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Quantum calculus on finite intervals and applications to impulsive difference equations

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Abstract
In this paper we initiate the study of quantum calculus on finite intervals. We define the q_k-derivative and q_k-integral of a function and prove their basic properties. As an application, we prove existence and uniqueness results for initial value problems for first- and second-order impulsive q_k-difference equations.

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1 Introduction

Quantum calculus is the modern name for the investigation of calculus without limits. The quantum calculus or q-calculus began with FH Jackson in the early twentieth century, but this kind of calculus had already been worked out by Euler and Jacobi. Recently it arose interest due to high demand of mathematics that models quantum computing. q-calculus appeared as a connection between mathematics and physics. It has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and other sciences quantum theory, mechanics and the theory of relativity. The book by Kac and Cheung [1] covers many of the fundamental aspects of quantum calculus. It has been shown that quantum calculus is a subfield of the more general mathematical field of time scales calculus. Time scales provide a unified framework for studying dynamic equations on both discrete and continuous domains. The text by Bohner and Peterson [2] collected much of the core theory in the calculus of time scales. In studying quantum calculus, we are concerned with a specific time scale, called the q-time scale, defined as follows: $\mathbb{T} := q^{\mathbb{N}_0} := \{q^t : t \in \mathbb{N}_0\}$, where $q > 1$.

In recent years, the topic of q-calculus has attracted the attention of several researchers, and a variety of new results can be found in the papers [3–15] and the references cited therein.

In this paper we initiate the study of quantum calculus on finite intervals. We define the q_k-derivative of a function $f : J_k := [t_k, t_{k+1}] \to \mathbb{R}$ and prove its basic properties such as the derivative of a sum, of a product or a quotient of two functions. Also, we define the q_k-integral and prove its basic properties. As an application, we prove existence and uniqueness results for initial value problems for first- and second-order impulsive q-difference equations.

The classical q-calculus cannot be used in problems with impulses because if an impulse point t_k for some $k \in \mathbb{N}$ appears between the points t and qt, then the definition of q-derivative does not work. However, this situation does not occur in impulsive problems on q-time scale because the points t and $qt = \rho(t)$ are consecutive points. In quantum calculus on finite intervals, the points t and $q_k t + (1 - q_k)t_k$ are considered only in an interval $[t_k, t_{k+1}]$. Therefore, q_k-calculus can be applied to systems with impulses at fixed times.

The rest of the paper is organized as follows. In Section 2 we recall some basic concepts of q-calculus. In Section 3 we give the new notions of q_k-derivative and q_k-integral on finite intervals and prove its basic properties. In Section 4 we apply the results of Section 3 to impulsive q_k-difference equations and prove existence and uniqueness results. Examples illustrating the abstract results are also presented.

2 Preliminaries

Let us recall some basic concepts of q-calculus [1, 16].

Definition 2.1 Let f be a function defined on a q-geometric set I, *i.e.*, $qt \in I$ for all $t \in I$. For $0 < q < 1$, we define the q-derivative as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \quad t \in I \setminus \{0\}, \qquad D_q f(0) = \lim_{t \to 0} D_q f(t).$$

Note that

$$\lim_{q \to 1} D_q f(t) = \lim_{q \to 1} \frac{f(qt) - f(t)}{(q-1)t} = \frac{df(t)}{dt}$$

if f is differentiable. The higher-order q-derivatives are given by

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}.$$

It is obvious that the q-derivative of a function is a linear operator. That is, for any constants a and b, we have

$$D_q\{af(t) + bg(t)\} = aD_q\{f(t)\} + bD_q\{g(t)\}.$$

The standard rules for differentiation of products and quotients apply in quantum calculus. Thus by Definition 2.1 we can easily prove that

$$\begin{aligned} D_q\{f(t)g(t)\} &= f(qt)D_q g(t) + g(t)D_q f(t) \\ &= f(t)D_q g(t) + g(qt)D_q f(t), \end{aligned} \tag{2.1}$$

$$D_q\left\{\frac{f(t)}{g(t)}\right\} = \frac{g(t)D_q f(t) - f(t)D_q g(t)}{g(qt)g(t)}. \tag{2.2}$$

For $t \geq 0$, we set $J_t = \{tq^n : n \in \mathbb{N} \cup \{0\}\} \cup \{0\}$ and define the definite q-integral of a function $f : J_t \to \mathbb{R}$ by

$$I_q f(t) = \int_0^t f(s)\, d_q s = \sum_{n=0}^{\infty} t(1-q)q^n f(tq^n)$$

provided that the series converges.

For $a, b \in J_t$, we set

$$\int_a^b f(s)\,d_q s = I_q f(b) - I_q f(a) = (1-q)\sum_{n=0}^{\infty} q^n\big[bf(bq^n) - af(aq^n)\big].$$

Note that for $a, b \in J_t$, we have $a = tq^{n_1}$, $b = tq^{n_2}$ for some $n_1, n_2 \in \mathbb{N}$, thus the definite integral $\int_a^b f(s)\,d_q s$ is just a finite sum, so no question about convergence is raised.

We note that

$$D_q I_q f(t) = f(t),$$

while if f is continuous at $t = 0$, then

$$I_q D_q f(t) = f(t) - f(0).$$

In q-calculus, the integration by parts formula is

$$\int_0^t f(x) D_q g(x)\,d_q x = \big[f(x)g(x)\big]_0^t - \int_0^t D_q f(x) g(qx)\,d_q x.$$

Further, reversing the order of integration is given by

$$\int_0^t \int_0^s f(r)\,d_q r\,d_q s = \int_0^t \int_{qr}^t f(r)\,d_q s\,d_q r.$$

In the limit $q \to 1$, the above results correspond to their counterparts in standard calculus.

3 Quantum calculus on finite intervals

In this section we extend the notions of q-derivative and q-integral of the previous section on finite intervals. For a fixed $k \in \mathbb{N} \cup \{0\}$, let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and $0 < q_k < 1$ be a constant. We define the q_k-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows.

Definition 3.1 Assume that $f : J_k \to \mathbb{R}$ is a continuous function, and let $t \in J_k$. Then the expression

$$\begin{aligned} D_{q_k} f(t) &= \frac{f(t) - f(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)}, \quad t \neq t_k, \\ D_{q_k} f(t_k) &= \lim_{t \to t_k} D_{q_k} f(t), \end{aligned} \tag{3.1}$$

is called the q_k-derivative of a function f at t.

We say that f is q_k-differentiable on J_k provided $D_{q_k} f(t)$ exists for all $t \in J_k$. Note that if $t_k = 0$ and $q_k = q$ in (3.1), then $D_{q_k} f = D_q f$, where D_q is the q-derivative of the function $f(t)$ defined in Definition 2.1.

