1). Denote the vector $[T^1]\in V(b,c^1)$ by fv_1, the vector $[T^1]\in V(b,c^2)$ by fv_2.

The space $(V(b,c^1)\otimes V(b,c^2))_\lambda$ is 2 dimensional with the basis $fv_1\otimes v_2$, $v_1\otimes fv_2$. The space $(V(b,c^2)\otimes V(b,c^1))_\lambda$ is 2 dimensional with the basis $fv_2\otimes v_1$, $v_2\otimes fv_1$.

The space $W_\lambda(b,c^1,c^2)$ is the 2 dimensional space of linear combinations of functions

$$f_{F_1}=\frac{-z_1}{t_1-c_1z_1},\quad f_{F_2}=\frac{-z_2}{t_1-c_2z_2},$$

where $F_1=(T^1,T^0)$, $F_2=(T^0,T^1)$ are the forests of weight λ admissible to b,c^1,c^2.

The tensor coordinates are given by the formulas:

$$L_\sigma(z_1,z_2)\colon fv_1\otimes v_2\mapsto\frac{-z_1}{t_1-c_1z_1},$$

$$L_\sigma(z_1,z_2)\colon v_1\otimes fv_2\mapsto\frac{c_1t_1-z_1}{t_1-c_1z_1}\,\frac{-z_2}{t_1-c_2z_2},$$

$$L_\nu(z_1,z_2)\colon fv_2\otimes v_1\mapsto\frac{-z_2}{t_1-c_2z_2},$$

$$L_\nu(z_1,z_2)\colon v_2\otimes fv_1\mapsto\frac{c_2t_1-z_2}{t_1-c_2z_2}\,\frac{-z_1}{t_1-c_1z_1}. \tag{1.7.1}$$

The transition function is given by:

$$L_{\sigma,\nu}(z_1,z_2)\colon fv_1\otimes v_2\mapsto\frac{((c_2)^2-1)z_1}{c_1c_2z_1-z_2}fv_2\otimes v_1+\frac{c_1z_1-c_2z_2}{c_1c_2z_1-z_2}v_2\otimes fv_1,$$

$$L_{\sigma,\nu}(z_1,z_2)\colon v_1\otimes fv_2\mapsto\frac{c_2z_1-c_1z_2}{c_1c_2z_1-z_2}fv_2\otimes v_1+\frac{((c_1)^2-1)z_2}{c_1c_2z_1-z_2}v_2\otimes fv_1. \tag{1.7.2}$$

This transformation is well known in the theory of quantum groups, see [M1-2], (3.1) and (3.4.4).

(1.8) Proof of Theorem (1.6.7). The theorem follows from the three lemmas formulated below.

Let $n\in\mathbb{N}_+$, $n>1$. Let $z, b, c_0, c_1, \dots, c_n$ be numbers, $t=(t_1, \dots, t_n)$. Consider the rational functions

$$F(t)=(t_1-a)(t_2-t_1)\dots(t_n-t_{n-1})(t_n-b),$$
$$g(t)=(c_0+c_1t_1+\dots+c_nt_n)/F(t),$$
$$f_1(t)=(t_1-a)/F(t), f_{n+1}(t)=(t_n-b)/F(t),$$
$$f_i(t)=(t_i-t_{i-1})/F(t) \text{ for } i=2,\dots,n.$$

(1.8.1) Lemma. *$g=m_1f_1+\dots+m_{n+1}f_{n+1}$ for some numbers $m_1, \dots, m_{n+1}$.*

The lemma is obvious.

Let $n\in\mathbb{N}_+$, $n>2$. Let a_i, b_i, c_i for $i=1, \dots, n$ be numbers. Consider the rational functions

$$F(t)=(a_1t_1-b_1t_2)(a_2t_2-b_2t_3)\dots(a_{n-1}t_{n-1}-b_{n-1}t_n)(a_nt_n-b_nt_1),$$
$$g(t)=(c_1t_1+\dots+c_nt_n)/F(t), \quad f_n(t)=(a_nt_n-b_nt_1)/F(t),$$
$$f_i(t)=(a_it_i-b_it_{i+1})/F(t) \text{ for } i=1,\dots,n-1.$$

(1.8.2) Lemma. *If the functions $(a_1t_1-b_1t_2), \dots, (a_{n-1}t_{n-1}-b_{n-1}t_n), (a_nt_n-b_nt_1)$ are linearly independent, then*

$$g=m_ff_1+\dots+m_nf_n$$

for some numbers $m_1, \dots, m_n$.

The lemma is obvious.

A *forest with artificial edges* of weight $\lambda\in\Lambda$ admissible to a weight form b and highest weights $c^1, \dots, c^n$ is a graph obtained from an admissible forest $F=(T_1, \dots, T_n)$ by attaching new (artificial) edges so that the total number of initial and artificial edges connecting any two vertices is not greater than one. The number of artificial edges will be called the *degree of the forest with artificial edges.*

For an artificial edge e connecting vertices (i), (j), where $i, j\in J(\lambda)$, define its *edge function* f_e by setting

$$f_e=D_{ij},$$

see (1.6.1). For an artificial edge e connecting vertices (j) and (0) where $j\in J(\lambda)$ and (0) is the root of the tree T_ℓ, define its *edge function* f_e by setting

$$f_e=D_j^\ell,$$

see (1.6.1). Define the *forest function of a forest F with artificial edges $\{e\}$* by setting

$$f_{F,\{e\}}=f_F\prod_e f_e,$$

where f_F is the untwisted forest function defined in (1.5.3) and the product is taken over all artificial edges.

(*1.8.3*) *Example.* A twisted forest function $f_{F,\sigma}$ defined in (1.6.5) is a forest function of a forest F with suitably attached artificial edges.

(1.8.4) Lemma. *For an arbitrary forest F with k artificial edges, there exist forests $F^1, \dots, F^N$ with $k-1$ artificial edges each, such that the forest function of F is a linear combination of forest functions of $F^1, \dots, F^N$.*

The lemma easily follows from Lemmas (1.8.1) and (1.8.2).
Theorem (1.6.7) follows from Lemma (1.8.4).

2. q-Hypergeometric Functions, Difference Equations for Jackson's Integrals

(*2.1*) *Hypergeometric Functions.* In this section we discuss some basic facts on multidimensional hypergeometric functions motivating our study of q-hypergeometric functions.

Fix $n, k \in \mathbb{N}_+$. Set $t=(t_1, \dots, t_k)$, $z=(z_1, \dots, z_n)$, and $dt = dt_1 \wedge \cdots \wedge dt_k$. Fix complex numbers $\{a_{ij}\}$ for $i,j \in \{1, \dots, k\}$, $i<j$, and $\{a_j^\ell\}$ for $j \in \{1, \dots, k\}$, $\ell \in \{1, \dots, n\}$. The function

$$F(t, z) = \prod_{i<j} (t_i - t_j)^{a_{ij}} \prod_{j,l} (t_j - z_\ell)^{a_j^\ell} \tag{2.1.1}$$

is a holomorphic multivalued function with singularities where $\{t_i = t_j\}$ or $\{t_j = z_\ell\}$.

A *general hypergeometric function associated with F* is an integral of the form

$$I(z; \varphi; \gamma) = \int_{\gamma(z)} \varphi F \, dt \, . \tag{2.1.2}$$

Here φ is a rational function of t, z regular outside singularities of F. $\gamma(z)$ is a family of suitable k-dimensional cycles continuously depending on z in a natural sense, see for example [SV, V].

For fixed γ and φ, the function I is a multivalued holomorphic function with singularities where some of $z_1, \dots, z_n$ coincide.

(*2.1.3*) *Example.*

$$I(z_1, z_2, z_3; 1; \gamma) = \int_{\gamma(z)} (t_1 - z_1)^{\alpha_1} (t_1 - z_2)^{\alpha_2} (t_1 - z_3)^{\alpha_3} \, dt_1 \, ,$$

where $\gamma(z)$ is a curve in $\mathbb{C}$ shown in Fig. 2.1.

Obvious homology reasons may be applied to studying hypergeometric functions. For a fixed z, the form $\varphi F \, dt$ is closed. Hence $I(z; \varphi; \gamma_1) = I(z; \varphi; \gamma_2)$ for all φ and homologous $\gamma_1(z)$ and $\gamma_2(z)$. If

$$\varphi_1 F \, dt - \varphi_2 F \, dt = d\psi \, ,$$

then $I(z, \varphi_1, \gamma) = I(z, \varphi_2, \gamma)$ for all γ.

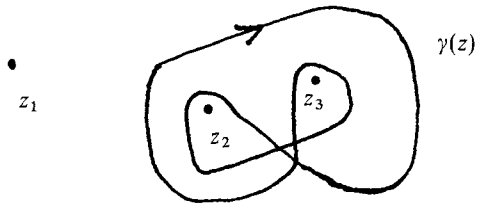

Fig. 2.1.

