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Quantum Affine Algebras, Graded Limits and Flags

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Abstract | In this survey, we review some of the recent connections between the representation theory of (untwisted) quantum affine algebras and the representation theory of current algebras. We mainly focus on the finite-dimensional representations of these algebras. This connection arises via the notion of the graded and classical limit of finite-dimensional representations of quantum affine algebras. We explain how this study has led to interesting connections with Macdonald polynomials and discuss a BGG-type reciprocity result. We also discuss the role of Demazure modules in this theory and several recent results on the presentation, structure and combinatorics of Demazure modules.

1 Introduction

Quantized enveloping algebras were introduced independently by Drinfeld (1985) and Jimbo (1986) in the context of integrable systems and solvable lattice models and give a systematic way to construct solutions to the quantum Yang–Baxter equation. The quantized algebra associated to an affine Lie algebra is called a quantum affine algebra. The representation theory of these has been intensively studied for nearly 35 years since its introduction. It has connections with many research areas of mathematics and physics, for example, statistical mechanics, cluster algebras, dynamical systems, the geometry of quiver varieties, Macdonald polynomials to name a few. In this survey, we mainly focus on the category of finite-dimensional representations $\mathcal{F}_q$ of quantum affine algebras and their connections to graded representations of current algebras. The fact that this category is not semi-simple gives a very rich structure and has many interesting consequences. The category is studied via the Drinfeld realization of quantum affine algebras and irreducible objects are parametrized in terms of Drinfeld polynomials. The classical version of $\mathcal{F}_q$ was studied previously in[19,32], and the irreducible finite-dimensional representations of the affine algebra and the loop algebra were classified in those papers.

However, we still have limited information on the structure of finite-dimensional representations of quantum affine algebras except for a few special cases. For example, we do not even know the dimension formulas in general. One way to study these representations is to go from quantum level to classical level by forming the classical limit, see for instance[38] for a necessary and sufficient condition for the existence of the classical limit. The classical limit (when it exists) is a finite-dimensional module for the corresponding affine Lie algebra. By restricting and suitably twisting this classical limit, we obtain the graded limit which is a graded representation of the corresponding current algebra, see Sect. 4 for more details. Most of the time we get a reducible indecomposable representation of affine Lie algebra (or current algebra) on passing to the classical (graded) limit. A similar phenomenon is observed in modular representation theory: an irreducible finite-dimensional representation in characteristic zero becomes reducible on passing to characteristic p. Many interesting families of representations from $\mathcal{F}_q$ admit this graded limit, for instance, the local Weyl modules, Kirillov–Reshetikhin modules, minimal affinizations, and some of the prime representations coming from the work of Hernandez and Leclerc on monoidal categorification.

In[38] the authors introduced the notion of local Weyl modules for a quantum affine algebra. They are given by generators and relations, are highest weight modules in a suitable sense and satisfy

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a canonical universal property. In particular, any irreducible module in $\mathcal{F}_q$ is a quotient of some Weyl module. It was conjectured in[38] (and proved there for $\mathfrak{sl}_2$) that any local Weyl module has a tensor product decomposition into fundamental local Weyl modules, see Sect. 3.2.6 for more details. This conjecture stimulated a lot of research on this topic and the general case was established through the work of[29,57,87]. The work of Kirillov and Reshetikhin[75] has a connection with the irreducible representations of quantum affine algebras corresponding to a multiple of a fundamental weight. These modules are referred as Kirillov–Reshetikhin modules in the literature, because they conjectured the classical decomposition of these modules in their paper. The study of Kirillov-Reshetikhin modules has been of immense interest in recent years due to their rich combinatorial structures and several applications to mathematical physics[63,64]. Many important conjectures on the character formulas of these modules and their fusion products were made from physical considerations and they stimulated lots of research, see[58,90,91] and the references therein.

One of the very natural questions that arises from the work of[60] is: what is the smallest representation from $\mathcal{F}_q$ that corresponds to a given finite-dimensional irreducible representation of the underlying simple Lie algebra $\mathfrak{g}$? The second author introduced the notion of a minimal affinization in[20] with this motivation and it was further studied in[20,33,34]. One introduces a poset for each dominant integral weight, such that each element of the poset determines a family of irreducible representations in $\mathcal{F}_q$. The irreducible modules that correspond to the minimal elements of this poset are minimal affinizations. Kirillov-Reshetikhin modules are the minimal affinizations of multiples of the fundamental weights. Our final example of graded limits comes from the work of Hernandez and Leclerc[66] on the monoidal categorification of cluster algebras. The authors defined an interesting monoidal subcategory of $\mathcal{F}_q$ in simply-laced type and proved that for $\mathfrak{g}$ of type A_n and D_4, it categorifies a cluster algebra of the same type, i.e., its Grothendieck ring admits a cluster algebra structure of the same type as $\mathfrak{g}$. The prime real representations of this subcategory are the cluster variables and these are called the HL-modules. A more detailed discussion of HL-modules can be found in Sect. 4.1.8.

Even though the study of graded representations of current algebras is mainly motivated by their connection with the representations of quantum affine algebras (via graded limits), they are now of independent interest as they have found many applications in number theory, combinatorics, and mathematical physics. They have connections with mock theta functions, cone theta functions[12,13,15], symmetric Macdonald polynomials[14,26,70], the $X = M$ conjecture[2,58,90], and Schur positivity[54,100] etc. One of the very important families of graded representations of current algebras comes from $\mathfrak{g}$ -stable Demazure modules. A Demazure module by definition is a module of the Borel subalgebra of the affine Lie algebra. If it is $\mathfrak{g}$-stable, then it naturally becomes a module of the maximal parabolic subalgebra which contains the current algebra. By restriction, we get a graded module of the current algebra. These modules are parametrized by pairs consisting of a positive integer and a dominant weight, and given such a pair (ℓ, λ), the corresponding $\mathfrak{g}$-stable Demazure module of the current algebra is denoted by $D(\ell, \lambda)$. These modules include all well-known families of graded representations of current algebras. For example, any local Weyl module of the current algebra is isomorphic to a level one Demazure module $D(1, \lambda)$ when $\mathfrak{g}$ is simply-laced.

The limit of a tensor product of quantum affine algebra modules is not necessarily isomorphic to the tensor product of their classical limits. So, we need to replace the tensor product with something else to study the limit of a tensor product of quantum affine algebra modules. Examples suggest that the fusion product introduced by Feigin and Loktev[49] is the correct notion that should replace the tensor product. It is a very important and seemingly very hard problem to understand the fusion products of $\mathfrak{g}$-stable Demazure modules of various levels. One would like to find the generators and relations and the graded character of these modules, but very limited cases are known[3,41,53,90].

The survey is organized as follows. We begin by stating the foundational results, including the definition of local Weyl modules and the classification of irreducible modules in Sect. 2. In Sect. 3, we discuss various well-studied families of finite-dimensional representations of quantum affine algebras and review the literature on the presentation of these modules, their classical limit, and the closely related graded limits. Later we move on to the study of graded finite-dimensional representations of current algebras. We relate the local Weyl modules to the $\mathfrak{g}$-stable Demazure modules and discuss the connection between the characters of the local Weyl modules and Macdonald polynomials. We also discuss the BGG-type reciprocity results and briefly mention some recent developments on tilting modules, generalized Weyl modules, and global Demazure modules. In the end, we collect together some results on Demazure modules.

2 The Simple and Untwisted Affine Lie Algebras

In this section we collect the notation and some well-known results on the structure and representation theory of affine Lie algebras.

2.1 Conventions

We let $\mathbb{C}$ (resp. $\mathbb{Q}, \mathbb{Z}, \mathbb{Z}_+, \mathbb{N}$) be the set of complex numbers (resp. rational numbers, integers, non-negative integers, positive integers). We adopt the convention that given two complex vector spaces V, W the corresponding tensor product $V \otimes_{\mathbb{C}} W$ will be just denoted as $V \otimes W$.

Given an indeterminate t we let $\mathbb{C}[t]$ (resp. $\mathbb{C}[t, t^{-1}]$, $\mathbb{C}(t)$) be the ring of polynomials (resp. Laurent polynomials, rational functions) in the variable t. For $s \in \mathbb{Z}, m, r \in \mathbb{Z}_+$ with $m \geq r$, set

$$[s]_t = \frac{t^s - t^{-s}}{t - t^{-1}}, \quad [m]_t! = [m]_t[m-1]_t \cdots [1]_t, \quad \begin{bmatrix} m \\ r \end{bmatrix}_t = \frac{[m]_t!}{[r]_t![m-r]_t!}.$$

For any complex Lie algebra $\mathfrak{a}$ we let $\mathbf{U}(\mathfrak{a})$ be the corresponding universal enveloping algebra. Given any commutative associative algebra A over $\mathbb{C}$ define a Lie algebra structure on $\mathfrak{a} \otimes A$ by

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab, \quad x, y \in \mathfrak{a}, \ a, b \in A.$$

In the special case when A is $\mathbb{C}[t]$ or $\mathbb{C}[t, t^{-1}]$ we set

$$\mathfrak{a}[t] = \mathfrak{a} \otimes \mathbb{C}[t], \quad L(\mathfrak{a}) = \mathfrak{a} \otimes \mathbb{C}[t^{\pm 1}].$$

2.2 Simple and Affine Lie Algebras

2.2.1 The Simple Lie Algebra $\mathfrak{g}$

Let $\mathfrak{g}$ denote a simple finite-dimensional Lie algebra over $\mathbb{C}$ and let $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$ and R the corresponding set of roots. Let $I = \{1, \ldots, n\}$ be an index set for the set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ of R and $\{\omega_1, \ldots, \omega_n\}$ a set of fundamental weights. Given $\lambda, \mu \in \mathfrak{h}^*$ we say that

$$\lambda \geq \mu \iff \lambda - \mu \in \sum_{i \in I} \mathbb{Z}_+ \alpha_i.$$

Let P, Q (resp. P^+, Q^+) be the $\mathbb{Z}$-span (resp. $\mathbb{Z}_+$-span) of the fundamental weights and simple roots respectively and let $R^+ = R \cap Q^+$. We denote by $\theta \in R^+$ the highest root in R^+ and let $(\, , \,)$ be the form on $\mathfrak{h}^*$ induced by the restriction of the Killing form $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ of $\mathfrak{g}$. We assume that it is normalized so that $(\theta, \theta) = 2$ and set

$$d_\alpha = 2/(\alpha, \alpha), \quad d_i = d_{\alpha_i}, \quad \alpha_i^\vee = d_i \alpha_i, \quad \omega_i^\vee = d_i \omega_i, \quad a_{i,j} = (\alpha_j, \alpha_i^\vee), \quad 1 \leq i, j \leq n.$$

Let W be the Weyl group of $\mathfrak{g}$; recall that it is the subgroup of $\mathrm{Aut}(\mathfrak{h}^*)$ generated by the reflections $s_i, i \in I$, defined by:

$$s_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i, \quad i \in I.$$

Fix a Chevalley basis $\{x_\alpha^\pm, h_i : \alpha \in R^+, i \in I\}$ of $\mathfrak{g}$ and set for simplicity $x_i^\pm = x_{\alpha_i}^\pm$. The elements $x_i^\pm, h_i$, $i \in I$ generate $\mathfrak{g}$ as a Lie algebra. Given $\alpha = \sum_{i=1}^n r_i \alpha_i \in R^+$ let $h_\alpha \in \mathfrak{h}$ be given by $h_\alpha = d_\alpha \sum_{i=1}^n \frac{r_i}{d_i} h_i$ and note that the elements $s_\alpha, \alpha \in R^+$, defined by $s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha$ are elements of W and we have $s_i = s_{\alpha_i}$.

Let $\mathfrak{n}^\pm$ be the subalgebra generated by the elements $\{x_i^\pm : i \in I\}$. Then,

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathbb{C} x_\alpha^\pm, \quad \mathfrak{b}^\pm = \mathfrak{h} \oplus \mathfrak{n}^\pm, \quad \mathfrak{g} = \mathfrak{b}^\pm \oplus \mathfrak{n}^\mp.$$

We have a corresponding decomposition of $\mathbf{U}(\mathfrak{g})$ as vector spaces

$$\mathbf{U}(\mathfrak{g}) \cong \mathbf{U}(\mathfrak{n}^-) \otimes \mathbf{U}(\mathfrak{h}) \otimes \mathbf{U}(\mathfrak{n}^+) \cong \mathbf{U}(\mathfrak{n}^-) \otimes \mathbf{U}(\mathfrak{b}^+).$$

2.2.2 The Affine Lie Algebra

The (untwisted) affine Lie algebra $\widehat{\mathfrak{g}}$ and its Cartan subalgebra $\widehat{\mathfrak{h}}$ are defined as follows:

$$\widehat{\mathfrak{g}} = L(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d, \quad \widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

with commutator given by requiring c to be central and

$$[x \otimes t^r, y \otimes t^s] = [x, y] \otimes t^{r+s} + r\delta_{r+s,0}\kappa(x, y)c, \quad [d, x \otimes t^r] = rx \otimes t^r.$$

Here $x, y \in \mathfrak{g}$, $r, s \in \mathbb{Z}$ and $\delta_{n,m}$ is the Kronecker delta symbol. Setting $h_0 = -h_\theta + c$ we see that the set $\{h_i, d : 0 \leq i \leq n\}$ is a basis for $\widehat{\mathfrak{h}}$.

Regard an element $\lambda \in \mathfrak{h}^*$ as an element of $\widehat{\mathfrak{h}}^*$ by setting $\lambda(c) = 0 = \lambda(d)$. Define elements δ, α_0 and the affine fundamental weights Λ_i, $0 \leq i \leq n$, of $\widehat{\mathfrak{h}}^*$ by:

$$\delta(d) = 1, \quad \delta(\mathfrak{h} \oplus \mathbb{C}c) = 0, \quad \alpha_0 = -\theta + \delta, \quad \Lambda_0(c) = 1, \quad \Lambda_0(\mathfrak{h} \oplus \mathbb{C}d) = 0, \quad \Lambda_i(h_j) = \delta_{i,j}, \quad \Lambda_i(d) = 0, \quad i \in I, \quad j \in \{0, \ldots, n\}.$$

The subset

$$\widehat{R} = \{\alpha + r\delta : \alpha \in R \cup \{0\}, r \in \mathbb{Z}\}\backslash\{0\} \subseteq \widehat{\mathfrak{h}}^*,$$

is called the set of affine roots. The set of affine simple roots is $\{\alpha_i : i \in \widehat{I}\}$ where $\widehat{I} = \{0, 1, \ldots, n\}$. The corresponding set of positive roots is given by:

$$\widehat{R}^+ = \{\pm\alpha + (r+1)\delta : \\ \alpha \in R^+ \cup \{0\}, r \in \mathbb{Z}_+\} \ \cup R^+.$$

Set

$$\widehat{P} = \sum_{i=0}^{n} \mathbb{Z}\Lambda_i + \mathbb{Z}\delta, \quad \widehat{P}^+ = \sum_{i=0}^{n} \mathbb{Z}_+\Lambda_i + \mathbb{Z}\delta$$

Let $\widehat{Q}$ (resp. $\widehat{Q}^+$) be the $\mathbb{Z}$-span (resp. $\mathbb{Z}_+$-span) of the affine simple roots. The affine Weyl group $\widehat{W}$ is the subgroup of $\mathrm{Aut}(\widehat{\mathfrak{h}}^*)$ generated by the set $\{s_i : i \in \widehat{I}\}$ where

$$s_i(\lambda) = \lambda - \lambda(h_i)\alpha_i, \quad i \in \widehat{I}, \quad \lambda \in \widehat{\mathfrak{h}}^*.$$

Clearly W is a subgroup of $\widehat{W}$ and we have an isomorphism

$$\widehat{W} \cong W \ltimes \sum_{i \in I} \mathbb{Z}\alpha_i^\vee.$$

It will also be convenient to introduce the extended affine Weyl group $\widetilde{W} = W \ltimes \sum_{i \in I} \mathbb{Z}\omega_i^\vee$.

Setting $x_0^\pm = x_\theta^\mp \otimes t^{\pm 1}$ we observe that $\widehat{\mathfrak{g}}$ is generated by the elements $\{x_i^\pm, h_i : i \in \widehat{I}\} \cup \{d\}$. The root space corresponding to an element $\pm\alpha + s\delta \in \widehat{R}$ with $\alpha \in R^+$ is $\mathbb{C}(x_\alpha^\pm \otimes t^s)$ and to an element $r\delta$ is $(\mathfrak{h} \otimes t^r)$, $s, r \in \mathbb{Z}$ and $r \neq 0$. We shall just denote a non-zero element of the one-dimensional root space corresponding to $\pm\alpha, \alpha \in \widehat{R}^+\backslash\mathbb{N}\delta$ by $x_\alpha^\pm$ and let $h_\alpha = [x_\alpha^+, x_\alpha^-]$. The subalgebras $\widehat{\mathfrak{n}}^\pm$ and $\widehat{\mathfrak{b}}$ are defined in the obvious way and we have

$$\widehat{\mathfrak{n}}^\pm = \mathfrak{g} \otimes t^{\pm 1}\mathbb{C}[t] \oplus \mathfrak{n}^\pm, \quad \widehat{\mathfrak{b}}^+ = \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}^+.$$

This gives rise to an analogous triangular decomposition

$$\mathbf{U}(\widehat{\mathfrak{g}}) \cong \mathbf{U}(\widehat{\mathfrak{n}}^-) \otimes \mathbf{U}(\widehat{\mathfrak{h}}) \otimes \mathbf{U}(\widehat{\mathfrak{n}}^+) \cong \mathbf{U}(\widehat{\mathfrak{n}}^-) \otimes \mathbf{U}(\widehat{\mathfrak{b}}^+).$$

2.2.3 The Loop Algebra $L(\mathfrak{g})$ and the Current Algebra $\mathfrak{g}[t]$

It is trivial to see that $L(\mathfrak{g}) \oplus \mathbb{C}d$ is the quotient of $\widehat{\mathfrak{g}}$ by the center $\mathbb{C}c$. The action of d obviously induces a $\mathbb{Z}$-grading on $L(\mathfrak{g})$ and the current algebra $\mathfrak{g}[t]$ is a graded subalgebra of $L(\mathfrak{g})$. Moreover $\mathfrak{g}[t] \oplus \mathbb{C}c \oplus \mathbb{C}d$ can also be regarded as a maximal parabolic subalgebra of $\widehat{\mathfrak{g}}$, namely

$$\mathfrak{g}[t] \oplus \mathbb{C}c \oplus \mathbb{C}d \cong \widehat{\mathfrak{b}^+} \oplus \mathfrak{n}^-.$$

We make the grading on $L(\mathfrak{g})$ and $\mathfrak{g}[t]$ explicit for the reader's convenience. For $r \in \mathbb{Z}$ we declare $\mathfrak{g} \otimes t^r$ to be the r-th graded piece and note that for $r, s \in \mathbb{Z}$, we have $[\mathfrak{g} \otimes t^r, \mathfrak{g} \otimes t^s] = \mathfrak{g} \otimes t^{r+s}$. This grading induces a grading on the corresponding enveloping algebras as well once we declare a monomial of the form $(a_1 \otimes t^{r_1}) \cdots (a_p \otimes t^{r_p})$, $a_s \in \mathfrak{g}$, $r_s \in \mathbb{Z}$, $1 \leq s \leq p$ to have grade $(r_1 + \cdots + r_p)$.

2.2.4 Ideals in $L(\mathfrak{g})$

The affine Lie algebra is clearly not simple; the center spans a one-dimensional ideal. One can prove using the explicit realization that this and the derived algebra $L(\mathfrak{g}) \oplus \mathbb{C}c$ are the only non trivial proper ideals in $\widehat{\mathfrak{g}}$. The following result is well-known, a proof can be found for instance in[24], Lemma 1].

Lemma For all $f \in \mathbb{C}[t, t^{-1}]$ the subspace $\mathfrak{g} \otimes f\mathbb{C}[t, t^{-1}]$ is an ideal in $L(\mathfrak{g})$. Moreover any ideal in $L(\mathfrak{g})$ must be of this form. In particular all ideals are of finite codimension. Writing $f = (t - a_1)^{r_1} \cdots (t - a_k)^{r_k}$ with $a_r \neq a_s$ for $1 \leq r \neq s \leq k$ we see that we have an isomorphism of Lie algebras

$$\frac{L(\mathfrak{g})}{\mathfrak{g} \otimes f\mathbb{C}[t, t^{-1}]} \cong \mathfrak{g} \otimes \frac{\mathbb{C}[t, t^{-1}]}{(f)} \\ \cong \bigoplus_{s=1}^{k} \left(\mathfrak{g} \otimes \frac{\mathbb{C}[t, t^{-1}]}{(t - a_s)^{r_s}} \right).$$

□

We shall sometimes refer to the finite-dimensional quotient of $L(\mathfrak{g})$ defined by $f \in \mathbb{C}[t, t^{-1}]$ as the truncation of $L(\mathfrak{g})$ at f.