Example 3.1 Let $f(t) = t^2$ for $t \in [1,4]$ and $q_k = \frac{1}{2}$. Now, we consider

$$\begin{aligned} D_{q_k} f(t) &= \frac{t^2 - (q_k t + (1-q_k)t_k)^2}{(1-q_k)(t-t_k)} \\ &= \frac{(1+q_k)t^2 - 2q_k t_k t - (1-q_k)t_k^2}{t-t_k} \\ &= \frac{3t^2 - 2t - 1}{2(t-1)}, \quad t \in (1,4] \end{aligned}$$

and $\lim_{t\to t_k} D_{q_k} f(t) = 2$, if $t = 1$. In particular, $D_{\frac{1}{2}} f(3) = 5$ can be interpreted as a difference quotient $\frac{f(3)-f(2)}{3-2}$.

Example 3.2 In classical q-calculus, we have $D_q t^n = [n]_q t^{n-1}$, where $[n]_q = \frac{1-q^n}{1-q}$. However, q_k-calculus gives $D_{q_k}(t-t_k)^n = [n]_{q_k}(t-t_k)^{n-1}$. Indeed, $f(t) = (t-t_k)^n$, $t \in J_k$, then

$$\begin{aligned} D_{q_k} f(t) &= \frac{(t-t_k)^n - (q_k t + (1-q_k)t_k - t_k)^n}{(1-q_k)(t-t_k)} \\ &= \frac{(t-t_k)^n - q_k^n (t-t_k)^n}{(1-q_k)(t-t_k)} \\ &= [n]_{q_k}(t-t_k)^{n-1}, \end{aligned}$$

where $[n]_{q_k} = \frac{1-q_k^n}{1-q_k}$.

Theorem 3.1 *Assume that $f, g : J_k \to \mathbb{R}$ are q_k-differentiable on J_k. Then:*

(i) *The sum $f + g : J_k \to \mathbb{R}$ is q_k-differentiable on J_k with*

$$D_{q_k}\big(f(t) + g(t)\big) = D_{q_k} f(t) + D_{q_k} g(t).$$

(ii) *For any constant α, $\alpha f : J_k \to \mathbb{R}$ is q_k-differentiable on J_k with*

$$D_{q_k}(\alpha f)(t) = \alpha D_{q_k} f(t).$$

(iii) *The product $fg : J_k \to \mathbb{R}$ is q_k-differentiable on J_k with*

$$\begin{aligned} D_{q_k}(fg)(t) &= f(t) D_{q_k} g(t) + g\big(q_k t + (1-q_k)t_k\big) D_{q_k} f(t) \\ &= g(t) D_{q_k} f(t) + f\big(q_k t + (1-q_k)t_k\big) D_{q_k} g(t). \end{aligned}$$

(iv) *If $g(t)g(q_k t + (1-q_k)t_k) \neq 0$, then $\frac{f}{g}$ is q_k-differentiable on J_k with*

$$D_{q_k}\left(\frac{f}{g}\right)(t) = \frac{g(t) D_{q_k} f(t) - f(t) D_{q_k} g(t)}{g(t) g(q_k t + (1-q_k)t_k)}.$$

Proof The proofs of (i)-(ii) are easy and omitted.

(iii) From Definition 3.1, we have

$$\begin{aligned}
D_{q_k}(fg)(t) &= \frac{f(t)g(t) - f(q_k t + (1-q_k)t_k)g(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)} \\
&= \bigl\{f(t)g(t) - f(t)g\bigl(q_k t + (1-q_k)t_k\bigr) + f(t)g\bigl(q_k t + (1-q_k)t_k\bigr) \\
&\quad - f\bigl(q_k t + (1-q_k)t_k\bigr)g\bigl(q_k t + (1-q_k)t_k\bigr)\bigr\}/(1-q_k)(t-t_k) \\
&= f(t)\left(\frac{g(t) - g(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)}\right) \\
&\quad + g\bigl(q_k t + (1-q_k)t_k\bigr)\left(\frac{f(t) - f(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)}\right) \\
&= f(t)D_{q_k}g(t) + g\bigl(q_k t + (1-q_k)t_k\bigr)D_{q_k}f(t).
\end{aligned}$$

The proof of the second equation in part (iii) is of a similar manner by interchanging the functions f and g.

(iv) For the q_k-derivative of a quotient, we can find that

$$\begin{aligned}
D_{q_k}\left(\frac{f}{g}\right)(t) &= \frac{\frac{f(t)}{g(t)} - \frac{f(q_k t + (1-q_k)t_k)}{g(q_k t + (1-q_k)t_k)}}{(1-q_k)(t-t_k)} \\
&= \frac{f(t)g(q_k t + (1-q_k)t_k) - g(t)f(q_k t + (1-q_k)t_k)}{g(t)g(q_k t + (1-q_k)t_k)(1-q_k)(t-t_k)} \\
&= \left\{g(t)\left(\frac{f(t) - f(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)}\right)\right. \\
&\quad \left. - f(t)\left(\frac{g(t) - g(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)}\right)\right\} \Big/ g(t)g\bigl(q_k t + (1-q_k)t_k\bigr) \\
&= \frac{g(t)D_{q_k}f(t) - f(t)D_{q_k}g(t)}{g(t)g(q_k t + (1-q_k)t_k)}.
\end{aligned}$$

□

Remark 3.1 In Example 3.2 we recall that in q-difference, if $f(t) = t^n$, then $D_q t^n = [n]t^{n-1}$. We cannot have a simple formula for q_k-difference. Using the derivative of a product, we have for some n:

$$\begin{aligned}
&D_{q_k} t = 1, \\
&D_{q_k} t^2 = D_{q_k}(t \cdot t) = (1+q_k)t + (1-q_k)t_k, \\
&D_{q_k} t^3 = D_{q_k}\bigl(t^2 \cdot t\bigr) = \bigl(1 + q_k + q_k^2\bigr)t^2 + \bigl(1 + q_k - 2q_k^2\bigr)tt_k + (1-q_k)^2 t_k^2, \\
&D_{q_k} t^4 = D_{q_k}\bigl(t^3 \cdot t\bigr) \\
&\qquad = \bigl(1 + q_k + q_k^2 + q_k^3\bigr)t^3 + \bigl(1 + q_k + q_k^2 - 3q_k^3\bigr)t_k t^2 \\
&\qquad\quad + \bigl(1 + q_k - 5q_k^2 + 3q_k^3\bigr)t_k^2 t + (1-q_k)^3 t_k^3.
\end{aligned}$$

In addition, we should define the higher q_k-derivative of functions.

Definition 3.2 Let $f : J_k \to \mathbb{R}$ be a continuous function. We call the second-order q_k-derivative $D_{q_k}^2 f$ provided $D_{q_k} f$ is q_k-differentiable on J_k with $D_{q_k}^2 f = D_{q_k}(D_{q_k} f) : J_k \to \mathbb{R}$. Similarly, we define the higher-order q_k-derivative $D_{q_k}^n : J_k \to \mathbb{R}$.