Therefore, instead of studying infinite-dimensional space of general hypergeometric functions associated with F, it suffices to study only a finite-dimensional family of hypergeometric functions $\{I(z, \varphi_\alpha, \gamma_\beta)\}_{\alpha,\beta}$ if for this family the differential forms $\{\varphi_\alpha F\, dt\}$ generate the corresponding cohomology group and the cycles $\{\varphi_\beta(z)\}$ generate the corresponding homology group for all z with pair-wise different coordinates.

Below we define such a family of differential forms.

The *Orlik–Solomon algebra* A associated with F is the finite dimensional exterior $\mathbb{C}$-algebra generated by differential forms $d(t_i-t_j)/(t_i-t_j)$, $d(t_i-z_\ell)/(t_i-z_\ell)$ for $i, j\in\{1, \ldots, k\}$, $\ell\in\{1, \ldots, n\}$. The algebra is graded:

$$A=\bigoplus A^p$$

where A^p is the space of p-forms.

A *hypergeometric differential form* associated with F is a differential form $F\omega$ for $\omega\in A$.

A *hypergeometric function associated with* F is an integral

$$J(z; \omega; \gamma)=\int_{\gamma(z)} F\omega \tag{2.1.4}$$

where $\omega\in A^k$, and $\gamma(z)$ is a family of cycles as in (2.1.2).

The finite-dimensional family of hypergeometric differential form $\{F\omega\}$ for $\omega\in A^k$ has two remarkable properties:

(2.1.5) Under certain conditions [ESV, SV], the forms $\{F\omega\}$, $\omega\in A^k$, generate the corresponding cohomology group, and, therefore, an arbitrary general hypergeometric function $I(z; \varphi, \gamma)$ of the form (2.1.2) may be represented as a linear combination

$$I(z; \varphi; \gamma)=\sum_\alpha c_\alpha(z) J(z; \omega_\alpha; \gamma)\,,$$

where $\omega_\alpha\in A^k$ and $\{c_\alpha(z)\}$ are rational functions of z independent of γ.

(2.1.6) For a basis $\omega_1, \ldots, \omega_r$ in A^k, there exists a set of constant $r\times r$-matrices $\{\Omega_{\ell m} | \ell, m\in\{1, \ldots, n\}$ and $\Omega_{\ell m}=\Omega_{m\ell}\}$ such that for any family of cycles γ the vector-function

$$I(z)=\left(\int_{\gamma(z)} F\omega_1, \ldots, \int_{\gamma(z)} F\omega_r\right)$$

satisfies the system of differential equations

$$\frac{\partial I}{\partial z_\ell} = \sum_{m \neq \ell} \frac{\Omega_{\ell m}}{z_\ell - z_m} I, \quad \ell = 1, \ldots, n,$$

see for example [SV].

The statement (2.1.6) says that, independently on (2.1.5), the subspace in the corresponding cohomology group, generated by hypergeometric forms $\{F\omega\}$ for a fixed z, is invariant with respect to the corresponding Gauss–Manin connection.

Interrelations of multidimensional hypergeometric functions with the representation theory of Kac–Moody Lie algebras come through this distinguished finite dimensional space of hypergeometric forms $\{F\omega\}$ [SV]. The space $\{F\omega\}$, $\omega \in A^k$, is interpreted as a weight component of the tensor product of n modules dual to Verma modules, $\Omega_{\ell m}$ as the Casimir operator acting in the ℓ^{th} and m^{th} factors of the product, and the system of differential equations (2.1.6) as the Knizhnik–Zamolodchikov equations in the Conformal Field Theory.

Analogously, there is a distinguished finite dimensional chain complex computing the homology groups. The cycles $\{\gamma_\beta(z)\}$ are constructed as linear combinations of cells of the chain complex. The chain complex is interpreted in terms of quantum groups, see [V].

The integration of hypergeometric forms over chains of the complex gives a pairing between the corresponding objects of the theory of Kac–Moody algebras and the theory of quantum groups.

Therefore generalizing the theory of hypergeometric functions to the case of q-hypergeometric functions, it is reasonable to look for an analog of the finite dimensional family of hypergeometric forms $\{F\omega\}$ with properties (2.1.5) and (2.1.6), and for an analog of the finite dimensional chain complex that would provide possible connections with representation theory.

In the next sections we describe such a q-analog of hypergeometric forms.

(2.2) q-Analogs of Differentiation and Integration [A, FR, M]. The q-analogs of differentiation and integration are given by the formulas

$$\frac{d_q f}{d_q t}(t) = \frac{1}{t} \frac{f(qt) - f(t)}{q-1},$$

$$\int_0^\xi f(t)\, d_q t = \xi(1-q) \sum_{n \geqq 0} f(\xi q^n) q^n,$$

$$\int_\xi^\infty f(t)\, d_q t = \xi(1-q) \sum_{n<0} f(\xi q^n) q^n. \tag{2.2.1}$$

when $q \to 1$, these operations become the usual differentiation and integration.

Set

$$\int_0^{\xi\infty} f(t) \frac{d_q t}{t} = (1-q) \sum_{n=-\infty}^{\infty} f(\xi q^n). \tag{2.2.2}$$

This sum is called the Jackson integral along a q-interval $[0, \xi\infty]_q$.

There is a q-analog of the Stokes theorem:

$$\int_0^{\xi\infty} t \frac{d_q f}{d_q t} \frac{d_q t}{t} = 0 . \tag{2.2.3}$$

The multiple Jackson integral is similarly defined where a q-cycle takes place instead of a q-interval.

$\mathbb{Z}^k$ acts on $\mathbb{C}^k$:

(2.2.4) For any $a=(a_1, \ldots, a_k) \in \mathbb{Z}^k$, set

$$M(a): \mathbb{C}^k \to \mathbb{C}^k ,$$

$$(\xi_1, \ldots, \xi_k) \mapsto (\xi_1 q^{a_1}, \ldots, \xi_k q^{a_k}) .$$

$\mathbb{Z}^k$ acts on functions on $\mathbb{C}^k$:

(2.2.5) For any $a \in \mathbb{Z}^k$, set

$$M(a): f(t_1, \ldots, t_k) \mapsto f(t_1 q^{a_1}, \ldots, t_k q^{a_k}) .$$

(2.2.6) For an arbitrary $\xi \in (\mathbb{C}^*)^k$ the $\mathbb{Z}^k$-orbit of ξ is called a *k-dimensional q-cycle* and denoted by $[0, \xi\infty]_q$. The *Jackson integral* of a function $f(t_1, \ldots, t_k)$ over a q-cycle $[0, \xi\infty]_\infty$

$$\int_{[0, \xi\infty]_q} f(t_1, \ldots, t_k) \Omega \tag{2.2.7}$$

for $\Omega = (d_q t_1/t_1) \wedge \cdots \wedge (d_q t_k/t_k)$ is the sum

$$(1-q)^k \sum_{-\infty < a_1, \ldots, a_k < \infty} f(\xi_1 q^{a_1}, \ldots, \xi_k q^{a_k}) \tag{2.2.8}$$

if it exists.

For any $a \in \mathbb{Z}^k$, we have

$$\int_{[0, \xi\infty]_q} M(a)(f) \Omega = \int_{[0, \xi\infty]_q} f \Omega . \tag{2.2.9}$$

(2.3) q-Cohomology and q-Hypergometric Functions, [A, AK]. Set

$$(t)_\infty = (t; q)_\infty = \prod_{n=0}^{\infty} (1-q^n t) ,$$

$$(t)_m = \prod_{n=0}^{m} (1-q^n t) . \tag{2.3.1}$$

A q-analog of the function $(1-t)^{2a}$ is the function

$$\frac{(q^{-a} t)_\infty}{(q^a t)_\infty} . \tag{2.3.2}$$

This function tends to $(1-t)^{2a}$ when $q \to 1$.

(2.3.3) Fix $n, k \in \mathbb{N}_+$. Fix complex numbers $\{b_{ij}, c_m^\ell \alpha_m\}$ where $i, j, m \in \{1, \ldots, k\}$, $i<j$, and $\ell \in \{1, \ldots, n\}$. We assume that $b_{ij} \neq 0$ and $c_m^\ell \neq 0$ for all i, j, ℓ, m. A number b_{ij} or c_m^ℓ will be called *essential* if it is not equal to 1.

Set

$$\Phi_{ij}(t_i, t_j)=\frac{((b_{ij})^{-1}t_i/t_j)_\infty}{(b_{ij}t_i/t_j)_\infty},$$

$$\Phi_m^\ell(t_m, z_\ell)=\frac{((c_m^\ell)^{-1}t_m/z_\ell)_\infty}{(c_m^\ell t_m/z_\ell)_\infty},$$

$$\Phi(t, z)=t_1^{\alpha_1}\cdots t_k^{\alpha_k}\prod_{i<j}\Phi_{ij}\prod_{\ell,m}\Phi_m^\ell. \tag{2.3.4}$$

Φ is a q-analog of the function $F(t, z)$ in (2.1.1).