2.3 Representations of Simple and Affine Lie Algebras

2.3.1 Finite-Dimensional Representations of $\mathfrak{g}$

We say that a $\mathfrak{g}$-module V is a weight module if,

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu, V_\mu = \{v \in V : hv = \mu(h)v, \ \forall h \in \mathfrak{h}\}$$

and we let $\mathrm{wt}(V) = \{\mu \in \mathfrak{h}^* : V_\mu \neq 0\}$. If $\mathrm{wt}(V) \subset P$ and $\dim V_\mu < \infty$ for all $\mu \in P$, we let $\mathrm{ch}(V)$ be the formal sum $\sum_{\mu\in P}(\dim V_\mu)\, e_\mu$ where e_μ varies over a basis of the group ring $\mathbb{Z}[P]$.

Given $\lambda \in P^+$ let $V(\lambda)$ be the $\mathfrak{g}$-module generated by an element v_λ with defining relations:

$$h_i v_\lambda = \lambda(h_i)v_\lambda, \quad x_i^+ v_\lambda = 0,$$
$$(x_i^-)^{\lambda(h_i)+1} v_\lambda = 0, \quad 1 \le i \le n.$$

It is well-known[68] that $V(\lambda)$ is a weight module with $\mathrm{wt}(V(\lambda)) \subseteq \lambda - Q^+$ and that it is also an irreducible and finite-dimensional $\mathfrak{g}$-module. The set $\mathrm{wt}(V(\lambda))$ is W-invariant and $\dim V(\lambda)_\mu = \dim V(\lambda)_{w\mu}$ for all $w \in W$.

Any finite-dimensional $\mathfrak{g}$-module is isomorphic to a direct sum of copies of $V(\lambda)$, $\lambda \in P^+$.

2.3.2 Integrable and Positive Level Representations of $\widehat{\mathfrak{g}}$

The notion of a weight module and its character for $\widehat{\mathfrak{g}}$ are defined as for simple Lie algebras with $\mathfrak{h}$ replaced by $\widehat{\mathfrak{h}}$. We say that a $\widehat{\mathfrak{g}}$-module V is integrable if it is a weight module and the elements $x_i^\pm$, $i \in \widehat{I}$, act locally nilpotently. We say that V is of level $r \in \mathbb{Z}$ if $cv = rv$ for all $v \in V$; if $r > 0$ (resp. $r < 0$) then we say that V is of positive level (resp. negative level).

Given $\lambda \in \widehat{P}^+$ let $V(\lambda)$ be the $\widehat{\mathfrak{g}}$-module generated by an element v_λ with defining relations:

$$h_i v_\lambda = \lambda(h_i)v_\lambda, \quad x_i^+ v_\lambda = 0,$$
$$(x_i^-)^{\lambda(h_i)+1} v_\lambda = 0, \quad 0 \le i \le n.$$

Again it is well-known[72] that $V(\lambda)$ is an integrable irreducible module with positive level $\lambda(c)$ and

$$\dim V(\lambda)_\mu \neq 0 \implies \mu \in \lambda - \widehat{Q}^+,$$
$$\dim V(\lambda)_\mu < \infty, \quad \mu \in \widehat{P},$$
$$\dim V(\lambda)_\mu = \dim V(\lambda)_{w\mu}, \quad w \in \widehat{W}, \quad \mu \in \widehat{P}.$$

Notice that the preceding properties show immediately that $V(\lambda)$ is infinite-dimensional as long as $\lambda \notin \mathbb{Z}\delta$. In the case when V is irreducible the following was proved in[19]. The complete reducibility statement was proved in[47].

Theorem *Any positive level integrable $\widehat{\mathfrak{g}}$-module V with the property that* $\dim V_\mu < \infty$ *for all* $\mu \in \widehat{P}$ *is isomorphic to a direct sum of modules of the form* $V(\lambda)$, $\lambda \in \widehat{P}^+$. □

There is a completely similar theory for negative level modules.

2.3.3 Demazure Modules

Given $\lambda \in P^+$ and $w \in W$ (resp. $\lambda \in \widehat{P}^+, w \in \widehat{W}$) the Demazure module $V_w(\lambda)$ is the $\mathfrak{b}$-submodule (resp. $\widehat{\mathfrak{b}}$-submodule) of $V(\lambda)$ generated by the one dimensional subspace $V(\lambda)_{w\lambda}$. The Demazure modules are always finite-dimensional; in the case of $\mathfrak{g}$ this statement is trivial while for $\widehat{\mathfrak{g}}$ the statement follows from the fact that $\mathrm{wt}(V(\lambda)) \subseteq \lambda - \widehat{Q}^+$. For $\lambda \in P^+$ and $w \in W$, (resp. $\lambda \in \widehat{P}^+$, $w \in \widehat{W}$) it is a result from[71,84] that as a $\mathbf{U}(\mathfrak{b})$-module (resp. $\mathbf{U}(\widehat{\mathfrak{b}})$-module) $V_w(\lambda)$ is generated by $v_{w\lambda}$ with relations: $hv_{w\lambda} = (w\lambda)(h)v_{w\lambda}$, for all $h \in \mathfrak{h}$ (resp. for all $h \in \widehat{\mathfrak{h}}$) and

$$(x_\alpha^+)^{p+1} v_{w\lambda} = 0, \quad p \ge \max\{0, -w\lambda(h_\alpha)\}, \quad \alpha \in R^+$$
$$(\text{resp.} (\mathfrak{h}\otimes t^{r+1})v_{w\lambda} = 0, \quad r \in \mathbb{Z}_+, \ (x_\alpha^+)^{p+1} v_{w\lambda} = 0,$$
$$p \ge \max\{0, -w\lambda(h_\alpha)\}, \quad \alpha \in \widehat{R}^+ \setminus \mathbb{N}\delta).$$

These relations were simplified in[41,79] and we refer to Theorem 6.1.1 for a precise statement.

Lemma Suppose that $\lambda \in P^+$, $w \in W$, and $i \in I$ are such that $(w\lambda)(h_i) \le 0$. Then $x_i^- V(\lambda)_{w\lambda} = 0$. An analogous statement holds for $V(\lambda)$ with $\lambda \in \widehat{P}^+$.

Proof If $w\lambda - \alpha_i \in \mathrm{wt}(V(\lambda))$, then $\lambda - w^{-1}\alpha_i \in \mathrm{wt}(V(\lambda))$. On the other hand our assumptions force $(w\lambda)(h_i) = 0$ or $w^{-1}\alpha_i \in R^-$ where the latter condition ends in a contradiction to $\mathrm{wt}(V(\lambda)) \subseteq \lambda - Q^+$. Hence $(w\lambda)(h_i) = 0$ and $x_i^- V(\lambda)_{w\lambda} = 0$ follows. □

As a consequence of this lemma we see immediately that

$$\lambda \in \widehat{P}^+, \quad w \in \widehat{W}, \quad (w\lambda)(h_i) \in -\mathbb{Z}_+,$$
$$\forall\, i \in I \implies \mathfrak{g}[t]V_w(\lambda) \subseteq V_w(\lambda).$$

We call these $\widehat{\mathfrak{b}}$-submodules the $\mathfrak{g}$-stable Demazure modules.

2.3.4 Level Zero Modules for $\widehat{\mathfrak{g}}$ and Finite-Dimensional Modules for $L(\mathfrak{g})$ and $\mathfrak{g}[t]$

A level zero module for $\widehat{\mathfrak{g}}$ is one on which the center acts trivially, in particular it can be regarded as a module for $L(\mathfrak{g}) \oplus \mathbb{C}d$. More generally given any representation V of $\mathfrak{g}$ one can define a $L(\mathfrak{g}) \oplus \mathbb{C}d$-module structure on $L(V) = V \otimes \mathbb{C}[t, t^{-1}]$ by:

$$(x \otimes t^r)(v \otimes t^s) = xv \otimes t^{r+s},$$
$$d(v \otimes t^r) = rv \otimes t^r, \quad r, s \in \mathbb{Z}, \ x \in \mathfrak{g}.$$

The only finite-dimensional representations of $\widehat{\mathfrak{g}}$ are those on which $L(\mathfrak{g}) \oplus \mathbb{C}c$ acts trivially. We give a proof of this fact for the reader's convenience. Note that by working with the Jordan–Holder series it suffices to prove this for irreducible finite-dimensional representations. Thus, let V be a finite-dimensional irreducible representation of $\widehat{\mathfrak{g}}$. Then it is easily seen that there exists a vector $0 \neq v \in V$ such that the following hold:

$$(x_\alpha^+ \otimes t^r)v = 0, \ \ (h_i \otimes t^s)v = a_{i,s}v, \ \ \alpha \in R^+,$$
$$i \in I, r \in \mathbb{Z}, \ \ s \in \mathbb{Z}_+, \ \ cv = \ell v, \ \ dv = av,$$

where $a, a_{i,s} \in \mathbb{C}$ for $s > 0$ and $a_{i,0} \in \mathbb{Z}_+$ and $\ell \in \mathbb{Z}$. Then

$$[d, h_i \otimes t^s]v = s(h_i \otimes t^s)v \implies 0$$
$$= sa_{i,s}v \implies a_{i,s} = 0, \ \ s > 0.$$

Working with the $\mathfrak{sl}_2$-triple $\{x_i^+ \otimes t^{-s}, x_i^- \otimes t^s, h_i - s\kappa(x_i^+, x_i^-)c\}$ we see that we must have $a_{i,0} - s\ell\kappa(x_i^+, x_i^-) \geq 0$ for all $s \geq 0$ and this forces $\ell = 0$. Hence V must be a finite-dimensional representation of $L(\mathfrak{g}) \oplus \mathbb{C}d$. Suppose that $a_{i,0} > 0$ for some $i \in I$. Then working again with the triple $\{x_i^+ \otimes t^{-s}, x_i^- \otimes t^s, h_i - s\kappa(x_i^+, x_i^-)\}$ we see that $(x_i^- \otimes t^s)v \neq 0$ for all $s > 0$. Since these elements have d-eigenvalues $a + s$ they must be linearly independent which is a contradiction. It follows that $a_{i,0} = 0$ for all $i \in I$ and then it is easily seen that V must be the trivial $L(\mathfrak{g}) \oplus \mathbb{C}c$-module.

Consider however, the commutator subalgebra $L(\mathfrak{g}) \oplus \mathbb{C}c$ of $\widehat{\mathfrak{g}}$. The preceding arguments show that the center must act trivially on any finite-dimensional representation. Hence it suffices to study finite-dimensional representations of $L(\mathfrak{g})$. To construct examples we introduce for each $a \in \mathbb{C}^\times$ the evaluation homomorphism $\mathrm{ev}_a : L(\mathfrak{g}) \to \mathfrak{g}$ which sends $x \otimes t^r \to a^r x$ for $x \in \mathfrak{g}$ and $r \in \mathbb{Z}$. Given a representation V of $\mathfrak{g}$ let $\mathrm{ev}_a V$ be the pull-back $L(\mathfrak{g})$-module. The following was proved in[19,31].

Proposition (i) *Any irreducible finite-dimensional representation of $L(\mathfrak{g})$ is isomorphic to a tensor product of the form $\mathrm{ev}_{a_1} V(\lambda_1) \otimes \cdots \otimes \mathrm{ev}_{a_k} V(\lambda_k)$ for some $k \geq 1$, pairwise distinct elements $a_1, \ldots, a_k \in \mathbb{C}^\times$, and elements $\lambda_1, \ldots, \lambda_k \in P^+$. Moreover any tensor product of irreducible finite-dimensional representations is either irreducible or completely reducible.*

(ii) *Suppose that V is an irreducible finite-dimensional module of $L(\mathfrak{g})$. Then $L(V)$ is a direct sum of irreducible modules for $L(\mathfrak{g}) \oplus \mathbb{C}d$. Any level zero integrable irreducible module for $\widehat{\mathfrak{g}}$ with finite-dimensional weight spaces is obtained as a direct summand of $L(V)$.*

(iii) *A similar result holds for finite-dimensional modules of $\mathfrak{g}[t]$ once we also allow the module $\mathrm{ev}_0 V(\lambda)$.* □

Level zero modules for $\widehat{\mathfrak{g}}$ are not completely reducible. The simplest example is the adjoint representation where the center is a proper submodule which does not have a complement. Finite-dimensional modules for $L(\mathfrak{g})$ and $\mathfrak{g}[t]$ are also not completely reducible. For instance the $\mathfrak{g}$-stable Demazure modules are usually reducible and indecomposable. A simple exercise, left to the reader, is to verify this in the case when $\widehat{\mathfrak{g}} = \widehat{\mathfrak{sl}_2}$, $\lambda = \Lambda_0$ and $w = s_1 s_0$.

We shall frequently be interested in $\mathbb{Z}$-graded modules for $\mathfrak{g}[t]$. These are $\mathbb{Z}$-graded vector spaces $V = \bigoplus_{m \in \mathbb{Z}} V[m]$ which admit an action of $\mathfrak{g}[t]$ satisfying $(\mathfrak{g} \otimes t^r)V[m] \subseteq V[m+r]$ for all $m \in \mathbb{Z}$ and $r \in \mathbb{Z}_+$. In particular each graded piece $V[m]$ is a module for $\mathfrak{g}$. If $\dim V[m] < \infty$ for all $m \in \mathbb{Z}$ we define the graded character as the formal sum

$$\mathrm{ch}_{\mathrm{gr}} V = \sum_{m \in \mathbb{Z}} \mathrm{ch}_{\mathfrak{g}} V[m] q^m.$$

The module $\mathrm{ev}_a V(\lambda)$ for $\mathfrak{g}[t]$ is graded if and only if $a = 0$. The $\mathfrak{g}$-stable Demazure modules are also graded where the grading is given by the action of d; namely $V_w(\lambda)[r] = \{v \in V_w(\lambda) : dv = rv\}$.

Given a $\mathbb{Z}$-graded vector space V and an integer $s \in \mathbb{Z}$ let $\tau_s V$ be the grade shifted vector space obtained by declaring $(\tau_s V)[r] = V[r - s]$.

2.3.5 The Monoid $\mathcal{P}^+$ and an Alternate Parametrization of Finite-Dimensional Modules

Let $\mathcal{P}^+$ be the free abelian multiplicative monoid generated by elements $\boldsymbol{\omega}_{i,a}, i \in I, a \in \mathbb{C}^\times$ and let $\mathcal{P}$ be the corresponding group generated by these elements.

For $\lambda \in P^+$ let $\boldsymbol{\omega}_{\lambda,a} = \prod_{i=1}^n \boldsymbol{\omega}_{i,a}^{\lambda(h_i)}$. Clearly any element of $\mathcal{P}^+$ can be written uniquely as a product $\boldsymbol{\omega}_{\lambda_1,a_1} \cdots \boldsymbol{\omega}_{\lambda_k,a_k}$ for some multisubset $\{\lambda_1, \ldots, \lambda_k\} \subseteq P^+$ and distinct elements $a_s \in \mathbb{C}^\times$, $1 \leq s \leq k$. Then part (i) of Proposition 2.3.4 can be reformulated as follows.

Proposition *There exists a bijective correspondence between $\mathcal{P}^+$ and isomorphism classes of finite-dimensional irreducible representations of $L(\mathfrak{g})$ given by*

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\lambda_1,a_1} \cdots \boldsymbol{\omega}_{\lambda_k,a_k} \longrightarrow [V(\boldsymbol{\omega})]$$
$$= [\mathrm{ev}_{a_1} V(\lambda_1) \otimes \cdots \otimes \ \mathrm{ev}_{a_k} V(\lambda_k)].$$

Moreover if $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{P}^+$ then $V(\boldsymbol{\omega}) \otimes V(\boldsymbol{\omega}')$ is completely reducible and has $V(\boldsymbol{\omega}\boldsymbol{\omega}')$ as a summand. □

2.3.6 Annihilating Ideals

Suppose that V is a finite-dimensional representation of $L(\mathfrak{g})$. Then the discussion in Sect. 2.2.4 shows that there exists $f \in \mathbb{C}[t, t^{-1}]$ such that

$$\{a \in L(\mathfrak{g}) : av = 0 \text{ for all } v \in V\} = \mathfrak{g} \otimes f\mathbb{C}[t, t^{-1}].$$

In particular V becomes a module for the truncated Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]/(f)$. In the case when V is irreducible the discussion so far proves that $f = (t - a_1) \cdots (t - a_k)$ for some distinct element $a_1, \ldots, a_k \in \mathbb{C}^\times$.

Suppose that V_1 and V_2 are modules for the truncation of $L(\mathfrak{g})$ at f_1 and f_2 respectively and let f be the least common multiple of the pair. Then $V_1 \otimes V_2$ is a module for the truncation at f.

3 The Quantized Simple and Affine Enveloping Algebras

3.1 Definitions and the Hopf Algebra Structure

3.1.1 The Drinfeld–Jimbo Presentation

Let q be an indeterminate and set $q_i = q^{d_i}$, $i \in I$, and $q_0 = q$. The quantized enveloping algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$ (also called the quantum affine algebra) is the associative algebra over $\mathbb{C}(q)$ generated by elements $X_i^\pm, K_i^{\pm 1}, D^{\pm 1}, i \in \widehat{I}$ and relations:

$$\begin{aligned}
&K_iK_i^{-1} = 1, \quad DD^{-1} = 1,\\
&K_iK_j = K_jK_i, \quad DK_i = K_iD, \quad i,j \in \widehat{I}\\
&K_iX_j^\pm K_i^{-1} = q_i^{\pm a_{ij}} X_j^\pm, \quad DX_j^\pm D^{-1} = q^{\pm \delta_{0,j}} X_j^\pm,\\
&[X_i^+, X_j^-] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad i,j \in \widehat{I},\\
&\sum_{m=0}^{1-a_{i,j}} (-1)^m \begin{bmatrix} 1 - a_{i,j} \\ m \end{bmatrix}_{q_i}\\
&(X_i^\pm)^{1-a_{i,j}-m} X_j^\pm (X_i^\pm)^m = 0,\\
&i,j \in \widehat{I}, \ i \neq j.
\end{aligned}$$

The Hopf structure on this algebra is given by

$$\begin{aligned}
&\Delta(D) = D \otimes D,\\
&\Delta(K_i) = K_i \otimes K_i,\\
&\Delta(X_i^+) = X_i^+ \otimes K_i + 1 \otimes X_i^+,\\
&\Delta(X_i^-) = X_i^- \otimes 1 + K_i^{-1} \otimes X_i^-,\\
&S(K_i) = K_i^{-1},\\
&S(D) = D^{-1}, \quad S(X_i^+) = -X_i^+ K_i^{-1},\\
&S(X_i^-) = -K_iX_i^-,\\
&\epsilon(K_i) = 1 = \epsilon(D), \quad \epsilon(X_i^\pm) = 0.
\end{aligned}$$

Set

$$C = K_0K_\theta, \quad K_\theta = \prod_{i \in I} K_i^{c_i},$$

where c_i are such that $h_\theta = \sum_{i \in I} c_i h_i$. The quantized enveloping algebra $\mathbf{U}_q(\mathfrak{g})$ is the Hopf subalgebra generated by the elements $X_i^\pm, K_i^{\pm 1}, i \in I$.

3.1.2 The Drinfeld Presentation of $\mathbf{U}_q(\widehat{\mathfrak{g}})$

An alternate presentation of the quantum affine algebra was given by Drinfeld.

