Quantum algorithm for a generalized hidden shift problem

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What is the power of quantum computers?

Quantum mechanical computers can efficiently solve problems that classical computers (apparently) cannot.

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Questions:

- What is the computational power of quantum mechanics?
- Is public-key cryptography possible in a quantum world?
  Shor's algorithm breaks RSA, elliptic curve cryptosystems, Diffie-Hellman key exchange, etc.
  What about, e.g., lattice cryptosystems?

Generalized hidden shift problem

Given: $f(b,x) : \{0,1,\ldots,M-1\} \times \mathbb{Z}_N \to S$

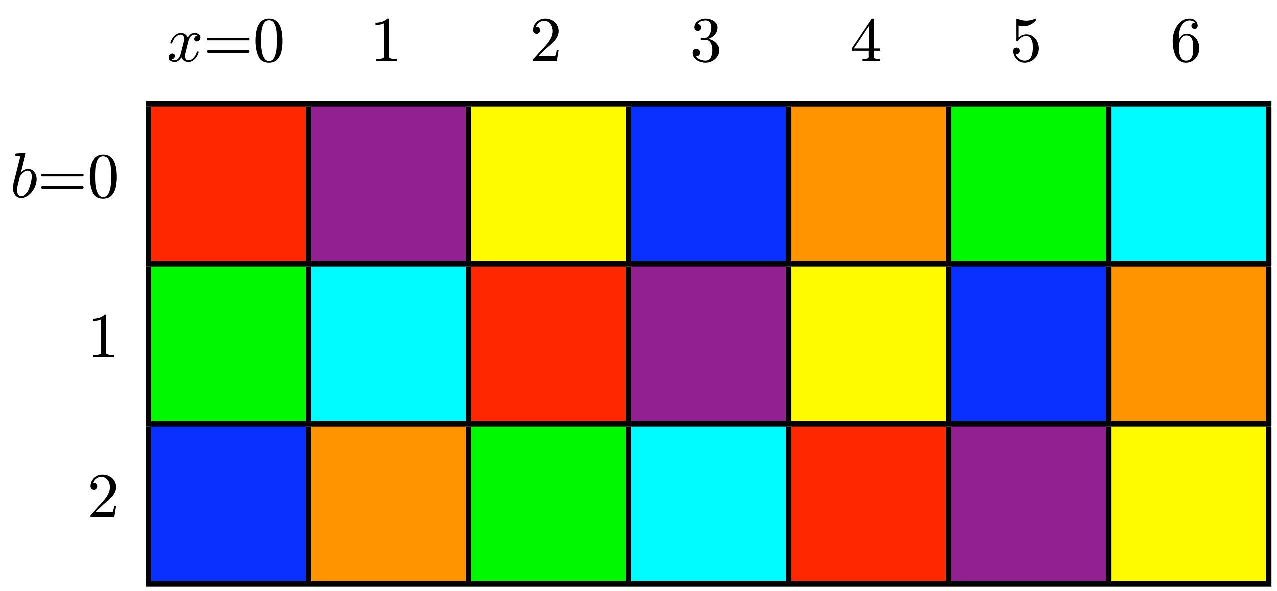
Satisfying: $f(0,x)$ injective

$$f(b+1, x+s) = f(b,x)$$

Find: s (the *hidden shift*)

$M=2$ (hardest), … , N (easiest)

Example. $N=7, M=3, s=2$



Classical complexity

Claim. To determine s, a classical, randomized algorithm must make exponentially many queries (in $\log N$) to f.

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Proof idea:

- Since the function values are arbitrary, they are not informative until we find two inputs that give the same output.
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Note: This holds independent of how big M is.

