Quantum Binary Polyhedral Groups And Their Actions On Quantum Planes

Chelsea Walton

Joint work with Kenneth Chan, Ellen Kirkman, and James Zhang

November 18, 2012

Goal

An investigation of noncommutative/ Hopf invariant theory...

Goal

An investigation of noncommutative/ Hopf invariant theory...
...quantizations of results in classical invariant theory

An investigation of noncommutative/ Hopf invariant theory...
...quantizations of results in classical invariant theory

Actions of finite subgroups of $SL_2(\mathbb{C})$

on

"planes" $\mathbb{C}[u, v]$

An investigation of noncommutative/ Hopf invariant theory...

...quantizations of results in classical invariant theory

Actions of quantum finite subgroups of $SL_2(\mathbb{C})$

on

"quantum planes": noncommutative $\mathbb{C}[u, v]$

Context

Let's recall some classical results.

Put $k = \mathbb{C}$

Take G a finite subgroup of $GL_2(k)$ acting faithfully on $k[u, v]$.

Context

Let's recall some classical results.

Put $k = \mathbb{C}$

Take G a finite subgroup of $GL_2(k)$ acting faithfully on $k[u, v]$.

[STC] $k[u, v]^G$ regular?

$k[u, v]^G \cong k[u', v'] \iff$ G is generated by reflections.

Context

Let's recall some classical results.

Put $k = \mathbb{C}$

Take G a finite subgroup of $GL_2(k)$ acting faithfully on $k[u, v]$.

[STC] $k[u, v]^G$ regular?

$k[u, v]^G \cong k[u', v'] \iff$
G is generated by reflections.

[Watanabe] $k[u, v]^G$ Gorenstein?

$G \leq SL_2(k) \implies k[u, v]^G$ Gorenstein

Context

Let's recall some classical results.

Put $k = \mathbb{C}$

Take G a finite subgroup of $GL_2(k)$ acting faithfully on $k[u, v]$.

[STC] $k[u, v]^G$ regular?

$k[u, v]^G \cong k[u', v'] \iff$

G is generated by reflections.

[Watanabe] $k[u, v]^G$ Gorenstein?

$G \leq SL_2(k) \implies k[u, v]^G$ Gorenstein

[Klein] Finite subgroups of $SL_2(k)$

are classified up to conjugation.

types: A_n D_n E_6 E_7 E_8

"binary polyhedral groups" =: G_{BPG}

...they are not generated by reflections

Context

Let's recall some classical results.

Put $k = \mathbb{C}$

Take G a finite subgroup of $GL_2(k)$ acting faithfully on $k[u, v]$.

[STC] $k[u, v]^G$ regular?

$k[u, v]^G \cong k[u', v'] \iff$

G is generated by reflections.

[Watanabe] $k[u, v]^G$ Gorenstein?

$G \leq SL_2(k) \implies k[u, v]^G$ Gorenstein

[Klein] Finite subgroups of $SL_2(k)$

are classified up to conjugation.

types: A_n D_n E_6 E_7 E_8

"binary polyhedral groups" =: G_{BPG}

...they are not generated by reflections

[DuVal-McKay] Geometry of $k[u, v]^{G_{BPG}}$.

The "Kleinian" or "DuVal" singularities

$X = \mathrm{Spec}(k[u, v]^{G_{BPG}})$

are precisely the rational double points

and the resolution graph of X is Dynkin.

Objects of Study

“quantum finite subgroups of $SL_2(k)$” acting on “quantum planes”

Objects of Study

"quantum finite subgroups of $SL_2(k)$" acting on "quantum planes"

Objects of Study

"quantum finite subgroups of $SL_2(k)$" acting on "quantum planes"

For $q \in k^\times$, categorically–

quantum groups - dual to - Hopf algs

$SL_q(2) \cdots\cdots\cdots\cdots \mathcal{O}_q(SL_2(k))$

G_q fin. subgrp $\cdots\cdots$ $\mathcal{O}_q(G)$ fin. Hopf quot.