The algebra $\mathbf{U}_q(\widehat{\mathfrak{g}})$ is isomorphic to the $\mathbb{C}(q)$ -associative algebra with unit given by generators $c^{\pm 1/2}$, $x_{i,r}^\pm$, $k_i^{\pm 1}$, $d^{\pm 1}$, $h_{i,s}$, for $i \in I$, $r, s \in \mathbb{Z}$ with $s \neq 0$ subject to the following relations:

$$\begin{aligned}
&c^{1/2}c^{-1/2} = 1 = dd^{-1} = k_ik_i^{-1} = k_i^{-1}k_i,\\
&c^{\pm 1/2} \text{ are central},\\
&k_ik_j = k_jk_i, \quad k_ih_{j,r} = h_{j,r}k_i,\\
&dk_i = k_id, \quad dh_{i,r}d^{-1} = q^r h_{i,r}\\
&k_ix_{j,r}^\pm k_i^{-1} = q_i^{\pm a_{i,j}} x_{j,r}^\pm,\\
&dx_{i,r}^\pm d^{-1} = q^r x_{i,r}^\pm,\\
&[h_{i,r}, h_{j,s}] = \delta_{r,-s} \frac{1}{r} [ra_{i,j}]_{q_i} \frac{c^r - c^{-r}}{q_j - q_j^{-1}},\\
&[h_{i,r}, x_{j,s}^\pm] = \pm \frac{1}{r} [ra_{i,j}]_{q_i} c^{\mp |r|/2} x_{j,r+s}^\pm,\\
&x_{i,r+1}^\pm x_{j,s}^\pm - q_i^{\pm a_{i,j}} x_{j,s}^\pm x_{i,r+1}^\pm = q_i^{\pm a_{i,j}} x_{i,r}^\pm x_{j,s+1}^\pm - x_{j,s+1}^\pm x_{i,r}^\pm,\\
&[x_{i,r}^+, x_{j,s}^-] = \delta_{i,j} \frac{c^{(r-s)/2}\phi_{i,r+s}^+ - c^{-(r-s)/2}\phi_{i,r+s}^-}{q_i - q_i^{-1}},\\
&\sum_{\sigma \in S_m} \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{q_i} x_{i,n_{\sigma(1)}}^\pm \ldots x_{i,n_{\sigma(k)}}^\pm x_{j,s}^\pm\\
&x_{i,n_{\sigma(k+1)}}^\pm \ldots x_{i,n_{\sigma(m)}}^\pm = 0, \ \text{if } i \neq j,
\end{aligned}$$

for all sequences of integers $n_1, \ldots, n_m$, where $m = 1 - a_{i,j}$, $i, j \in I$, S_m is the symmetric group on m letters, and the $\phi_{i,r}^\pm$ are determined by equating powers of u in the formal power series

$$\begin{aligned}
\phi_i^\pm(u) &= \sum_{r=0}^{\infty} \phi_{i,\pm r}^\pm u^{\pm r}\\
&= k_i^{\pm 1} \exp\left(\pm(q_i - q_i^{-1}) \sum_{r=1}^{\infty} h_{i,\pm r} u^{\pm r} \right).
\end{aligned}$$

Note that $\phi_{i,\mp r}^\pm = 0$ for $r > 0$. The quantized loop algebra $\mathbf{U}_q(L(\mathfrak{g}))$ is the algebra with generators $x_{i,r}^\pm$, $k_i^{\pm 1}$, $h_{i,s}$, $r, s \in \mathbb{Z}$ with $s \neq 0$ and $i \in I$ and the same relations as above where we replace $c^{1/2}$ by 1. Moreover we have a canonical inclusion $\mathbf{U}_q(\mathfrak{g}) \hookrightarrow \mathbf{U}_q(L(\mathfrak{g}))$ given by mapping $X_i^\pm \to x_{i,0}^\pm$, $k_i^\pm \to K_i^\pm$ $i \in I$.

Explicit formulae for the Hopf algebra structure in terms of these generators are not known.

However the following partial information on the coproduct is often enough for our purposes[36,43].

Proposition *Let*

$$X^{\pm} = \sum_{i\in I,\, r\in\mathbb{Z}} \mathbb{C}(q)x^{\pm}_{i,r},$$

$$X^{\pm}(i) = \sum_{j\in I\setminus\{i\},\, r\in\mathbb{Z}} \mathbb{C}(q)x^{\pm}_{j,r}, \quad i \in I.$$

Then

(i) *Modulo* $\mathbf{U}_q(\widehat{\mathfrak{g}})X^- \otimes \mathbf{U}_q(\widehat{\mathfrak{g}})(X^+)^2 + \mathbf{U}_q(\widehat{\mathfrak{g}})X^- \otimes \mathbf{U}_q(\widehat{\mathfrak{g}})X^+(i)$, *we have*

$$\Delta(x^+_{i,k}) = x^+_{i,k} \otimes 1 + k_i \otimes x^+_{i,k} + \sum_{j=1}^{k} \phi^+_{i,j} \otimes x^+_{i,k-j}, \quad k \geq 0,$$

$$\Delta(x^+_{i,-k}) = x^+_{i,-k} \otimes 1 + k_i^{-1} \otimes x^+_{i,-k} + \sum_{j=1}^{k-1} \phi^-_{i,-j} \otimes x^+_{i,-k+j}, \quad k > 0.$$

(ii) *Modulo* $\mathbf{U}_q(\widehat{\mathfrak{g}})(X^-)^2 \otimes \mathbf{U}_q(\widehat{\mathfrak{g}})X^+ + \mathbf{U}_q(\widehat{\mathfrak{g}})X^- \otimes \mathbf{U}_q X^+(i)$, *we have*

$$\Delta(x^-_{i,k}) = x^-_{i,k} \otimes k_i + 1 \otimes x^-_{i,k} + \sum_{j=1}^{k-1} x^-_{i,k-j} \otimes \phi^+_{i,j}, \quad k > 0,$$

$$\Delta(x^-_{i,-k}) = x^-_{i,-k} \otimes k_i^{-1} + 1 \otimes x^-_{i,-k} + \sum_{j=1}^{k} x^-_{i,-k+j} \otimes \phi^-_{i,-j}, \quad k \geq 0.$$

(iii) *Modulo* $\mathbf{U}_q(\widehat{\mathfrak{g}})X^- \otimes \mathbf{U}_q(\widehat{\mathfrak{g}})X^+$, *we have*

$$\Delta(h_{i,k}) = h_{i,k} \otimes 1 + 1 \otimes h_{i,k}, \quad k \in \mathbb{Z}\setminus\{0\}.$$

□

3.2 Representations of Quantum Algebras

The classification of finite-dimensional and integrable representations of the quantum algebras is essentially the same as that of the corresponding Lie algebras, once we impose certain restrictions. Thus we shall only be interested in *type 1* modules for these algebras; namely we require that the elements $K_i^{\pm 1}$ act semi-simply on the module with eigenvalues in $q^{\mathbb{Z}}$. The character of such a representation V is given by $\mathrm{ch}(V) = \sum_\mu \dim(V_\mu)e_\mu$ where $V_\mu = \{v \in V : K_i^{\pm 1}v = q_i^{\pm\mu(h_i)}v,\ i \in I\}$.

3.2.1 The Irreducible Modules

Let $\mathcal{P}_q^+$ (resp. $\mathcal{P}_q$) be the free abelian monoid (resp. group) generated by elements $\boldsymbol{\omega}_{i,a}$ with $a \in \mathbb{C}(q)^\times$. Clearly we can regard $\mathcal{P}^+$ as a submonoid of $\mathcal{P}_q^+$. Define wt: $\mathcal{P}_q^+ \to P^+$ by extending the assignment wt $\boldsymbol{\omega}_{i,a} = \omega_i$ to a morphism of monoids.

Part (i) of the next result was proved in[32] while part (ii) was proved in[82,97].

Theorem (i) *There is a bijection* $\boldsymbol{\omega} \to [V_q(\boldsymbol{\omega})]$ *between elements of* $\mathcal{P}_q^+$ *and isomorphism classes of finite-dimensional irreducible representations of* $\mathbf{U}_q(L(\mathfrak{g}))$. *Moreover if* $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{P}_q^+$ *then* $V_q(\boldsymbol{\omega}\boldsymbol{\omega}')$ *is a subquotient of* $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\omega}')$.

(ii) *Given* $\lambda \in P^+$ *there exists a unique (up to isomorphism) finite-dimensional irreducible* $\mathbf{U}_q(\mathfrak{g})$*-module* $V_q(\lambda)$. *Moreover* $\mathrm{ch}\, V_q(\lambda) = \mathrm{ch}\, V(\lambda)$ *and for* $\lambda, \mu \in P^+$ *we have that* $V_q(\lambda + \mu)$ *is a summand of* $V_q(\lambda) \otimes V_q(\mu)$. *An analogous statement holds for positive level integrable representations of* $\mathbf{U}_q(\widehat{\mathfrak{g}})$. □

Remark 1. It is worth emphasizing that in general $\mathrm{ch}\, V_q(\boldsymbol{\omega}) \neq \mathrm{ch}\, V(\boldsymbol{\omega})$ for $\boldsymbol{\omega} \in \mathcal{P}^+$.
2. The elements $\boldsymbol{\omega}_{i,a} \in \mathcal{P}_q^+$ are called the fundamental weights and the associated representations are called fundamental representations.

3.2.2 The Category $\mathcal{F}_q$

We shall be interested primarily in the category $\mathcal{F}_q$ of finite-dimensional representations of $\mathbf{U}_q(L(\mathfrak{g}))$. The Hopf algebra structure of $\mathbf{U}_q(L(\mathfrak{g}))$ ensures that $\mathcal{F}_q$ is a monoidal tensor category. Roughly speaking this means that $\mathcal{F}_q$ is an abelian category which is closed under taking tensor products and duals. However, since the Hopf algebra is not co-commutative it is not true in general that the modules $V \otimes W$ and $W \otimes V$ are isomorphic. One has also to be careful to distinguish between left and right duals say V^* and ${}^*V$; in one case we have an inclusion of $\mathbb{C} \hookrightarrow V \otimes V^*$ and in the other case a projection ${}^*V \otimes V \to \mathbb{C} \to 0$. The tensor product defines a ring structure on the Grothendieck group of this category. A very interesting fact proved in[59] is that the Grothendieck ring is always commutative.

We shall say that an irreducible representation in $\mathcal{F}_q$ is prime if it cannot be written in a non-trivial way as a tensor product of two objects of $\mathcal{F}_q$. It is trivially true that any irreducible representation is isomorphic to a tensor product of prime representations. It follows that to understand irreducible representations it is enough to understand the prime ones. However, outside $\mathfrak{g} = \mathfrak{sl}_2$ the classification of prime objects seems to be a very hard and perhaps wild problem. We shall nevertheless, give various examples of families of prime representations in this section and the next, including some coming from the connection with cluster algebras.

Our next definition is entirely motivated by the connection with cluster algebras. Namely we shall say that an object V of $\mathcal{F}_q$ is real if $V^{\otimes r}$ is irreducible for all $r \geq 1$. As a consequence of the main result of[65] it is enough to require $V^{\otimes 2}$ to be irreducible. Again, it is hard to characterize real representations. A well-known example of Leclerc shows in[80] that there are prime representations which are not real.

The notion of prime and real objects can obviously be defined for the finite-dimensional module category of any Hopf algebra, and in particular for irreducible finite-dimensional representations of $\mathfrak{g}$ and $L(\mathfrak{g})$. For $\mathfrak{g}$ it is an exercise to prove that the representations $V(\lambda)$, $\lambda \in P^+$ are prime and not real if $\lambda \neq 0$. In the case of $L(\mathfrak{g})$ it then follows from Proposition 2.3.4 that the prime irreducible representations are precisely $\mathrm{ev}_a V(\lambda)$, $\lambda \in P^+$, $a \in \mathbb{C}$. Moreover, it also follows that these representations are never real if $\lambda \neq 0$. So these notions are uninteresting in these examples.

As in the case of $L(\mathfrak{g})$ the objects of $\mathcal{F}_q$ are not completely reducible. These categories behave more like the category $\mathcal{O}$ for simple Lie algebras and some of these similarities are explored in this article.

3.2.3 Representations of Quantum Loop $\mathfrak{sl}_2$

In this case the study of the irreducible objects in $\mathcal{F}_q$ is well-understood and we briefly review the main results from[32]. Given $r \in \mathbb{Z}_+$ and $a \in \mathbb{C}(q)^\times$ let

$$\boldsymbol{\omega}_{1,a,r} = \boldsymbol{\omega}_{1,aq^{r-1}}\boldsymbol{\omega}_{1,aq^{r-3}} \cdots \boldsymbol{\omega}_{1,aq^{-r+1}}.$$

Note that $\boldsymbol{\omega}_{1,a,0}$ is the unit element of the monoid $\mathcal{P}_q^+$ for all $a \in \mathbb{C}(q)^\times$. Then

$$V_q(\boldsymbol{\omega}_{1,a,r}) \cong_{\mathbf{U}_q(\mathfrak{sl}_2)} V_q(r\omega_1).$$

Moreover

$$V_q(\boldsymbol{\omega}_{1,a,r}) \otimes V_q(\boldsymbol{\omega}_{1,b,s}) \cong V_q(\boldsymbol{\omega}_{1,a,r}\boldsymbol{\omega}_{1,b,s}) \iff ab^{-1} \notin \{q^{\pm(r+s-2p)} : 0 \leq p < \min\{r,s\}\}.$$

In particular, the modules $V_q(\boldsymbol{\omega}_{1,a,r})$ are prime and real. If $ab^{-1} = q^{\pm(r+s-2p)}$ for some $0 \leq p < \min\{r,s\}$ then we have a non-split short exact sequence,

$$0 \to V_1 \to V_q(\boldsymbol{\omega}_{1,a,r}) \otimes V_q(\boldsymbol{\omega}_{1,b,s}) \to V_2 \to 0,$$

where

$$V_1 \cong V_q(\boldsymbol{\omega}_{1,aq^{(r-p)},p}) \otimes V_q(\boldsymbol{\omega}_{1,bq^{(p-r)},r+s-p}),$$
$$V_2 \cong V_q(\boldsymbol{\omega}_{1,aq^{-p-1},r-p-1}) \otimes V_q(\boldsymbol{\omega}_{1,aq^{p+1},s-p-1})$$

if $ab^{-1} = q^{-(r+s-2p)}$ while if $ab^{-1} = q^{(r+s-2p)}$ then

$$V_1 \cong V_q(\boldsymbol{\omega}_{1,aq^{p+1},r-p-1}) \otimes V_q(\boldsymbol{\omega}_{1,aq^{-p-1},s-p-1}),$$
$$V_2 \cong V_q(\boldsymbol{\omega}_{1,aq^{(p-r)},p}) \otimes V_q(\boldsymbol{\omega}_{1,bq^{(r-p)},r+s-p}).$$

Any irreducible module in $\mathcal{F}_q$ is isomorphic to a tensor product of representations of the form $V_q(\boldsymbol{\omega}_{1,a,r})$, $r \in \mathbb{Z}_+$, $a \in \mathbb{C}(q)^\times$. More precisely, if $\boldsymbol{\omega} \in \mathcal{P}_q^+$ then

$$V(\boldsymbol{\omega}) \cong V_q(\boldsymbol{\omega}_{1,a_1,r_1}) \otimes \cdots \otimes V_q(\boldsymbol{\omega}_{1,a_k,r_k}),$$

for a unique choice of $k \geq 1$ and pairs (a_s, r_s), $1 \leq s \leq k$ satisfying $a_s a_m^{-1} \neq q^{\pm(r_s+r_m-2p)}$ for any $0 \leq p < \min\{r_s, r_m\}, 1 \leq s \neq m \leq k$.

3.2.4 Local Weyl Modules for Quantum Loop Algebras

We identify the monoid $\mathcal{P}_q^+$ with the monoid consisting of I-tuples of polynomials $(\pi_i(u))_{i\in I}$, $\pi_i(u) \in \mathbb{C}(q)[u], \pi_i(0) = 1$, via

$$\boldsymbol{\omega}_{i,a} \mapsto (1 - \delta_{i,j}au)_{j\in I}.$$

For $\boldsymbol{\omega} \in \mathcal{P}_q^+$ the local Weyl module $W_q(\boldsymbol{\omega})$ is the $\mathbf{U}_q(L(\mathfrak{g}))$-module generated by an element $v_{\boldsymbol{\omega}}$ with relations

$$x_{i,r}^+ v_{\boldsymbol{\omega}} = 0 = (x_{i,0}^-)^{\deg \pi_i(u)+1} v_{\boldsymbol{\omega}},$$
$$\phi_{i,r}^\pm v_{\boldsymbol{\omega}} = \gamma_{i,r}^\pm v_{\boldsymbol{\omega}}, \quad r \in \mathbb{Z}$$

where $\gamma_{i,r}^\pm \in \mathbb{C}(q)$ are defined by

$$\sum_{r=0}^{\infty} \gamma_{i,\pm r}^\pm u^{\pm r} = q_i^{\deg \pi_i} \frac{\pi_i(q_i^{-1}u)}{\pi_i(q_i u)}, \quad \boldsymbol{\omega} = (\pi_i(u))_{i\in I}.$$

The following was proved in[38].

Proposition *Let $\boldsymbol{\omega} \in \mathcal{P}_q^+$. Then* $\dim W_q(\boldsymbol{\omega}) < \infty$ *and $W_q(\boldsymbol{\omega})$ has a unique irreducible quotient $V_q(\boldsymbol{\omega})$. If $\boldsymbol{\omega}' \in \mathcal{P}_q^+$ then the modules $W_q(\boldsymbol{\omega})$ and $W_q(\boldsymbol{\omega}')$ are isomorphic if and only if $\boldsymbol{\omega} = \boldsymbol{\omega}'$. The module $V_q(\boldsymbol{\omega}\boldsymbol{\omega}')$ is a subquotient of $W_q(\boldsymbol{\omega}) \otimes W_q(\boldsymbol{\omega}')$.* □

3.2.5 The Fundamental Local Weyl Modules

It was proved in[21] that $W_q(\boldsymbol{\omega}_{i,a}) \cong V_q(\boldsymbol{\omega}_{i,a})$. In general it is not true that local Weyl modules are irreducible but we will discuss conditions for these later in the paper.

3.2.6 Local Weyl Modules and Tensor Products

Suppose that $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{P}_q^+$ and let M, M' be any quotient of $W_q(\boldsymbol{\omega})$ and $W_q(\boldsymbol{\omega}')$ respectively. Let $v_{\boldsymbol{\omega}}$ and $v_{\boldsymbol{\omega}'}$ also denote the images of these elements in M and M'. Then using the formulae for comultiplication one can prove that we have the following sequence of surjective maps:

$$W_q(\boldsymbol{\omega}\boldsymbol{\omega}') \twoheadrightarrow \mathbf{U}_q(L(\mathfrak{g}))(v_{\boldsymbol{\omega}} \otimes v_{\boldsymbol{\omega}'}) \twoheadrightarrow V_q(\boldsymbol{\omega}\boldsymbol{\omega}').$$

In particular $V_q(\boldsymbol{\omega}\boldsymbol{\omega}')$ is a subquotient of $M \otimes M'$.

Suppose that $\boldsymbol{\omega} = \boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k} \in \mathcal{P}_q^+$; the discussion so far establishes the existence of a map of $\mathbf{U}_q(L(\mathfrak{g}))$-modules

$$\phi_{\boldsymbol{\omega}} : W_q(\boldsymbol{\omega}) \to V_q(\boldsymbol{\omega}_{i_1,a_1}) \otimes \cdots \otimes V_q(\boldsymbol{\omega}_{i_k,a_k}).$$

If we assume that $a_1, \ldots, a_k$ are such that $a_j/a_s \notin q^{\mathbb{N}}$ for all $1 \le j < s \le k$, then the results of[1,22,99] show that this map is surjective. It was conjectured in[38] that $\phi_{\boldsymbol{\omega}}$ is an isomorphism. Clearly to prove the conjecture it suffices to establish an equality of dimensions. This equality was proved in that paper for $\mathfrak{sl}_2$. In the general case the conjecture was established through the work of[29,57,87]. We will discuss this further in the next section.

Assuming from now on that $\phi_{\boldsymbol{\omega}}$ is an isomorphism we study the question of the irreducibility of $W_q(\boldsymbol{\omega})$. A sufficient condition for $W_q(\boldsymbol{\omega})$ to be irreducible is to require that $a_j/a_s \notin q^{\mathbb{Z}}$ for all $1 \le j \ne s \le k$; this was known through the work of[1,22]. A precise statement was given in[22], Corollary 5.1] when $\mathfrak{g}$ is of classical type and for some of the exceptional nodes. The following result summarizes the results of[22] for classical cases.

Theorem (i) *Suppose that $\mathfrak{g}$ is of classical type and $i, j \in I$ and $a, b \in \mathbb{C}(q)^\times$. Then*

$$ab^{-1} \notin q^{\pm S(i,j)} \implies V_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}) \cong V_q(\boldsymbol{\omega}_{i,a}) \otimes V_q(\boldsymbol{\omega}_{j,b}) \cong W_q(\boldsymbol{\omega}_{i,a}\boldsymbol{\omega}_{j,b}),$$

where $S(i,j)$ is given as follows:

If $\mathfrak{g}$ *is of type* A_n:

$$S(i,j) = \{2 + 2k - i - j : \max\{i,j\} \le k \le \min\{i+j-1, n\}\}.$$

If $\mathfrak{g}$ is of type B_n, and α_1 is short:

- $S(1,1) = \{4k - 2 : 1 \le k \le n\}$,
- $S(i,1) = S(i,1) = \{4k - 2i + 1 : i \le k \le n\}$,
- $S(i,j) = \{4 + 4k - 2i - 2j : \max\{i,j\} \le k \le n\} \cup \{4k - 2 - 2|j-i| : \max\{i,j\} \le k \le n\}, i,j > 1$.

- *If $\mathfrak{g}$ is of type C_n, and α_1 is long:*

- $S(1,1) = \{2k + 2 : 1 \le k \le n\}$
- $S(i,1) = S(1,i) = \{2k - i + 3 : 1 \le k \le n\}$ $i > 1$,
- $S(i,j) = \{2 + 2k - i - j : \max\{i,j\} \le k \le n\} \cup \{2 + 2k - |i-j| : \max\{i,j\} \le k \le n\}, i,j > 1$.