For example, if $f : J_k \to \mathbb{R}$, then we have

$$\begin{aligned} D_{q_k}^2 f(t) &= D_{q_k}\big(D_{q_k} f(t)\big) \\ &= \frac{D_{q_k} f(t) - D_{q_k} f(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)} \\ &= \frac{\frac{f(t)-f(q_k t+(1-q_k)t_k)}{(1-q_k)(t-t_k)} - \frac{f(q_k t+(1-q_k)t_k)-f(q_k^2 t+(1-q_k^2)t_k)}{(1-q_k)(t-t_k)}}{(1-q_k)(t-t_k)} \\ &= \frac{f(t) - 2f(q_k t + (1-q_k)t_k) + f(q_k^2 t + (1-q_k^2)t_k)}{(1-q_k)^2(t-t_k)^2}, \quad t \neq t_k, \end{aligned}$$

and $D_{q_k}^2 f(t_k) = \lim_{t\to t_k} D_{q_k}^2 f(t)$.

To construct the q_k-antiderivative $F(t)$, we define a shifting operator by

$$E_{q_k} F(t) = F\big(q_k t + (1-q_k)t_k\big).$$

It is easy to prove by using mathematical induction that

$$E_{q_k}^n F(t) = E_{q_k}\big(E_{q_k}^{n-1} F\big)(t) = F\big(q_k^n t + \big(1-q_k^n\big)t_k\big),$$

where $n \in \mathbb{N}$ and $E_{q_k}^0 F(t) = F(t)$.

Then we have by Definition 3.1 that

$$\frac{F(t) - F(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)} = \frac{1 - E_{q_k}}{(1-q_k)(t-t_k)} F(t) = f(t).$$

Therefore, the q_k-antiderivative can be expressed as

$$F(t) = \frac{1}{1-E_{q_k}}\big((1-q_k)(t-t_k)f(t)\big).$$

Using the geometric series expansion, we obtain

$$\begin{aligned} F(t) &= (1-q_k)\sum_{n=0}^{\infty} E_{q_k}^n (t-t_k)f(t) \\ &= (1-q_k)\sum_{n=0}^{\infty}\big(q_k^n t + \big(1-q_k^n\big)t_k - t_k\big) f\big(q_k^n t + \big(1-q_k^n\big)t_k\big) \\ &= (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f\big(q_k^n t + \big(1-q_k^n\big)t_k\big). \end{aligned} \tag{3.2}$$

It is clear that the above calculus is valid only if the series in the right-hand side of (3.2) is convergent.

Definition 3.3 Assume that $f : J_k \to \mathbb{R}$ is a continuous function. Then the q_k-integral is defined by

$$\int_{t_k}^{t} f(s)\, d_{q_k} s = (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f\big(q_k^n t + \big(1-q_k^n\big)t_k\big) \tag{3.3}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$, then the definite q_k-integral is defined by

$$\begin{aligned}\int_a^t f(s)\,d_{q_k}s &= \int_{t_k}^t f(s)\,d_{q_k}s - \int_{t_k}^a f(s)\,d_{q_k}s \\ &= (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n t + \bigl(1-q_k^n\bigr)t_k\bigr) \\ &\quad - (1-q_k)(a-t_k)\sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n a + \bigl(1-q_k^n\bigr)t_k\bigr).\end{aligned}$$

Note that if $t_k = 0$ and $q_k = q$, then (3.3) reduces to q-integral of a function $f(t)$, defined by $\int_0^t f(s)\,d_q s = (1-q)t\sum_{n=0}^{\infty} q^n f(q^n t)$ for $t \in [0,\infty)$ (see Section 2).

Example 3.3 Let $f(t) = t$ for $t \in J_k$, then we have

$$\begin{aligned}\int_{t_k}^t f(s)\,d_{q_k}s &= \int_{t_k}^t s\,d_{q_k}s \\ &= (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n\bigl(q_k^n t + \bigl(1-q_k^n\bigr)t_k\bigr) \\ &= \frac{(t-t_k)(t+q_k t_k)}{1+q_k}.\end{aligned}$$

Theorem 3.2 *For $t \in J_k$, the following formulas hold*:

(i) $D_{q_k}\int_{t_k}^t f(s)\,d_{q_k}s = f(t)$;

(ii) $\int_{t_k}^t D_{q_k} f(s)\,d_{q_k}s = f(t)$;

(iii) $\int_a^t D_{q_k} f(s)\,d_{q_k}s = f(t) - f(a)$ *for* $a \in (t_k, t)$.

Proof (i) Using Definitions 3.1 and 3.3, we get

$$\begin{aligned}D_{q_k}\int_{t_k}^t f(s)\,d_{q_k}s &= D_{q_k}\Biggl[(1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n t + \bigl(1-q_k^n\bigr)t_k\bigr)\Biggr] \\ &= \frac{(1-q_k)}{(1-q_k)(t-t_k)}\Biggl[(t-t_k)\sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n t + \bigl(1-q_k^n\bigr)t_k\bigr) \\ &\quad - \bigl(q_k t + (1-q_k)t_k - t_k\bigr) \\ &\quad \times \sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n\bigl(q_k t + (1-q_k)t_k\bigr) + \bigl(1-q_k^n\bigr)t_k\bigr)\Biggr] \\ &= \frac{1}{(t-t_k)}\Biggl[(t-t_k)\sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n t + \bigl(1-q_k^n\bigr)t_k\bigr) \\ &\quad - q_k(t-t_k)\sum_{n=0}^{\infty} q_k^n f\bigl(q_k^{n+1} t + \bigl(1-q_k^{n+1}\bigr)t_k\bigr)\Biggr] \\ &= \sum_{n=0}^{\infty} q_k^n f\bigl(q_k^n t + \bigl(1-q_k^n\bigr)t_k\bigr) - \sum_{n=0}^{\infty} q_k^{n+1} f\bigl(q_k^{n+1} t + \bigl(1-q_k^{n+1}\bigr)t_k\bigr) \\ &= f(t).\end{aligned}$$

(ii) By computing directly, we have

$$\begin{aligned}\int_{t_k}^{t} D_{q_k} f(s)\, d_{q_k} s &= \int_{t_k}^{t} \frac{f(s) - f(q_k s + (1-q_k)t_k)}{(1-q_k)(s-t_k)}\, d_{q_k} s \\ &= (1-q_k)(t-t_k)\sum_{n=0}^{\infty} q_k^n \\ &\quad \times \frac{f(q_k^n t + (1-q_k^n)t_k) - f(q_k(q_k^n t + (1-q_k^n)t_k) + (1-q_k)t_k)}{(1-q_k)(q_k^n t + (1-q_k^n)t_k - t_k)} \\ &= (t-t_k)\sum_{n=0}^{\infty} q_k^n \frac{f(q_k^n t + (1-q_k^n)t_k) - f(q_k^{n+1} t + (1-q_k^{n+1})t_k)}{q_k^n (t-t_k)} \\ &= \sum_{n=0}^{\infty} f\bigl(q_k^n t + \bigl(1-q_k^n\bigr)t_k\bigr) - f\bigl(q_k^{n+1} t + \bigl(1-q_k^{n+1}\bigr)t_k\bigr) \\ &= f(t).\end{aligned}$$