Objects of Study

"quantum finite subgroups of $SL_2(k)$" acting on "quantum planes"

Finite dim'l Hopf algebras H

For $q \in k^\times$, categorically–

quantum groups - dual to - Hopf algs

$SL_q(2) \cdots\cdots\cdots\cdots\cdots \mathcal{O}_q(SL_2(k))$

G_q fin. subgrp $\cdots\cdots$ $\mathcal{O}_q(G)$ fin. Hopf quot.

Objects of Study

"quantum finite subgroups of $SL_2(k)$" acting on "quantum planes"

Finite dim'l Hopf algebras H

...that are not necessarily finite quotients of $\mathcal{O}_q(SL_2(k))$

with structure: $(H, m, \Delta, u, \epsilon, S)$

Objects of Study

"quantum finite subgroups of $SL_2(k)$" acting on "quantum planes"

Finite dim'l Hopf algebras H	AS regular algebras R of gldim 2
...that are not necessarily finite quotients of $\mathcal{O}_q(SL_2(k))$	AS = Artin-Schelter
with structure: $(H, m, \Delta, u, \epsilon, S)$	* R is graded with $R_0 = k$
	* global dimension 2
	* AS-Gorenstein
	* polynomial growth

Objects of Study

"quantum finite subgroups of $SL_2(k)$" acting on "quantum planes"

Finite dim'l Hopf algebras H

...that are not necessarily finite quotients of $\mathcal{O}_q(SL_2(k))$

with structure: $(H, m, \Delta, u, \epsilon, S)$

AS regular algebras R of gldim 2

AS = Artin-Schelter

* R is graded with $R_0 = k$
* global dimension 2
* AS-Gorenstein
* polynomial growth

Viewed as 'noncommutative $k[u, v]$' in Noncommutative Projective AG "quantum finite subgroups of $SL_2(k)$" acting on "quantum planes"

Finite dim'l Hopf algebras H	AS regular algebras R of gldim 2

...that are not necessarily finite quotients of $\mathcal{O}_q(SL_2(k))$

with structure: $(H, m, \Delta, u, \epsilon, S)$

AS = Artin-Schelter
* R is graded with $R_0 = k$
* global dimension 2
* AS-Gorenstein
* polynomial growth

Classified up to isomorphism:

$$k_q[u,v] := k\langle u,v\rangle/(vu - quv), \; q \in k^\times$$
$$k_J[u,v] := k\langle u,v\rangle/(vu - uv - u^2)$$

Objects of Study

"quantum finite subgroups of $SL_2(k)$" acting on "quantum planes"

| Finite dim'l Hopf algebras H |

...that are not necessarily finite quotients of $\mathcal{O}_q(SL_2(k))$

with structure: $(H, m, \Delta, u, \epsilon, S)$

| AS regular algebras R of gldim 2 |

* R is graded with $R_0 = k$
* global dimension 2
* AS-Gorenstein
* polynomial growth

Classified up to isomorphism:

$$k_q[u, v] := k\langle u, v\rangle/(vu - quv), \ q \in k^\times$$
$$k_J[u, v] := k\langle u, v\rangle/(vu - uv - u^2)$$

H acts on R if R is a left H-*module algebra*: R is a left H-module and $h \cdot (ab) = \sum(h_1 \cdot a)(h_2 \cdot b)$ and $h \cdot 1_R = \epsilon(h)1_R$ for all $h \in H$, and for all $a, b \in R$

Setting of Study

Let $H \neq k$ be a finite dimensional Hopf algebra acting on an AS regular algebra R of global dimension 2.

(H1) [notion of faithfulness]

.

(H2) H preserves the grading of R

(H3) [notion of H-action having 'determinant 1']
... as results involving G with $\det(G)$ =1 motivate our results.
See [DuVal-McKay] for instance.

Setting of Study

Let $H \neq k$ be a finite dimensional Hopf algebra acting on an AS regular algebra R of global dimension 2.