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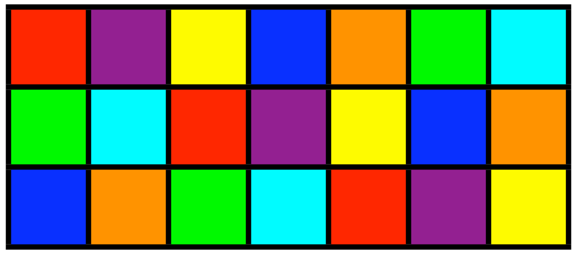

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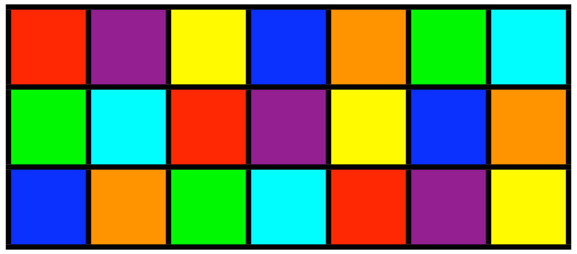
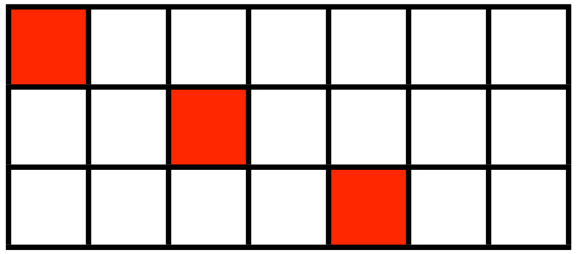


Measure function value: obtain (with equal probability)



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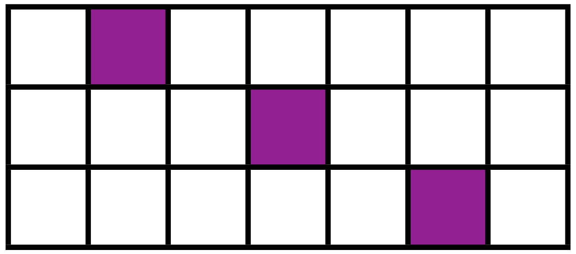

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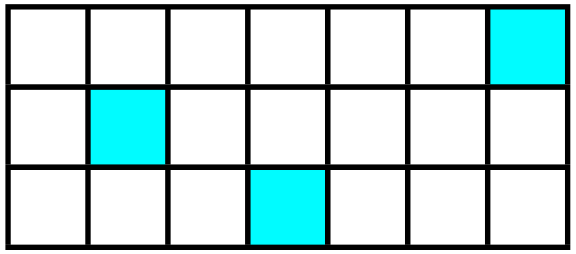

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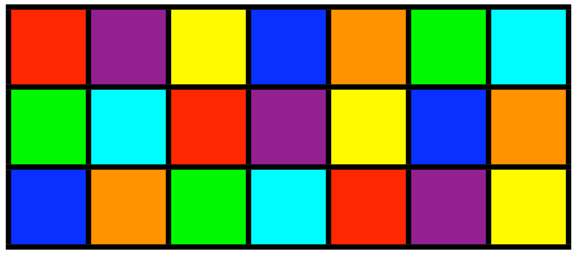

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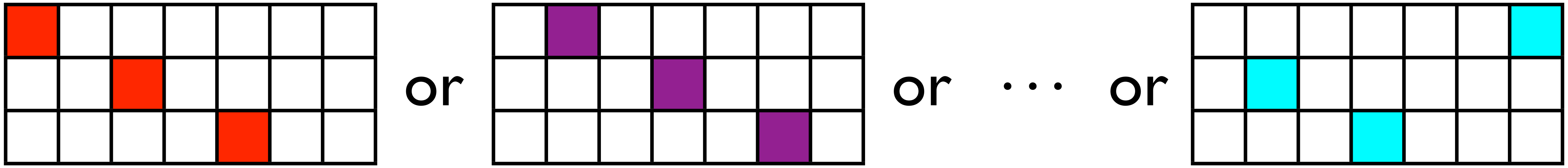

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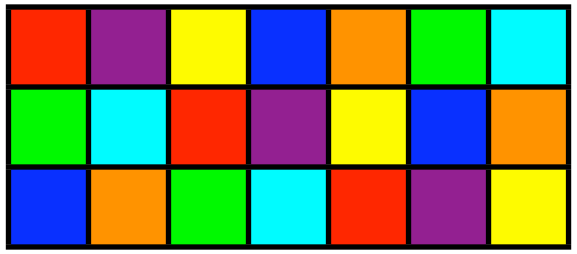