Functions Φ_{ij} and Φ_m^ℓ have the properties:

$$\Phi_{ij}=1 \quad \text{if } b_{ij}=1. \quad \Phi_m^\ell=1 \text{ if } \quad c_m^\ell=1. \tag{2.3.5}$$

$$\Phi_{ij}(qt_i, qt_j)=\Phi_{ij}(t_i, t_j),$$

$$\Phi_{ij}(qt_i, t_j)=b_{ij}D_{ij}(t_i, t_j)\,\Phi_{ij}(t_i, t_j),$$

$$D_{ij}(t_i, qt_j)\Phi_{ij}(t_i, qt_j)=(b_{ij})^{-1}\Phi_{ij}(t_i, t_j), \tag{2.3.6}$$

where D_{ij} is defined in (1.6.1).

$$\Phi_m^\ell(qt_m, qz_\ell)=\Phi_m^\ell(t_m, z_\ell),$$

$$\Phi_m^\ell(qt_m, z_\ell)=c_m^\ell D_m^\ell(t_m, z_\ell)\Phi_m^\ell(t_m, z_\ell),$$

$$D_m^\ell(t_m, qz_\ell)\Phi_m^\ell(t_m, qz_\ell)=(c_m^\ell)^{-1}\Phi_m^\ell(t_m, z_\ell). \tag{2.3.7}$$

where D_m^ℓ is defined in (1.6.1).

(2.3.8) For $a\in\mathbb{Z}^k$ let $T(a)$ be the operator defined in (2.2.5). Then

$$M(a)(\Phi)=r_a\Phi,$$

where r_a is a rational function of t and z.

A *general q-hypergeometric function* associated with Φ is a Jackson integral of the form

$$I(z;\varphi;\xi)=\int_{[0,\xi\infty]_q}\varphi\Phi\Omega, \tag{2.3.9}$$

where φ is a rational function of t and z, and $[0, \xi\infty]_q$ is a k-dimensional q-cycle.

(2.3.10) Remark. We will discuss formal algebraic properties of such integrals and will not discuss their convergence.

For $a\in\mathbb{Z}^k$ set

$$\nabla_a(\varphi)=\varphi-r_aM(a)(\varphi). \tag{2.3.11}$$

Then by the Stokes theorem we have

$$\int_{[0,\xi\infty]_q}\nabla_a(\varphi)\Phi\Omega=0. \tag{2.3.12}$$

We will restrict the class of admissible functions φ in (2.3.9) and will consider only functions that belong to the vector space $\mathscr{V}$ defined below. Roughly speaking, $\mathscr{V}$ is the space of functions that have no poles outside singularities of Φ.

More precisely, $\mathscr{V}$ consists of all rational functions of t and z having the form

$$\varphi = P\left(\prod_{i=1}^{k} t_i^{a_i} \prod_{i<j} ((b_{ij})^{-1} t_i/t_j)_{r_{ij}} \prod_{i<j} (q^{-r'_{ij}} b_{ij} t_i/t_j)_{r'_{ij}} \right.$$
$$\left. \times \prod_{\ell,m} ((c_m^\ell)^{-1} t_m/z_\ell)_{s_{\ell m}} \prod_{\ell,m} (q^{-s'_{\ell m}} t_m/z_\ell)_{s'_{\ell m}} \right)^{-1}, \tag{2.3.13}$$

where P is a polynomial of t and z, $\{a_i, r_{ij}, r'_{ij}, s_{\ell m}, s'_{\ell m}\}$ are arbitrary natural numbers.

Now let $z^0=(z_1^0, \ldots, z_n^0)$ be a set of pair-wise different non-zero numbers. Define the space of rational functions of t by the formula

$$\mathscr{V}(z^0)=\{\varphi|_{z=z^0}, \text{ where } \varphi\in\mathscr{V}\}\ .$$

The space $V(z^0)$ contains 1 and is invariant with respect to the operators $\{\nabla_a|_{z=z^0}\}$, see [A, AK].

(2.3.14) The k^{th} *cohomology group* associated with Φ for $z=z^0$ is

$$H^k(\Phi, z^0)=\mathscr{V}(z^0)\Big/\sum_{a\in\mathbb{Z}^k} \nabla_a|_{z=z^0}\,\mathscr{V}(z^0)\ .$$

(2.3.15) The k^{th} *homology group* associated with Φ for $z=z^0$ is the group $H_k(\Phi, z^0)$ dual to $H^k(\Phi, z^0)$.

It turns out that the group $H^k(\Phi, z^0)$ is finite dimensional, and, moreover, the dimension of the group is equal to the number of suitable forests. Namely:

(2.3.16) Let $\lambda=(\lambda_1, \lambda_2, \ldots)$ be the sequence in Λ such that $\lambda_j=1$ for $j\leqq k$ and $\lambda_j=0$ for $j>k$. Let $b=(b_{ij})$ be a weight form such that the numbers $\{b_{ij}\}$ for $i<j\leqq k$ coincide with the numbers $\{b_{ij}\}$ in (2.3.3). Let $c^1=(c_1^1, c_2^1, \ldots), \ldots, c^n=(c_1^n, c_2^n, \ldots)$ be highest weights such that the numbers $\{c_m^\ell\}$ for $\ell\in\{1, \ldots, n\}$ and $m\in\{1, \ldots, k\}$ coincide with the numbers $\{c_m^\ell\}$ in (2.3.3).

(2.3.17) **Theorem** [A, AK]. *Under generality conditions on z^0, on the numbers $\{\alpha_m\}$, and on the essential numbers $\{b_{ij}, c_m^\ell\}$ in* (2.3.3), *the dimension of the group $H^k(\Phi, z^0)$ is equal to the number of n-forests of weight λ admissible with respect to the weight form b and the highest weights $c^1, \ldots, c^n$.*

For the proof, see [A, AK]. In [A, AK2] the theorem is proved only in the case where all numbers $\{b_{ij}, c_m^\ell\}$ in (2.3.3) are essential. However, a similar proof is valid in the present situation.

(2.4) q-Analog of Hypergeometric Forms. Let $\mathscr{F}$ be the set of all n-forests of weight λ admissible with respect to the weight form b and highest weights $c^1, \ldots, c^n$, where $\lambda, b, c^1, \ldots, c^n$ are defined in (2.3). Fix a set of pair-wise different non-zero numbers $z=(z_1, \ldots, z_n)$.

A forest $F=(T_1, \ldots, T_n)$ has k edges. There are 2^k different orientations of the edges. Assume that the edges of the forest are oriented.

Define the *edge function f_e of an oriented edge e* of F, see [A].

(2.4.1) Let T_ℓ be a tree of F, e its oriented edge. Let e connect vertices (i) and (j) such that $i<j$ and $i,j\in\{1,\dots,k\}$. If e is oriented from (i) to (j), then set

$$f_e=z_\ell/(b_{ij}t_j-t_i)\ .$$

If e is oriented from (j) to (i), then set

$$f_e=z_\ell/(qt_j-b_{ij}t_i)\ ,$$

cf. (1.5.1).

Let e connect vertices (0) and (m). If e is oriented from (0) to (m), then set

$$f_e=z_\ell/(c_m^\ell z_\ell-t_m)\ .$$

If e is oriented from (m) to (0), then set

$$f_e=z_\ell/(qz_\ell-c_m^\ell t_m)\ .$$

(2.4.2) For an oriented forest F, define its *oriented forest function* by

$$f_F=\prod_e f_e\ ,$$

where f_e is the function of an oriented edge, and the product is taken over all edges of F, see [A].

There are two distinguished orientations for a forest.

The orientation is called *natural* if each edge is oriented from the vertex with the smaller number to the vertex with the greater number.

For a naturally oriented forest F, its oriented forest function defined in (2.4.2) coincides with the forest function of an unoriented forest F defined in (1.5.3).

The orientation is called *terminal* if each edge of each tree of the forest is oriented in the direction opposite to the direction to the root of the tree.

Let us consider three finite dimensional spaces of functions: the space B of all $\mathbb{C}$-linear combinations of all oriented functions of forests in $\mathscr{F}$, the space B_0 of all $\mathbb{C}$-linear combinations of all terminally oriented forest functions of forests in $\mathscr{F}$, and the space B_1 of all $\mathbb{C}$-linear combinations of all naturally oriented forests in $\mathscr{F}$.

