Quantum capacity of dephasing channels with memory

Giuliano Benenti

Center for Nonlinear and Complex Systems

Università degli Studi dell'Insubria

Como, Italy

http://scienze-como.uninsubria.it/benenti
http://scienze-como.uninsubria.it/complexcomo

In collaboration with:
Antonio D'Arrigo, Giuseppe Falci (Catania)

Motivations and Outline

- Non-Markovian effects in open quantum systems
- Can memory effects enhance the capacity of quantum channels?

Dephasing channels with memory: quantum capacity maximized by separable input states

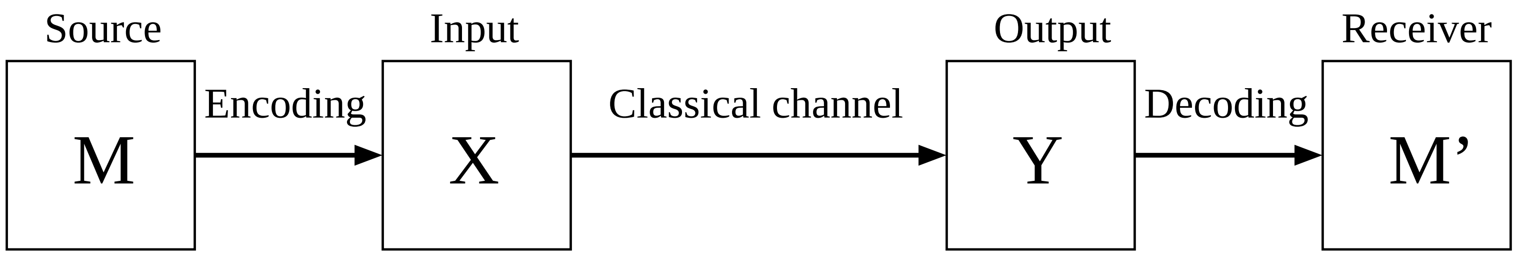
1) Markov chain model: explicit computation of the quantum capacity

2) Bosonic bath of oscillators

Non-Markovian effects

- Low-frequency noise noise in solid-state devices (for instance, $1/f$ noise)
- Fluctuating birefringe in optical fibers
- Quantum information transmission across spin chains
- Sending atoms through a resonant cavity

Capacity of a classical channel



MUTUAL INFORMATION $I(X:Y) = H(X) + H(Y) - H(X,Y)$

$H(X) = -\sum_x p_x \log_2 p_x$ Shannon information of the random variable X

CAPACITY: maximum rate at which classical information can be reliably transmitted down the channel

$$C = \max_{p_x} H(X:Y)$$

Quantum channels

QUANTUM SOURCE: quantum states chosen from the ensemble $\{\rho_0, ..., \rho_k\}$ with a priori probabilities $\{p_1, ..., p_k\}$ are sent through the channel

QUANTUM CHANNEL described by a linear, completely positive, trace preserving (CPT) map $\mathcal{E}$:

$$\rho' = \mathcal{E}(\rho), \quad \rho = \sum_{x=1}^{k} p_x \rho_x$$

Use quantum states to reliably transmit classical information (classical capacity) or quantum information (quantum capacity)

Entanglement fidelity

How to measure the reliability in the transmission of quantum information?

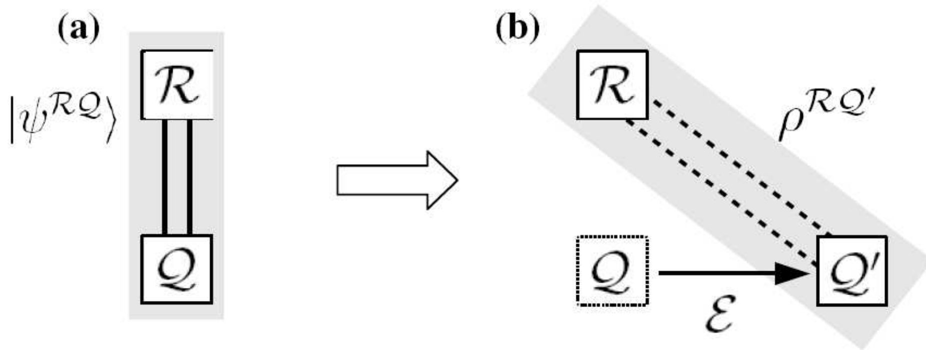
It is not sufficient to verify that the input state ρ is transmitted with high fidelity
Ex: send a member of a Bell pair through a completely dephasing channel

$$|\psi\rangle_{12} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle), \quad \rho = \mathrm{Tr}_2(|\psi\rangle_{12}\langle\psi|) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\mathcal{E}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & 0 \\ 0 & \rho_{11} \end{pmatrix}$$

$$\mathcal{E}(\rho) = \rho, \quad \text{but} \quad (\mathcal{E}\otimes\mathcal{I})(|\psi\rangle_{12}\langle\psi|) = \frac{1}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|)$$