(H1) H acts on R *inner faithfully*:
there is not an induced action of H/I on R for any nonzero Hopf ideal I of H

(H2) H preserves the grading of R

(H3) [notion of H-action having 'determinant 1']
... as results involving G with det(G) =1 motivate our results.
See [DuVal-McKay] for instance.

Setting of Study

Let $H \neq k$ be a finite dimensional Hopf algebra acting on an AS regular algebra R of global dimension 2.

(H1) H acts on R *inner faithfully*:
there is not an induced action of H/I on R for any nonzero Hopf ideal I of H

(H2) H preserves the grading of R

(H3) H-action of R have trivial "homological determinant".
here, $\text{hdet}_H R$: $H \to k$ and it is *trivial* if equal to the counit map ϵ

Setting of Study

Let $H \neq k$ be a finite dimensional Hopf algebra acting on an AS regular algebra R of global dimension 2.

(H1) H acts on R *inner faithfully*:
there is not an induced action of H/I on R for any nonzero Hopf ideal I of H

(H2) H preserves the grading of R

(H3) H-action of R have trivial "homological determinant".

Definition. A Hopf algebra H satisfying the conditions above is called a quantum binary polyhedral group, denoted by H_{QBPG}.

Main Result

Theorem. [CKWZ] The pairs (H_{QBPG}, R_{ASreg2}) are classified as follows.

Main Result

Theorem. [CKWZ] The pairs (H_{QBPG}, R_{ASreg2}) are classified as follows.

H noncom & s.s.

$(kG_{BPG}, k[u,v])$
G_{BPG} nonabelian

$(kD_{2n}, k_{-1}[u,v])$
$n \geq 3$

$(\mathcal{D}(G_{BPG})^{\circ}, k_{-1}[u,v])$
$\mathcal{D}(G_{BPG})$: Hopf deformation
of nonabelian b.p.g. [BN]

Main Result

Theorem. [CKWZ] The pairs (H_{QBPG}, R_{ASreg2}) are classified as follows.

H noncom & s.s.	H comm (& s.s.)
$(kG_{BPG}, k[u,v])$ G_{BPG} nonabelian	$(kC_2, \text{any } R)$ diagonal action
$(kD_{2n}, k_{-1}[u,v])$ $n \geq 3$	$(kC_2, k_{-1}[u,v])$ non-diagonal action
$(\mathcal{D}(G_{BPG})^{\circ}, k_{-1}[u,v])$ $\mathcal{D}(G_{BPG})$: Hopf deformation of nonabelian b.p.g. [BN]	$(kC_n, k_q[u,v])$ $n \geq 3$
	$((kD_{2n})^{\circ}, k_{-1}[u,v])$ $n \geq 3$

Theorem. [CKWZ] The pairs (H_{QBPG}, R_{ASreg2}) are classified as follows.

H noncom & s.s.

$(kG_{BPG}, k[u,v])$
G_{BPG} nonabelian

$(kD_{2n}, k_{-1}[u,v])$
$n \geq 3$

$(\mathcal{D}(G_{BPG})^{\circ}, k_{-1}[u,v])$
$\mathcal{D}(G_{BPG})$: Hopf deformation of nonabelian b.p.g. [BN]

H comm (& s.s.)

$(kC_2, \text{any } R)$
diagonal action

$(kC_2, k_{-1}[u,v])$
non-diagonal action

$(kC_n, k_q[u,v])$
$n \geq 3$

$((kD_{2n})^{\circ}, k_{-1}[u,v])$
$n \geq 3$

H nonsemisimple

For q is a root of 1, $q^2 \neq 1$

$((T_{q,\alpha,n})^{\circ}, k_{q^{-1}}[u,v])$
$T_{q,\alpha,n}$: generalized Taft alg.