(iii) The part (ii) of this theorem implies that

$$\begin{aligned}\int_{a}^{t} D_{q_k} f(s)\, d_{q_k} s &= \int_{t_k}^{t} D_{q_k} f(s)\, d_{q_k} s - \int_{t_k}^{a} D_{q_k} f(s)\, d_{q_k} s \\ &= f(t) - f(a).\end{aligned}$$

□

Theorem 3.3 *Assume that $f, g : J_k \to \mathbb{R}$ are continuous functions, $\alpha \in \mathbb{R}$. Then, for $t \in J_k$,*

(i) $\int_{t_k}^{t} [f(s) + g(s)]\, d_{q_k} s = \int_{t_k}^{t} f(s)\, d_{q_k} s + \int_{t_k}^{t} g(s)\, d_{q_k} s$;

(ii) $\int_{t_k}^{t} (\alpha f)(s)\, d_{q_k} s = \alpha \int_{t_k}^{t} f(s)\, d_{q_k} s$;

(iii) $\int_{t_k}^{t} f(s) D_{q_k} g(s)\, d_{q_k} s = (fg)(t) - \int_{t_k}^{t} g(q_k s + (1-q_k)t_k) D_{q_k} f(s)\, d_{q_k} s$.

Proof The results of (i)-(ii) follow from Definition 3.3.

(iii) From Theorem 3.1 part (iii), we have

$$f(t) D_{q_k} g(t) = D_{q_k}(fg)(t) - g\bigl(q_k t + (1-q_k)t_k\bigr) D_{q_k} f(t).$$

Taking q_k-integral for the above equation and applying Theorem 3.2 part (ii), we get the result in (iii) as required. □

Theorem 3.4 (Reversing the order of q_k-integration) *Let $f \in C(J_k, \mathbb{R})$, then the following formula holds*:

$$\int_{t_k}^{t} \int_{t_k}^{s} f(r)\, d_{q_k} r\, d_{q_k} s = \int_{t_k}^{t} \int_{q_k r + (1-q_k)t_k}^{t} f(r)\, d_{q_k} s\, d_{q_k} r.$$

Proof By Definition 3.3, we have

$$\begin{aligned}\int_{t_k}^{t} \int_{t_k}^{s} f(r)\, d_{q_k} r\, d_{q_k} s &= \int_{t_k}^{t} (1-q_k)(s-t_k) \sum_{n=0}^{\infty} \bigl[q_k^n f\bigl(q_k^n s + \bigl(1-q_k^n\bigr)t_k\bigr)\bigr]\, d_{q_k} s \\ &= (1-q_k)\sum_{n=0}^{\infty} q_k^n \biggl[\int_{t_k}^{t} (s-t_k) f\bigl(q_k^n s + \bigl(1-q_k^n\bigr)t_k\bigr)\, d_{q_k} s\biggr]\end{aligned}$$

$$\begin{aligned}&= (1-q_k)\sum_{n=0}^{\infty}\int_{t_k}^{t}\big[\big(q_k^n s + (1-q_k^n)t_k\big)f\big(q_k^n s + (1-q_k^n)t_k\big)\\&\quad - t_k f\big(q_k^n s + (1-q_k^n)t_k\big)\big]\,d_{q_k}s.\end{aligned}$$

Since

$$\int_{t_k}^{tq_k^n+(1-q_k^n)t_k} f(u)\,du = (1-q_k)q_k^n(t-t_k)\sum_{m=0}^{\infty}q_k^m f\big(tq_k^{n+m} + (1-q_k^{n+m})t_k\big),$$

then we get that

$$\begin{aligned}\int_{t_k}^{t}\int_{t_k}^{s} f(r)\,d_{q_k}r\,d_{q_k}s &= (1-q_k)^2(t-t_k)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_k^m f\big(q_k^{n+m}t + (1-q^{n+m})t_k\big)\\&\quad\times\big[q^{n+m}t + (1-q_k^{n+m})t_k - t_k\big]\\&= (1-q_k)^2(t-t_k)^2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_k^{n+2m}f\big(q_k^{n+m}t + (1-q^{n+m})t_k\big).\end{aligned}$$

Now, we consider

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q_k^{n+2m}f\big(q_k^{n+m}t + (1-q^{n+m})t_k\big)\\&\quad= \sum_{n=0}^{\infty}\big[q_k^n f\big(q_k^n t + (1-q_k^n)t_k\big) + q_k^{n+2}f\big(q_k^{n+1}t + (1-q_k^{n+1})t_k\big)\\&\qquad + q_k^{n+4}f\big(q_k^{n+2}t + (1-q_k^{n+2})t_k\big) + q_k^{n+6}f\big(q_k^{n+3}t + (1-q_k^{n+3})t_k\big) + \cdots\big]\\&\quad= f(t) + q_k^2 f\big(q_k t + (1-q_k)t_k\big) + q_k^4 f\big(q_k^2 t + (1-q_k^2)t_k\big) + \cdots\\&\qquad + q_k f\big(q_k t + (1-q_k)t_k\big) + q_k^3 f\big(q_k^2 t + (1-q_k^2)t_k\big) + q_k^5 f\big(q_k^3 t + (1-q_k^3)t_k\big) + \cdots\\&\qquad + q_k^2 f\big(q_k^2 t + (1-q_k^2)t_k\big) + q_k^4 f\big(q_k^3 t + (1-q_k^3)t_k\big) + q_k^6 f\big(q_k^4 t + (1-q_k^4)t_k\big) + \cdots\\&\quad= f(t) + q_k(1+q_k)f\big(q_k t + (1-q_k)t_k\big) + q_k^2(1+q_k+q_k^2)f\big(q_k^2 t + (1-q_k^2)t_k\big) + \cdots\\&\quad= \sum_{n=0}^{\infty}q_k^n\left(\frac{1-q_k^{n+1}}{1-q_k}\right)f\big(q_k^n t + (1-q_k^n)t_k\big).\end{aligned}$$

It follows that

$$\begin{aligned}\int_{t_k}^{t}\int_{t_k}^{s} f(r)\,d_{q_k}r\,d_{q_k}s &= (1-q_k)(t-t_k)^2\sum_{n=0}^{\infty}q_k^n(1-q_k^{n+1})f\big(q_k^n t + (1-q_k^n)t_k\big)\\&= (1-q_k)(t-t_k)\sum_{n=0}^{\infty}q_k^n(1-q_k^{n+1})(t-t_k)f\big(q_k^n t + (1-q_k^n)t_k\big)\\&= \int_{t_k}^{t}\big(t - qr - (1-q_k)t_k\big)f(r)\,d_{q_k}r\\&= \int_{t_k}^{t}\int_{q_k r+(1-q_k)t_k}^{t} f(r)\,d_{q_k}s\,d_{q_k}r.\end{aligned}$$

This completes the proof. □

4 Impulsive q_k-difference equations

Let $J = [0, T]$, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k = 1, 2, \dots, m$. Let $PC(J, \mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1, 2, \dots, m\}$. $PC(J, \mathbb{R})$ is a Banach space with the norms $\|x\|_{PC} = \sup\{|x(t)|; t \in J\}$.