B_1 coincides with $W_\lambda(b,c^1,\dots,c^n,z_1,\dots,z_n)$ defined in (1.5.4). We have

$$B\supset B_0,\ B\supset B_1\ .$$

If N is the number of forests in $\mathscr{F}$, then

$$\dim B=2^kN,\ \dim B_0=\dim B_1=N$$

for $b,c^1,\dots,c^n,z_1,\dots,z_n$ in general position.

Obviously, $B\subset\mathscr{V}(z)$, where $\mathscr{V}(z)$ is defined in (2.3.13). This inclusion induces homomorphisms

$$i_0\colon B_0\to H^k(\Phi,z)\ ,$$

$$i_1\colon B_1\to H^k(\Phi,z)\ . \tag{2.4.3}$$

(2.4.4) Theorem [A]. *Under generality conditions on z, on the numbers $\{\alpha_m\}$, and on the essential numbers $\{b_{ij},c_m^\ell\}$ in (2.3.3), the homomorphism i_0 is an isomorphism.*

In [A] the theorem is proved only in the case where all numbers are essential and $n=1$. However, a similar proof is valid in the present situation.

(2.4.5) Theorem. *Under generality conditions on z, on the numbers* $\{\alpha_m\}$, *and on the essential numbers* $\{b_{ij}, c_m^\ell\}$ *in* (2.3.3), *the homomorphism* i_1 *is an isomorphism.*

It suffices to prove that i_1 is a monomorphism under generality conditions. It may be done similarly to Lemma 5.5 in [A] proving that a suitable determinant has a nontrivial asymptotics. The proof will be published elsewhere.

(2.5) q-Difference Equations for q-Hypergeometric Functions. For a function $h(z_1, \ldots, z_n)$, set

$$Z_j\colon h(z_1, \ldots, z_n) \mapsto h(z_1, \ldots, qz_j, \ldots, z_n)\,, \tag{2.5.1}$$

where $j=1, \ldots, n$. We will describe the action of the operators $Z_1, \ldots, Z_n$ on q-hypergeometric functions $I(z, \varphi, \xi)$ for $\varphi \in B_1$. We will define some linear operators

$$A_j(z_1, \ldots, z_n)\colon B_1 \to B_1$$

for $j=1, \ldots, n$ so that

$$Z_j\colon I(z, \varphi, \xi) \mapsto I(z, A_j(z)\varphi, \xi)$$

for all k-dimensional q-cycles $[0, \xi\infty]_q$.

In (1.6) we have constructed tensor coordinates on B_1. In particular, we have identified B_1 with the space

$$V_\lambda = (V(b, c^1) \otimes \cdots \otimes V(b, c^n))_\lambda$$

of all n-forests of weight λ admissible to $b, c^1, \ldots, c^n$, where $\lambda, b, c^1, \ldots, c^n$ are defined in (2.3.3). We'll define the operators $A_j(z)$ as linear operators on V_λ, the operators will be defined in terms of transition functions for those tensor coordinates.

First, define n diagonal operators $D_1, \ldots, D_n$ on V_λ. Set

$$V_i = V(b, c^i) \tag{2.5.2}$$

for $i=1, \ldots, n$. We have

$$V_\lambda = \bigoplus_{\lambda_1 + \cdots + \lambda_n = \lambda} (V_1)_{\lambda_1} \otimes \cdots \otimes (V_n)_{\lambda_n}\,. \tag{2.5.3}$$

(2.5.4) For any $r \in \{1, \ldots, n\}$ define the linear operator

$$D_r\colon V_\lambda \to V_\lambda$$

by the stipulation that for any $\lambda_1, \ldots, \lambda_n$ the operator D_r restricted to $(V_1)_{\lambda_1} \otimes \cdots \otimes (V_n)_{\lambda_n}$ is the operator of multiplication by the number $d(\lambda, \lambda_r)$ defined below.

Let $J(\lambda)$ and $J(\lambda_r) \subset \mathbb{N}_+$ be the subsets corresponding to λ and λ_r, see (1.2.4). Set

$$d(\lambda, \lambda_r) = \left(\prod q^{\alpha_u}\right)\left(\prod_{ij} b_{ij}\right)\left(\prod_{i'j'} b_{i'j'}\right)^{-1}\left(\prod_{\ell m} c_m^\ell\right)\left(\prod_{m'} c_{m'}^r\right)^{-1}. \tag{2.5.5}$$

Here the first product is over $u \in J(\lambda_r)$. The second product is over all $i<j$ such that $i \in J(\lambda_r)$ and $j \in J(\lambda)-J(\lambda_r)$. The third product is over all $i'<j'$ such that $j' \in J(\lambda_r)$ and $i' \in J(\lambda)-J(\lambda_r)$. The fourth product is over all $m \in J(\lambda_r)$ and $\ell \in \{1, \ldots, n\}$, $\ell \neq r$. The fifth product is over all $m' \in J(\lambda)-J(\lambda_r)$.

(*2.5.6*) *Example.* For $J(\lambda_1)=\{1\}$ and $J(\lambda)=\{1, 2\}$, we have

$$d(\lambda, \lambda_1)=q^{\alpha_1} b_{12} c_2^1/c_1^2 .$$

For $i, j \in \{1, \ldots, n\}$, $i \neq j$, let

$$R(i, j, z_1, z_2): (V_i \otimes V_j)_\Lambda \to (V_j \otimes V_i)_\Lambda \tag{2.5.7}$$

be the transition function introduced in (1.6.12).

Let

$$P: (V_j \otimes V_i)_\Lambda \to (V_i \otimes V_j)_\Lambda \tag{2.5.8}$$

be the transposition of factors. Set

$$\bar{R}(i, j, z_1, z_2)=PR(i, j, z_1, z_2): (V_i \otimes V_j)_\Lambda \to (V_i \otimes V_j)_\Lambda . \tag{2.5.9}$$

For any $i, j \in \{1, \ldots, n\}$, $i<j$, defined the linear operator

$$R_{ij}(z_i, z_j): V_\lambda \to V_\lambda , \tag{2.5.10}$$

acting as $\bar{R}(i, j, z_i, z_j)$ on $V_i \otimes V_j$ and as the identity on other factors of V_λ. R_{ij} homogeneously depends on z_i, z_j, see (1.6.10).

For any $j \in \{1, \ldots, n\}$, set

$$A_j(z_1, \ldots, z_n)=R_{j,j+1}^{-1}(z_j, z_{j+1}) \ldots R_{j,n}^{-1}(z_j, z_n) D_j R_{1,j}(z_1, qz_j) \ldots R_{j-1,j}(z_{j-1}, qz_j) \tag{2.5.11}$$

Let

$$L(z)=L_{\mathrm{id}}(z_1, \ldots, z_n): V_\lambda \to B_1=W_\lambda(b, c^1, \ldots, c^n, z_1, \ldots, z_n) \tag{2.5.12}$$

be the isomorphism constructed in (1.6.6). Here $\mathrm{id} \in S_n$ is the identity permutation. This isomorphism sends an admissible forest F to the twisted forest function $f_{F, \mathrm{id}}$.

(2.5.13) Theorem. *For any $x \in V_\lambda$ and any $j \in \{1, \ldots, n\}$, we have*

$$Z_j: I(z, L(z)x, \xi) \mapsto I(z, L(z)A_j(z)x, \xi)$$

for all k-dimensional q-cycles $[0, \xi\infty]$. More precisely, we have

$$Z_j L(z)x - L(z)A_j(z)x \in \sum_{a \in \mathbb{Z}^k} \nabla_a|_z \mathscr{V}(z) ,$$

see (2.3.14).

This theorem is a q-analog of (2.1.6).

The theorem is proved in (2.6).

(2.5.14) Example, cf. [M1-2]. Let $n=2$, $k=1$. *Set*

$$\Phi(t_1, z_1, z_2)=t_1^{\alpha_1} t_2^{\alpha_2} \frac{(t_1/c_1 z_1)_\infty}{(c_1 t_1/z_1)_\infty} \frac{(t_1/c_2 z_2)_\infty}{(c_2 t_1/z_2)_\infty},$$

$$\varphi_1(t_1, z_1, z_2)=-z_1/(t_1-c_1/z_1),$$

$$\varphi_2(t_1, z_1, z_2)=\frac{ct_1-z_1}{t_1-c_1 z_1} \frac{-z_2}{t_1-c_2 z_2},$$

cf. (1.7.2). Set

$$I_j(z_1, z_2)=I(z_1, z_2, \varphi_j, \xi)=\int_{[0, \xi\infty]_q} \varphi_j \Phi d_q t_1/t_1$$

for a 1-dimensional q-cycle $[0, \xi\infty]_q$. Then

$$I_1(qz_1, z_2)=\left(\frac{c_1 z_2-c_2 z_1}{c_1 c_2 z_2-z_1} I_1(z_1, z_2)+\frac{((c_2)^2-1) z_1}{c_1 c_2 z_2-z_1} I_2(z_1, z_2)\right) q^{\alpha_1} c_2,$$

$$I_2(qz_1, z_2)=\left(\frac{((c_1)^2-1) z_2}{c_1 c_2 z_2-z_1} I_1(z_1, z_2)+\frac{c_2 z_2-c_1 z_1}{c_1 c_2 z_2-z_1} I_2(z_1, z_2)\right)(c_1)^{-1}.$$

This transformation is inverse to the transformation $PL_{\sigma, \nu}$ given in (1.7.2), up to the factors $q^{\alpha_1} c_2$ and $(c_1)^{-1}$.