- *If $\mathfrak{g}$ is of type D_n, and 1 and 2 denote the spin nodes:*

- $S(1,1) = S(2,2) = \{2k - 2 : 2 \le k \le n, k \equiv 0 \mod 2\}$
- $S(1,2) = S(2,1) = \{2k - 2 : 3 \le k \le n, k \equiv 1 \mod 2\}$
- $S(1,j) = S(2,j) = \{2k - j : j \le k \le n\} = S(j,2) = S(j,1), j \ge 3$.
- $S(i,j) = \{2 + 2k - i - j : \max\{i,j\} \le k \le n\} \cup \{-2 + 2k - |i-j| : \max\{i,j\} \le k \le n\}$, $i,j \ge 3$.

(ii) *Given $\boldsymbol{\omega} = \boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k}$ the module $W_q(\boldsymbol{\omega})$ is irreducible if*

$$a_r a_s^{-1} \notin q^{\pm S(i_r,i_s)}, \quad 1 \le r < s \le k.$$

□

Remark More recently an alternative approach to describing the set $S(i,j)$ was given in[61], Theorem 2.10] and[62], Section 6] by relating it to the poles of the universal R-matrix. It is nontrivial to see that those conditions are equivalent to the explicit description given in the preceding theorem.

3.2.7 A-Forms and Classical Limits

Let $\mathbf{A} = \mathbb{Z}[q, q^{-1}]$ and define $\mathbf{U}_{\mathbf{A}}(\widehat{\mathfrak{g}})$ to be the $\mathbf{A}$-subalgebra of $\mathbf{U}_q(\widehat{\mathfrak{g}})$ generated by the elements $(X_i^{\pm})^r/[r]_q!, i \in \widehat{I}$. Then

$$\mathbf{U}_q(\widehat{\mathfrak{g}}) \cong \mathbf{U}_{\mathbf{A}}(\widehat{\mathfrak{g}}) \otimes_{\mathbf{A}} \mathbb{C}(q).$$

For $\epsilon \in \mathbb{C}^\times$ we let $\mathbb{C}_\epsilon$ be the $\mathbf{A}$-module obtained by letting q act as ϵ and set

$$\mathbf{U}_\epsilon(\widehat{\mathfrak{g}}) = \mathbf{U}_{\mathbf{A}}(\widehat{\mathfrak{g}}) \otimes_{\mathbf{A}} \mathbb{C}_\epsilon.$$

The algebra $\mathbf{U}(\widehat{\mathfrak{g}})$ is isomorphic to the quotient of $\mathbf{U}_1(\widehat{\mathfrak{g}})$ by the ideal generated by $K_i - 1, D - 1,\ i \in \widehat{I}$. Similar assertions hold for $\mathbf{U}_q(\mathfrak{g})$ and $\mathbf{U}_q(L(\mathfrak{g}))$ as well. Part (i) of the following was proved in[82,97] while part (ii) was proved in[38].

Theorem (i) *Suppose that $\lambda \in \widehat{P}^+$ and $\epsilon \in \mathbb{C}^\times$. There exists a $\mathbf{U}_\mathbf{A}(\widehat{\mathfrak{g}})$-submodule $V_\mathbf{A}(\lambda)$ of $V_q(\lambda)$ such that*

$$V_q(\lambda) \cong V_\mathbf{A}(\lambda) \otimes_\mathbf{A} \mathbb{C}(q).$$

In particular $V_\epsilon(\lambda) = V_\mathbf{A}(\lambda) \otimes_\mathbf{A} \mathbb{C}_\epsilon$ is a module for $\mathbf{U}_\epsilon(\widehat{\mathfrak{g}})$, and if $\epsilon = 1$ or if ϵ is not a root of unity then we have $\operatorname{ch} V_\epsilon(\lambda) = \operatorname{ch} V_q(\lambda)$. Analogous statements hold for the $\mathbf{U}_q(\mathfrak{g})$ representations $V_q(\lambda), \lambda \in P^+$.

(ii) *Let $\mathcal{P}^+_\mathbf{A}$ be the submonoid of $\mathcal{P}^+$ generated by elements $\boldsymbol{\omega}_{i,a}, a \in \mathbf{A}$ and let $\boldsymbol{\omega} \in \mathcal{P}^+_\mathbf{A}$. Then $W_q(\boldsymbol{\omega})$ admits a $\mathbf{U}_\mathbf{A}(L(\mathfrak{g}))$-submodule $W_\mathbf{A}(\boldsymbol{\omega})$ and*

$$W_q(\boldsymbol{\omega}) \cong W_\mathbf{A}(\boldsymbol{\omega}) \otimes_\mathbf{A} \mathbb{C}(q).$$

In particular $W_\epsilon(\boldsymbol{\omega}) = W_\mathbf{A}(\boldsymbol{\omega}) \otimes_\mathbf{A} \mathbb{C}_\epsilon$ is a $\mathbf{U}_\epsilon(L(\mathfrak{g}))$-module, and $\operatorname{ch} W_\epsilon(\boldsymbol{\omega}) = W_q(\boldsymbol{\omega})$ if $\epsilon = 1$ or if ϵ is not a root of unity. If M_q is any $\mathbf{U}_q(L(\mathfrak{g}))$-module quotient of $W_q(\boldsymbol{\omega})$ let $M_\mathbf{A}$ be the image of $W_\mathbf{A}(\boldsymbol{\omega})$. Then,

$$M_q \cong M_\mathbf{A} \otimes_\mathbf{A} \mathbb{C}(q),\quad M_\epsilon \cong M_\mathbf{A} \otimes_\mathbf{A} \mathbb{C}_\epsilon,$$

and M_ϵ is a canonical quotient of $W_\epsilon(\boldsymbol{\omega})$.

□

- *Remark* The modules $V_1(\lambda),\ \lambda \in \widehat{P}^+$ and $V_1(\boldsymbol{\omega}), \boldsymbol{\omega} \in \mathcal{P}^+_\mathbf{A}$ are modules for the universal enveloping algebra of $\mathbf{U}(L(\mathfrak{g}))$ and are called the classical limit of $V_q(\lambda)$ and $V_q(\boldsymbol{\omega})$ respectively.
- It is worth noting again, that in part (i) of the theorem the module $V_\epsilon(\lambda)$ is irreducible for $\mathbf{U}_\epsilon(\widehat{\mathfrak{g}})$ if ϵ is not a root of unity. This is false in part (ii).

4 Classical and Graded Limits

In this section we discuss various well-studied families of finite-dimensional representations of quantum affine algebras. We review the literature on the presentation of these modules, their classical limits and the closely related *graded limits*.

4.1 Classical and Graded Limits of the Quantum Local Weyl Modules

4.1.1 Relations in the Classical Limit

Suppose that $\boldsymbol{\omega} \in \mathcal{P}^+_\mathbf{A}$ and let $V_q(\boldsymbol{\omega})$ be the unique irreducible quotient of $W_q(\boldsymbol{\omega})$ (see Sect. 3.2.4). Since $W_q(\boldsymbol{\omega})$ is finite-dimensional the corresponding classical limits (see Theorem 3.2.7) $W_1(\boldsymbol{\omega})$ and $V_1(\boldsymbol{\omega})$ are finite-dimensional modules for $L(\mathfrak{g})$. Let $\bar{v}_\omega = v_\omega \otimes 1 \in W_1(\boldsymbol{\omega})$. The following was proved in[38].

Lemma Suppose that $\boldsymbol{\omega} = \boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k} \in \mathcal{P}^+_\mathbf{A}$. The following relations hold in $W_1(\boldsymbol{\omega})$:

$$h\bar{v}_\omega = \operatorname{wt} \boldsymbol{\omega}(h)\bar{v}_\omega,\ \ \Big(h \otimes (t - a_1(1)) \cdots (t - a_k(1))\mathbb{C}[t, t^{-1}]\Big)\bar{v}_\omega = 0,$$
$$(x^+_\alpha \otimes \mathbb{C}[t, t^{-1}])\bar{v}_\omega = 0,\ \ h \in \mathfrak{h},\ \ \alpha \in R^+.$$

□

Remark With a little more work one can actually prove that there exists an integer $N \in \mathbb{N}$ such that $W_1(\boldsymbol{\omega})$ is a module for the truncation of $L(\mathfrak{g})$ by the polynomial $(t - a_1(1))^N \cdots (t - a_k(1))^N$.

4.1.2 Graded Limits: The Modules V_{loc}

In the rest of this section we shall restrict our attention to the submonoid $\mathcal{P}^+_\mathbb{Z}$ which is generated by the elements $\boldsymbol{\omega}_{i,q^r}$ for $i \in I$ and $r \in \mathbb{Z}$.

In this case the results of Sect. 4.1.1 imply that $W_1(\boldsymbol{\omega})$ is a module for the truncation of $L(\mathfrak{g})$ at $(t-1)^N$ for some N sufficiently large. Using the isomorphism of Lie algebras

$$\mathfrak{g} \otimes \frac{\mathbb{C}[t, t^{-1}]}{(t-1)^N} \cong \mathfrak{g} \otimes \frac{\mathbb{C}[t]}{(t-1)^N} \cong \mathfrak{g} \otimes \frac{\mathbb{C}[t]}{(t^N)}$$

we see that we can regard $W_1(\boldsymbol{\omega})$ as a module for the truncation of $\mathfrak{g}[t]$ at t^N. We shall denote this module by $W_{\mathrm{loc}}(\boldsymbol{\omega})$. We call this the *graded limit* of $W_q(\boldsymbol{\omega})$. If M is any quotient of $W_q(\boldsymbol{\omega})$ we can define a corresponding module M_{loc} for $\mathfrak{g}[t]$ using the isomorphisms of Lie algebras and call this the graded limit of M.

This terminology of course requires justification. Recall that the action of d on $\mathfrak{g}[t]$ defines a $\mathbb{Z}_+$-grading on it: the r-th graded piece is $\mathfrak{g} \otimes t^r$. The adjoint action of d on $\mathbf{U}(\mathfrak{g}[t])$ also gives a $\mathbb{Z}_+$-grading. Hence one can define the notion of a graded $\mathfrak{g}[t]$-module V to be one which admits a compatible $\mathbb{Z}$-grading namely:

$$V = \bigoplus_{s\in\mathbb{Z}} V[s], \quad (\mathfrak{g}\otimes t^r)V[s] \subseteq V[r+s].$$

The general belief is that when M is a quotient of $W_q(\boldsymbol{\omega})$ with $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{Z}}^+$ then $M_{\rm loc}$ is a graded $\mathfrak{g}[t]$-module. This is far from clear in general and is hard to prove even in specific cases. In the rest of the section we will discuss certain families of modules where the corresponding graded limit is a graded $\mathfrak{g}[t]$-module.

Remark In the discussion that follows we shall see that the classical or graded limit depends only on wt $\boldsymbol{\omega}$. So there is a substantial loss of information when we go to the limits. However, the character and the underlying $\mathbf{U}_q(\mathfrak{g})$-module is the same and this is one reason for our interest in this study.

4.1.3 Kirillov–Reshetikhin Modules

We begin by discussing this particular family of modules since this was essentially the motivation for the interest in graded limits.

Given $i\in I, r\in\mathbb{N}$ and $s\in\mathbb{Z}$ set

$$\boldsymbol{\omega}_{i,s,r} = \boldsymbol{\omega}_{i,q_i^{s+r-1}}\boldsymbol{\omega}_{i,q_i^{s+r-3}}\cdots\boldsymbol{\omega}_{i,q_i^{s-r+1}}.$$

Notice that in the case $i=1$ these elements were introduced in Sect. 3.2.3 in the case of $\mathfrak{sl}_2$ where they were denoted as $\boldsymbol{\omega}_{1,q^s,r}$ since we were working in a more general situation. The corresponding irreducible $\mathbf{U}_q(L(\mathfrak{g}))$-module is called a Kirillov–Reshetikhin module. This is because of an important conjecture that they had made; they predicted the existence of certain modules for the quantum loop algebra with a specific decomposition as $\mathbf{U}_q(\mathfrak{g})$-modules (see also[63]). In[21] it was proved that the conjectured modules were of the form $V_q(\boldsymbol{\omega}_{i,s,r})$ for all classical Lie algebras and for some $i\in I$ in the exceptional cases. Moreover the following presentation was given (see Corollary 2.1 of[21]) for the module $V_1(\boldsymbol{\omega}_{i,s,r})$.

Theorem *The $L(\mathfrak{g})$-module $V_1(\boldsymbol{\omega}_{i,s,r})$ is generated by an element $v_{i,s,r}$ with relations:*

$$x_\alpha^+ v_{i,s,r} = 0, (h\otimes t^k)v_{i,s,r} = r\omega_i(h)v_{i,s,r},$$
$$((x_i^-\otimes t^k) - x_i^-\otimes 1)v_{i,s,r} = 0,$$
$$(x_\alpha^-\otimes 1)^{r\omega_i(h_\alpha)+1}v_{i,s,r} = 0,$$

where $\alpha\in R^+, h\in\mathfrak{h}$ and $k\in\mathbb{Z}$. □

Here we have used the fact that wt $\boldsymbol{\omega}_{i,s,r} = r\omega_i$. Notice that these relations are independent of s. Moreover,

$$((h\otimes t^k) - h)v_{i,s,r} = 0$$
$$\implies (h\otimes(t-1)^k)v_{i,s,r} = 0, \quad 0\neq k\in\mathbb{Z},$$

and similarly for the third relation in the presentation above. It follows that $V_{\rm loc}(\boldsymbol{\omega}_{i,s,r})$ is the $\mathfrak{g}[t]$-quotient of $W_{\rm loc}(\boldsymbol{\omega}_{i,s,r})$ by imposing the additional relation: $(x_i^-\otimes t)v_{\boldsymbol{\omega}_{i,s,r}} = 0$.

Later, in[30], a more systematic self contained study of these modules was developed and the graded $\mathfrak{g}$-module decomposition of these modules was calculated. One can think of this as a graded version of the Kirillov–Reshetikhin character formula. The results of[30] led to the definition of graded limits and more generally resulted in the development of the subject of graded (not necessarily finite-dimensional) representations of $\mathfrak{g}[t]$. We say more about this study in later sections of the paper.

4.1.4 Minimal Affinizations

The notion of minimal affinizations was introduced and further studied in[20,33,34]. Perhaps the simplest place to explain what this notion means is in the case of $\mathfrak{sl}_2$. Since we have only one simple root we denote the generators of $\mathcal{P}_q^+$ by $\boldsymbol{\omega}_{1,a}$. If $\boldsymbol{\omega} = \boldsymbol{\omega}_{1,a_1}\cdots\boldsymbol{\omega}_{1,a_k}\in\mathcal{P}_q^+$ then it is not hard to see that there exists a $\mathbf{U}_q(\mathfrak{g})$-module M such that

$$V_q(\boldsymbol{\omega}) \cong_{\mathbf{U}_q(\mathfrak{g})} V_q(k\omega)\oplus M.$$

Moreover, it was shown in[32] that $M\neq 0$ unless $V_q(\boldsymbol{\omega})$ is a Kirillov–Reshetikhin module:

$$M = 0 \iff$$
$$\boldsymbol{\omega} = \boldsymbol{\omega}_{1,aq^{k-1}}\cdots\boldsymbol{\omega}_{1,aq^{-k+1}}, \ a\in\mathbb{C}(q)^\times.$$

Using the results of[32] we can also give precise conditions under which $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\omega}')$ are isomorphic as $\mathbf{U}_q(\mathfrak{g})$-modules.

It is natural to ask what analogs of these results hold in the higher rank case. It was known essentially from the beginning (see[44]) that if $\mathfrak{g}$ is not of type A, there does not exist a corresponding $\mathbf{U}_q(L(\mathfrak{g}))$-module structure on $V_q(\lambda)$. On the other hand it is also clear that there were many pairs $\boldsymbol{\omega},\boldsymbol{\omega}'\in\mathcal{P}_q^+$ with $V_q(\boldsymbol{\omega})\cong_{\mathbf{U}_q(\mathfrak{g})} V_q(\boldsymbol{\omega}')$. So this motivated the question: given $\lambda\in P^+$, is there a "smallest" $\mathbf{U}_q(\mathfrak{g})$-module containing a copy of $V_q(\lambda)$ which admits an action of the quantum loop algebra. This question can be more formally stated as follows.

Given $\boldsymbol{\omega},\boldsymbol{\omega}'\in\mathcal{P}_q^+$ we say that $V_q(\boldsymbol{\omega})$ is equivalent to $V_q(\boldsymbol{\omega}')$ if they are isomorphic as $\mathbf{U}_q(\mathfrak{g})$

-modules. Denote the equivalence class corresponding to $\boldsymbol{\omega}$ by $[V_q(\boldsymbol{\omega})]_{\mathfrak{g}}$. In particular,

$$[V_q(\boldsymbol{\omega})]_{\mathfrak{g}} = [V_q(\boldsymbol{\omega}')]_{\mathfrak{g}} \implies \operatorname{wt}\boldsymbol{\omega} = \operatorname{wt}\boldsymbol{\omega}'.$$

The converse statement is definitely false, this is already the case in $\mathfrak{sl}_2$.

Define a partial order on the set of equivalence classes by: $[V_q(\boldsymbol{\omega})]_{\mathfrak{g}} \le [V_q(\boldsymbol{\omega}')]_{\mathfrak{g}}$ if for all $\mu \in P^+$ either

$$\dim \operatorname{Hom}_{\mathbf{U}_q(\mathfrak{g})}(V_q(\mu), V_q(\boldsymbol{\omega})) \le \dim \operatorname{Hom}_{\mathbf{U}_q(\mathfrak{g})}(V_q(\mu), V_q(\boldsymbol{\omega}'))$$

or there exists $\nu > \mu$ (i.e. $\nu - \mu \in Q^+\backslash\{0\}$) such that

$$\dim \operatorname{Hom}_{\mathbf{U}_q(\mathfrak{g})}(V_q(\nu), V_q(\boldsymbol{\omega})) < \dim \operatorname{Hom}_{\mathbf{U}_q(\mathfrak{g})}(V_q(\nu), V_q(\boldsymbol{\omega}')).$$

It was proved in[20] that minimal elements exist in this order and an irreducible representation corresponding to a minimal element was called a minimal affinization. When $\mathfrak{g}$ is not of type D or E the explicit expression for the elements $\boldsymbol{\omega} \in \mathcal{P}_q^+$ which give a minimal affinization are given by

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{i_1,s_1,r_1} \cdots \boldsymbol{\omega}_{i_k,s_k,r_k}, \;\; i_1 < i_2 < \cdots < i_k, \\ s_{p+1} - s_p = \epsilon \left(d_{i_p} r_p + d_{i_{p+1}} r_{p+1} + \sum_{j=i_p}^{i_{p+1}-1} (d_j - 1 - a_{j,j+1}) \right), \\ 1 \le p \le k-1,$$

where either $\epsilon = 1$ for all p or $\epsilon = -1$ for all p and $\boldsymbol{\omega}_{i,s,r}$ is the element of $\mathcal{P}_q^+$ which was introduced in Sect. 4.1.3. In types D and E the preceding formulae still correspond to minimal affinizations under suitable restrictions. Unfortunately these are far from being all of them; the difficulty lies in the existence of the trivalent node (see[35,36]). The problem of classifying all the minimal elements was studied in[94] but the full details are still to appear.

The equivalence classes of Kirillov–Reshetikhin modules are clearly minimal affinizations. The following result was conjectured in[85] and proved in[81,88,89]. It again justifies the use of the term graded limit. We do not state the result in full generality in type D but restrict our attention to the minimal affinizations discussed here.

Theorem *Assume that $\boldsymbol{\omega} \in \mathcal{P}_q^+$ is as in the discussion just preceding the theorem. Then $V_{\mathrm{loc}}(\boldsymbol{\omega})$ is the $\mathfrak{g}[t]$-module generated by an element $v_{\boldsymbol{\omega}}$ with relations:*

$$x_i^+ v_{\boldsymbol{\omega}} = 0, \;\; (h_i \otimes t^k) v_{\boldsymbol{\omega}} = \delta_{k,0} \operatorname{wt}\boldsymbol{\omega}(h_i) v_{\boldsymbol{\omega}} \\ (x_\beta^- \otimes t) v_{\boldsymbol{\omega}} = 0 \;\; (x_i^- \otimes 1)^{\operatorname{wt}\boldsymbol{\omega}(h_i)+1} v_{\boldsymbol{\omega}} = 0,$$

for all $i \in I$ and for all $\beta = \sum_{i=1}^n s_i \alpha_i \in R^+$ with $s_i \le 1$. □

The $\mathfrak{g}$-module decomposition of the minimal affinizations was computed in rank two in[20]. In the case of A_n the graded limit is irreducible and so its character is just the character of $V_q(\operatorname{wt}\boldsymbol{\omega})$. The $\mathfrak{g}$-module decomposition in type B and D_4 was partially given in[85] and the result in complete generality is in[88,89] for types B, C and for certain minimal affinizations in type D. Moreover, in[98] Sam proved a conjecture made in[25] that the character of minimal affinizations in types BCD are given by a Jacobi–Trudi determinant.