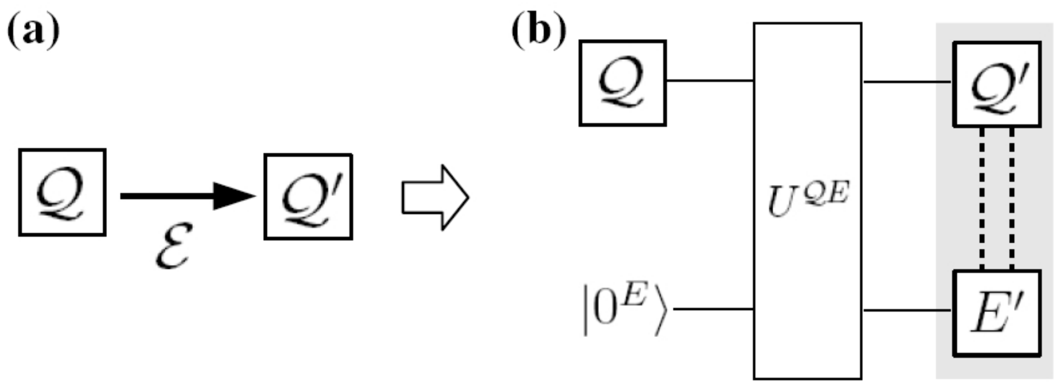
Entanglement is lost



$$
\begin{aligned}
F_e = F_e(\rho, \mathcal{E}) &= F\big(|\psi^{\mathcal{RQ}}\rangle, \rho^{\mathcal{RQ}'}\big) = \langle\psi^{\mathcal{RQ}}|\,\rho^{\mathcal{RQ}'}\,|\psi^{\mathcal{RQ}}\rangle \\
&= \langle\psi^{\mathcal{RQ}}|\,(\mathcal{E}^{\mathcal{Q}} \otimes \mathcal{I}^{\mathcal{R}})\big(|\psi^{\mathcal{RQ}}\rangle\langle\psi^{\mathcal{RQ}}|\big)\,|\psi^{\mathcal{RQ}}\rangle
\end{aligned}
$$

The ENTANGLEMENT FIDELITY F_e is independent of the purification $\mathcal{R}$ of the quantum system $\mathcal{Q}$

Entropy exchange



$$S_e = S_e(\rho, \mathcal{E}) = S(\rho^{E'}), \quad S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho) \text{ von Neumann entropy}$$

The ENTROPY EXCHANGE S_e is the entropy of the final state $\rho^{E'}$ of a "mock" environment, initially in a pure state $|0^E\rangle$

Coherent information

Analogous to mutual information but for quantum information

$$I_c\big(\rho,\mathcal{E}\big) \;=\; S\big(\mathcal{E}(\rho)\big) \;-\; S_e(\rho,\mathcal{E})$$

The COHERENT INFORMATION $I_c = S(\rho^{\mathcal{Q}'}) - S(\rho^{\mathcal{R}\mathcal{Q}'})$ can never be positive for classical systems

I_c deals with the entanglement transmission through the channel
Ex: if the channel is noiseless, $I_c = 0$ is ρ pure, I_c is maximum if ρ is maximally mixed

Quantum data-processing inequality:

$$I_c(\rho, \mathcal{E}_1) \geq I(\rho, \mathcal{E}_2 \circ \mathcal{E}_1)$$

We cannot increase the coherent information acting on the output

In contrast to mutual information, I_c in general is not subadditive
Using entangled input states $\rho_{12} \neq \rho_1 \otimes \rho_2$ $[\rho_1 = \mathrm{Tr}_2(\rho_{12}), \rho_2 = \mathrm{Tr}_1(\rho_{12}))]$ we can obtain

$$I_c(\rho_{12}, \mathcal{E} \otimes \mathcal{E}) > I_c(\rho_1, \mathcal{E}) + I_c(\rho_2, \mathcal{E})$$

Quantum capacity

The QUANTUM CAPACITY Q measures the maximum number of qubits (per channel use) that can be reliably trasmitted down a noisy channel