Theorem (2.5.13) may be reformulated as follows.

Let V_λ^* be the space dual to V_λ. A basis in V_λ is formed by vectors $\{[T_1] \otimes \cdots \otimes [T_n]\}$, where $F=(T_1, \ldots, T_n)$ runs through the set $\mathscr{F}$ of all admissible forests of weight λ. Let $\{\delta_F\}$, $F \in \mathscr{F}$, be the dual basis in V_λ^*. Let $A_j^*(z)$: $V_\lambda^* \to V_\lambda^*$ be the linear operator dual to $A_j(z)$. We have

$$A_j^*(z)(z_1, \ldots, z_n)$$

$$=R_{j-1, j}^*(z_{j-1}, qz_j) \ldots R_{1, j}^*(z_1, qz_j) D_j^* R_{j, n}^{-1 *}(z_j, z_n) \ldots R_{j, j+1}^{-1 *}(z_j, z_{j+1}) \quad (2.5.15)$$

for $j=1, \ldots, n$. Define the system of q-difference equations a V_λ^*-valued function $\psi(z_1, \ldots, z_n)$ by the formulas

$$Z_j \psi(z)=A_j^* \psi(z) \quad (2.5.16)$$

for $j=1, \ldots, n$.

(2.5.17) Corollary of Theorem (2.5.13). *For any k-dimensional q-cycle* $[0, \xi\infty]_q$ *the* V_λ^**-valued function*

$$\psi(z, \xi)=\sum_{F \in \mathscr{F}} I(z, L(z)[F], \xi) \delta_F$$

is a solution to the system (2.5.16).

Denote by N the number $\#\mathscr{F}=\dim V_\lambda$.

(2.5.18) Corollary of (2.3.17) and (2.4.5). *Under generality conditions on z, on the numbers* $\{\alpha_m\}$, *and on the essential numbers* $\{b_{ij}, c_m^\ell\}$ *in* (2.3.3), *there exist such q-cycles* $[0, \xi_1 \infty]_q, \ldots, [0, \xi_N \infty]_q$ *that the vectors* $\psi(z, \xi_1), \ldots, \psi(z, \xi_N)$ *form a basis in* V_λ^*.

(2.5.19) Remarks.

1. I. Frenkel and N. Reshetikhin in [FR] derived a q-difference system of equations for the matrix coefficients of the product of intertwining operators for a quantum affine group. Their system is a q-deformation of the KZ equation. The Frenkel–Reshetikhin system of equations has the form (2.5.15–16), where the function ψ takes values in the tensor product of finite dimensional representations of the quantum group and $\{R_{ij}\}$ are the R-matrices acting in the corresponding factors, see [FR] and Sect. 3.

An interesting open problem is to compare the two systems of q-difference equations. According to A. Matsuo [M1], system (2.5.16) for $k=1$ coincides with the Frenkel–Reshetikhin system for $U_q(s\hat{\ell}_2)$ restricted to the first non-trivial weight subspace, see also Sect. 3.

Integral solutions for the Frenkel–Reshetikhin system in the $U_q(s\hat{\ell}_2)$ case observe in [M1-2] and Sect. 3, see also [R]. Integral solutions for the $U_q(g\hat{\ell}_N)$-case were found resently in [TV].

2. In [A, AK, AKM] asymptotics of q-hypergeometric functions are considered. Matrices connecting q-hypergeometric functions with different asymptotics are constructed. The connection matrices are linear operators in spaces of forests satisfying the Yang–Baxter equation. Their entries are q-periodic (elliptic) functions. This solution for the Yang–Baxter equation is different from the solution described in Sect. 1, our R-matrices are rational functions of parameters. The situation here is similar to the situation with the Kniznik–Zamolodchikov equation: the equation is given in terms of Casimir operators of a Lie algebra, and its monodromy is described in terms of the universal R-matrix of the corresponding quantum group.

(2.6) Proof of Theorem (2.5.13). First, we prove the theorem for the operator Z_1.

Let id, $\sigma \in S_n$ be the identity permutation and the permutation $(\sigma(1), \ldots, \sigma(n)) = (2, 3, \ldots, n, 1)$, resp.

Let $F = (T_1, \ldots, T_n) \in \mathcal{F}$ be an admissible forest of multi-weight $(\lambda_1, \ldots, \lambda_n)$. Let $f_{F,\mathrm{id}}$ and $f_{F,\sigma}$ be its twisted forest functions defined in (1.6.5).

Let U_F be the transformation sending a function h of $\{t_i\}$, $i \in \{1, \ldots, k\}$, to the new function $U_F h$ obtained from h by the substitution $t_i \to q t_i$ if $i \in J(\lambda_1)$ and $t_i \to t_i$ if $i \notin J(\lambda_1)$.

(2.6.1) Lemma.

$$Z_1 U_F(f_{F,\mathrm{id}} \Phi) = d(\lambda, \lambda_1) f_{F,\sigma} \Phi ,$$

where $d(\lambda, \lambda_1)$ *is defined by* (2.5.5).

The lemma is a direct corollary of (2.3.6), (2.3.7) and definitions of twisted forest functions.

(2.6.2) Corollary.

$$Z_1(f_{F,\mathrm{id}} \Phi)/\Phi - d(\lambda, \lambda_1) f_{F,\sigma} \in \sum_{a \in \mathbb{Z}^k} \nabla_a|_z \mathcal{V}(z) .$$

The map

$$M\colon B_1 \to B_1, f_{F,\mathrm{id}} \mapsto d(\lambda, \lambda_1) f_{F,\sigma}\,, \tag{2.6.3}$$

may be described as follows.

(2.6.4) Lemma. *Let*

$$Q\colon (V_1 \otimes V_2 \otimes \cdots \otimes V_n)_\lambda \to (V_2 \otimes \cdots \otimes V_n \otimes V_1)_\lambda$$

be the permutation of factors. Then

$$M = L_\sigma(z_1, \ldots, z_n) Q D_1 L_{\mathrm{id}}^{-1}(z_1, \ldots, z_n)$$

where $L_{\mathrm{id}}, L_\sigma$ are the coordinate maps constructed in (1.6.7), *D_1 is the operator defined in* (2.5.4).

The lemma is a direct corollary of the definitions of the maps $L_\sigma, L_{\mathrm{id}}$.

(2.6.5) Lemma. *For the map*

$$L_{\mathrm{id}}^{-1}(z_1, \ldots, z_n) M L_{\mathrm{id}}(z_1, \ldots, z_n)\colon V_\lambda \to V_\lambda\,,$$

we have the following formula:

$$L_{\mathrm{id}}^{-1} M L_{\mathrm{id}} = R_{12}^{-1}(z_1, z_2) \ldots R_{1,n}^{-1}(z_1, z_n) D_1 = A_1(z_1, \ldots, z_n)\,.$$

The lemma is a direct corollary of the definitions of operators $\{R_{ij}\}$, the locality property (1.6.13), and the unitarity properties (1.1.6) and (1.3.3) of the tensor coordinate constructed in (1.6).

The proof of the theorem for Z_j, $j>1$, is analogous to the proof for Z_1. Let $\mu, \nu \in S_n$ be the permutations given by the formulas: $(\mu(1), \ldots, \mu(n)) = (j, 1, 2, \ldots, j-1, j+1, \ldots, n)$ and $(\nu(1), \ldots, \nu(n)) = (1, 2, \ldots, j-1, j+1, \ldots, n, j)$. Let $F = (T_1, \ldots, T_n) \in \mathscr{F}$ be an admissible forest of multi-weight $\lambda_1, \ldots, \lambda_n$. Let U'_F be the transformation sending a function h of $\{t_i\}$, $i \in \{1, \ldots, k\}$, to the new function $U'_f h$ obtained from h by the substitution $t_i \to q t_i$ if $i \in J(\lambda_j)$ and $t_i \to t_i$ if $i \notin J(\lambda_j)$.

(2.6.6) Lemma.

$$Z_j U'_f(f_{F,\mu}\Phi) = d(\lambda, \lambda_j) f_{F,\nu}\Phi\,.$$

Theorem (2.5.13) for Z_j easily follows from Lemma (2.6.7) as Theorem (2.5.13) for Z_1 follows from Lemma (2.6.1).