4.1 First-order impulsive q_k-difference equations

In this subsection, we study the existence and uniqueness of solutions for the following initial value problem for first-order impulsive q_k-difference equation:

$$\begin{aligned} &D_{q_k}x(t) = f\big(t, x(t)\big), \quad t \in J, t \neq t_k,\\ &\Delta x(t_k) = I_k\big(x(t_k)\big), \quad k = 1, 2, \dots, m,\\ &x(0) = x_0, \end{aligned} \tag{4.1}$$

where $x_0 \in \mathbb{R}$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k \in C(\mathbb{R}, \mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $k = 1, 2, \dots, m$ and $0 < q_k < 1$ for $k = 0, 1, 2, \dots, m$.

Lemma 4.1 *If $x \in PC(J, \mathbb{R})$ is a solution of* (4.1), *then for any $t \in J_k$, $k = 0, 1, 2, \dots, m$,*

$$\begin{aligned} x(t) = x_0 &+ \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} f\big(s, x(s)\big)\, d_{q_{k-1}}s \\ &+ \sum_{0<t_k<t} I_k\big(x(t_k)\big) + \int_{t_k}^{t} f\big(s, x(s)\big)\, d_{q_k}s, \end{aligned} \tag{4.2}$$

with $\sum_{0<0}(\cdot) = 0$, is a solution of (4.1). *The converse is also true.*

Proof For $t \in J_0$, q_0-integrating (4.1), it follows

$$x(t) = x_0 + \int_0^t f\big(s, x(s)\big)\, d_{q_0}s,$$

which leads to

$$x(t_1) = x_0 + \int_0^{t_1} f\big(s, x(s)\big)\, d_{q_0}s.$$

For $t \in J_1$, taking q_1-integral to (4.1), we have

$$x(t) = x\big(t_1^+\big) + \int_{t_1}^{t} f\big(s, x(s)\big)\, d_{q_1}s.$$

Since $x(t_1^+) = x(t_1) + I_1(x(t_1))$, then we have

$$x(t) = x_0 + \int_0^{t_1} f\big(s, x(s)\big)\, d_{q_0}s + \int_{t_1}^{t} f\big(s, x(s)\big)\, d_{q_1}s + I_1\big(x(t_1)\big).$$

Again q_2-integrating (4.1) from t_2 to t, where $t \in J_2$, then

$$\begin{aligned} x(t) &= x(t_2^+) + \int_{t_2}^{t} f(s, x(s))\, d_{q_2}s \\ &= x_0 + \int_0^{t_1} f(s, x(s))\, d_{q_0}s + \int_{t_1}^{t_2} f(s, x(s))\, d_{q_1}s + \int_{t_2}^{t} f(s, x(s))\, d_{q_2}s \\ &\quad + I_1(x(t_1)) + I_2(x(t_2)). \end{aligned}$$

Repeating the above procession, for $t \in J$, we obtain (4.2).

On the other hand, assume that $x(t)$ is a solution of (4.1). Applying the q_k-derivative on (4.2) for $t \in J_k$, $k = 0, 1, 2, \ldots, m$, it follows that

$$D_{q_k} x(t) = f(t, x(t)).$$

It is easy to verify that $\Delta x(t_k) = I_k(x(t_k))$, $k = 1, 2, \ldots, m$ and $x(0) = x_0$. This completes the proof. □

Theorem 4.1 *Assume that the following assumptions hold*:

(H_1) $f : J \times \mathbb{R} \to \mathbb{R}$ *is a continuous function and satisfies*

$$|f(t, x) - f(t, y)| \le L|x - y|, \quad L > 0, \forall t \in J, x, y \in \mathbb{R};$$

(H_2) $I_k : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \ldots, m$, *are continuous functions and satisfy*

$$|I_k(x) - I_k(y)| \le M|x - y|, \quad M > 0, \forall x, y \in \mathbb{R}.$$

If

$$LT + mM \le \delta < 1,$$

then the nonlinear impulsive q_k-*difference initial value problem* (4.1) *has a unique solution on* J.

Proof We define an operator $\mathcal{A} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ by

$$\begin{aligned} (\mathcal{A}x)(t) &= x_0 + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} f(s, x(s))\, d_{q_{k-1}}s \\ &\quad + \sum_{0<t_k<t} I_k(x(t_k)) + \int_{t_k}^{t} f(s, x(s))\, d_{q_k}s, \end{aligned}$$

with $\sum_{0<0}(\cdot) = 0$. Assume that $\sup_{t\in J} |f(t, 0)| = N_1$ and $\max\{|I_k(0)| : k = 1, 2, \ldots, m\} = N_2$; we choose a constant r such that

$$r \ge \frac{1}{1 - \varepsilon}\left[|x_0| + N_1 T + mN_2\right],$$

where $\delta \leq \varepsilon < 1$. Now, we will show that $\mathcal{A}B_r \subset B_r$, where a ball $B_r = \{x \in PC(J,\mathbb{R}) : \|x\| \leq r\}$. For any $x \in B_r$ and for each $t \in J$, we have

$$\begin{aligned}
\bigl|(\mathcal{A}x)(t)\bigr| &\leq |x_0| + \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \bigl|f\bigl(s,x(s)\bigr)\bigr| \, d_{q_{k-1}}s \\
&\quad + \sum_{0<t_k<t} \bigl|I_k\bigl(x(t_k)\bigr)\bigr| + \int_{t_k}^{t} \bigl|f\bigl(s,x(s)\bigr)\bigr| \, d_{q_k}s \\
&\leq |x_0| + \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} \bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr|\bigr) \, d_{q_{k-1}}s \\
&\quad + \sum_{0<t_k<T} \bigl(\bigl|I_k\bigl(x(t_k)\bigr) - I_k(0)\bigr| + \bigl|I_k(0)\bigr|\bigr) \\
&\quad + \int_{t_m}^{T} \bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr|\bigr) \, d_{q_m}s \\
&\leq |x_0| + (Lr + N_1) \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} d_{q_{k-1}}s \\
&\quad + \sum_{0<t_k<T} (Mr + N_2) + (Lr + N_1) \int_{t_m}^{T} d_{q_m}s \\
&\leq |x_0| + (Lr + N_1)T + m(Mr + N_2) \\
&\leq (\delta + 1 - \varepsilon)r \leq r.
\end{aligned}$$

This implies that $\mathcal{A}B_r \subset B_r$.