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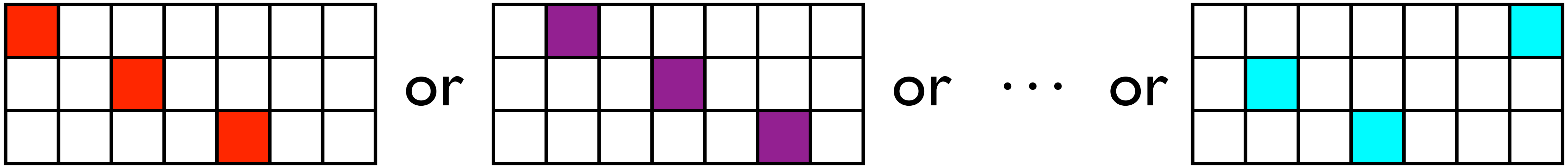

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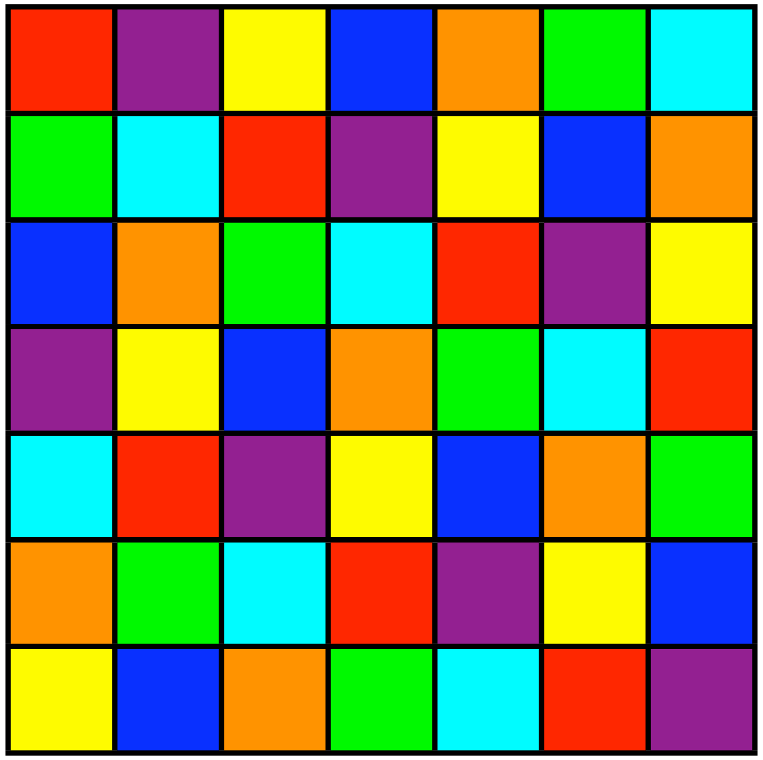
Main question: *Can we do it in* $\mathrm{poly}(\log N)$ *time?*

$M = N$: An abelian hidden subgroup problem

Easiest hidden shift problem:



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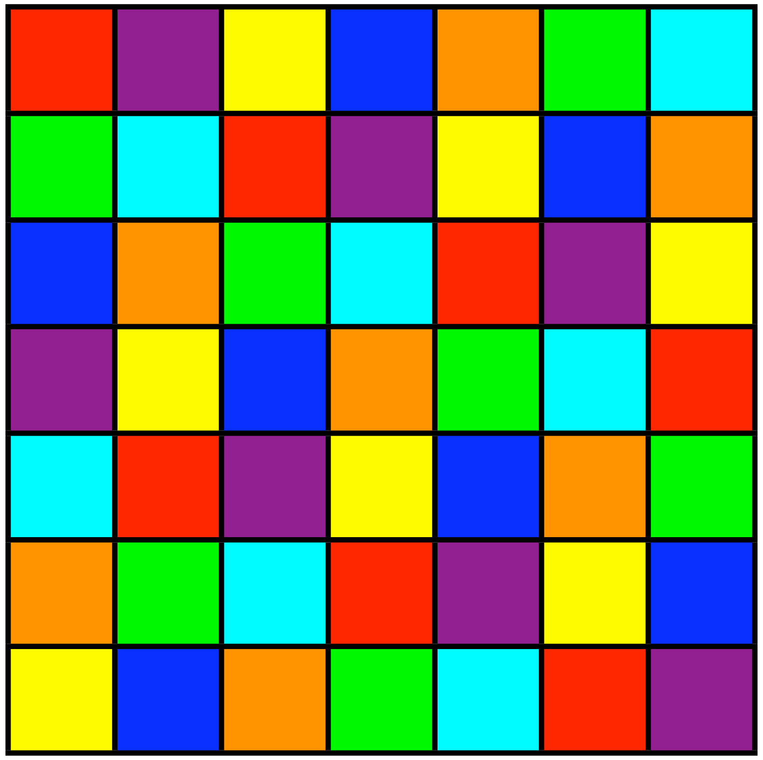

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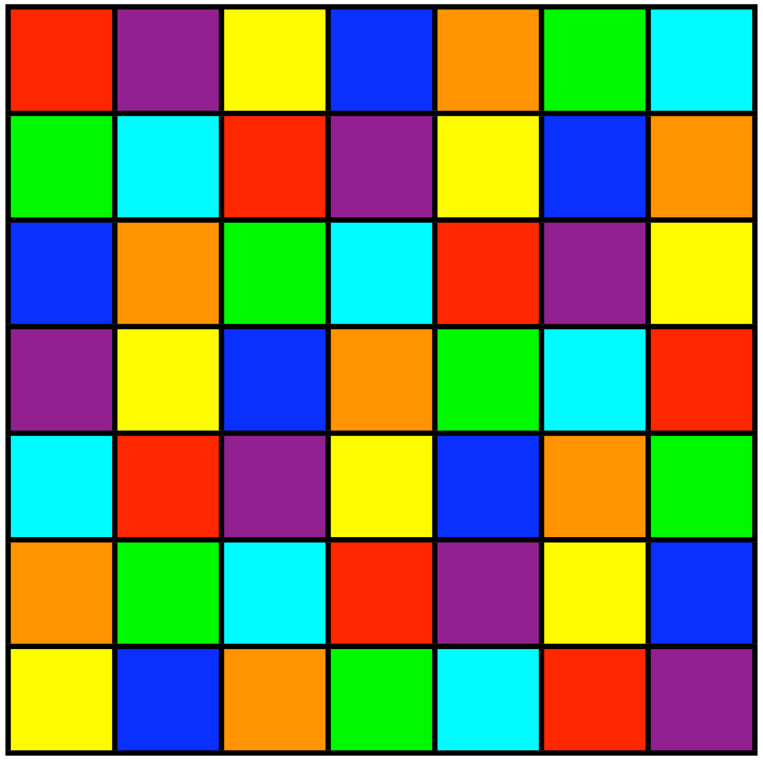

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The same approach works for any $M \geq N/\mathrm{poly}(\log N)$, but not smaller!

$M=2$: The dihedral hidden subgroup problem

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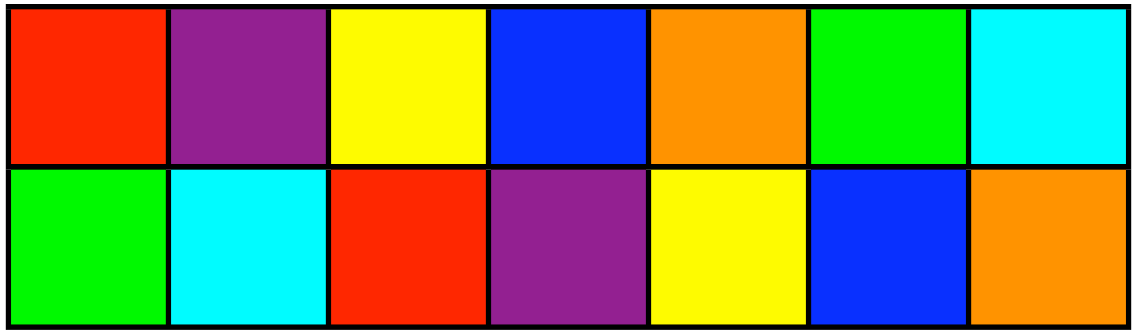