$(H, k_{q^{-1}}[u,v])$ ord(q) odd
$1 \to (kG_{BPG})^{\circ} \to H^{\circ} \to \overline{\mathcal{O}_q(SL_2)} \to 1$

$(H, k_{q^{-1}}[u,v])$ ord(q) even
$1 \to (kG_{PG})^{\circ} \to H^{\circ} \to \overline{\mathcal{O}_q(SL_2)} \to 1$

Main Result

Theorem. [CKWZ] The pairs (H_{QBPG}, R_{ASreg2}) are classified as follows.

$$R = k[u,v] \implies H = kG_{BPG}, \text{ no "new" } H$$

H noncom & s.s.	H comm (& s.s.)	H nonsemisimple
$(kG_{BPG}, k[u,v])$ G_{BPG} nonabelian	$(kC_2, k[u,v])$ diagonal action	For q is a root of 1, $q^2 \neq 1$
$(kD_{2n}, k_{-1}[u,v])$ $n \geq 3$	$(kC_2, k_{-1}[u,v])$ non-diagonal action	$((T_{q,\alpha,n})^\circ, k_{q^{-1}}[u,v])$ $T_{q,\alpha,n}$: generalized Taft alg.
$(\mathcal{D}(G_{BPG})^\circ, k_{-1}[u,v])$ $\mathcal{D}(G_{BPG})$: Hopf deformation of nonabelian b.p.g. [BN]	$(kC_n, k[u,v])$ $n \geq 3$	$(H, k_{q^{-1}}[u,v])$ ord(q) odd $1 \to (kG_{BPG})^\circ \to H^\circ \to \overline{O_q(SL_2)} \to 1$
	$((kD_{2n})^\circ, k_{-1}[u,v])$ $n \geq 3$	$(H, k_{q^{-1}}[u,v])$ ord(q) even $1 \to (kG_{PG})^\circ \to H^\circ \to \overline{O_q(SL_2)} \to 1$

Main Result

Theorem. [CKWZ] The pairs (H_{QBPG}, R_{ASreg2}) are classified as follows.

For $R = k_{-1}[u, v]$

H noncom & s.s.	H comm (& s.s.)	H nonsemisimple
$(kG_{BPG}, k[u, v])$ G_{BPG} nonabelian	$(kC_2, k_{-1}[u, v])$ diagonal action	For q is a root of 1, $q^2 \neq 1$
$(kD_{2n}, k_{-1}[u, v])$ $n \geq 3$	$(kC_2, k_{-1}[u, v])$ non-diagonal action	$((T_{q,\alpha,n})^{\circ}, k_{q^{-1}}[u, v])$ $T_{q,\alpha,n}$: generalized Taft alg.
$(\mathcal{D}(G_{BPG})^{\circ}, k_{-1}[u, v])$ $\mathcal{D}(G_{BPG})$: Hopf deformation of nonabelian b.p.g. [BN]	$(kC_n, k_{-1}[u, v])$ $n \geq 3$	$(H, k_{q^{-1}}[u, v])$ ord(q) odd $1 \to (kG_{BPG})^{\circ} \to H^{\circ} \to \overline{O_q(SL_2)} \to 1$
	$((kD_{2n})^{\circ}, k_{-1}[u, v])$ $n \geq 3$	$(H, k_{q^{-1}}[u, v])$ ord(q) even $1 \to (kG_{PG})^{\circ} \to H^{\circ} \to \overline{O_q(SL_2)} \to 1$

Main Result

Theorem. [CKWZ] The pairs (H_{QBPG}, R_{ASreg2}) are classified as follows.

For $R = k_q[u,v]$ with q a root of unity, $q^2 \neq 1$

H noncom & s.s.

$(kG_{BPG}, k[u,v])$
G_{BPG} nonabelian

$(kD_{2n}, k_{-1}[u,v])$
$n \geq 3$

$(\mathcal{D}(G_{BPG})^{\circ}, k_{-1}[u,v])$
$\mathcal{D}(G_{BPG})$: Hopf deformation of nonabelian b.p.g. [BN]

H comm (& s.s.)