For $x, y \in PC(J,\mathbb{R})$ and for each $t \in J$, we have

$$\begin{aligned}
\bigl|(\mathcal{A}x)(t) - (\mathcal{A}y)(t)\bigr| &\leq \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} \bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr| \, d_{q_{k-1}}s \\
&\quad + \sum_{0<t_k<t} \bigl|I_k\bigl(x(t_k)\bigr) - I_k\bigl(y(t_k)\bigr)\bigr| \\
&\quad + \int_{t_k}^{t} \bigl|f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr| \, d_{q_k}s \\
&\leq \sum_{0<t_k<T} \int_{t_{k-1}}^{t_k} \bigl(L\bigl|x(s) - y(s)\bigr|\bigr) \, d_{q_{k-1}}s \\
&\quad + \sum_{0<t_k<T} M\bigl|x(t_k) - y(t_k)\bigr| \\
&\quad + \int_{t_m}^{T} \bigl(L\bigl|x(s) - y(s)\bigr|\bigr) \, d_{q_m}s \\
&\leq (LT + mM)\|x - y\|.
\end{aligned}$$

It follows that

$$\|\mathcal{A}x - \mathcal{A}y\| \leq (LT + mM)\|x - y\|.$$

As $LT + mM < 1$, by the Banach contraction mapping principle, $\mathcal{A}$ is a contraction. Therefore, $\mathcal{A}$ has a fixed point which is a unique solution of (4.1) on J. □

Example 4.1 Consider the following first-order impulsive q_k-difference initial value problem:

$$\begin{aligned} &D_{\frac{1}{2+k}} x(t) = \frac{e^{-t}|x(t)|}{(t+\sqrt{5})^2(1+|x(t)|)}, \quad t \in J = [0,1], t \neq t_k = \frac{k}{10}, \\ &\Delta x(t_k) = \frac{|x(t_k)|}{12 + |x(t_k)|}, \quad k = 1,2,\ldots,9, \\ &x(0) = 0. \end{aligned} \tag{4.3}$$

Here $q_k = 1/(2+k)$, $k = 0,1,2,\ldots,9$, $m = 9$, $T = 1$, $f(t,x) = (e^{-t}|x|)/((t+\sqrt{5})^2(1+|x|))$ and $I_k(x) = |x|/(12+|x|)$. Since $|f(t,x) - f(t,y)| \leq (1/5)|x-y|$ and $|I_k(x) - I_k(y)| \leq (1/12)|x-y|$, then (H_1), (H_2) are satisfied with $L = (1/5)$, $M = (1/12)$. We can show that

$$LT + mM = \frac{1}{5} + \frac{9}{12} = \frac{19}{20} < 1.$$

Hence, by Theorem 4.1, the initial value problem (4.3) has a unique solution on $[0,1]$.

4.2 Second-order impulsive q_k-difference equations

In this subsection, we investigate the second-order initial value problem of impulsive q_k-difference equation of the form

$$\begin{aligned} &D_{q_k}^2 x(t) = f(t, x(t)), \quad t \in J, t \neq t_k, \\ &\Delta x(t_k) = I_k(x(t_k)), \quad k = 1,2,\ldots,m, \\ &D_{q_k} x(t_k^+) - D_{q_{k-1}} x(t_k) = I_k^*(x(t_k)), \quad k = 1,2,\ldots,m, \\ &x(0) = \alpha, \qquad D_{q_0} x(0) = \beta, \end{aligned} \tag{4.4}$$

where $\alpha, \beta \in \mathbb{R}$, $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k, I_k^* \in C(\mathbb{R}, \mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$ for $k = 1,2,\ldots,m$ and $0 < q_k < 1$ for $k = 0,1,2,\ldots,m$.

Lemma 4.2 *The unique solution of problem* (4.4) *is given by*

$$\begin{aligned} x(t) = {}& \alpha + \beta t + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} (t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}) f(s, x(s))\, d_{q_{k-1}}s + I_k(x(t_k)) \right) \\ &+ t \left[\sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f(s, x(s))\, d_{q_{k-1}}s + I_k^*(x(t_k)) \right) \right] \\ &- \sum_{0<t_k<t} t_k \left(\int_{t_{k-1}}^{t_k} f(s, x(s))\, d_{q_{k-1}}s + I_k^*(x(t_k)) \right) \\ &+ \int_{t_k}^{t} (t - q_k s - (1-q_k)t_k) f(s, x(s))\, d_{q_k}s, \end{aligned} \tag{4.5}$$

with $\sum_{0<0}(\cdot) = 0$.

Proof For $t \in J_0$, taking q_0-integral for the first equation of (4.4), we get

$$D_{q_0}x(t) = D_{q_0}x(0) + \int_0^t f(s,x(s))\, d_{q_0}s = \beta + \int_0^t f(s,x(s))\, d_{q_0}s, \tag{4.6}$$

which yields

$$D_{q_0}x(t_1) = \beta + \int_0^{t_1} f(s,x(s))\, d_{q_0}s. \tag{4.7}$$

For $t \in J_0$, we obtain, by q_0-integrating (4.6),

$$x(t) = \alpha + \beta t + \int_0^t \int_0^s f(\sigma,x(\sigma))\, d_{q_0}\sigma\, d_{q_0}s,$$

which, on changing the order of q_0-integral, takes the form

$$x(t) = \alpha + \beta t + \int_0^t (t - q_0 s) f(s,x(s))\, d_{q_0}s. \tag{4.8}$$

In particular, for $t = t_1$,

$$x(t_1) = \alpha + \beta t_1 + \int_0^{t_1} (t_1 - q_0 s) f(s,x(s))\, d_{q_0}s. \tag{4.9}$$

For $t \in J_1 = (t_1, t_2]$, q_1-integrating (4.4), we have

$$D_{q_1}x(t) = D_{q_1}x(t_1^+) + \int_{t_1}^t f(s,x(s))\, d_{q_1}s.$$

Using the third condition of (4.4) with (4.7) yields that

$$D_{q_1}x(t) = \beta + \int_0^{t_1} f(s,x(s))\, d_{q_0}s + I_1^*(x(t_1)) + \int_{t_1}^t f(s,x(s))\, d_{q_1}s. \tag{4.10}$$

For $t \in J_1$, taking q_1-integral for (4.10) and changing the order of q_1-integral, we obtain

$$\begin{aligned} x(t) &= x(t_1^+) + \left[\beta + \int_0^{t_1} f(s,x(s))\, d_{q_0}s + I_1^*(x(t_1))\right](t - t_1) \\ &\quad + \int_{t_1}^t \bigl(t - q_1 s - (1 - q_1)t_1\bigr) f(s,x(s))\, d_{q_1}s. \end{aligned} \tag{4.11}$$