For memoryless channels

$$Q = \lim_{n\to\infty} \frac{Q_n}{n}, \quad Q_n = \max_{\rho} I_c(\mathcal{E}_n, \rho)$$

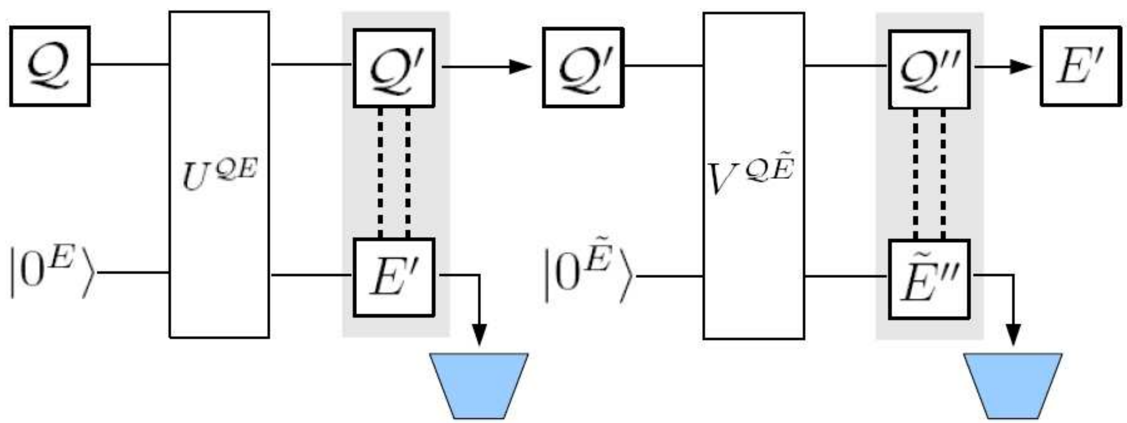
$$I_c(\mathcal{E}_n, \rho) = S[\mathcal{E}_n(\rho)] - S[\tilde{\mathcal{E}}_n(\rho)], \quad \mathcal{E}_n = \mathcal{E}^{\otimes n}$$

$$\rho' = \mathcal{E}_n(\rho) = \mathrm{Tr}_E[U_n(\rho \otimes |0\rangle_E\langle 0|)U_n^\dagger], \quad \rho'_E = \tilde{\mathcal{E}}_n(\rho) = \mathrm{Tr}_S[U_n(\rho \otimes |0\rangle_E\langle 0|)U_n^\dagger]$$

The regularization $n \to \infty$ is necessary since I_c in general fails to be subadditive

Degradable channels

Degradable channels: the final state ρ'_E of the environment can be reconstructed from the final sate ρ' of the system



In this case $I_c = S(\rho^{\mathcal{Q}'E'}) - S(\rho^{E'}) = S(\mathcal{Q}'|E')$ is subadditive
$Q = Q_1$ ("single-letter" formula)

Dephasing channels

Single-use dephasing channel:

$$\mathcal{E}\begin{pmatrix} \rho_{00} & \rho_{01} \\ \rho_{10} & \rho_{11} \end{pmatrix} = \begin{pmatrix} \rho_{00} & g\rho_{01} \\ g\rho_{10} & \rho_{11} \end{pmatrix}, \quad g \text{ dephasing factor}$$

Generalized n-uses dephasing channel

$$U_n|i\rangle|0\rangle_E = |i\rangle|\phi_i\rangle_E, \quad |i\rangle = |i_1, ..., i_n\rangle \text{ preferential basis}$$

$$\rho' = \mathcal{E}_n(\rho) = \sum_\alpha A_\alpha \rho A_\alpha^\dagger, \quad A_\alpha = {}_E\langle\alpha|U_n|0\rangle_E \text{ diagonal Kraus operators}$$

$$(A_\alpha)_{ij} = {}_E\langle\alpha|\phi_i\rangle_E \delta_{ij}$$

Degradability of the generalized dephasing channel

$$\rho = \sum_{i,j} c_{ij} |i\rangle\langle j| \text{ generic input state}$$

The final environmental state depends only on the populations of ρ

$$\rho'_E = \tilde{\mathcal{E}}_n(\rho) = \sum_i |c_i|^2 |\phi_i\rangle_E \langle \phi_i|$$