$(kC_2, k_q[u,v])$
diagonal action

$(kC_2, k_{-1}[u,v])$
non-diagonal action

$(kC_n, k_q[u,v])$
$n \geq 3$

$((kD_{2n})^{\circ}, k_{-1}[u,v])$
$n \geq 3$

H nonsemisimple

For q is a root of 1, $q^2 \neq 1$

$((T_{q,\alpha,n})^{\circ}, k_{q^{-1}}[u,v])$
$T_{q,\alpha,n}$: generalized Taft alg.

$(H, k_{q^{-1}}[u,v])$ ord(q) odd
$1 \to (kG_{BPG})^{\circ} \to H^{\circ} \to \overline{O_q(SL_2)} \to 1$

$(H, k_{q^{-1}}[u,v])$ ord(q) even
$1 \to (kG_{PG})^{\circ} \to H^{\circ} \to \overline{O_q(SL_2)} \to 1$

Main Result

Theorem. [CKWZ] The pairs (H_{QBPG}, R_{ASreg2}) are classified as follows.

For $R = k_q[u, v]$ for q not a root of 1

H noncom & s.s.	H comm (& s.s.)	H nonsemisimple
$(kG_{BPG}, k[u, v])$ G_{BPG} nonabelian	$(kC_2, k_q[u, v])$ diagonal action	For q is a root of 1, $q^2 \neq 1$
$(kD_{2n}, k_{-1}[u, v])$ $n \geq 3$	$(kC_2, k_{-1}[u, v])$ non-diagonal action	$((T_{q,\alpha,n})^\circ, k_{q^{-1}}[u, v])$ $T_{q,\alpha,n}$: generalized Taft alg.
$(\mathcal{D}(G_{BPG})^\circ, k_{-1}[u, v])$ $\mathcal{D}(G_{BPG})$: Hopf deformation of nonabelian b.p.g. [BN]	$(kC_n, k_q[u, v])$ $n \geq 3$	$(H, k_{q^{-1}}[u, v])$ ord(q) odd $1 \to (kG_{BPG})^\circ \to H^\circ \to \overline{O_q(SL_2)} \to 1$
	$((kD_{2n})^\circ, k_{-1}[u, v])$ $n \geq 3$	$(H, k_{q^{-1}}[u, v])$ ord(q) even $1 \to (kG_{PG})^\circ \to H^\circ \to \overline{O_q(SL_2)} \to 1$

Main Result

Theorem. [CKWZ] The pairs (H_{QBPG}, R_{ASreg2}) are classified as follows.

For $R = k_J[u, v]$

H noncom & s.s.	H comm (& s.s.)	H nonsemisimple
$(kG_{BPG}, k[u, v])$ G_{BPG} nonabelian	$(kC_2, k_J[u, v])$ diagonal action	For q is a root of 1, $q^2 \neq 1$
$(kD_{2n}, k_{-1}[u, v])$ $n \geq 3$	$(kC_2, k_{-1}[u, v])$ non-diagonal action	$((T_{q,\alpha,n})^\circ, k_{q^{-1}}[u, v])$ $T_{q,\alpha,n}$: generalized Taft alg.
$(\mathcal{D}(G_{BPG})^\circ, k_{-1}[u, v])$ $\mathcal{D}(G_{BPG})$: Hopf deformation of nonabelian b.p.g. [BN]	$(kC_n, k_q[u, v])$ $n \geq 3$	$(H, k_{q^{-1}}[u, v])$ ord(q) odd $1 \to (kG_{BPG})^\circ \to H^\circ \to \overline{O_q(SL_2)} \to 1$
	$((kD_{2n})^\circ, k_{-1}[u, v])$ $n \geq 3$	$(H, k_{q^{-1}}[u, v])$ ord(q) even $1 \to (kG_{PG})^\circ \to H^\circ \to \overline{O_q(SL_2)} \to 1$