3. Integral Solutions to the Frankel–Reshetikhin Equations for $U_q\hat{s\ell}_2$

In this section we prove the Matsuo conjecture which gives solutions to the Frenkel–Reshetikhin q-difference equations for the $U_q(\hat{s\ell}_2)$ case. We follow [M2].

(3.1) The Frenkel–Reshetikhin Equations for $U_q(\hat{s\ell}_2)$ [M-2, FR]. The quantum group $\hat{U}_q = U_q(\widehat{s\ell_2})$ is the algebra generated by

$$X_0^{\pm}, X_1^{\pm}, K_0^{\pm 1}, K_1^{\pm 1} \tag{3.1.1}$$

subject to the relations

$$K_0K_1=K_1K_0,\ K_0K_0^{-1}=K_1K_1^{-1}=1\ ,$$
$$K_iX_i^{\pm}K_i^{-1}=q^{\pm 2}X_i^{\pm}\ ,$$
$$K_iX_j^{\pm}K_i^{-1}=q^{\mp 2}X_j^{\pm} \tag{3.1.2}$$

for $i\neq j$,

$$[X_i^+, X_j^-]=\delta_{ij}\frac{K_i-K_i^{-1}}{q-q^{-1}}\ ,$$
$$(X_i^{\pm})^3X_j^{\pm}-(q^2+1+q^{-2})(X_i^{\pm})^2X_j^{\pm}X_i^{\pm}$$
$$+(q^2+1+q^{-2})X_i^{\pm}X_j^{\pm}(X_i^{\pm})^2-X_j^{\pm}(X_i^{\pm})^3=0$$

for $i\neq j$. Here q is a generic complex parameter.

The comultiplication $\Delta\colon \hat{U}_q\to\hat{U}_q\otimes\hat{U}_q$ is defined by

$$\Delta(X_i^+)=X_i^+\otimes K_i+1\otimes X_i^+\ ,$$
$$\Delta(X_i^-)=X_i^-\otimes 1+K_i^{-1}\otimes X_i^-\ ,$$
$$\Delta(K_i)=K_i\otimes K_i\ . \tag{3.1.3}$$

Set $\Delta'=\sigma\Delta$ where $\sigma(a\otimes b)=b\otimes a$ in $\hat{U}_q\otimes\hat{U}_q$.

The subalgebra $U_q=U_q(s\ell_2)$ is generated by $X^{\pm}=X_1^{\pm}$, $K^{\pm}=K_1^{\pm}$.

For each $x\in\mathbb{C}$, there is an algebra homomorphism $\varphi_x\colon \hat{U}_q\to U_q$ defined by

$$\varphi_x(X_0^{\pm})=x^{\pm 1}X^{\mp},\ \varphi_x(X_1^{\pm})=X^{\pm}\ ,$$
$$\varphi_x(K_0)=K^{-1},\ \varphi_x(K_1)=K\ . \tag{3.1.4}$$

Let $\{(V_i,\pi_i)\}$ be Verma modules of U_q with highest weights $\{\lambda_i\}$. Then $\{(V_i(x),\hat{\pi}_i)=(V_i,\pi_i\circ\varphi_x)\}$ are representations of $\hat{U}_q$.

There is the trigonometric R-matrix

$$R_{V_iV_j}(x)\colon V_i(x)\otimes V_j(1)\to V_i(x)\otimes V_j(1) \tag{3.1.5}$$

such that

$$\Delta'(a)R_{V_iV_j}(x)=R_{V_iV_j}(x)\Delta(a),\quad a\in\hat{U}_q\ , \tag{3.1.6}$$

normalized by the condition

$$R_{V_iV_j}(x)v_i\otimes v_j=v_i\otimes v_j\ , \tag{3.1.7}$$

where v_i denotes the highest weight vector in V_i for any i.

If $R_{V_iV_j}(x)=\sum_d R_i^{(d)}(x)\otimes R_j^{(d)}(x)$, then, following Matsuo, we let it act on $V_1\otimes\cdots\otimes V_n$ by $1\otimes\cdots\otimes R_i^{(d)}(x)\otimes\cdots\otimes R_j^{(d)}(x)\otimes\cdots\otimes 1$, where $R_i^{(d)}(x)$ stands in the i^{th} factor and $R_j^{(d)}(x)$ stands in the j^{th} factor. Note that i might be greater than j.

For a weight λ, define the operator $q^{\pi_i(\lambda)}$ by the formula:

$$q^{\pi_i(\lambda)}(X^-)^kv_i=q^{(\lambda,\lambda_i-k\alpha)}(X^-)^kv_i\ , \tag{3.1.8}$$

where α is the simple root.

Let $q^{\pi_i(\lambda)}$ act on the i^{th} component of $V_1\otimes\cdots\otimes V_n$.

Let p be a complex parameter. Suppose that

$$p^{-\mu}=q\,. \tag{3.1.9}$$

Let Z_j denote the p-shift operator:

$$Z_j\colon\ \Psi(z_1,\dots,z_n)\to\Psi(z_1,\dots,pz_j,\dots,z_n)\,. \tag{3.1.10}$$

The Frenkel–Reshetikhin equations for a $V_1\otimes\cdots\otimes V_n$-valued function $\Psi(z_1,\dots,z_n)$ is the system of equations

$$\begin{aligned} Z_j\Psi=R_{V_jV_{j-1}}\left(\frac{pz_j}{z_{j-1}}\right)\cdots R_{V_jV_1}\left(\frac{pz_j}{z_1}\right)q^{\pi_j(\lambda+\alpha)-(\lambda,\lambda_j)}R^{-1}_{V_nV_j}\left(\frac{z_n}{z_j}\right)\\ \cdots R^{-1}_{V_{j+1}V_j}\left(\frac{z_{j+1}}{z_j}\right)\Psi\,, \end{aligned} \tag{3.1.11}$$

where $j=1,\dots,n$. Here a weight λ is a parameter of the equations, see [FR, M1-2].

(3.2) Action of Symmetric Group. The symmetric group S_k is generated by the standard generators $\sigma_1,\dots,\sigma_{k-1}$, where σ_i permutes i and $i+1$.

Set

$$D_{ij}=\frac{t_i-q^2t_j}{q^2t_i-t_j}=\frac{t_i-p^{-2\nu}t_j}{p^{-2\nu}t_i-t_j}\,. \tag{3.2.1}$$

For $f(t_1,\dots,t_n)$, define

$$(\sigma_i f)(t_i,\dots,t_k)=f(t_1,\dots,t_{i+1},t_i,\dots,t_k)D_{ii+1}\,. \tag{3.2.2}$$

This formula induces an action of S_k on the space of functions of $t_1,\dots,t_k$.

For a permutation $\sigma=(\sigma(1),\dots,\sigma(n))$,

$$(\sigma f)(t_1,\dots,t_k)=f(t_{\sigma(1)},\dots,t_{\sigma(k)})\prod D_{\sigma(j)\sigma(i)}\,, \tag{3.2.3}$$

where the product is taken over all pairs (i,j) such that $1\leqq i<j\leqq k$ and $\sigma(i)>\sigma(j)$.

(3.3) The Matsuo Conjecture. Set

$$M_i=(\lambda_i,\alpha)\ \text{and}\ M=(\lambda+\alpha,\alpha)\,. \tag{3.3.1}$$

Fix $k\in\mathbb{N}_+$ and set

$$\begin{aligned} \Phi_p(z,t)=&\prod_{\substack{1\leqq i\leqq n\\ 1\leqq j\leqq k}}\left(\frac{z_i}{t_j}\right)^{-M_i\mu}\frac{(p^{M_i\mu}t_j/z_i;p)_\infty}{(p^{-M_i\mu}t_j/z_i;p)_\infty}\\ &\times\prod_{1\leqq i<j\leqq k}\left(\frac{t_j}{t_i}\right)^{2\mu}\frac{(p^{-2\mu}t_i/t_j;p)_\infty}{(p^{2\mu}t_i/t_j;p)_\infty}\prod_{1\leqq i\leqq n}z_i^{-M_i\mu}\prod_{1\leqq j\leqq k}t_j^{M\mu}\,. \end{aligned} \tag{3.3.2}$$

Set

$$A_{ij}=\frac{p^{M_i\mu}z_i-t_j}{z_i-p^{M_i\mu}t_j},\quad B_{ij}=\frac{z_i}{z_i-p^{M_i\mu}t_j}\,, \tag{3.3.3}$$

(3.3.4) Let $\ell=(\ell_1,\dots,\ell_n)\in\mathbb{N}^n$ be a multi-index such that $\ell_1+\cdots+\ell_n=k$.