4.1.5 Tensor Products and Fusion Products

Before continuing with our justification for the term graded limit, we discuss the following natural question. Suppose that M and M' are $\mathbf{U}_q(L(\mathfrak{g}))$-modules which have a classical (resp. graded) limit. Is it true that $M \otimes M'$ has a classical limit and how does this relate to the tensor product of the classical (resp. graded) limits? The answer to this question is far from straightforward; even if $M \otimes M'$ does have a classical limit it is easy to generate examples where $(M \otimes M')_1$ is not isomorphic to $M_1 \otimes M_1'$ as $L(\mathfrak{g})$-modules. For instance if we take $\mathfrak{g} = \mathfrak{sl}_2$ and $M = V_q(\boldsymbol{\omega}_{1,q^2})$, $M' = V_q(\boldsymbol{\omega}_{1,1})$ then the module $M \otimes M'$ is a cyclic $\mathbf{U}_q(L(\mathfrak{sl}_2))$-module and so the classical limit is a cyclic indecomposable module for $L(\mathfrak{sl}_2)$. However the tensor product of the classical limits is $V(\boldsymbol{\omega}_1) \otimes V(\boldsymbol{\omega}_1)$ (equivalently $\operatorname{ev}_1 V(\omega) \otimes \operatorname{ev}_1 V(\omega)$) which is completely reducible.

The $\mathfrak{g}$-module structure however is unchanged in the process. This is because it is known that the process of taking classical limits preserves tensor products for the simple Lie algebras. If we work with the graded limit then again it is false that the graded character of $(M \otimes M')_{\mathrm{loc}}$ is the same as $M_{\mathrm{loc}} \otimes M'_{\mathrm{loc}}$. However in many examples the graded limit of the tensor product coincides with an operation called the fusion product defined on graded $\mathfrak{g}[t]$-modules. This notion was introduced in[49] and we now recall this construction.

Let V be a finite-dimensional cyclic $\mathfrak{g}[t]$-module generated by an element v and for $r \in \mathbb{Z}_+$ set

$$F^r V = \left(\bigoplus_{0 \le s \le r} \mathbf{U}(\mathfrak{g}[t])[s] \right) \cdot v$$

Clearly $F^r V$ is a $\mathfrak{g}$-submodule of V and we have a finite $\mathfrak{g}$-module filtration

$$0 \subseteq F^0 V \subseteq F^1 V \subseteq \cdots \subseteq F^k V = V,$$

for some $k \in \mathbb{Z}_+$. The associated graded vector space $\mathrm{gr} V$ acquires a graded $\mathfrak{g}[t]$-module structure in a natural way and is generated by the image of v in $\mathrm{gr} V$. Given a $\mathfrak{g}[t]$-module V and $z \in \mathbb{C}$, let V^z be the $\mathfrak{g}[t]$-module with action

$$(x \otimes t^r)w = (x \otimes (t+z)^r)w,$$
$$x \in \mathfrak{g}, \ \ r \in \mathbb{Z}_+, \ w \in V.$$

Let V_s, $1 \le s \le p$, be cyclic finite-dimensional $\mathfrak{g}[t]$-modules with cyclic vectors v_s, $1 \le s \le p$ and let $z_1, \ldots, z_p$ be distinct complex numbers. Then the module $V_1^{z_1} \otimes \cdots \otimes V_p^{z_p}$ is cyclic with cyclic generator $v_1 \otimes \cdots \otimes v_p$. The fusion product $V_1^{z_1} * \cdots * V_p^{z_p}$ is defined to be $\mathrm{gr} V_1^{z_1} \otimes \cdots \otimes V_p^{z_p}$. For ease of notation we shall use $V_1 * \cdots * V_k$ for $V_1^{z_1} * \cdots * V_k^{z_k}$.

It is conjectured in[49] that under some suitable conditions on V_s and v_s, the fusion product is independent of the choice of the complex numbers z_s, $1 \le s \le k$, and this conjecture is verified in many special cases by various people (see for instance[29,40,48,50,57,74,90]). In all these cases the conjecture is proved by exhibiting a graded presentation of the fusion product which is independent of all parameters. This is much like what we have been doing to justify the use of *graded limit* and the coincidence is not accidental. In almost all of these papers the proof of the Feigin–Loktev conjecture involves giving a presentation of the graded limit of certain $\mathbf{U}_q(L(\mathfrak{g}))$-modules.

4.1.6 A Presentation of $W_{\mathrm{loc}}(\boldsymbol{\omega})$

We return to our discussion in Sect. 3.2.6. Recall that we had discussed that given $\boldsymbol{\omega} \in \mathcal{P}_q^+$ we can write $\boldsymbol{\omega} = \boldsymbol{\omega}_{i_1,a_1} \cdots \boldsymbol{\omega}_{i_k,a_k}$ so that there is a surjective map of $\mathbf{U}_q(L(\mathfrak{g}))$-modules

$$W_q(\boldsymbol{\omega}) \to V_q(\boldsymbol{\omega}_{i_1,a_1}) \otimes \cdots \otimes V_q(\boldsymbol{\omega}_{i_k,a_k}) \to 0.$$

It has been conjectured in[37] that

$$\begin{aligned} \dim W_q(\boldsymbol{\omega}) &= \prod_{s=1}^{k} \dim V_q(\boldsymbol{\omega}_{i_s,a_s}) \\ &= \prod_{s=1}^{k} \dim W_q(\boldsymbol{\omega}_{i_s,a_s}), \end{aligned} \tag{4.1}$$

where the second equality is a consequence of Sect. 3.2.5. Since the dimension is unchanged when passing to the graded limit it suffices to prove that

$$\dim W_{\mathrm{loc}}(\boldsymbol{\omega}) = \prod_{s=1}^{k} \dim V_{\mathrm{loc}}(\boldsymbol{\omega}_{i_s,a_s}). \tag{4.2}$$

Using Lemma 4.1.1 and the discussion in Section 4.1.2 we see that $W_{\mathrm{loc}}(\boldsymbol{\omega})$ is the quotient of the module $\tilde{W}_{\mathrm{loc}}(\mathrm{wt}\boldsymbol{\omega})$ which is generated as a $\mathfrak{g}[t]$-module by an element $w_{\boldsymbol{\omega}}$ with defining relations:

$$(h \otimes t\mathbb{C}[t])w_{\boldsymbol{\omega}} = 0, \ \ hv_{\boldsymbol{\omega}} = \mathrm{wt}\boldsymbol{\omega}(h)w_{\boldsymbol{\omega}},$$
$$x_i^+ w_{\boldsymbol{\omega}} = 0, \ (x_i^-)^{\mathrm{wt}\boldsymbol{\omega}(h_i)+1} v_{\boldsymbol{\omega}} = 0.$$

Notice that by Theorem 4.1.3 we know that

$$\tilde{W}_{\mathrm{loc}}(\omega_i) \cong W_{\mathrm{loc}}(\boldsymbol{\omega}_{i,a}).$$

Choosing distinct scalars $z_1, \ldots, z_k$ consider the fusion product $\tilde{W}_{\mathrm{loc}}(\omega_{i_1})^{z_1} * \cdots * \tilde{W}_{\mathrm{loc}}(\omega_{i_k})^{z_k}$. It is not too hard to prove that this module is a quotient of $\tilde{W}_{\mathrm{loc}}(\mathrm{wt}\boldsymbol{\omega})$. We get

$$\begin{aligned} \dim \tilde{W}_{\mathrm{loc}}(\mathrm{wt}\boldsymbol{\omega}) &\ge \dim W_{\mathrm{loc}}(\boldsymbol{\omega}) \\ &\ge \prod_{s=1}^{k} \dim \tilde{W}_{\mathrm{loc}}(\omega_{i_s}). \end{aligned}$$

The following result was established in[38] for $\mathfrak{sl}_2$. Using this the result was established in[29] in the case of $\mathfrak{sl}_{n+1}$ where a Gelfand–Tsetlin type basis was also given for $W_{\mathrm{loc}}(\boldsymbol{\omega})$. These bases were further studied in[95,96]. In[57] the theorem was proved for simply-laced Lie algebras. Finally in[87] the result was established for non-simply laced types.

Theorem *We have an isomorphism of* $\mathfrak{g}[t]$ *-modules:*

$$\tilde{W}_{\mathrm{loc}}(\mathrm{wt}\boldsymbol{\omega}) \cong W_{\mathrm{loc}}(\boldsymbol{\omega})$$
$$\cong \tilde{W}_{\mathrm{loc}}(\omega_{i_1})^{z_1} * \cdots * \tilde{W}_{\mathrm{loc}}(\omega_{i_k})^{z_k},$$
$$\mathrm{wt}\boldsymbol{\omega} = \omega_{i_1} + \cdots + \omega_{i_k}.$$

□

Clearly this theorem establishes the conjecture in[21] and also the conjecture of Feigin–Loktev for this particular family of modules.

Remark Although the preceding theorem is uniformly stated the methods of proof are very different. In[29,38] the proof goes by writing down a basis and then doing a dimension count. In[57] the proof proceeds by showing that $\tilde{W}_{\rm loc}(\mathrm{wt}\boldsymbol{\omega})$ is isomorphic to a stable Demazure module in a level one representation of the affine Lie algebra (see Sect. 2.3.2 for the relevant definitions). This isomorphism fails in the non-simply laced case. Instead it is proved in[87] that the module has a flag by stable level one Demazure modules and this plays a key role in the proof. We return to these ideas in the later sections of this paper.

4.1.7 Tensor Products of Kirillov–Reshetikhin Modules

It was proved in[22] that the tensor products of Kirillov–Reshetikhin modules $V_q(\boldsymbol{\omega}_{i_1,s_1,r_1})\otimes\cdots\otimes V_q(\boldsymbol{\omega}_{i_k,s_k,r_k})$ is irreducible as long as s_i-s_p, $1\le i\ne p\le k$ lie outside a finite set. A precise description of this set was also given in that paper when $\mathfrak{g}$ is classical. Set

$$\mathbf{V}=V_q(\boldsymbol{\omega}_{i_1,s_1,r_1})\otimes\cdots\otimes V_q(\boldsymbol{\omega}_{i_k,s_k,r_k}),\;\;\lambda=\sum_{s=1}^k r_s\omega_{i_s}.$$

We now discuss the results of[90] on the structure of $\mathbf{V}_{\rm loc}$. Thus, let $\tilde{\mathbf{V}}_{\rm loc}$ be the $\mathfrak{g}[t]$-module generated by a vector v satisfying the relations:

$$\mathfrak{n}^+[t]v=0=(\mathfrak{h}\otimes t\mathbb{C}[t])v,\quad hv=\lambda(h)v,\;\;h\in\mathfrak{h},$$
$$\left(F_i(z)^r\right)_s v=0,\;\;i\in I,\;r>0,\;s<-\sum_{p:\,i_p=i}\min\{r,r_p\},$$

where $(F_i(z)^r)_s$ denotes the coefficient of z^s in the r-th power of

$$F_i(z)=\sum_{m=0}^{\infty}(x_i^-\otimes t^m)z^{-m-1}\in\mathbf{U}(\mathfrak{g}[t])[[z^{-1}]].$$

The following is the main result of[90].

Theorem *We have an isomorphism of graded* $\mathfrak{g}[t]$*-modules*

$$\mathbf{V}_{\rm loc}\cong V_{\rm loc}(\boldsymbol{\omega}_{i_1,s_1,r_1})^{z_1}*\cdots*V_{\rm loc}(\boldsymbol{\omega}_{i_k,s_k,r_k})^{z_k}\cong\tilde{\mathbf{V}}_{\rm loc}.$$

□

Again, the conjecture of Feigin–Loktev for this family of modules is a consequence of this presentation. The proof of the Feigin–Loktev conjecture when $r_1=r_2=\cdots=r_k$ and $\mathfrak{g}$ simply-laced was proved earlier in[57] by identifying the fusion product with a $\mathfrak{g}$-stable Demazure module. In general the connection with Demazure modules or the existence of a Demazure flag (as in the case of local Weyl modules) is not known.

4.1.8 Monoidal Categorification and HL-Modules

Our final example of graded limits comes from the work of David Hernandez and Bernard Leclerc on monoidal categorification of cluster algebras. We refer the reader to[102] for a quick introduction to cluster algebras. For the purposes of this article it is enough for us to recall that a cluster algebra is a commutative ring with certain distinguished generators called *cluster variables* and certain algebraically independent subsets of cluster variables called *clusters*. Monomials in the cluster variables belonging to a cluster are called *cluster monomials*. There is also an operation called mutation; this is a way to produce a new cluster by replacing exactly one element of the original cluster by another cluster variable.

The remarkable insight of Hernandez–Leclerc was to relate these ideas to the representation theory of quantum affine algebras associated to simply-laced Lie algebras. Broadly speaking they prove that the Grothendieck ring of a suitable tensor subcategory admits the structure of a cluster algebra. A cluster variable is a prime real representation in this category (see Sect. 3.2.2 for the definitions) and we call these the HL-modules. Suppose that V,V' are irreducible modules in this subcategory. Assume that their isomorphism classes correspond to cluster variables which belong to the same cluster. Then $V\otimes V'$ is an irreducible module. The operation of mutation in this language corresponds to the Jordan–Holder decomposition of the corresponding tensor product.

We now give one specific example of their work and relate it to our study of graded limits. We assume that $\mathfrak{g}$ is of type A_n. Let $\kappa:\{1,\dots,n\}\to\mathbb{Z}$ be a height function; namely it satisfies $|\kappa(i+1)-\kappa(i)|=1$ for $1\le i\le n$. Let $\mathcal{P}_\kappa^+$ be the submonoid of $\mathcal{P}_q^+$ generated by elements $\boldsymbol{\omega}_{i,q^{\kappa(i)\pm1}}$, $i\in I$. Let $\mathcal{F}_\kappa$ be the full subcategory of $\mathcal{F}_q$ consisting of finite-dimensional $\mathbf{U}_q(L(\mathfrak{g}))$-modules whose Jordan–Holder constituents are isomorphic to $V_q(\boldsymbol{\omega})$ for some $\boldsymbol{\omega}\in\mathcal{P}_\kappa^+$. It was shown in[66] that $\mathcal{F}_\kappa$ is closed under taking tensor products and that its Grothendieck ring has the structure of a cluster algebra of type A_n. The following result was proved in[66] when $\kappa(i)=i \mod 2$, in[67] when $\kappa(i)=i$ and in complete generality in[16].

Theorem *Suppose that* $V_q(\boldsymbol{\omega})$ *is a prime real object of* $\mathcal{F}_\kappa$*. Then* $\boldsymbol{\omega}$ *must be one of the following:*

$$\boldsymbol{\omega}_{i,q^{\kappa(i)\pm1}}, \boldsymbol{\omega}_{i,q^{\kappa(i)+1}}\boldsymbol{\omega}_{i,q^{\kappa(i)-1}}, i \in I,$$
$$\boldsymbol{\omega}_{i,a_1}\boldsymbol{\omega}_{i_2,a_2}\cdots\boldsymbol{\omega}_{i_{k-1},a_{k-1}}\boldsymbol{\omega}_{j,a_k},\ 1 \le i < j \le n,$$

where $i_2 < \cdots < i_{k-1}$ is an ordered enumeration of $\{p : i < p < j,\ \kappa(p-1) = \kappa(p+1)\}$ and $a_1 = q^{\kappa(i)\pm1}$ if $\kappa(i+1) = \kappa(i) \mp 1$ and $a_s = q^{\kappa(i_s)\pm1}$ if $\kappa(i_s) = \kappa(i_s - 1) \pm 1$ for $s \ge 2$. Conversely the irreducible representation associated to any $\boldsymbol{\omega}$ as above is a real prime object of $\mathcal{F}_\kappa$. □

4.1.9 Graded Limits of HL-Modules in $\mathcal{F}_\kappa$

Continue to assume that $\mathfrak{g}$ is of type A_n and for $1 \le i \le j \le n$ set $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j \in R^+$. It follows from the discussion in Sect. 3.2.5 that $V_{\rm loc}(\boldsymbol{\omega}_{i,q^{\kappa(i)\pm1}}) \cong W_{\rm loc}(\boldsymbol{\omega}_{i,q^{\kappa(i)\pm1}})$. The discussion in Sect. 4.1.3 gives a presentation for $V_{\rm loc}(\boldsymbol{\omega}_{i,q^{\kappa(i)+1}}\boldsymbol{\omega}_{i,q^{\kappa(i)-1}})$ since this is a special example of a Kirillov–Reshetikhin module. The following was proved in[17] and shows that the graded limits of HL-modules are indeed graded.

Theorem *Suppose that $\mathfrak{g}$ is of type A_n and $\boldsymbol{\omega} = \boldsymbol{\omega}_{i,a_1}\cdots\boldsymbol{\omega}_{j,a_k} \in \mathcal{P}^+$ is as in Theorem 4.1.8. Then $V_{\rm loc}(\boldsymbol{\omega})$ is the quotient of $W_{\rm loc}(\boldsymbol{\omega})$ by the submodule generated by the additional relations:*

$$(x_\alpha^- \otimes t)w_{\boldsymbol{\omega}} = 0,\ \ \alpha \in \{\alpha_{i,i_2}, \alpha_{i_2,i_3}, \cdots, \alpha_{i_{k-1},j}\}.$$

□

We remark that the result in[17] is more general in the sense that it gives a presentation of the graded limit of the tensor product of an HL-module with the Kirillov–Reshetikhin modules in this category. Here again the result shows that tensor products specialize to fusion products. A problem that has not been studied so far is to understand the graded limit of a tensor product of $V_q(\boldsymbol{\omega}) \otimes V_q(\boldsymbol{\omega}')$ for an arbitrary pair $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{P}_\kappa^+$ and the connection with the fusion product of the graded limits of $V_q(\boldsymbol{\omega})$ and $V_q(\boldsymbol{\omega}')$.

The graded characters of the limits of HL-modules have been studied in[4,14] in different ways. In the first paper a character formula was given as an explicit linear combination of Macdonald polynomials. In[4] the authors studied the $\mathfrak{g}$-module decomposition of the graded limit. The multiplicity of a particular $\mathfrak{g}$-type is given by the number of certain lattice points in a convex polytope. Moreover, considering a particular face of that polytope encodes the graded multiplicity.

A comparable study of HL-modules in other types is only partially explored. A first step was taken in[23] in type D_n but it does not capture all the prime objects in the category $\mathcal{F}_\kappa$. There are important differences from the A_n case and some new ideas seem to be necessary.

4.1.10 Further Remarks

As we said, there are other subcategories of representations of $\mathcal{F}_q$ which were shown by Hernandez-Leclerc to be monoidal categorifications of (infinite rank) cluster algebras. However, it is far from clear what subset of $\mathcal{P}_q^+$ is an index set for the prime representations corresponding to the cluster variables. Hence little is known about the characters or the graded limits of these representations.

Another example of prime representations comes from the theory of snake modules studied in types A_n and B_n in[18,86]. Again the problem of studying the graded limits of these modules is wide open.

5 Demazure Modules, Projective Modules and Global Weyl Modules

Our focus in this section will be on the study of graded representations of $\mathfrak{g}[t]$. We begin by establishing the correct category $\mathcal{G}$ of representations of the current algebra and introduce the projective objects and the global Weyl modules. We then relate the study of local Weyl modules in Sect. 4 to the $\mathfrak{g}$-stable Demazure modules introduced in Sect. 2.3.3. Next we discuss the characters of the local Weyl modules and relate them to Macdonald polynomials. Finally, we discuss BGG-type reciprocity results. We conclude the section with some comments on the more recent work of[45,46,51,73].

5.1 The Category $\mathcal{G}$

The study of this category was initiated in[25] and we recall several ideas from that paper. Recall from Sect. 4.1.2 that we have a $\mathbb{Z}_+$-grading on $\mathfrak{g}[t]$ and its universal enveloping algebra. Define $\mathcal{G}$ to be the category whose objects are $\mathbb{Z}$-graded representations $V = \oplus_{m\in\mathbb{Z}} V[m]$ of $\mathfrak{g}[t]$ with $\dim V[m] < \infty$ for all $m \in \mathbb{Z}$. The morphisms in the category are grade preserving maps of $\mathfrak{g}[t]$ -modules.

Define the restricted dual of an object V in $\mathcal{G}$ by

$$V^* = \bigoplus_{m\in\mathbb{Z}} V^*[m],\quad V^*[m] = V[-m]^*.$$

Clearly V^* is again an object of $\mathcal{G}$.