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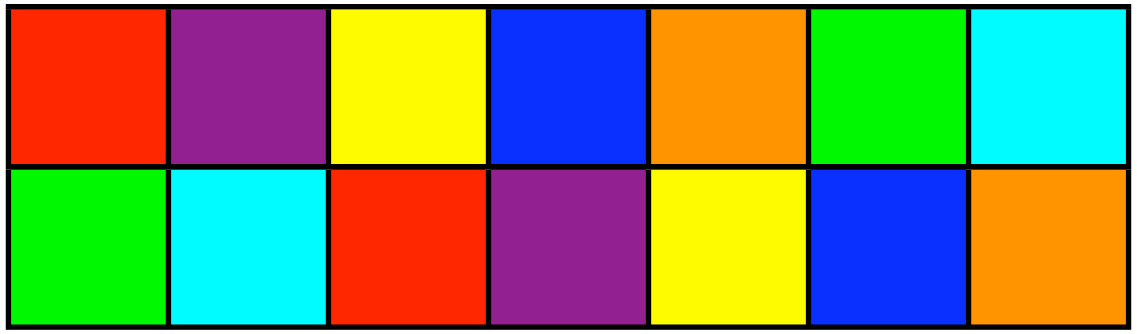

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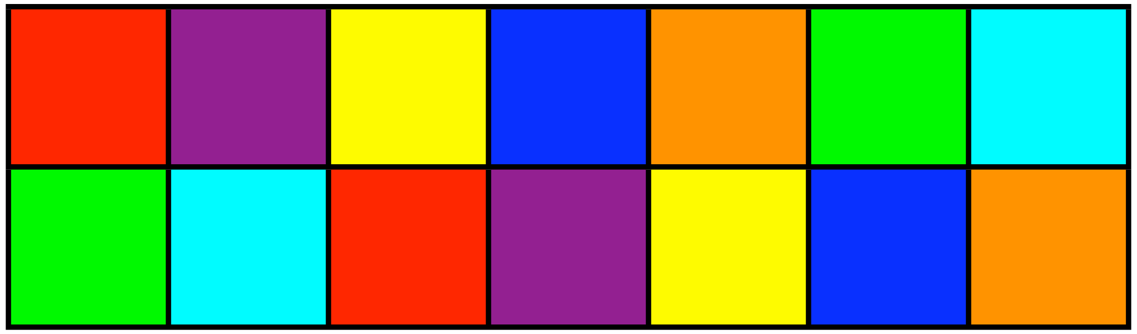

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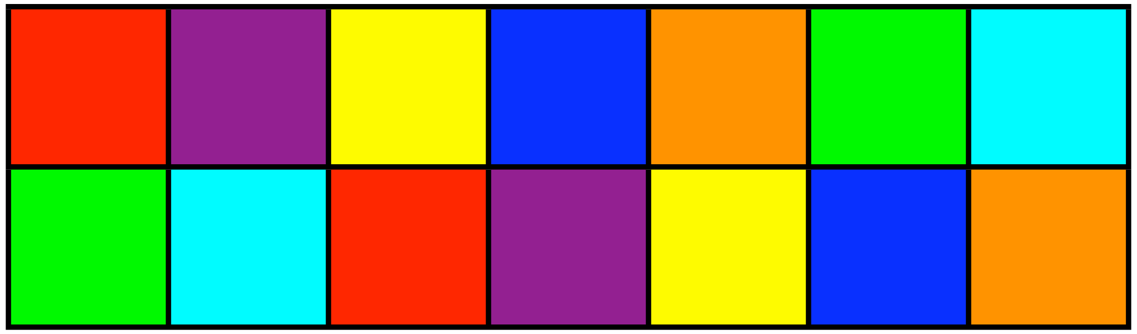