Since the dephasing channel $\mathcal{E}_n$ does not affect populations,

$$\tilde{\mathcal{E}}_n = \tilde{\mathcal{E}}_n \circ \mathcal{E}_n$$

Maximization of the coherent information

The coherent information $I_c(\mathcal{E}_n, \rho)$ of a generalized dephasing channel is maximized by SEPARABLE input states DIAGONAL in the preferential basis $\{|i\rangle\}$

$$\rho_k = \frac{\rho_{k-1} + \sigma_z^{(k)} \rho_{k-1} \sigma_z^{(k)}}{2}, \quad (k = 1, ..., n)$$

- the Kraus operators commute with $\sigma_z^{(k)}$
- the coherent information is concave for degradable channels

$$I_c(\mathcal{E}_n, \rho_n) \geq I_c(\mathcal{E}_n, \rho_{n-1}) \geq \cdots \geq I_c(\mathcal{E}_n, \rho_0)$$

Forgetful channels

Memory effects vanish exponentially fast with time

DOUBLE-BLOCKING strategy:

- consider blocks of $n + l$ uses of the channel
- do the actual coding and decoding for the first n uses
- let $n \to \infty$
- quantum capacity $Q = \lim_{n\to\infty} \frac{Q_n}{n}$

A phenomenological noise model

$$\rho' = \mathcal{E}_n(\rho) = \sum_{i_1,\ldots,i_n=0,z} A_{i_1\ldots i_n} \rho A^\dagger_{i_1\ldots i_n}, \quad A_{i_1\ldots i_n} = \sqrt{p_{i_1\ldots i_n}} \sigma^{(1)}_{i_1} \otimes \cdots \otimes \sigma^{(n)}_{i_n},$$

$p_{i_1\ldots i_n}$ probability that the the ordered sequence $\sigma^{(1)}_{i_1}, \ldots, \sigma^{(n)}_{i_n}$ of Pauli operators (I or σ_z) is applied to the n qubits crossing the channel

- Dephasing probability stationary: $p_{i_k=z} = p_z$ $[p_{i_k=0} = p_0 = 1 - p_z]$ for all k $(p_{i_k} = \sum_{i_1,\ldots,i_{k-1},i_{k+1},\ldots,i_n} p_{i_1\ldots i_n})$

- Forgetful channel: $|p_{i_{k'} i_k} - p_{i_{k'}} p_{i_k}|$ decays exponentially with $|k' - k|$

The maximum of coherent information in this model is obtained for the maximally mixed input state $\rho_I \equiv \frac{1}{2^n} I^{\otimes n}$

$$\rho_k = \frac{\rho_{k-1} + \sigma_x^{(k)} \rho_{k-1} \sigma_x^{(k)}}{2}, \quad (k = 1, ..., n)$$

Starting from a diagonal ρ_0 we can prove

$$I_c(\mathcal{E}_n, \rho_n = \rho_I) \geq I_c(\mathcal{E}_n, \rho_0)$$

Markov-chain model

$$p_{i_1,\ldots,i_n} = p_{i_1} p_{i_2|i_1} \cdots p_{i_n|i_n-1}, \quad p_{i_k|i_{k-1}} = (1-\mu) p_{i_k} + \mu \delta_{i_k,i_{k-1}}$$

μ measures the partial memory of the channel
$\mu = 0$ memoryless channel
$\mu = 1$ perfect memory

μ might depend on the time interval τ between two consecutive channel uses, compared with the memory time scale τ_c

Quantum capacity of a Markov chain

$$S[\mathcal{E}_n(\rho_I)] = S(\rho_I) = n$$

$$S_e = -\sum_{i_1,\ldots,i_n} p_{i_1\ldots i_n} \log_2 p_{i_1\ldots i_n} \equiv H(X_1,...,X_n)$$

$H(X_1,...,X_n)$ Shannon entropy of the collection of random variables $X_1,...,X_n$ (characterized by the joint probabilities $p_{i_1\ldots i_n}$)

For a stationary Markov chain

$$\lim_{n\to\infty} \frac{1}{n} H(X_1,...,X_n) = H(X_2|X_1) = p_0 H(q_0) + p_z H(q_z), \quad q_i \equiv p(i|i) = (1-\mu)p_i + \mu$$