For any object V of $\mathcal{G}$, each graded subspace $V[m]$ is a finite-dimensional $\mathfrak{g}$-module and we define the graded $\mathfrak{g}$-character of V to be the element of $\mathbb{Z}[P][[q^{\pm 1}]]$:

$$\operatorname{ch}_{\rm gr} V = \sum_{\lambda\in P^+}\sum_{m\in\mathbb{Z}} \dim \operatorname{Hom}_{\mathfrak{g}}(V(\lambda),$$
$$V[m])q^m \operatorname{ch} V(\lambda) = \sum_{\mu\in P}\sum_{m\in\mathbb{Z}} \dim V[m]_\mu q^m e_\mu$$
$$= \sum_{\mu\in P} p_\mu(q) e_\mu, p_\mu(q)\in\mathbb{Z}_+[[q^{\pm 1}]].$$

It is clear that for all $r\in\mathbb{Z}$ we have $\operatorname{ch}_{\rm gr}(\tau_r V) = q^r \operatorname{ch}_{\rm gr} V$, where τ_r is as defined in Sect. 2.3.4).

Finally, note that $\mathcal{G}$ is an abelian category and is closed under taking restricted duals. If V and V' are objects of $\mathcal{G}$ then $V\otimes V'$ is again an object in $\mathcal{G}$ if $\dim V<\infty$.

5.1.1 Finite-Dimensional Objects of $\mathcal{G}$

It is straightforward that if V is a simple object of $\mathcal{G}$, then V is concentrated in a single grade. In particular V must be a finite-dimensional irreducible $\mathfrak{g}$-module. In other words $V\cong \tau_m \mathrm{ev}_0 V(\lambda)$ for some $\lambda\in P^+$ where ev_0 is the evaluation $\mathfrak{g}[t]\to\mathfrak{g}, x\otimes t^r\mapsto \delta_{0,r}x$. From now we set

$$V(\lambda,m)=\tau_m \mathrm{ev}_0 V(\lambda).$$

Another example of finite-dimensional modules in $\mathcal{G}$ are the $\mathfrak{g}$-stable Demazure modules $V_w(\lambda)$, $\lambda\in\widehat{P}^+$ (see Sect. 2.3.3) and the local Weyl modules studied in Sect. 4.1.6. We give a direct definition of those objects as $\mathfrak{g}[t]$-modules here for the reader's convenience, and we also drop the ˜ for ease of notation.

Given $\lambda\in P^+$ the local Weyl module $W_{\rm loc}(\lambda)$ is the $\mathfrak{g}[t]$-module generated by an element v_λ and relations:

$$x_i^+ v_\lambda = 0,\ \ (h\otimes t^r)v_\lambda = \delta_{r,0}\lambda(h)v_\lambda,$$
$$(x_i^-)^{\lambda(h_i)+1}v_\lambda = 0,\ \ i\in I,\ \ h\in\mathfrak{h}.$$

Setting $\mathrm{gr} v_\lambda = r$ we see that $W_{\rm loc}(\lambda)$ can be regarded as an object of $\mathcal{G}$ and we denote this as $W_{\rm loc}(\lambda,r)$. Clearly $W_{\rm loc}(\lambda,r)=\tau_r W_{\rm loc}(\lambda,0)$. It was proved in[38] that the local Weyl modules are finite-dimensional with unique irreducible quotients.

Given $\mu\in P^+$ and $\ell\in\mathbb{N}$, let $D(\ell,\mu)$ be the quotient of $W_{\rm loc}(\mu)$ by the submodule generated by elements

$$(x_\alpha^-\otimes t^{s_\alpha-1})^{m_\alpha+1}v_\mu,\ \text{ if } m_\alpha<d_\alpha\ell,$$
$$(x_\alpha^-\otimes t^{s_\alpha})v_\mu,\ \ \alpha\in R^+$$

where s_α and m_α are determined by

$$\mu(h_\alpha)=(s_\alpha-1)d_\alpha\ell+m_\alpha,\ \ 0<m_\alpha\le d_\alpha\ell.$$

The following was proved in[41].

Proposition *Suppose that $\lambda\in\widehat{P}^+$ and $w\in\widehat{W}$ is such that $w\lambda(h_i)\le 0$ for all $i\in I$ and assume that $\lambda(c)=\ell$. The module $V_w(\lambda)$ is isomorphic to $\tau_r D(\ell,\mu)$ where $\mu\in P^+$ is given by $\mu(h_i)=-w_\circ w\lambda(h_i)$ and $r=w\lambda(d)$.* □

Remark An analogous presentation of non-stable Demazure modules is given in[79] and we discuss this in the next section. These modules however are not objects of $\mathcal{G}$. □

5.1.2 Relation Between Local Weyl and Demazure Modules

The following corollary of Proposition 5.1.1 is easily established.

Corollary *If $\mathfrak{g}$ is simply-laced then*

$$D(1,\mu)\cong W_{\rm loc}(\mu),\ \ \mu\in P^+.$$

Proof It suffices to prove that the following relation holds in $W_{\rm loc}(\mu)$:

$$(x_\alpha^-\otimes t^{\mu(h_\alpha)})v_\mu = 0,\ \ \alpha\in R^+.$$

But this follows by using

$$(x_\alpha^-\otimes t^{\mu(h_\alpha)})v_\mu = (x_\alpha^+\otimes t)^{\mu(h_\alpha)}(x_\alpha^-)^{\mu(h_\alpha)+1}v_\mu = 0.$$

Here the first equality is established by a simple calculation and using the relations in $W_{\rm loc}(\mu)$. The second equality holds since $W_{\rm loc}(\mu)$ is finite-dimensional and

$$x_\alpha^+ v_\mu = 0 \implies (x_\alpha^-)^{\mu(h_\alpha)+1}v_\mu = 0.$$

□

In the non-simply laced case it is not true in general that $W_{\rm loc}(\mu)\cong D(1,\mu)$. However it was proved in[87] that $W_{\rm loc}(\mu)$ admits a decreasing filtration where the successive quotients are isomorphic to $\tau_r D(1,\mu_r)$ for some $r\in\mathbb{Z}$ and $\mu_r\in P^+$. One can make a more precise statement which can be found in Sect. 6.2.

5.1.3 Projective Modules and Global Weyl

Modules

Given $(\lambda, r) \in P^+ \times \mathbb{Z}$ set

$$P(\lambda, r) = \mathbf{U}(\mathfrak{g}[t]) \otimes_{\mathbf{U}(\mathfrak{g})} V(\lambda, r).$$

It is not hard to check that $P(\lambda, r)$ is an indecomposable projective object of $\mathcal{G}$ and that there exists a surjective map $P(\lambda, r) \to V(\lambda, r) \to 0$ of $\mathfrak{g}[t]$-modules. Equivalently $P(\lambda, r)$ is the $\mathfrak{g}[t]$-module generated by an element v_λ of grade r subject to the relations:

$$x_i^+ v_\lambda = 0, \quad hv_\lambda = \lambda(h)v_\lambda,$$
$$(x_i^-)^{\lambda(h_i)+1} v_\lambda = 0, \quad i \in I, h \in \mathfrak{h}.$$

The global Weyl module $W(\lambda, r)$ is the maximal quotient of $P(\lambda, r)$ such that $\mathrm{wt}\, W(\lambda, r) \subseteq \lambda - Q^+$. Equivalently it is the quotient of $P(\lambda, r)$ obtained by imposing the additional relations $(x_i^+ \otimes t^k)v_\lambda = 0$ for all $i \in I$ and $k \geq 0$. Clearly we have the following sequence of surjective maps

$$P(\lambda, r) \twoheadrightarrow W(\lambda, r) \twoheadrightarrow W_{\mathrm{loc}}(\lambda, r) \twoheadrightarrow V(\lambda, r).$$

5.1.4 The Algebra $\mathbb{A}_\lambda$ and the Bimodule Structure on $W(\lambda, r)$

Let $\mathfrak{h}[t]_+ = \mathfrak{h} \otimes t\mathbb{C}[t]$ and for $\lambda \in P^+$ and $v_\lambda \in W(\lambda, r)_\lambda$ non-zero of grade r let

$$\mathbb{I}_\lambda = \{u \in \mathbf{U}(\mathfrak{h}[t]_+) : uv_\lambda = 0\},$$
$$\mathbb{A}_\lambda = \mathbf{U}(\mathfrak{h}[t]_+)/\mathbb{I}_\lambda.$$

Clearly $\mathbb{A}_\lambda$ is commutative and graded. Moreover $W(\lambda, r)$ is a $(\mathfrak{g}[t], \mathbb{A}_\lambda)$-bimodule where the right action of $\mathbb{A}_\lambda$ is given by:

$$(gv_\lambda)a = gav_\lambda, \quad g \in \mathbf{U}(\mathfrak{g}[t]), \quad a \in \mathbb{A}_\lambda.$$

To see that the action is well-defined, one must prove that

$$(\mathfrak{n}^+ \otimes \mathbb{C}[t])(h \otimes f)v_\lambda = 0,$$
$$(h' - \lambda(h'))(h \otimes f)v_\lambda = 0,$$
$$(x_i^-)^{\lambda(h_i)+1}(h \otimes f)v_\lambda = 0$$

for all $i \in I$, $h, h' \in \mathfrak{h}$ and $f \in \mathbb{C}[t]$. However, all relations are immediate to check. It was proved in[38] (for the loop algebra; the proof is essentially the same for the current algebra) that $\mathbb{A}_\lambda$ can be realized as a ring of invariants as follows. Consider the polynomial ring $\mathbb{C}[x_{i,r} : i \in I, 1 \leq r \leq \lambda(h_i)]$. The direct product of symmetric groups

$$\mathcal{S}_\lambda = S_{\lambda(h_1)} \times \cdots \times S_{\lambda(h_n)}$$

acts on this ring in an obvious way and we have

$$\mathbb{A}_\lambda \cong \mathbb{C}[x_{i,r} : i \in I, 1 \leq r \leq \lambda(h_i)]^{\mathcal{S}_\lambda}.$$

The grading on $\mathbb{A}_\lambda$ is given by requiring the grade of $x_{i,r}$ being r. Let $\mathbf{I}_\lambda$ be the maximal graded ideal in $\mathbb{A}_\lambda$. The local Weyl module can then be realized as follows:

$$W_{\mathrm{loc}}(\lambda, r) = W(\lambda, r) \otimes_{\mathbb{A}_\lambda} \mathbb{A}_\lambda/\mathbf{I}_\lambda.$$

A nontrivial consequence of the dimension conjecture discussed in Sect. 4.1.6 (see[24,38] for more details) is the following result.

Proposition *The global Weyl module $W(\lambda)$ is a free $\mathbb{A}_\lambda$-module of rank equal to the dimension of $W_{\mathrm{loc}}(\lambda)$.* □

The algebra $\mathbb{A}_\lambda$ plays an important role in the rest of the section.

5.2 The Category $\mathcal{O}$ for $\mathfrak{g}$

Before continuing our study of the category $\mathcal{G}$, we discuss briefly, the resemblance of the theory with that of the well-known category $\mathcal{O}$ for semi-simple Lie algebras.

The objects of $\mathcal{O}$ are finitely generated weight modules (with finite-dimensional weight spaces) for $\mathfrak{g}$ which are locally nilpotent for the action of $\mathfrak{n}^+$. The morphisms are just $\mathfrak{g}$-module maps. Given $\lambda \in \mathfrak{h}^*$ one can associate to it a Verma module $M(\lambda)$ which is defined as

$$M(\lambda) = \mathbf{U}(\mathfrak{g}) \otimes_{\mathbf{U}(\mathfrak{b})} \mathbb{C}v_\lambda,$$

where $\mathbb{C}v_\lambda$ is the one-dimensional $\mathfrak{b}$-module given by $hv_\lambda = \lambda(h)v_\lambda$ and $\mathfrak{n}^+ v_\lambda = 0$. It is not hard to prove that $M(\lambda)$ is infinite-dimensional and has a unique irreducible quotient denoted by $V(\lambda)$ and any irreducible object in $\mathcal{O}$ is isomorphic to some $V(\lambda)$. Moreover $V(\lambda)$ is finite-dimensional if and only if $\lambda \in P^+$; in particular $M(\lambda)$ is reducible if $\lambda \in P^+$.

The modules $M(\lambda)$ have finite length and the multiplicity of $V(\mu)$ in the Jordan–Hölder series of $M(\lambda)$ is denoted by $[M(\lambda) : V(\mu)]$. The study of these multiplicities has been of great interest and there is extensive literature on the subject. Perhaps the starting point for this study is the famous result of Bernstein–Gelfand–Gelfand (BGG) which we now recall.

The category $\mathcal{O}$ has enough projectives, which means that for $\lambda \in \mathfrak{h}^*$ there exists an indecomposable module $P(\lambda)$ which is projective in $\mathcal{O}$ and we have surjective maps

$$P(\lambda) \twoheadrightarrow M(\lambda) \twoheadrightarrow V(\lambda).$$

The following theorem (known as BGG-reciprocity) was proved in[10].

Theorem *Given $\lambda_0 \in \mathfrak{h}^*$ there exist $\lambda_1, \dots, \lambda_r \in \mathfrak{h}^*$ such that the module $P(\lambda_0)$ has a decreasing filtration $P_0 = P(\lambda_0) \supseteq P_1 \supseteq P_2 \supseteq \cdots \supseteq P_r \supseteq P_{r+1} = \{0\}$, and*

$$P_i/P_{i+1} \cong M(\lambda_i), \;\; 0 \le i \le r.$$

Moreover if we let $[P(\lambda) : M(\mu)]$ be the multiplicity of $M(\mu)$ in this filtration then we have $[P(\lambda) : M(\mu)] = [M(\mu) : V(\lambda)]$. □

Remark Although the filtration is not unique in general, a comparison of formal characters shows that the filtration length and the multiplicity $[P(\lambda), M(\mu)]$ (see[69], Section 3.7]) is independent of the choice of the filtration.

More generally a module in $\mathcal{O}$ which admits a decreasing sequence of submodules where the successive quotients are Verma modules is said to admit a standard filtration. In the rest of this section we shall discuss an analog of this result for current algebras.

We will also explore other ideas stemming from the formal similarity between $\mathcal{O}$ and $\mathcal{G}$. For instance it is known that $\dim \operatorname{Hom}_{\mathfrak{g}}(M(\lambda), M(\mu)) \le 1$ and that any nonzero map between Verma modules is injective and we shall discuss its analog for current algebras. We shall also discuss an analog of tilting modules; in the category $\mathcal{O}$ these are defined to be modules which admit a filtration where the successive quotients are Verma modules and also a filtration where the successive quotients are the restricted duals of Verma modules. It is known that for each $\lambda \in \mathfrak{h}^*$ there exists a unique indecomposable tilting module which contains a copy of $M(\lambda)$.

5.3 BGG Reciprocity in $\mathcal{G}$

In the category $\mathcal{G}$ the role of the Verma module is played by the global Weyl module. However, in general the global Weyl module $W(\lambda, r)$, $\lambda \in P^+$ does not have a unique finite-dimensional quotient in $\mathcal{G}$; for instance the modules $W_{\text{loc}}(\lambda, r)$ and $V(\lambda, r)$ are usually not isomorphic and we have

$$W(\lambda, r) \twoheadrightarrow W_{\text{loc}}(\lambda, r) \twoheadrightarrow V(\lambda, r).$$

However both quotients have a uniqueness property; $W_{\text{loc}}(\lambda, r)$ is unique in the sense that any finite-dimensional quotient of $W(\lambda, r)$ is actually a quotient of $W_{\text{loc}}(\lambda, r)$ and $V(\lambda, r)$ is the unique irreducible quotient of $W(\lambda, r)$. The further difference from the category $\mathcal{O}$ situation is that the global Weyl module is not of finite length. In spite of these differences, one is still able to formulate the appropriate version of BGG-reciprocity. Such a formulation was first conjectured in[9] and proved there for $\mathfrak{sl}_2[t]$. The result was proved in complete generality in[26] for twisted and untwisted current algebras; as usual the case of $A_{2n}^{(2)}$ is much more difficult and one has to work with the hyperspecial current algebra. A key ingredient in the proof is to relate the character of the local Weyl module to specializations of (non)symmetric Macdonald polynomials (see Sect. 5.4.1 for a brief review).

The following is the main result of[26].

Theorem *Let $(\lambda, r) \in P^+ \times \mathbb{Z}_+$. The module $P(\lambda, r)$ admits a decreasing series of submodules: $P_0 = P(\lambda, r) \supseteq P_1 \supseteq P_2 \supseteq \cdots$ such that*

$$P_i/P_{i+1} \cong W(\mu_i, s_i), \;\; \text{for some } \; (\mu_i, s_i) \in P^+ \times \mathbb{Z}_+,$$

and

$$[P(\lambda, r) : W(\mu_i, s_i)] = [W_{\text{loc}}(\mu_i, s_i) : V(\lambda, r)].$$

□

5.3.1 Tilting Modules

We discuss the construction of tilting modules and some of their properties. These ideas were developed in[5–7] and one works in a suitable subcategory of $\mathcal{G}$. Thus, let $\mathcal{G}_{\text{bdd}}$ be the full subcategory of objects M of $\mathcal{G}$ such that $M[j] = 0$ for all $j \gg 0$ and

$$\operatorname{wt}(M) \subseteq \bigcup_{i=1}^{s} \operatorname{conv} W\mu_i, \;\; \mu_1, \dots, \mu_s \in P^+$$

where conv $W\mu$ denotes the convex hull of the Weyl group orbit $W\mu$. An object M in the category $\mathcal{G}_{\text{bdd}}$ is called *tilting* if it admits two increasing filtrations:

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M_r \subseteq \cdots,$$
$$M^0 \subseteq M^1 \subseteq \cdots \subseteq M^r \subseteq \cdots$$
$$M = \bigcup_{r \ge 0} M_r = \bigcup_{r \ge 0} M^r,$$

such that M_{i+1}/M_i (resp. M^{i+1}/M^i) is isomorphic to a finite direct sum of modules of the form

$W_{\mathrm{loc}}(\lambda, r)$ (resp. to a sum of dual global Weyl modules $W(\lambda, r)^*$) where $(\lambda, r) \in P^+ \times \mathbb{Z}$. One can also work with a dual definition of tilting modules, where one requires that the module has decreasing filtrations and the successive quotients are isomorphic to the dual local Weyl modules and the global Weyl modules respectively.

The following was proved in[6], Section 2].

Theorem *For $(\lambda, r) \in P^+ \times \mathbb{Z}$ there exists an indecomposable tilting module $T(\lambda, r)$ in $\mathcal{G}_{\mathrm{bdd}}$ which maps onto the local Weyl module $W_{\mathrm{loc}}(\lambda, r)$ and such that*

$$\tau_r T(\lambda, 0) = T(\lambda, r),$$
$$T(\lambda, r) \cong T(\mu, s) \Leftrightarrow (\lambda, r) = (\mu, s).$$

Any indecomposable tilting module in $\mathcal{G}_{\mathrm{bdd}}$ is isomorphic to $T(\mu, s)$ for some $(\mu, s) \in P^+ \times \mathbb{Z}$ and any tilting module in $\mathcal{G}_{\mathrm{bdd}}$ is isomorphic to a direct sum of indecomposable tilting modules. □

The proof of the theorem relies on the following necessary and sufficient condition for an object of $\mathcal{G}_{\mathrm{bdd}}$ to admit a filtration by dual global Weyl modules. Namely:

M admits a filtration by costandard modules if and only if $\mathrm{Ext}^1_{\mathcal{G}}(W_{\mathrm{loc}}(\mu, s), M) = 0$, $\forall (\mu, s) \in P^+ \times \mathbb{Z}$.

Remark This equivalence was established in[6] in the case when $\mathfrak{g}$ is of type A. This is because the proof depended on knowing Theorem 5.3 which at that time had only been proved when $\mathfrak{g}$ is of type A. However the proof given there goes through verbatim for any $\mathfrak{g}$.

5.3.2 Tilting Modules for $\mathfrak{sl}_2$ and in Serre Subcategories

The existence of tilting modules is proved in a very abstract way. In[7] an explicit realization of the dual modules was given in the case of $\mathfrak{sl}_2$. In this case we identify P^+ with $\mathbb{Z}_+$. Recall also the algebra $\mathbb{A}_\lambda$ defined in Sect. 5.1.4; in this special case it is just the ring of symmetric polynomials in λ-variables.

Theorem *Suppose that $\mathfrak{g} = \mathfrak{sl}_2$ and let $(\lambda, r) \in \mathbb{Z}_+ \times \mathbb{Z}_+$. The module $T(\lambda, r)^*$ is a free right $\mathbb{A}_\lambda$-module and $T(\lambda, r) \cong \tau_r\, T(\lambda, 0)$. Moreover,*

$$T(1, 0)^* \cong W(1, 0), \quad T(\lambda, 0)^* \cong \tau_{-r_\lambda}$$
$$\bigwedge^{\lambda} W(1, 0), \quad r_\lambda = \binom{\lambda}{2}, \quad \lambda \geq 2.$$
$$\mathrm{ch}_{\mathrm{gr}} T(\lambda, 0)^* = \sum_{s=0}^{\lfloor \frac{\lambda}{2} \rfloor} t^{s(s-\lambda)} (1:t)_s \mathrm{ch}_{\mathrm{gr}}$$
$$W(\lambda - 2s, 0) = t^{r_\lambda} (1:t)_\lambda \mathrm{ch}_{\mathrm{gr}} W_{\mathrm{loc}}(\lambda, r_\lambda)^*$$

where

$$(1:t)_n = \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)}$$

□

Very little is known about the structure or the character of the tilting modules in general.