Applying the second equation of (4.4) with (4.9) and (4.11), we get

$$\begin{aligned} x(t) &= \alpha + \beta t_1 + \int_0^{t_1} (t_1 - q_0 s) f(s,x(s))\, d_{q_0}s + I_1(x(t_1)) \\ &\quad + \left[\beta + \int_0^{t_1} f(s,x(s))\, d_{q_0}s + I_1^*(x(t_1))\right](t - t_1) \\ &\quad + \int_{t_1}^t \bigl(t - q_1 s - (1 - q_1)t_1\bigr) f(s,x(s))\, d_{q_1}s \end{aligned}$$

$$\begin{aligned}
&= \alpha + \beta t + \int_0^{t_1} (t_1 - q_0 s) f\bigl(s, x(s)\bigr)\, d_{q_0} s + I_1\bigl(x(t_1)\bigr) \\
&\quad + \left[\int_0^{t_1} f\bigl(s, x(s)\bigr)\, d_{q_0} s + I_1^*\bigl(x(t_1)\bigr)\right](t - t_1) \\
&\quad + \int_{t_1}^{t} \bigl(t - q_1 s - (1 - q_1) t_1\bigr) f\bigl(s, x(s)\bigr)\, d_{q_1} s.
\end{aligned}$$

Repeating the above process, for $t \in J$, we obtain (4.5) as required. □

Next, we prove the existence and uniqueness of a solution to the initial value problem (4.4). We shall use the Banach fixed point theorem to accomplish this.

Theorem 4.2 *Assume that* (H_1) *and* (H_2) *hold. In addition, we suppose that*:

(H_3) $I_k^* : \mathbb{R} \to \mathbb{R}$, $k = 1, 2, \dots, m$, *are continuous functions and satisfy*

$$\bigl|I_k^*(x) - I_k^*(y)\bigr| \le M^* |x - y|, \quad M^* > 0, \forall x, y \in \mathbb{R}.$$

If

$$\theta := L(\nu_1 + T\nu_2 + \nu_3) + mM + (mT + \nu_4)M^* \le \delta < 1,$$

where

$$\nu_1 = \sum_{k=1}^{m+1} \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}}, \qquad \nu_2 = \sum_{k=1}^{m} (t_k - t_{k-1}), \qquad \nu_3 = \sum_{k=1}^{m} t_k (t_k - t_{k-1}), \qquad \nu_4 = \sum_{k=1}^{m} t_k,$$

then the initial value problem (4.4) *has a unique solution on* J.

Proof Firstly, in view of Lemma 4.2, we define an operator $\mathcal{F} : PC(J, \mathbb{R}) \to PC(J, \mathbb{R})$ as

$$\begin{aligned}
(\mathcal{F}x)(t) &= \alpha + \beta t + \sum_{0 < t_k < t} \left(\int_{t_{k-1}}^{t_k} \bigl(t_k - q_{k-1} s - (1 - q_{k-1}) t_{k-1}\bigr) f\bigl(s, x(s)\bigr)\, d_{q_{k-1}} s + I_k\bigl(x(t_k)\bigr)\right) \\
&\quad + t\left[\sum_{0 < t_k < t} \left(\int_{t_{k-1}}^{t_k} f\bigl(s, x(s)\bigr)\, d_{q_{k-1}} s + I_k^*\bigl(x(t_k)\bigr)\right)\right] \\
&\quad - \sum_{0 < t_k < t} t_k \left(\int_{t_{k-1}}^{t_k} f\bigl(s, x(s)\bigr)\, d_{q_{k-1}} s + I_k^*\bigl(x(t_k)\bigr)\right) \\
&\quad + \int_{t_k}^{t} \bigl(t - q_k s - (1 - q_k) t_k\bigr) f\bigl(s, x(s)\bigr)\, d_{q_k} s,
\end{aligned}$$

with $\sum_{0<0}(\cdot) = 0$.

Setting $\sup_{t \in J} |f(t, 0)| = \Omega_1$, $\max\{I_k(0) : k = 1, 2, \dots, m\} = \Omega_2$ and $\max\{I_k^*(0) : k = 1, 2, \dots, m\} = \Omega_3$, we will show that $\mathcal{F}B_R \subset B_R$, where $B_R = \{x \in PC(J, \mathbb{R}) : \|x\| \le R\}$ and a constant R satisfies

$$R \ge \frac{|\alpha| + |\beta| T + \Omega_1(\nu_1 + T\nu_2 + \nu_3) + m\Omega_2 + (mT + \nu_4)\Omega_3}{1 - \varepsilon},$$

where $\delta \le \varepsilon < 1$. For $x \in B_R$, taking into account Example 3.3, we have

$$
\begin{aligned}
&\bigl|(\mathcal{F}x)(t)\bigr| \\
&\quad\le |\alpha| + |\beta| t \\
&\qquad + \sum_{0<t_k<t} \biggl(\int_{t_{k-1}}^{t_k} \bigl(t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}\bigr) \bigl|f\bigl(s,x(s)\bigr)\bigr| \, d_{q_{k-1}}s + \bigl|I_k\bigl(x(t_k)\bigr)\bigr| \biggr) \\
&\qquad + t \biggl[\sum_{0<t_k<t} \biggl(\int_{t_{k-1}}^{t_k} \bigl|f\bigl(s,x(s)\bigr)\bigr| \, d_{q_{k-1}}s + \bigl|I_k^*\bigl(x(t_k)\bigr)\bigr| \biggr) \biggr] \\
&\qquad + \sum_{0<t_k<t} t_k \biggl(\int_{t_{k-1}}^{t_k} \bigl|f\bigl(s,x(s)\bigr)\bigr| \, d_{q_{k-1}}s + \bigl|I_k^*\bigl(x(t_k)\bigr)\bigr| \biggr) \\
&\qquad + \int_{t_k}^{t} \bigl(t - q_k s - (1-q_k)t_k\bigr) \bigl|f\bigl(s,x(s)\bigr)\bigr| \, d_{q_k}s \\
&\quad\le |\alpha| + |\beta| T \\
&\qquad + \sum_{0<t_k<T} \biggl(\int_{t_{k-1}}^{t_k} \bigl(t_k - q_{k-1}s - (1-q_{k-1})t_{k-1}\bigr) \bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| \\
&\qquad + \bigl|f(s,0)\bigr| \bigr) \, d_{q_{k-1}}s + \bigl(\bigl|I_k\bigl(x(t_k)\bigr) - I_k(0)\bigr| + \bigl|I_k(0)\bigr| \bigr) \biggr) \\
&\qquad + T \biggl[\sum_{0<t_k<T} \biggl(\int_{t_{k-1}}^{t_k} \bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr| \bigr) \, d_{q_{k-1}}s \\
&\qquad + \bigl(\bigl|I_k^*\bigl(x(t_k)\bigr) - I_k^*(0)\bigr| + \bigl|I_k^*(0)\bigr| \bigr) \biggr) \biggr] \\
&\qquad + \sum_{0<t_k<T} t_k \biggl(\int_{t_{k-1}}^{t_k} \bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr| \bigr) \, d_{q_{k-1}}s \\
&\qquad + \bigl(\bigl|I_k^*\bigl(x(t_k)\bigr) - I_k^*(0)\bigr| + \bigl|I_k^*(0)\bigr| \bigr) \biggr) \\
&\qquad + \int_{t_m}^{T} \bigl(T - q_m s - (1-q_m)t_m\bigr) \bigl(\bigl|f\bigl(s,x(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr| \bigr) \, d_{q_m}s \\
&\quad\le |\alpha| + |\beta| T + \sum_{k=1}^{m} \biggl(\frac{(t_k - t_{k-1})^2 (LR + \Omega_1)}{(1+q_{k-1})} + (MR + \Omega_2) \biggr) \\
&\qquad + T \Biggl[\sum_{k=1}^{m} \bigl((LR+\Omega_1)(t_k - t_{k-1}) + \bigl(M^*R + \Omega_3\bigr) \bigr) \Biggr] \\
&\qquad + \sum_{k=1}^{m} t_k \bigl((LR+\Omega_1)(t_k - t_{k-1}) + \bigl(M^*R + \Omega_3\bigr) \bigr) + \frac{(LR+\Omega_1)(T - t_m)^2}{1+q_m} \\
&\quad= |\alpha| + |\beta| T + (LR + \Omega_1)(\nu_1 + T\nu_2 + \nu_3) \\
&\qquad + (MR + \Omega_3)(mT + \nu_4) + m(MR + \Omega_2) \\
&\quad\le (\delta + 1 - \varepsilon) R \le R.
\end{aligned}
$$