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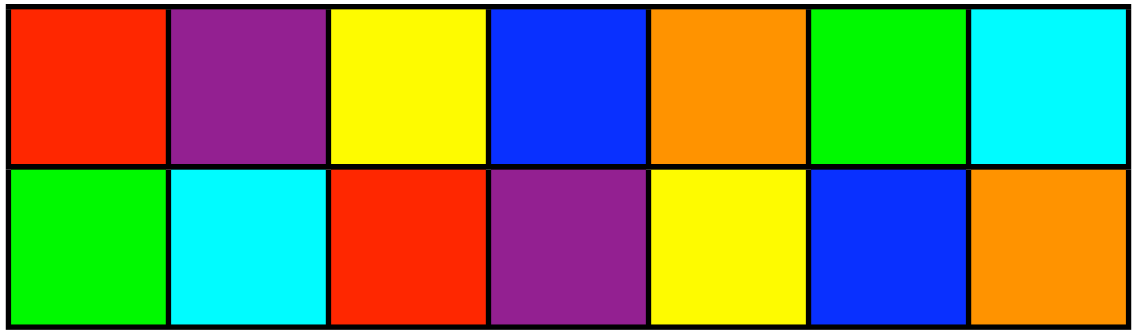

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Regev's reduction also works for larger M. Is this any easier?

Main result

Theorem. Let $M = N^\epsilon$ for any fixed $\epsilon > 0$. Then there is an efficient (i.e., run time $\mathrm{poly}(\log N)$) quantum algorithm for the generalized hidden shift problem, using entangled measurements on $k = \max\{3, \log \frac{1}{\epsilon}\}$ registers.

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Tools:

- "Pretty good measurement" on hidden shift states, à la Bacon, Childs, van Dam 2005.
- Integer programming in constant dimensions (Lenstra 1983).

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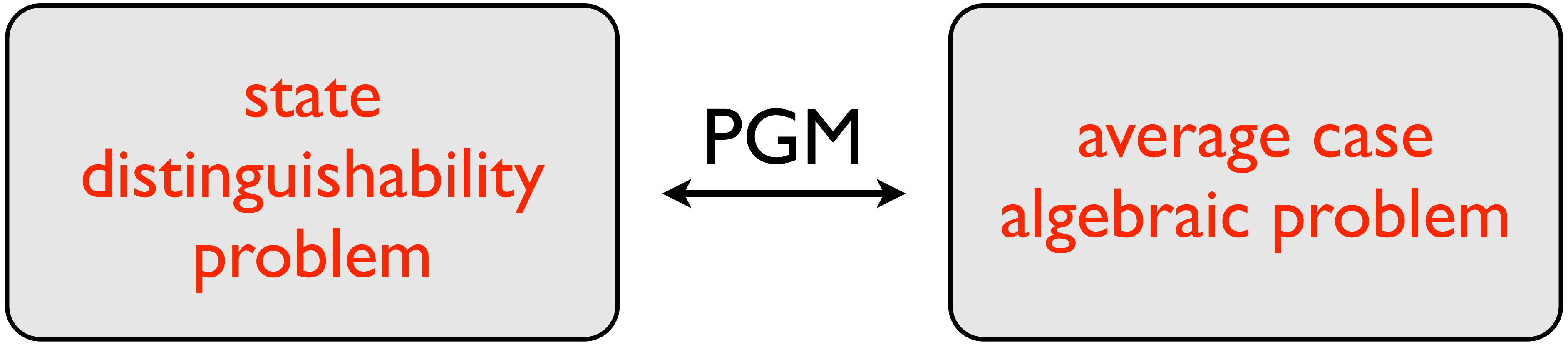