Set

$$\ell \cdot \ell = \sum_{1 \leqq i < j \leqq n} \ell_i \ell_j \,. \tag{3.3.5}$$

Let $a_i = \ell_1 + \cdots + \ell_i$ for $i = 1, \ldots, n$ and $a_0 = 0$.
Set

$$\varphi_\ell(z, t) = q^{\ell \cdot \ell} \prod_{i=1}^{n} \left(\prod_{a_{i-1}+1 \leqq j \leqq a_i} A_{1j} \ldots, A_{i-1\,j} B_{ij} \right),$$

$$\tilde{\varphi}_\ell(z, t) = \sum_{\sigma \in S_k} (\sigma \varphi_\ell)(z, t) \,. \tag{3.3.6}$$

Fix a k-dimensional p-cycle $[0, \xi\infty]_p$. For any multi-index ℓ with property (3.3.4), set

$$F_\ell(z) = \int_{[0, \xi\infty]_p} \Phi_p(z, t) \tilde{\varphi}_\ell(z, t) \frac{d_p t_1}{t_1} \wedge \cdots \wedge \frac{d_p t_k}{t_k} \,. \tag{3.3.7}$$

We assume that these Jackson integrals are well-defined for all ℓ.

Define the $V_1 \otimes \cdots \otimes V_n$-valued function $F(z)$ by the formula:

$$F(z) = \sum_{\substack{\ell = (\ell_1, \ldots, \ell_n) \\ \ell_1 + \cdots + \ell_n = k}} F_\ell(z) v_1^{(\ell_1)} \otimes \cdots \otimes v_n^{(\ell_n)} \,,$$

where v_i is the highest weight vector of V_i,

$$v_i^{(m)} = \frac{(X^-)^m}{[m]!} v_i \,,$$

$$[m]! = [m][m\text{-}1] \ldots [1] \,,$$

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}} \,.$$

(3.3.10) Conjecture [M2]. *The function $F(z)$ is a solution of the* FR*-equations* (3.1.11).

(3.3.11) Theorem [M1-2]. *The conjecture is true if $k = 1$ or $n = 2$.*

(3.3.12) Theorem. *The conjecture is true for arbitrary n and k.*

Theorem (3.3.12) is proved in (3.5)

(3.4) The Matsuo Results for $n = 2$ [M2]. Let $n = 2$. Let $\ell = (\ell_1, \ell_2) \in \mathbb{N}^2$ be a multi-index such that $\ell_1 + \ell_2 = k$. Then the function $\tilde{\varphi}_\ell(z, t)$ is given by (3.3.6):

$$\varphi_\ell(z, t) = q^{\ell_1 \ell_2} \left(\prod_{j=1}^{\ell_1} B_{1j} \right) \left(\prod_{j=\ell_1+1}^{k} A_{1j} B_{2j} \right), \tag{3.4.1}$$

$$\tilde{\varphi}_\ell(z, t) = q^{\ell_1 \ell_2} \sum_{\substack{\sigma \in S_k \\ \sigma = (\sigma(1), \ldots, \sigma(k))}} \left(\prod_{j=1}^{\ell_1} B_{1\sigma(j)} \right) \left(\prod_{j=\ell_1+1}^{k} A_{1\sigma(j)} B_{2\sigma(j)} \right) \prod_{\substack{1 \leqq i < j \leqq k \\ \sigma(i) > \sigma(j)}} D_{\sigma(j)\sigma(i)} \,.$$

Introduce the function $\tilde{\psi}_\ell(z, t)$ by the formula

$$\psi_\ell(z, t) = q^{\ell_1 \ell_2} \left(\prod_{j=1}^{\ell_2} B_{2j} \right) \left(\prod_{j=\ell_2+1}^{k} A_{2j} B_{1j} \right), \tag{3.4.2}$$

$$\tilde{\psi}_\ell(z, t) = \sum_{\sigma \in S_k} (\sigma \psi_\ell)(z, t)$$

$$= q^{\ell_1 \ell_2} \sum_{\substack{\sigma \in S_k \\ \sigma = (\sigma(1), \ldots, \sigma(k))}} \left(\prod_{j=1}^{\ell_2} B_{2(j)} \right) \left(\prod_{j=\ell_2+1}^{k} A_{(j)} B_{1(j)} \right) \prod_{\substack{1 \leqq i < j \leqq k \\ \sigma(i) > \sigma(j)}} D_{\sigma(j)\sigma(i)} .$$

Let

$$R_{V_1 V_2}(x) \colon V_1(x) \otimes V_2(1) \to V_1(x) \otimes V_2(1)$$

be the trigonometric R-matrix, see (3.1.5). The R-matrix preserves the weight decomposition. Introduce its matrix coefficients $\{R^{\ell_1 \ell_2}_{m_1 m_2}(x)\}$ on the level k by the formula: for any (ℓ_1, ℓ_2) such that $\ell_1 + \ell_2 = k$, set

$$R_{V_1 V_2}(x) v_1^{(\ell_1)} \otimes v_2^{(\ell_2)} = \sum_{m_1 + m_2 = k} R^{\ell_1 \ell_2}_{m_1 m_2}(x) v_1^{(m_1)} \otimes v_2^{(m_2)} . \tag{3.4.3}$$

(3.4.4) Theorem [M2, Lemma 5.2.2]. *For any* (ℓ_1, ℓ_2),

$$\tilde{\psi}_{\ell_1, \ell_2}(z, t) = \sum_{m_1 + m_2 = k} R^{m_1 m_2}_{\ell_1 \ell_2}(z_1/z_2)\, \tilde{\varphi}_{m_1, m_2}(z, t) .$$

In other words, $\{\tilde{\psi}_\ell\}$ and $\{\tilde{\psi}_m\}$ are connected by the matrix transposed to the R-matrix.

Let $\sigma = (\sigma(1), \ldots, \sigma(k)) \in S_k$. Let $\tilde{Z}_1[\sigma\varphi_\ell)(z, t) \Phi_p(z, t)]$ be the function obtained from the function $(\sigma\varphi_\ell)(z, t) \Phi_p(z, t)$ by the transformation

$$z_1, z_2, t_{\sigma(1)}, \ldots, t_{\sigma(k)} \to p z_1, z_2, p t_{\sigma(1)}, \ldots, p t_{\sigma(\ell_1)}, t_{\sigma(\ell_1+1)}, \ldots, t_{\sigma(k)} .$$

By the Stokes theorem we have

$$Z_1 \left(\int_{[0, \xi\infty]_p} (\sigma\varphi_\ell)(z, t) \Phi_p(z, t) \frac{d_p t_1}{t_1} \wedge \cdots \wedge \frac{d_p t_k}{t_k} \right)$$

$$= \int_{[0, \xi\infty]_p} \tilde{Z}_1[(\sigma\varphi_\ell)(z, t) \Phi_p(z, t)] \frac{d_p t_1}{t_1} \wedge \cdots \wedge \frac{d_p t_k}{t_k} \tag{3.4.5}$$

for any $[0, \xi\infty]_p$.

(3.4.6) Lemma. *For any* $\ell = (\ell_1, \ell_2)$,

$$\tilde{Z}_1[(\sigma\varphi_\ell)(z, t) \Phi_p(z, t)] = q^{M_1 - \ell_1 M} (\sigma\nu\psi_\ell)(z, t) \Phi_p(z, t) ,$$

where $\nu \in S_k$ *is defined by the formula*

$$\nu = (\nu(1), \ldots, \nu(k)) = (\ell_1 + 1, \ell_1 + 2, \ldots, k, 1, 2, \ldots, \ell_1) ,$$

and $\sigma\nu \in S_k$ *is the product of two permutations.*

This lemma is a straightforward generalization of the formula in [M2] which is the next after (5.2.6).

Theorem (3.4.4) and Lemma (3.4.6) prove Conjecture (3.3.10) for $n=2$, see [M2].

(*3.5*) *Generalized Tensor Coordinates.* We consider the situation described in (3.3) for arbitrary n and k. In this case our functions depend on $z_1, \ldots, z_n, t_1, \ldots, t_k$. Two symmetric groups S_n and S_k will appear in our considerations.

Let $\ell=(\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$ be a multi-index such that $\ell_1 + \cdots + \ell_n = k$. Let $a_i = \ell_1 + \cdots + \ell_i$ for $i=1, \ldots, n$ and $a_0 = 0$. Let $\omega=(\omega(1), \ldots, \omega(n))$ be an element of S_n. Introduce the function $\tilde{\varphi}_{\omega,\ell}$ by the formula

$$\varphi_{\omega,\ell}(z, t) = q^{\ell \cdot \ell} \prod_{i=1}^{n} \left(\prod_{a_{i-1}+1 \leqq j \leqq a_i} A_{\omega(1)j} \cdots A_{\omega(i-1)j} B_{\omega(i)j} \right),$$

$$\tilde{\varphi}_{\omega,\ell}(z, t) = \sum_{\sigma \in S_k} (\sigma \varphi_{\omega,\ell})(z, t) . \tag{3.5.1}$$

(*3.5.2*) *Example.* Let $n=2$. Let σ and ν be the trivial and nontrivial elements of the symmetric group S_2. Then for any $\ell=(\ell_1, \ell_2)$, we have $\tilde{\varphi}_{\sigma,\ell} = \tilde{\varphi}_\ell$ and $\tilde{\varphi}_{\nu,\ell} = \tilde{\psi}_\ell$, where $\tilde{\varphi}_\ell$ and $\tilde{\psi}_\ell$ are defined by (3.4.1) and (3.4.2).