A theory of tilting modules was also developed for Serre subcategories of $\mathcal{G}$ which are defined as follows. Given a subset $\Gamma \subseteq P^+ \times \mathbb{Z}$, we define a full subcategory $\mathcal{G}(\Gamma)$ whose objects M satisfy additionally

$$[M : V(\lambda, r)] \neq 0 \Rightarrow (\lambda, r) \in \Gamma.$$

The category $\mathcal{G}(\Gamma)_{\mathrm{bdd}}$ is now defined in an obvious way. If $\Gamma = P^+ \times J$, where J is an (possibly infinite) interval in $\mathbb{Z}$, then the existence of tilting modules holds with $\mathcal{G}_{\mathrm{bdd}}$ and $P^+ \times \mathbb{Z}$ replaced by $\mathcal{G}(\Gamma)_{\mathrm{bdd}}$ and Γ respectively (see[5], Proposition 4.2 and Theorem 4.3]). The local and dual global Weyl modules in this setting are obtained by applying a certain natural functor to the standard and costandard modules in $\mathcal{G}$.

5.3.3 Socle and Radical Filtration for Local Weyl Modules

The local Weyl module $W_{\mathrm{loc}}(\lambda)$ has a natural increasing grading filtration induced from its graded module structure. This filtration coincides with the radical filtration (see[76], Proposition 3.5]) which is defined as follows. For a module M of $\mathbf{U}(\mathfrak{g}[t])$ the radical filtration is given by

$$\cdots \subseteq \mathrm{rad}^k(M) \subseteq \cdots \subseteq \mathrm{rad}^1(M) \subseteq \mathrm{rad}^0(M) = M$$

where $\mathrm{rad}(M)$ is the smallest submodule of M such that the quotient $M/\mathrm{rad}(M)$ is semi-simple and $\mathrm{rad}^k(M)$ is defined inductively by

$$\mathrm{rad}^k(M) = \mathrm{rad}(\mathrm{rad}^{k-1}(M)).$$

In particular,

$$\mathrm{rad}^1(W_{\mathrm{loc}}(\lambda)) = \bigoplus_{s>0} \mathbf{U}(\mathfrak{g}[t])[s] v_\lambda.$$

There is another natural filtration on a module M, called the socle filtration. It is given as follows

$$0 = \mathrm{soc}^0(M) \subseteq \mathrm{soc}^1(M) \subseteq \cdots \subseteq \mathrm{soc}^k(M) \subseteq \cdots,$$

where $\mathrm{soc}(M) = \mathrm{soc}^1(M)$ is the largest semi-simple submodule of M and $\mathrm{soc}^k(M)$ is defined inductively by

$$\mathrm{soc}^k(M)/\mathrm{soc}^{k-1}(M) = \mathrm{soc}(M/\mathrm{soc}^{k-1}(M)).$$

A module M of $\mathbf{U}(\mathfrak{g}[t])$ is called *rigid* if the socle filtration coincides with the radical filtration. This is in particular the case, if M is a finite-dimensional graded module such that $M/\mathrm{rad}(M)$ and $\mathrm{soc}(M)$ are both simple. We remind the reader that when $\mathfrak{g}$ is of type *ADE* the local Weyl module is isomorphic to a level one Demazure module and hence embeds in a highest weight module for the affine Lie algebra. Given $\lambda \in P^+$, let $w \in \widehat{W}$ be such that $\Lambda = w^{-1}(w_0\lambda + \Lambda_0) \in \widehat{P}^+$. Since $V(\Lambda)$ is an irreducible integrable module for $\widehat{\mathfrak{g}}$, it follows that any $\widehat{\mathfrak{b}}$-submodule of $M = V_w(\Lambda)$ must contain the highest weight vector v_Λ. Hence any $\mathfrak{g}[t]$-submodule of M contains the $\mathfrak{g}[t]$-module $U(\mathfrak{g}[t])v_\Lambda$. In other words $\mathrm{soc}(M)$ must be simple and we must have $\mathrm{soc}(M) = U(\mathfrak{g}[t])v_\Lambda \cong V(\Lambda|_{\mathfrak{h}}, 0)$. So, we get

Lemma Let $\mathfrak{g}$ be of type *ADE*. Then $\mathrm{soc}(W_{\mathrm{loc}}(\lambda)) \cong V(\Lambda|_{\mathfrak{h}}, 0)$. □

It was proved in[76] that when $\mathfrak{g}$ is of type *ADE* the local Weyl module is rigid. However, in general the socle of the local Weyl module is not simple and we give the counterexample given in[76], Example 3.12]. In type C_2, the socle of the local Weyl module $W_{\mathrm{loc}}(2\omega_1 + \omega_2)$ (the short root is α_1) is isomorphic to $V(0,3) \oplus V(\omega_2, 2)$.

5.3.4 Maps Between Local Weyl Modules

We apply the discussion on the socle of $W_{\mathrm{loc}}(\lambda, r)$ to study morphisms between local Weyl modules when $\mathfrak{g}$ is of type *ADE*. For the purposes of this section it will be convenient to think of $W_{\mathrm{loc}}(\lambda, r)$ as modules for the subalgebra $\mathfrak{g}[t] \oplus \mathbb{C}c \oplus \mathbb{C}d$ of $\widehat{\mathfrak{g}}$ where we let c act as 1 and the action of d is given by the grading. Recall the Bruhat order on $\widehat{W}$ given by $u \le w$ if some substring of some reduced word for w is a reduced word for u.

Proposition *Assume that $\mathfrak{g}$ is of type ADE and let $(\lambda, r), (\mu, s) \in P^+ \times \mathbb{Z}$. Then,*

$$\dim \mathrm{Hom}_{\mathcal{G}}(W_{\mathrm{loc}}(\lambda, r), W_{\mathrm{loc}}(\mu, s)) \le 1$$

with equality holding if and only if there exist $w_1, w_2 \in \widehat{W}$ and $\Lambda \in \widehat{P}^+$ such that the following hold:

$$w_2 \le w_1,$$
$$w_1(\lambda + \Lambda_0 + r\delta) = \Lambda = w_2(\mu + \Lambda_0 + s\delta).$$

Moreover, any non-zero map between local Weyl modules is injective.

Proof Let $\varphi : W_{\mathrm{loc}}(\lambda, r) \to W_{\mathrm{loc}}(\mu, s)$ be a non-zero homomorphism. We first prove that this implies that there exist $w_1, w_2 \in \widehat{W}$ and $\Lambda \in \widehat{P}^+$ with $w_1(\lambda + \Lambda_0 + r\delta) = \Lambda = w_2(\mu + \Lambda_0 + s\delta)$. To see this assume that $w_1(\lambda + \Lambda_0 + r\delta) = \Lambda$ and $w_2(\mu + \Lambda_0 + s\delta) = \Lambda'$ with $\Lambda, \Lambda' \in \widehat{P}^+$ and let $W_{\mathrm{loc}}(\mu, s) \hookrightarrow V(\Lambda')$ be the inclusion which exists since $W_{\mathrm{loc}}(\mu, s)$ is isomorphic to a stable Demazure module in $V(\Lambda')$. Since the image of φ is non-zero it must include the simple socle of $W_{\mathrm{loc}}(\mu, s)$. This in turn implies that Λ' is a weight of $W_{\mathrm{loc}}(\lambda, r) \hookrightarrow V(\Lambda)$. It follows that $\Lambda - \Lambda'$ must be a sum of affine positive roots. On the other hand since $\varphi(w_\lambda) \neq 0$ it follows that $\lambda + \Lambda_0 + r\delta$ must be a weight of $V(\Lambda')$ and hence Λ is also a weight of $V(\Lambda')$. This forces $\Lambda' - \Lambda$ to be a sum of positive affine roots and also shows that $\Lambda = \Lambda'$. To see that φ is injective, we note that otherwise both the kernel and the cokernel of φ would have to contain v_Λ which is absurd.

Finally to see that the dimension of the homomorphism space is at most one, it suffices to note that $\dim V(\Lambda)_{w\Lambda} = 1$ for all $w \in \widehat{W}$. □

5.3.5 Morphisms Between Global Weyl Modules

The study of the homomorphism space between global Weyl modules has also been studied in[8] and confirms further the phenomenon that the global Weyl module plays a role similar to that of the Verma modules in category $\mathcal{O}$. The following result can be found in[8], Theorem 3].

Theorem *Let $\lambda, \mu \in P^+$ and assume that $\mu(h_i) = 0$ for all $i \in I$ with $\omega_i(h_\theta) \neq 1$. Then*

$$\mathrm{Hom}_{\mathcal{G}}(W(\lambda), W(\mu)) = 0, \ \text{ if } \lambda \neq \mu,$$
$$\mathrm{Hom}_{\mathcal{G}}(W(\mu), W(\mu)) \cong \mathbb{A}_\mu.$$

Moreover any non-zero map $\varphi : W(\mu) \to W(\mu)$ is injective. □

The restriction on μ is necessary (see[8], Remark 6.1]). For instance, in types B_n and D_n ($n \ge 6$) we have $\mathrm{Hom}_{\mathcal{G}}(W(\omega_2), W(\omega_4)) \neq 0$. However the second statement, namely the injectivity of any non-zero map, is still expected to hold in general.

Remark This theorem is quite unlike the analogous theorem for local Weyl modules which was discussed in the preceding section.

5.4 Generalized Weyl Modules, Global Demazure Modules and Other Directions

We now discuss generalizations of some of the ideas presented earlier in this section. This is a brief and far from complete discussion of the papers of[45,46,51] and we refer the interested readers to those papers for greater detail. We begin by elucidating the connection between local Weyl modules and specializations of Macdonald polynomials which was briefly mentioned in Sect. 5.3. These polynomials are those associated with (anti) dominant weights. We then discuss the work of[51] who introduced the notion of generalized Weyl modules for $\widehat{\mathfrak{n}}^+$ and showed that their characters are again related to specializations of Macdonald polynomials associated with any integral weight.

We then move on to discuss the notion of global Demazure modules introduced by Dumanski and Feigin and state a few open problems regarding the homomorphism spaces between these objects. The aim is to generalize the global-local picture of Weyl modules for wider families of modules and develop some modifications of results in this broader setting.

5.4.1 Local Weyl Modules, Generalized Weyl Modules and Macdonald Polynomials

Let

$$\mathcal{R}_{q,t} = \mathbb{Q}(q,t)[e_\lambda : \lambda \in P] \text{ and}$$
$$\mathcal{R}_q = \mathbb{Q}(q)[e_\lambda : \lambda \in P]$$

respectively be the group algebra of the weight lattice with coefficients in $\mathbb{Q}(q,t)$ and $\mathbb{Q}(q)$ respectively. Consider $R^W_{q,t}$ the subring of W-invariants where the action is induced from the action of W on P and define R^W_q similarly. For $f \in R^W_{q,t}$, we denote by $[f]$ its constant term (i.e. the coefficient in front of e_0) and set

$$\nabla(q,t) = \prod_{\alpha \in (R + \mathbb{Z}_+\delta)} \frac{1-e_\alpha}{1-t^{-1}e_\alpha},$$
$$e_\delta = q^{-1}, \quad \Delta(q,t) = \frac{\nabla(q,t)}{[\nabla(q,t)]}.$$

This ring $R^W_{q,t}$ and R^W_q both admit a scalar product

$$\langle f,g\rangle_{q,t} = [f\overline{g}\Delta(q,t)] \; f,g \in R^W_{q,t},$$
$$\langle f,g\rangle_q = [f\iota(g)\Delta(q,\infty)] \; f,g \in R^W_q$$

where $\bar{\cdot}$ is the involution on $R^W_{q,t}$ given by $t \mapsto t^{-1}$, $q \mapsto q^{-1}$, $e_\lambda \mapsto e_{-\lambda}$ and ι is the involution of $\mathcal{R}_q$ fixing q and mapping e_λ to $e_{-w_\circ\lambda}$. Moreover, we have a natural basis $\{m_\lambda\}_{\lambda\in P^+}$ given by

$$m_\lambda(q,t) = \sum_{\mu \in W\cdot\lambda} e_\mu$$

The *symmetric Macdonald polynomials* $\{P_\lambda(q,t)\}_{\lambda\in P^+}$ are uniquely defined by the following two properties

(1) $$P_\lambda(q,t) = m_\lambda(q,t) + \sum_{\substack{\mu < \lambda \\ \mu \in P^+}} c_{\lambda,\mu} m_\mu(q,t), \quad c_{\lambda,\mu} \in \mathbb{Q}(q,t),$$

(2) $$\langle P_\lambda(q,t), P_\mu(q,t)\rangle_{q,t} = 0, \quad \lambda \neq \mu$$

These polynomials have the property that the limit $t \to \infty$ exists which we denote by $P_\lambda(q,\infty) = \lim_{t\to\infty} P_\lambda(q,t)$. The following result can be found in[26], Theorem 4.2].

Theorem *The family $\{P_\lambda(q,\infty)\}_{\lambda\in P^+}$ forms an orthogonal basis of R^W_q with respect to the form $\langle\cdot,\cdot\rangle_q$. Moreover*

$$P_\lambda(q,\infty) = \mathrm{ch_{gr}}\, W_{\mathrm{loc}}(\lambda), \quad \lambda \in P^+.$$

□

In the case when $\mathfrak{g}$ is simply-laced it was already proved in[70] that the graded character of the stable Demazure module was given by the specialization of the Macdonald polynomial as in the above theorem; recall the connection between local Weyl modules and stable Demazure modules first made in[29] in the case of $\mathfrak{sl}_{n+1}$ and then in[57] for $\mathfrak{g}$ simply-laced.

At that time it was also known that this formula could not hold when $\mathfrak{g}$ was not simply-laced. In the non-simply laced case the result of[87] showed that the local Weyl module had a flag where the successive quotients were stable Demazure modules. The corresponding results for the twisted current algebras were studied in[56]. However in the case of the twisted $A^{(2)}_{2n}$ one has to work with a different current algebra[27], called the hyperspecial current algebra.

5.4.2 Nonsymmetric Macdonald Polynomials

There is another family of polynomials $\{E_\lambda(q,t)\}_{\lambda\in P}$ indexed by the weight lattice called the *nonsymmetric Macdonald polynomials*. They were introduced by Opdam[92] and Cherednik[42]. First we define a new order on the set of weights P. Consider the level one action of $\widehat{W}$ on $\mathfrak{h}^*$ defined as follows (the action differs only for s_0)

$$s_0 \circ \mu := s_\theta(\mu) + \theta,\ \mu \in \mathfrak{h}^*.$$

Given $\lambda \in P$, we denote by w_λ the unique minimal length element of $\widehat{W}$ such that $w_\lambda \circ \lambda$ is either miniscule or zero. For $\lambda, \mu \in P$, we say $\mu <_b \lambda$ if and only if $w_\mu < w_\lambda$ with respect to the Bruhat order.

Again we can define a scalar product $(\cdot,\cdot)_{q,t}$ on $\mathcal{R}_{q,t}$ and the family $\{E_\lambda(q,t)\}_{\lambda\in P}$ is uniquely determined by the following two properties

$$(1)\, E_\lambda(q,t) = e_\lambda + \sum_{\mu<_b\lambda} c_{\lambda,\mu} e_\mu,\ \ c_{\lambda,\mu} \in \mathbb{Q}(q,t),$$

$$(2)\qquad (E_\lambda(q,t), e_\mu)_{q,t} = 0 \ \text{ if } \mu <_b \lambda.$$

For a dominant weight λ we have $P_\lambda(q,\infty) = \lim_{t\to\infty} E_{w_\circ\lambda}(q,t) = E_{w_\circ\lambda}(q,\infty)$ and hence by the above theorem the characters of local Weyl modules appear also as specializations of nonsymmetric Macdonald polynomials for anti-dominant weights

$$E_{w_\circ\lambda}(q,\infty) = \mathrm{ch}_{\mathrm{gr}}\, W_{\mathrm{loc}}(\lambda),\quad \lambda \in P^+$$

The natural question is whether other specializations are also meaningful in the sense that they have a representation theoretic interpretation. This leads to the definition of generalized local Weyl modules which can be found in[51].

The reader should be warned that there are several notions of generalized Weyl modules, e.g. in[24,55,77] when the polynomial algebra is replaced by an arbitrary commutative algebra. But these are not the modules under consideration in this discussion.

Definition Given $\mu \in P$ let W_μ be the $\widehat{\mathfrak{n}}^+$-module generated by v_μ with relations,

$$(h \otimes t^{r+1})v_\mu = 0,\ r \geq 0,$$
$$(x_\alpha^+ \otimes 1)^{\max\{-\mu(h_\alpha),0\}+1} v_\mu = 0,$$
$$(x_\alpha^- \otimes t)^{\max\{\mu(h_\alpha),0\}+1} v_\mu = 0,\ \ \alpha \in R^+.$$

These are called the *generalized local Weyl modules*.

Note that for anti-dominant weights we obviously have $W_\mu \cong W_{\mathrm{loc}}(w_\circ\mu)$ as $\widehat{\mathfrak{n}}^+$-modules and hence the character is again a specialized nonsymmetric Macdonald polynomial. The characters of W_λ for $\lambda \in P^+$ are related to the Orr–Shimozono specialization of $E_{w_0\lambda}(q,t)$. The first part of the next proposition is proved in[93] and the second part in[51].

Proposition *Let $\lambda \in P^+$.*

(1) *The limit $E_{w_\circ\lambda}(q^{-1},0) := \lim_{t\to 0} E_{w_\circ\lambda}(q^{-1},t)$ exists and admits an explicit combinatorial formula in terms of quantum alcove paths.*

(2) *The character of W_λ is given by $w_\circ E_{w_\circ\lambda}(q^{-1},0)$.*

□

5.4.3 Recovering the Global Weyl Module from the Local Weyl Module

We recall a general construction which was first introduced in[8]. Namely, if V is any $\mathfrak{g}[t]$-module, one can define an action of $\mathfrak{g}[t]$ on $V[t] := V \otimes \mathbb{C}[t]$, by

$$(x\otimes t^r)(v\otimes t^s) = \sum_{j=0}^{r} \binom{r}{j} ((x\otimes t^{r-j})v) \otimes t^{s+j}.$$

It was introduced in a more general context in Section 4 of[8] by replacing $\mathbb{C}[t]$ by any commutative associative Hopf algebra A. Notice that if V is generated by an element v then $V[t]$ is generated by $v \otimes 1$. It was shown also that the fundamental global Weyl modules could be realized in this way by taking V to be $W_{\mathrm{loc}}(\omega_i)$ for some $i \in I$. Moreover, it was shown in Proposition 6.2 of that paper that if $\mu = \sum_{i=1}^n s_i\omega_i$ and $\mu(h_i) = 0$ if $\omega_i(h_\theta) \neq 1$ then there exists an injective map

$$W(\mu) \to W(\omega_1)^{\otimes s_1} \otimes \cdots \otimes W(\omega_n)^{\otimes s_n},$$

extending the assignment $w_\mu \to w_{\omega_1}^{s_1} \otimes \cdots \otimes w_{\omega_n}^{s_n}$. This result was then established without the restriction on μ in[73] by different methods.

5.4.4 A Generalization

Suppose now that $W_1,\dots,W_r$ are graded $\mathfrak{g}[t]$-modules with generators $w_1,\dots,w_r$. Then the associated global module is defined to be the submodule of $W_1[t] \otimes \cdots \otimes W_r[t]$ generated by the tensor product of $(w_1 \otimes 1) \otimes \cdots \otimes (w_r \otimes 1)$. This notion was introduced in[52], Section 1.3] and the resulting module is denoted as $R(W_1, \dots, W_r)$. In the case when the additional relations

$$(\mathfrak{h} \otimes t\mathbb{C}[t])w_i = 0, \ \ hw_i = \lambda_i(h)w_i, \qquad (5.1)$$

hold we can define (as in the case of global Weyl modules) a right action of $\mathbf{U}(\mathfrak{h}[t])$ on $R(W_1, \dots, W_r)$. The algebra $\mathcal{A}(\lambda_1, \dots, \lambda_r)$ is as usual the quotient of $\mathbf{U}(\mathfrak{h}[t]_+)$ by the annihilating ideal of the cyclic vector $(w_1 \otimes 1) \otimes \cdots \otimes (w_r \otimes 1)$. This algebra is harder to understand although one does have an embedding

$$\mathcal{A}(\lambda_1, \dots, \lambda_r) \hookrightarrow \bigotimes_{i=1}^{r} \mathcal{A}(\lambda_i), \ \ \mathcal{A}(\lambda_i) \cong \mathbb{C}[z_i]$$

given by

$$(h \otimes t^m) \mapsto \sum_{i=1}^{r} (1 \otimes \cdots \otimes ht^m \otimes \cdots \otimes 1)$$
$$\mapsto \lambda_1(h)z_1^m + \cdots + \lambda_r(h)z_r^m.$$

Given $u \in \mathbb{C}$, set

$$W_i(u) = W_i[t] \otimes_{\mathcal{A}(\lambda_i)} \mathbb{C},$$

where we regard $\mathbb{C}$ as an $\mathcal{A}(\lambda_i)$-module by letting z_i act as u.