Given a pair ($H = H_{QBPG}$, $R = R_{ASreg2}$) in the main theorem, to say:

a finite dimensional Hopf algebra H acts inner faithfully and preserves the grading of an AS regular algebra R of gldim 2, with H-action having trivial homological determinant

we have the following results.

$$R^H = \{r \in R \mid h \cdot r = \epsilon(h) r \text{ for all } h \in H\}$$

[On the regularity of the invariant subring R^H, motivated by [STC]]

[On the Gorenstein condition for the invariant subring R^H, motivated by [Watanabe]]

Given a pair ($H = H_{QBPG}$, $R = R_{ASreg2}$) in the main theorem, to say:

a finite dimensional Hopf algebra H acts inner faithfully and preserves the grading of an AS regular algebra R of gldim 2, with H-action having trivial homological determinant

we have the following results.

$$R^H = \{r \in R \mid h \cdot r = \epsilon(h)r \text{ for all } h \in H\}$$

Theorem. [CKWZ] Let (H, R) be as above. If $R^H \neq R$, then R^H is *not* AS-regular. (R^H has ∞ gldim.)

[On the Gorenstein condition for the invariant subring R^H, motivated by [Watanabe]]

Given a pair ($H = H_{QBPG}$, $R = R_{ASreg2}$) in the main theorem, to say:

a finite dimensional Hopf algebra H acts inner faithfully and preserves the grading of an AS regular algebra R of gldim 2, with H-action having trivial homological determinant

we have the following results.

$$R^H = \{r \in R \mid h \cdot r = \epsilon(h)r \text{ for all } h \in H\}$$

Theorem. [CKWZ] Let (H, R) be as above. If $R^H \neq R$, then R^H is *not* AS-regular. (R^H has ∞ gldim.)

Proposition. [CKWZ] Let (H, R) be as above. The invariant subring R^H is AS-Gorenstein. (semisimple case by [KKZ])

Future Work

(1) Since R^H is Gorenstein and is not regular ...

Motivated by [DuVal-McKay] and others:

Study the geometry of 'noncommutative Gorenstein singularities' R^H for (H, R) in the main theorem, particularly with H semisimple.

Future Work

(1) Since R^H is Gorenstein and is not regular ...
Motivated by [DuVal-McKay] and others:

Study the geometry of 'noncommutative Gorenstein singularities' R^H for (H, R) in the main theorem, particularly with H semisimple.

(2) Motivated by [STC] and others:

Study finite dimensional Hopf algebra actions on AS regular algebras of gldim 2 with *arbitrary* homological determinant.

Future Work

(1) Since R^H is Gorenstein and is not regular ...
Motivated by [DuVal-McKay] and others:

Study the geometry of 'noncommutative Gorenstein singularities' R^H for (H, R) in the main theorem, particularly with H semisimple.

(2) Motivated by [STC] and others:

Study finite dimensional Hopf algebra actions on AS regular algebras of gldim 2 with *arbitrary* homological determinant.

(3) Since AS regular algebras of gldim 3 have been classified...

Study finite dim'l Hopf algebra actions on AS reg. algs of gldim 3.

... AS regular algebras of gldim > 3 have not been classified

References:

[CKWZ] K. Chan, E. Kirkman, C. Walton, J. Zhang, *Quantum binary polyhedral groups and their actions on quantum planes*, in preparation.

[Ben93] D. J. Benson. Polynomial invariants of finite groups, 1993

[BN] J. Bichon and S. Natale, Hopf algebra deformations of binary polyhedral groups, 2011.

[DuVal-McKay] P. du Val, On isolated singularities of surfaces which do not affect the conditions of adjunction, 1934; J. McKay, Graphs, singularities, and finite groups, 1980.

[Klein] F. Klein, Ueber binäre Formen mit linearen Transformationen in sich selbst., 1875; Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade, 1884.

[STC] = [Ben93, Theorem 7.2.1]

[Watanabe] = [Ben93, Theorem 4.6.2]

Thank you for listening!