Then we get that $\mathcal{F}B_R \subset B_R$.

For any $x, y \in PC(J, \mathbb{R})$, we have

$$\begin{aligned}
\left|(\mathcal{F}x)(t) - (\mathcal{F}y)(t)\right| &\le \sum_{k=1}^{m} \biggl(\int_{t_{k-1}}^{t_k} \bigl(t_k - q_{k-1}s - (1 - q_{k-1})t_{k-1}\bigr) \\
&\quad \times \bigl|f\bigl(s, x(s)\bigr) - f\bigl(s, y(s)\bigr)\bigr| \, d_{q_{k-1}}s + \bigl|I_k\bigl(x(t_k)\bigr) - I_k\bigl(y(t_k)\bigr)\bigr| \biggr) \\
&\quad + T\Biggl[\sum_{k=1}^{m} \biggl(\int_{t_{k-1}}^{t_k} \bigl|f\bigl(s, x(s)\bigr) - f\bigl(s, y(s)\bigr)\bigr| \, d_{q_{k-1}}s \\
&\quad + \bigl|I_k^*\bigl(x(t_k)\bigr) - I_k^*\bigl(y(t_k)\bigr)\bigr| \biggr) \Biggr] \\
&\quad + \sum_{k=1}^{m} t_k \biggl(\int_{t_{k-1}}^{t_k} \bigl|f\bigl(s, x(s)\bigr) - f\bigl(s, y(s)\bigr)\bigr| \, d_{q_{k-1}}s \\
&\quad + \bigl|I_k^*\bigl(x(t_k)\bigr) - I_k^*\bigl(y(t_k)\bigr)\bigr| \biggr) \\
&\quad + \int_{t_m}^{T} \bigl(t - q_m s - (1 - q_m)t_m\bigr) \bigl|f\bigl(s, x(s)\bigr) - f\bigl(s, y(s)\bigr)\bigr| \, d_{q_m}s \\
&\le \sum_{k=1}^{m} \biggl(\frac{(t_k - t_{k-1})^2}{(1 + q_{k-1})} L + M \biggr) \|x - y\| \\
&\quad + T\Biggl[\sum_{k=1}^{m} \bigl(L(t_k - t_{k-1}) + M^*\bigr) \Biggr] \|x - y\| \\
&\quad + \sum_{k=1}^{m} t_k \bigl(L(t_k - t_{k-1}) + M^*\bigr) \|x - y\| + L \frac{(T - t_m)^2}{1 + q_m} \|x - y\| \\
&= \theta \|x - y\|,
\end{aligned}$$

which implies that $\|\mathcal{F}x - \mathcal{F}y\| \le \theta\|x - y\|$. As $\theta < 1$, by the Banach contraction mapping principle, $\mathcal{F}$ has a fixed point which is a unique solution of (4.4) on J. □

Example 4.2 Consider the following second-order impulsive q_k-difference initial value problem:

$$\begin{aligned}
&D_{\frac{2}{3+k}}^2 x(t) = \frac{e^{-\sin^2 t}|x(t)|}{(7 + t)^2(1 + |x(t)|)}, \quad t \in J = [0, 1], t \neq t_k = \frac{k}{10}, \\
&\Delta x(t_k) = \frac{|x(t_k)|}{5(6 + |x(t_k)|)}, \quad k = 1, 2, \ldots, 9, \\
&D_{\frac{2}{3+k}} x\bigl(t_k^+\bigr) - D_{\frac{2}{3+k-1}} x(t_k) = \frac{1}{9} \tan^{-1}\biggl(\frac{1}{5} x(t_k)\biggr), \quad k = 1, 2, \ldots, 9, \\
&x(0) = 0, \qquad D_{\frac{2}{3}} x(0) = 0.
\end{aligned} \tag{4.12}$$

Here $q_k = 2/(3 + k)$, $k = 0, 1, 2, \ldots, 9$, $m = 9$, $T = 1$, $f(t, x) = (e^{-\sin^2 t}|x|)/((7 + t)^2(1 + |x|))$, $I_k(x) = |x|/(5(6 + |x|))$ and $I_k^*(x) = (1/9)\tan^{-1}(x/5)$. Since $|f(t, x) - f(t, y)| \le (1/49)|x - y|$, $|I_k(x) - I_k(y)| \le (1/30)|x - y|$ and $|I_k^*(x) - I_k^*(y)| \le (1/45)|x - y|$, then (H_1), (H_2) and (H_3)

are satisfied with $L = (1/49)$, $M = (1/30)$, $M^* = (1/45)$. We find that

$$\nu_1 = \sum_{k=1}^{m+1} \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} = \frac{1{,}380{,}817}{180{,}180}, \qquad \nu_2 = \sum_{k=1}^{m} (t_k - t_{k-1}) = \frac{9}{10},$$
$$\nu_3 = \sum_{k=1}^{m} t_k (t_k - t_{k-1}) = \frac{45}{100}, \qquad \nu_4 = \sum_{k=1}^{m} t_k = \frac{45}{10}.$$

Clearly,

$$L(\nu_1 + T\nu_2 + \nu_3) + mM + (mT + \nu_4)M^* = 0.7839 < 1.$$

Hence, by Theorem 4.2, the initial value problem (4.12) has a unique solution on $[0,1]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally in this article. They read and approved the final manuscript.

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