Fix $z_1, \ldots, z_n$. For a fixed $\omega \in S_n$, introduce the space

$$W_{k,\omega}(z_1, \ldots, z_n) \tag{3.5.3}$$

as the linear space consisting of all linear combinations $\sum_\ell a_\ell \tilde{\varphi}_{\omega,\ell}(z, t)$ where ℓ runs through the set of all multi-indices $\ell \in \mathbb{N}^n$ such that $\ell_1 + \cdots + \ell_n = k$ and $\{a_\ell\}$ are complex coefficients.

For $\omega=(\omega(1), \ldots, \omega(n)) \in S_n$, consider the tensor product

$$V_\omega = V_{\omega(1)} \otimes \cdots \otimes V_{\omega(n)} . \tag{3.5.4}$$

V_ω has the weight decomposition

$$V_\omega = \bigoplus_{k \geqq 0} V_{\omega,k} , \tag{3.5.5}$$

where $V_{\omega,k}$ consists of all elements x such that $Kx = q^{M_1 + \cdots + M_n - 2k} x$. $V_{\omega,k}$ has a basis consisting of the monomials

$$v^{(\ell)} := v_1^{(\ell_1)} \otimes \cdots \otimes v_n^{(\ell_n)}, \quad \ell_1 + \cdots + \ell_n = k .$$

Consider the dual space $V_{\omega,k}^*$ with the dual basis $\{v^{(\ell)*}\}$. Define a linear map

$$L_\omega : V_{\omega,k}^* \to W_{k,\omega}(z_1, \ldots, z_n) \tag{3.5.6}$$

by the formula

$$L_\omega : v^{(\ell)*} \mapsto \tilde{\varphi}_{\omega,\ell} . \tag{3.5.7}$$

For any i, j, let

$$R^{\vee}_{V_i V_j}(x) : V_i(x) \otimes V_j(1) \to V_j(1) \otimes V_i(x) \tag{3.5.8}$$

be the linear map defined by $PR_{V_i V_j}(x)$, where $R_{V_i V_j}(x)$ is introduced in (3.1.5) and P is the transposition of factors. Let

$$R^{\vee}_{V_j^* V_i^*}(x) : (V_j(1) \otimes V_i(x))^* \to (V_i(x) \otimes V_j(1))^* . \tag{3.5.9}$$

be the dual map.

If $R^{\vee}_{V_j^* V_i^*}(x)=\sum_j R_i^{(d)}(x)\otimes R_j^{(d)}(x)$, then let it act on $V^*_{\omega(1)}\otimes\cdots\otimes V^*_{\omega(k)}$ by

$$1\otimes\cdots\otimes R_i^{(d)}(x)\otimes\cdots\otimes R_j^{(d)}(x)\otimes\cdots\otimes 1\,,$$

where $R_i^{(d)}(x)$ stands in the $V_i^{*\,\text{th}}$ factor and $R_j^{(d)}(x)$ stands ion the $V_j^{*\,\text{th}}$ factor.

(3.5.10) Theorem.

1. *The space $W_{k,\omega}(z_1,\ldots,z_n)$ is independent on ω, that is, for any $\omega, \nu\in S_n$ we have*

$$W_{k,\omega}(z_1,\ldots,z_n)=W_{k,\nu}(z_1,\ldots,z_n)\,.$$

2. *For any $\omega=(\omega(1),\ldots,\omega(n))\in S_n$ and $i=1,\ldots,n-1$, the following diagram is commutative*

$$\begin{array}{ccc} & & V_{\omega(1)}\otimes\ldots V_{\omega(i)}\otimes V_{\omega(i+1)}\cdots\otimes V_{\omega(n))}^* \\ L_{\omega(1),\ldots,\omega(n)} & & \\ W_k(z_1,\ldots,z_n) & & R^{\vee}_{V^*_{\omega(i)}V^*_{\omega(i+1)}}(z_{\omega(i)}/z_{\omega(i+1)}) \\ L_{\omega(1),\ldots,\omega(i+1)\,\omega(i),\ldots,\omega(n)} & & \\ & & (V_{\omega(1)}\otimes\ldots V_{\omega(i+1)}\otimes V_{\omega(i)}\cdots\otimes V_{\omega(n)})^*_k \end{array}$$

For $n=2$ Theorem (3.5.10) coincides with Theorem (3.4.4). For $n>2$ Theorem (3.5.10) easily follows from Theorem (3.4.4).

(3.5.11) Remark. I do not know whether $\{L_\omega\}$ are isomorphisms for generic $q, z_1,\ldots,z_n$. If $\{L_\omega\}$ are isomorphisms, then $\{L_\omega\}$ form a system of local tensor coordinates on $W_k(z_1,\ldots,z_n)$ in the sense of (1.3). In any case, we call $\{L_\omega\}$ the *generalized tensor coordinates* on $W_k(z_1,\ldots,z_n)$.

Let $\sigma=(\sigma(1),\ldots,\sigma(k))\in S_k$, $\omega=(\omega(1),\ldots,\omega(n))\in S_n$. Let $\ell=(\ell_1,\ldots,\ell_n)\in\mathbb{N}^n$ be a multi-index such that $\ell_1+\cdots+\ell_k=n$. Consider the function

$$(\sigma\varphi_{\omega,\ell})(z,t)$$
$$=q^{\ell\cdot\ell}\prod_{i=1}^{n}\left(\prod_{a_{i-1}+1\leqq j\leqq a_i} A_{\omega(1),\sigma(j)}\cdots A_{\omega(i-1)\sigma(j)}B_{\omega(i),\sigma(j)}\right)\prod_{\substack{1\leqq i<j\leqq k\\ \sigma(i)>\sigma(j)}} D_{\sigma(j),\sigma(i)}\,, \tag{3.5.12}$$

see (3.5.1).

Let $\tilde{Z}_{\omega(1)}[(\sigma\varphi_{\omega,\ell})(z,t)\Phi_p(z,t)]$ be the function obtained from the function $(\sigma\varphi_{\omega,\ell})(z,t)\Phi_p(z,t)$ by the transformation

$$z_{\omega(1)},\ldots,z_{\omega(n)},t_{\sigma(1)},\ldots,t_{\sigma(k)}\to pz_{\omega(1)},z_{\omega(2)},\ldots,z_{\omega(n)},$$
$$pt_{\sigma(1)},\ldots,pt_{\sigma(\ell_1)},t_{\sigma(\ell_{1+1})},\ldots,t_{\sigma(k)}\,.$$

By the Stokes theorem we have

$$Z_{\omega(1)}\left(\int_{[0,\xi\infty]_p}(\sigma\varphi_{\omega,\ell})(z,t)\Phi_p(z,t)\frac{d_pt_1}{t_1}\wedge\cdots\wedge\frac{d_pt_k}{t_k}\right)$$
$$=\int_{[0,\xi\infty]_p}\tilde{Z}_{\omega(1)}[(\sigma\varphi_{\omega,\ell})(z,t)\Phi_p](z,)\frac{d_pt_1}{t_1}\wedge\cdots\wedge\frac{d_pt_k}{t_k} \tag{3.5.13}$$

for any $[0,\xi\infty]_p$.

(3.5.14) Lemma.

$$\tilde{Z}_{\omega(1)}[(\sigma\varphi_{\omega,\ell})(z,t)\Phi_p(z,t)] = q^{M_{\omega(1)}-\ell_1 M}(\sigma^{\nu}\hat{r}\varphi_{\omega\mu,\ell})(z,t), \Phi_p(z,t)\,,$$

where $\nu_\ell \in S_k$ *is defined by the formula*

$$\nu_\ell = (\nu(1), \ldots, \nu(k)) = (\ell_1+1, \ell_1+2, \ldots, k, 1, 2, \ldots, \ell_1)\,,$$

and $\omega \in S_n$ *is defined by the formula*

$$\mu = (\mu(1), \ldots, \mu(n)) = (2, \ldots, n, 1)\,.$$

This formula is proved by an easy direct calculation.

(3.5.15) Theorem (3.3.12) is a direct corollary of Theorem (3.5.10) and Lemma (2.5.14), cf. the deduction of Theorem (3.3.11) for $n=2$ from Theorem (3.4.4) and Lemma (3.4.6) in [M2].

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