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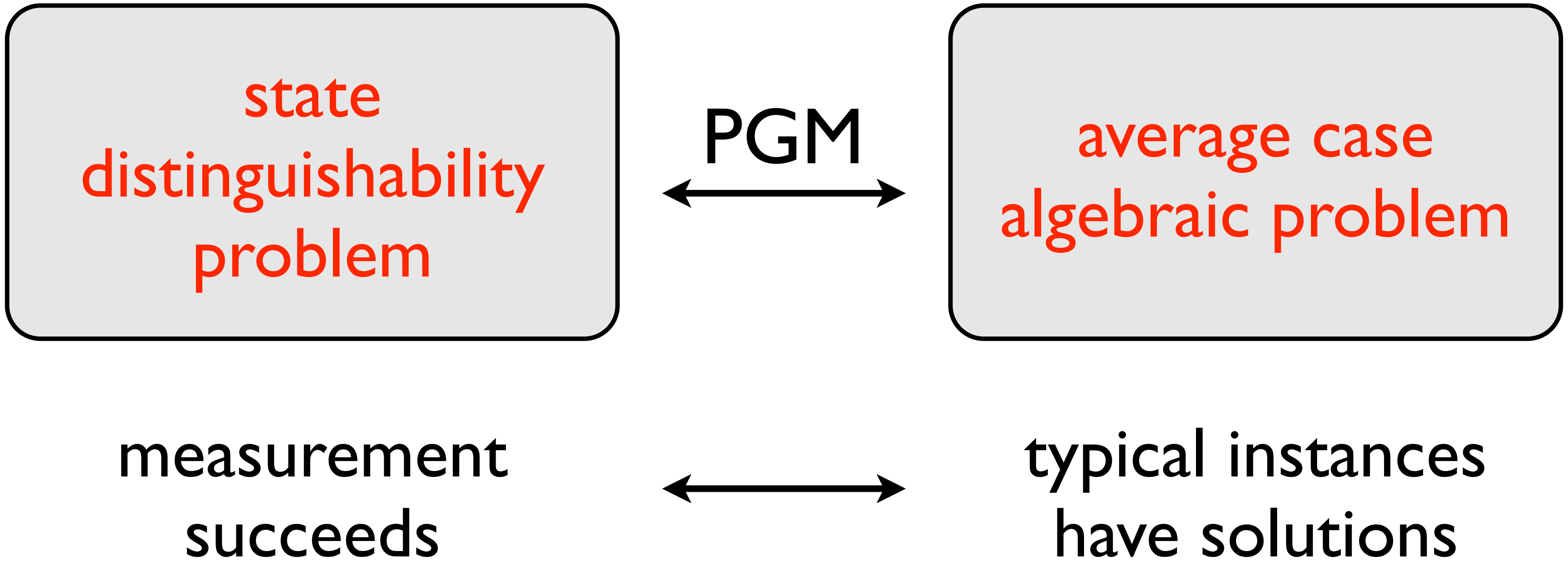