The following was proved in[45,46].

Theorem *Let $W_1, \dots, W_r$ be as above.*

(i) *The algebra $\mathcal{A}(\lambda_1, \dots, \lambda_r)$ is isomorphic to the subalgebra in $\mathbb{C}[z_1, \dots, z_r]$ generated by the polynomials $\lambda_1(h)z_1^m + \cdots + \lambda_r(h)z_r^m$, $h \in \mathfrak{h}$, $m \geq 0$.*

(ii) *There exists a nonempty Zariski open subset U of $\mathbb{C}^r$ (in particular $0 \notin U$) such that for all $\mathbf{u} = (u_1, \dots, u_r) \in U$*

$$R(W_1, \cdots, W_r) \otimes_{\mathcal{A}(\lambda_1, \dots, \lambda_r)} \mathbb{C}_{\mathbf{u}} \ \cong_{\mathfrak{g}[t]} \ \bigotimes_{i=1}^{r} W_i(u_i)$$

where $\mathbb{C}_{\mathbf{u}}$ denotes the quotient of $\mathcal{A}(\lambda_1, \dots, \lambda_r)$ by the maximal ideal corresponding to $\mathbf{u}$.

(iii) *The module $R(W_1, \dots, W_r)$ is a finitely-generated $\mathcal{A}(\lambda_1, \dots, \lambda_r)$-module.*

□

The following conjecture of[45] generalizes the known results for local Weyl modules.

Conjecture The $\mathfrak{g}[t]$-module

$$R(W_1, \dots, W_r) \otimes_{\mathcal{A}(\lambda_1, \dots, \lambda_r)} \mathbb{C}_0$$

is isomorphic to the fusion product of the modules $W_i(c_i)$ for $(c_1, \dots, c_r)$ in some Zariski open subset of $\mathbb{C}^r$.

In[45] the authors prove this conjecture for a certain families of Demazure modules when $\mathfrak{g}$ is of type *ADE*, and in[46] they drop the assumption on the type of $\mathfrak{g}$. Given a collection of dominant integral weights $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$ we set

$$\mathbb{D}_{\ell, \ell\underline{\lambda}} = R(D(\ell, \ell\lambda_1), \dots, D(\ell, \ell\lambda_r))$$

and let v be the generating vector of $\mathbb{D}_{\ell, \ell\underline{\lambda}}$. The following theorem has been proved for the tuple $\underline{\lambda} = (\omega_1, \dots, \omega_1, \dots, \omega_n, \dots, \omega_n)$ by Dumanski–Feigin and extended later by Dumanski–Feigin–Finkelberg to arbitrary tuples.

Theorem *Let $\lambda \in P^+$ and $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$ be such that $\lambda = \sum_{i=1}^{r} \lambda_i$. Then, we have an isomorphism*

$$D(\ell, \ell\lambda) \to \mathbb{D}_{\ell, \ell\underline{\lambda}} \otimes_{\mathcal{A}(\ell\lambda_1, \dots, \ell\lambda_r)} \mathbb{C}_0.$$

□

An interesting question would be to determine the generators and relations for global Demazure modules.

Remark Dumanski–Feigin–Finkelberg also prove that $\mathbb{D}_{\ell, \ell\underline{\lambda}}$ is free over $\mathcal{A}(\ell\lambda_1, \dots, \ell\lambda_r)$ and that there exists a tensor product decomposition

$$\mathbb{D}_{\ell, \ell(\underline{\lambda} \cup \underline{\mu})} \otimes \mathbb{C}_{(c,d)} \cong (\mathbb{D}_{\ell, \ell\underline{\lambda}} \otimes \mathbb{C}_c) \otimes (\mathbb{D}_{\ell, \ell\underline{\mu}} \otimes \mathbb{C}_d)$$

provided that c and d have no common entries. This is analogous to the well-known factorization of local Weyl modules which was proved in[37].

An interesting direction of research would be to study the homomorphisms between global Demazure modules and observe the analogues to homomorphisms between global Weyl modules discussed earlier in this section.

As we mentioned earlier the current algebra is the derived algebra of the standard maximal parabolic subalgebra of the affine Lie algebra. A natural problem is to develop an analogous theory for other parabolic subalgebras. This has been attacked for the first time in[28] and the two important families of local and global Weyl modules have been intensively studied, but many problems are still open. The global Weyl modules continue to be parametrized by dominant integral weights of a semi-simple subalgebra of $\mathfrak{g}$ depending on the choice of the maximal parabolic algebra.

However, the following interesting differences appear.

- The algebra $\mathbb{A}_\lambda$ (modulo its Jacobson radical) is a Stanley–Reisner ring; in particular it has relations and is not a polynomial algebra (see[28], Theorem 1]).
- The algebra $\mathbb{A}_\lambda$ and the global Weyl module can be finite-dimensional and this happens if and only if $\mathbb{A}_\lambda$ is a local ring.
- The global Weyl module is not a free $\mathbb{A}_\lambda$ module in general. However we expect the global Weyl module to be free over a suitable quotient algebra of $\mathbb{A}_\lambda$ corresponding to the coordinate ring of one of the irreducible subvarieties of $\mathbb{A}_\lambda$.

The dimension of the local Weyl module depends on the choice of the maximal ideal of $\mathbb{A}_\lambda$. This was in the current algebra case one of the key observations together with the Quillen–Suslin theorem to obtain the freeness of global Weyl modules. It is still an open and interesting question to find the maximal ideals of $\mathbb{A}_\lambda$ producing the local Weyl modules of maximal dimension. An example has been discussed in[28], Section 7.1].

6 Fusion Product Decompositions, Demazure Flags and Connections to Combinatorics and Hypergeometric Series

In this section we collect together several results on Demazure modules which are of independent interest.

6.1 Demazure Modules Revisited

6.1.1 A Simplified Presentation of Demazure Modules

Recall that following[71,84], we gave in Sect. 2.3.3 a presentation of Demazure modules involving infinitely many relations. On the other hand we also discussed in Sect. 5.1.2 that when $\mathfrak{g}$ is simply-laced the local Weyl module $W_{\rm loc}(\mu, r)$ is isomorphic to a Demazure module occurring in a level one highest weight representation. The local Weyl module by definition has only finitely many relations. It turns out that this remains true for arbitrary Demazure modules. The following result was first proved in[41] for $\mathfrak{g}$-stable Demazure modules (see Proposition 5.1.1) and was recently proved for arbitrary Demazure modules in[79].

Theorem *Suppose that $(\lambda, w) \in \widehat{P}^+ \times \widehat{W}$ and assume that $\lambda(c) = \ell$, $\lambda(d) = r$ and $w\lambda|_{\mathfrak{h}} = \mu$. The module $V_w(\lambda)$ is isomorphic to a cyclic $\mathbf{U}(\widehat{\mathfrak{b}})$-module generated by a non-zero vector v with the following relations:*

$$(h \otimes t^s)v = \delta_{s,0} \cdot \mu(h)v, \text{ for all } h \in \mathfrak{h},$$
$$dv = rv, \ \ cv = \ell v,$$

and for $\alpha \in R^{\mp}(\mu) = \{\alpha \in R^+ : \mu(h_\alpha) \in \mp\mathbb{Z}_+\}$ we have

$$\left(x_\alpha^\pm \otimes t^{s_\alpha^\pm - 1}\right)^{m_\alpha^\pm + 1} v = 0,$$
$$\text{if } m_\alpha^\pm < d_\alpha \ell; \ \ \left(x_\alpha^\pm \otimes t^{s_\alpha^\pm}\right)v = 0,$$
$$\left(x_\alpha^+ \otimes \mathbb{C}[t]\right)v = 0,$$
$$\left(x_\alpha^- \otimes t\right)^{\max\{0,\, \mu(h_\alpha) - d_\alpha \ell\}+1} v = 0, \ \ \text{if } \alpha \in R^+(\mu)$$
$$\left(x_\alpha^- \otimes t\mathbb{C}[t]\right)v = 0,$$
$$\left(x_\alpha^+ \otimes 1\right)^{-\mu(h_\alpha)+1} v = 0, \ \ \text{if } \alpha \in R^-(\mu)$$

where $s_\alpha^\pm, m_\alpha^\pm \in \mathbb{Z}_+$ are the unique integers such that

$$\mp\mu(h_\alpha) = (s_\alpha^\pm - 1)d_\alpha \ell + m_\alpha^\pm, \ \ 0 < m_\alpha^\pm \le d_\alpha \ell.$$

□

6.1.2 A Tensor Product Theorem for $\mathfrak{g}$-Stable Demazure Modules

Recall that we discussed in Sect. 4.1.6 the realization of local Weyl modules as a fusion product of fundamental local Weyl modules. We also discussed in Sect. 4.1.7 the results of[90] which gave the generators of the fusion products of Kirillov–Reshetikhin modules. These modules in the simply-laced case are known to be just $\mathfrak{g}$-stable Demazure modules associated to weights of the form $\ell\lambda$ with $\ell \in \mathbb{N}$ and $\lambda \in P^+$. We remark here that in the simply-laced case, the results of[90] are a vast generalization of the results of[57] where a presentation was given of the fusion product of Demazure modules of a fixed level. This was achieved by showing that the fusion product was isomorphic to a Demazure module. The following theorem which may be viewed as a Steinberg type decomposition theorem for $\mathfrak{g}$-stable Demazure modules was proved in[40] (see also[101]) and completes the picture studied in[57].

Theorem *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. Given $k \in \mathbb{N}$, let $\lambda \in P^+$, $\ell \in \mathbb{N}$ and suppose that $\lambda = \ell\,(\sum_{i=1}^k \lambda_i) + \lambda_0$ with $\lambda_0 \in P^+$ and λ_i in the $\mathbb{Z}_+$-span of the $\omega_j^\vee$ for $1 \le j \le k$. Then there is an isomorphism of $\mathfrak{g}[t]$-modules*

$$D(\ell, \lambda) \cong D(\ell, \lambda_0)^{z_0} * D(\ell, \ell\lambda_1)^{z_1} * \cdots * D(\ell, \ell\lambda_k)^{z_k}$$

where $z_0, \dots, z_k$ are distinct complex parameters. In particular the fusion product is independent of the choice of parameters. □

The proof of the theorem relies on the simplified presentation of $D(\ell, \lambda)$ given in Theorem 6.1.1 and a character computation, using the Demazure character formula. This allows one to show that both modules in the theorem have the same $\mathfrak{h}$-characters which is the crucial step to establish the theorem. The analogous theorem for twisted current algebras was proved in[78].

We discuss an interesting consequence of Theorem 6.1.2. Say that a $\mathfrak{g}[t]$-module is *prime* if it is not isomorphic to a fusion product of non-trivial $\mathfrak{g}[t]$-modules. The interested reader should compare this definition with that given in Sect. 3.2.2 where an analogous definition was made in the context of quantum affine algebras. The following factorization result is a consequence of Theorem 6.1.2:

Corollary *Given $\ell \geq 1$ and $\lambda \in P^+$ write*

$$\lambda = \ell\left(\sum_{i=1}^{n} d_i m_i \omega_i\right) + \lambda_0,$$

$$\lambda_0 = \sum_{i=1}^{n} r_i \omega_i, \;\; 0 \leq r_i < d_i \ell, \;\; m_i \in \mathbb{Z}_+.$$

Then $D(\ell, \lambda)$ has the following fusion product factorization:

$$D(\ell,\lambda) \cong_{\mathfrak{g}[t]} D(\ell, \ell d_1 \omega_1)^{*m_1} * D(\ell, \ell d_2 \omega_2)^{*m_2} * \cdots * D(\ell, \ell d_n \omega_n)^{*m_n} * D(\ell, \lambda_0). \tag{6.1}$$

In addition, if we assume that $\mathfrak{g}$ is simply-laced then 6.1 gives prime factorization of $D(\ell, \lambda)$ (i.e., each module on the ritht hand side is prime).

6.2 Demazure Flags

In this section we explain a connection between modules admitting Demazure flags and combinatorics and hypergeometric series.

Say that a finite-dimensional $\mathfrak{g}[t]$-module M has a level m-Demazure flag if it admits a decreasing family of submodules, $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_r \supseteq 0$, such that

$$M_j/M_{j+1} \cong \tau_{r_j} D(m, \mu_j), \;\; r_j \in \mathbb{Z}, \;\; \mu_j \in P^+.$$

It is not hard to see by working with graded characters, that if $M = M'_0 \supseteq M'_1 \supseteq \cdots \supset M'_s \supseteq 0$ is another level m-Demazure flag then $r = s$ and the multiplicity of $\tau_r D(m, \mu)$ in both flags is the same. Hence we define $[M : \tau_r D(m, \mu)]$ to be the number of times $\tau_r D(m, \mu)$ occurs in a level m-Demazure flag of M.

The study of Demazure flags goes back to the work of Naoi[87] on local Weyl modules in the non-simply laced case. It was proved in that paper that these modules admit a level one Demazure flag. This was done by first showing that in the simply-laced case every $\mathfrak{g}[t]$-stable Demazure module of level ℓ admits a Demazure flag of level m if $m \geq \ell \geq 1$. In the case when $\ell = 1$ and $m = 2$ the multiplicities occurring in this flag can be explicitly related to the multiplicity of the level one flag of the local Weyl module for non-simply laced Lie algebras. However, the methods do not lead to precise formulae for the multiplicity. In this section we discuss how one might approach this problem using different kinds of generating series.

6.2.1 The Case of $\mathfrak{sl}_2$ and Level Two Flags

A first step to calculate the multiplicity was taken in[39] for the Lie algebra $\mathfrak{sl}_2$ when $m = 2$ and $\ell = 1$. Then the graded multiplicities can be expressed by q-binomial coefficients[39], Theorem 3.3]:

$$\begin{aligned}
&[D(1, \mu\omega) : D(2, \nu\omega)]_q \\
&= \sum_{p \geq 0} [D(1, \mu\omega) : \tau_p D(2, \nu\omega)]\, q^p \\
&= \begin{cases} q^{(\mu-\nu)/2 \lceil \mu/2 \rceil} \begin{bmatrix} \lfloor \mu/2 \rfloor \\ (\mu-\nu)/2 \end{bmatrix}_q, & \mu - \nu \in 2\mathbb{Z}_+, \\ 0 \;\; \text{otherwise.} \end{cases}
\end{aligned}$$

6.2.2 The Case of $\mathfrak{sl}_2$ and Arbitrary Level

A more general approach in the $\mathfrak{sl}_2$ case was taken in the articles[12,15] and in the $A_2^{(2)}$ case in[11]. In those papers, the authors found a connection to algebraic combinatorics and number theory. We first need to introduce more notation. Define a family of generating series by

$$A_n^{\ell \to m}(x, q) = \sum_{k \geq 0} [D(\ell, (n+2k)\omega) : D(m, n\omega)]_q \cdot x^k, \;\; n \geq 0.$$

Introduce the partial theta function $\theta(q, z) = \sum_{k=0}^{\infty} q^{k^2} z^k$ and let

$$P_n(x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k} x^k$$

and note that the polynomials $P_n(x)$ are related to the Chebyshev polynomials $U_n(x)$ of the second kind as follows $P_n(x^2) = x^n\, U_n((2x)^{-1})$.

The following theorem can be found in[12], Theorem 1.6] and[12], Corollary 1.3] respectively.

Theorem (i) *Let $n, m \in \mathbb{Z}_+$ and write $n = ms + r$ with $s \in \mathbb{Z}_+$ and $0 \le r < m$. Then*

$$A_n^{1\to m}(x,1) = \frac{P_{m-r-1}(x)}{P_m(x)^{s+1}}.$$

(ii) *The specializations*

$$A_1^{1\to 3}(1,q),\quad q\cdot A_1^{1\to 3}(q,q),\quad A_0^{1\to 3}(1,q^2) + qA_2^{1\to 3}(1,q^2),\quad q^4 A_2^{1\to 3}(q^2,q^2) + qA_0^{1\to 3}(q^2,q^2)$$

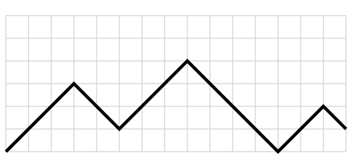
coincide with fifth order mock theta functions of Ramanujan.

(iii) *The series $A_n^{1\to 2}(x,q)$ and $A_n^{2\to 3}(x,q)$ can be expressed as a linear combination of specializations of the partial theta function θ whose coefficients are given by products of q-binomial coefficients.*

□

6.2.3 Combinatorics of Dyck Paths and the Functions $A_n^{1\to m}$

We further discuss the $\mathfrak{sl}_2$ case and its connection to the combinatorics of Dyck paths. In[15] a combinatorial formula has been obtained whose ingredients we will now explain. A *Dyck path* is a diagonal lattice path from the origin (0, 0) to (s, n) for some non-negative integers $s, n \in \mathbb{Z}_+$, such that the path never goes below the x-axis. We encode such a path by a 01-word, where 1 encodes the up-steps and 0 the down-steps. We denote by $\mathcal{D}_n^N$ the set of Dyck paths that end at height n and which do not cross the line $y = N$. The following picture is an example of an element in $\mathcal{D}_1^4$.



For $n, m \in \mathbb{Z}_+$ let $n_0, n_1 \in \mathbb{Z}_+$ be such that $n_0 < m$ and $n = mn_1 + n_0$. If $n < m$ we set $A(m,n) = \emptyset$ and otherwise define

$$A(m,n) := \{(i_1, m), (i_2, m+1), \ldots, (i_{n-m+1}, n)\} \subseteq \mathbb{Z}_+^2$$

where $i_1 < \cdots < i_{n-m+1}$ is the natural ordering of the set

$$\{0,\ldots,n\}\backslash\{pn_1 + n_0 + \min\{0,(p-1)-n_0\},\ 1 \le p \le m\}.$$

Given a pair of non-negative integers $(a,b) \in \mathbb{Z}_+^2$, we say that $P \in \mathcal{D}_n^{\max\{m-1,n\}}$ is $(a,\, b)$-admissible if and only if P satisfies the following property. If P has a peak at height b, the subsequent path is strictly above the line $y = a$. For example, the path above is not (0, 3)-admissible.

Let $\mathcal{D}_{m,n}$ be the set Dyck paths in $\mathcal{D}_n^{\max\{m-1,n\}}$, which are (a,b)-admissible for all $(a,b) \in A(m,n)$.

The major statistics of a Dyck path was studied first by MacMahon[83] in his interpretation of the q-Catalan numbers. Let $P = a_1 \cdots a_s$, $a_i \in \{0,1\}$ be a Dyck path of length s. The major and comajor index are defined by

$$\operatorname{maj}(P) = \sum_{\substack{1\le i<s,\\ a_i > a_{i+1}}} i, \quad \operatorname{comaj}(P) = \sum_{\substack{1\le i<s,\\ a_i > a_{i+1}}} (s-i).$$

The following was proved in[15], Theorem 4].

Theorem *Let $m \in \mathbb{N}, n \in \mathbb{Z}_+$. We have,*

$$A_n^{1\to m}(x,q) = \sum_{P\in\mathcal{D}_{m,n}} q^{\operatorname{comaj}(P)}\, x^{d(P)}$$

where d(P) denotes the number of down-steps of P.

□

In the twisted case graded and weighted generating functions encode again the multiplicity of a given Demazure module. For small ranks these generating functions are completely determined in[11] and they define hypergeometric series and are related to the q-Fibonacci polynomials defined by Carlitz. For more details we refer the reader to[11], Section 2].

6.2.4 The General Case

It is still an open problem to come up with closed or even recursive formulas for the generating series for other finite-dimensional simply-laced Lie algebras; the multiplicities and generating functions are defined in the obvious way. However, some progress has been made in[14] for the Lie algebra $\mathfrak{sl}_{n+1}$ and the connection to Macdonald polynomials was established. The following result can be derived from[14].

Theorem *Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and $\lambda, \mu \in P^+$ such that $\lambda - \mu = \sum_{i=1}^n k_i\alpha_i$, $k_i \in \mathbb{Z}_+$. Then,*

$$[D(1,\lambda):D(2,\mu)]_q = \prod_{i=1}^{n} \left[D(1,(\mu(h_i)+2k_i)\omega) : D(2,\mu(h_i)\omega)\right]_q$$

where ω is the corresponding fundamental weight for $\mathfrak{sl}_2$. □

So combining the above theorem with the combinatorial formula in Theorem 6.2.3 gives a combinatorial formula for graded multiplicities of level 2 Demazure modules in level one flags.

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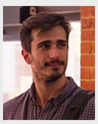

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