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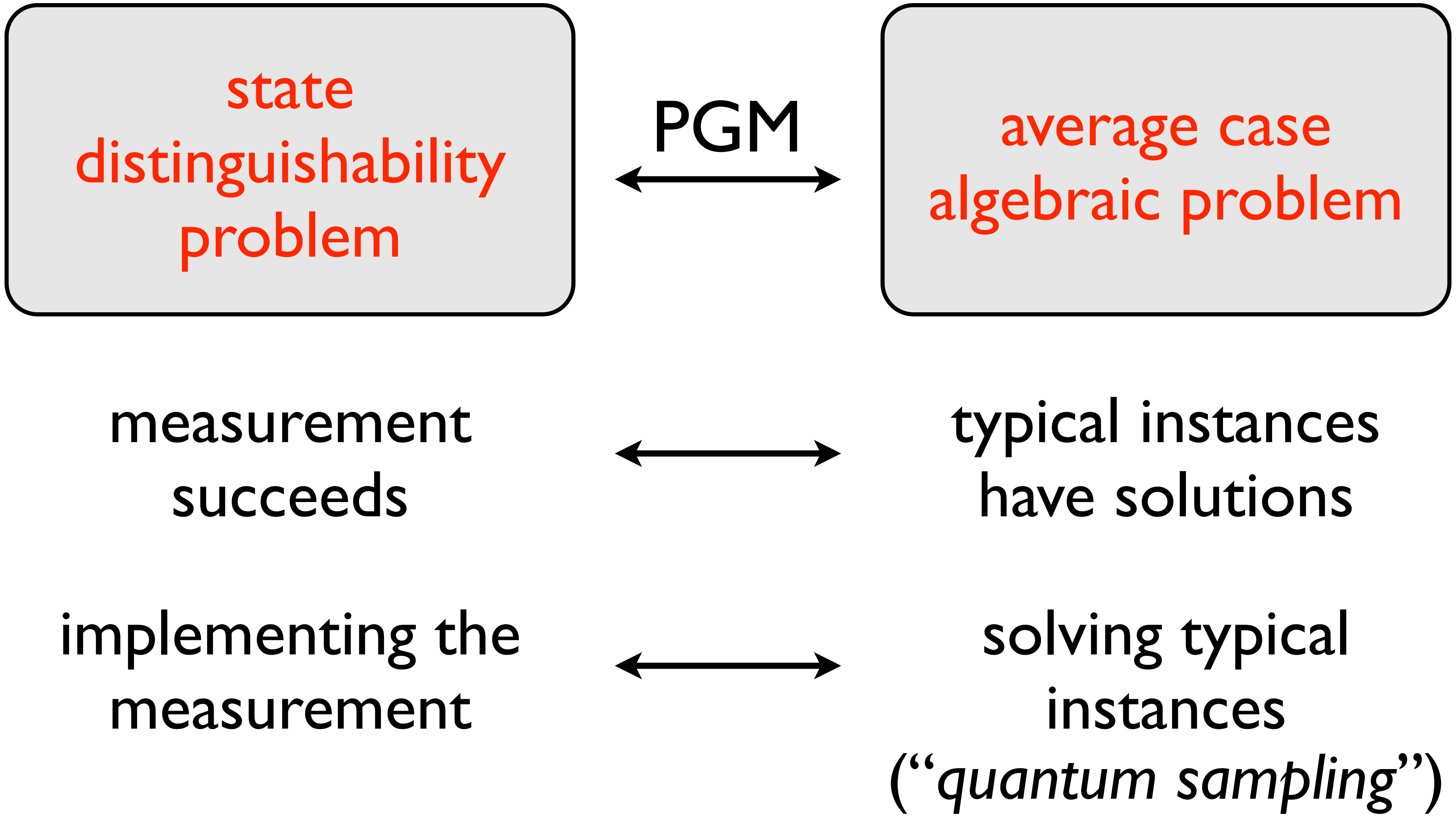

state distinguishability problem ↔ PGM ↔ average case algebraic problem

measurement succeeds ↔ typical instances have solutions

implementing the measurement ↔ solving typical instances ("*quantum sampling*")

The algebraic problem

Given: random $x \in \mathbb{Z}_N^k$

random $w \in \mathbb{Z}_N$

Find: $b \in \{0, 1, \ldots, M-1\}^k$

such that $b \cdot x = w \bmod N$

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Key observation: This is a k-dimensional integer program.

- Solutions of $b \cdot x = w$ over $\mathbb{Z}$ form a shifted integer lattice
- "$\mathrm{mod}\ N$" can be enforced by adding a component
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Lenstra 1983: $2^{O(k^3)}$ time algorithm for integer programming in k dimensions (using LLL lattice basis reduction)

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□

Questions

- Is the quantum solvability of the generalized hidden shift problem with $M = \Omega(N^\epsilon)$ useful for any problems going beyond factoring/discrete log?
- Can we solve the problem efficiently for smaller M?
  Can we at least interpolate with Kuperberg's algorithm?
- What if we replace $\mathbb{Z}_N$ by a nonabelian group?
  (Then even $M=2$ is not a hidden subgroup problem.)
  Can we solve this even for very large M?