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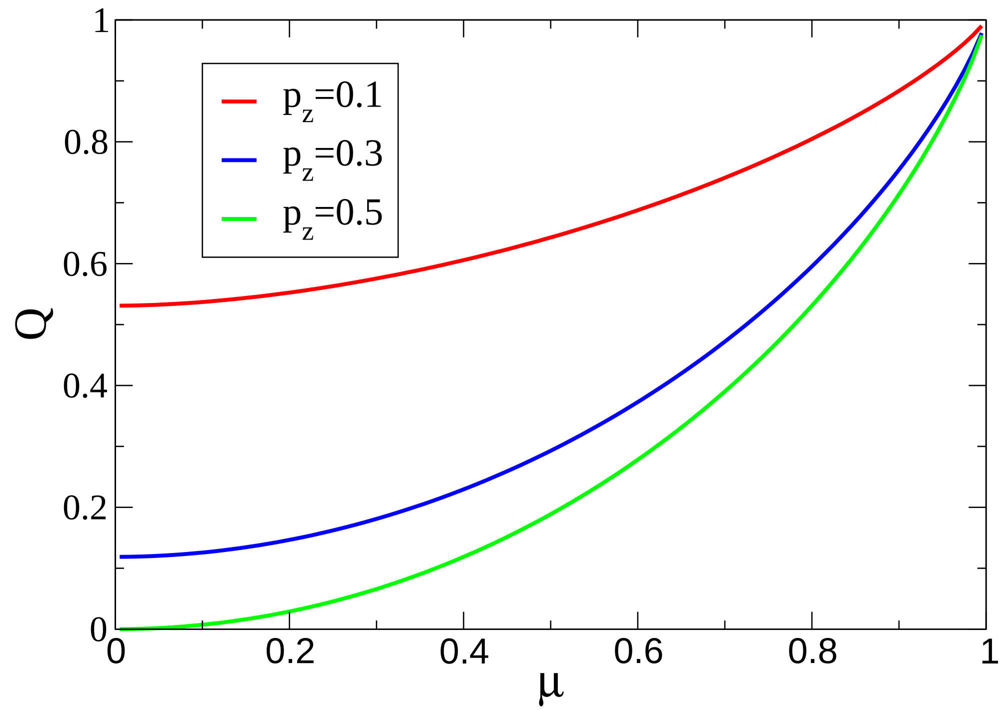
$$Q = 1 - p_0 H(q_0) - p_z H(q_z)$$

$Q = 1 - H(p_0)$ memoryless limit

$Q = 1$ with perfect memory (noiseless channel)



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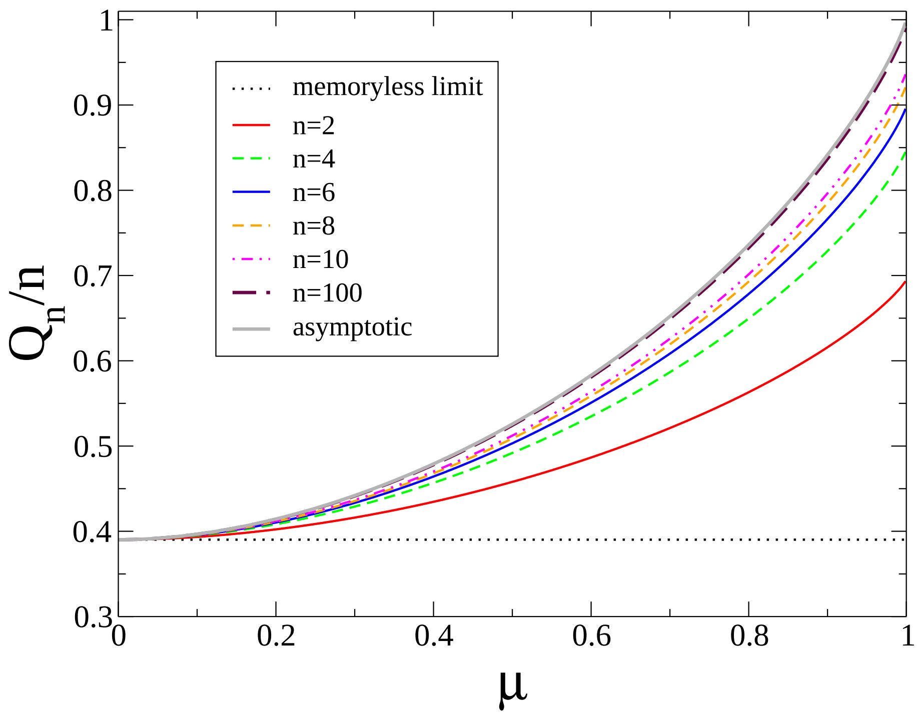
Convergence of Q_n/n to Q

$$Q_n = n - (n-1)[p_0 H(q_0) + p_z H(q_z)] - H(p_0)$$



$\epsilon_n \equiv Q - \frac{Q_n}{n}$ growing function of μ

$$\epsilon_n(\mu = 0) = 0$$

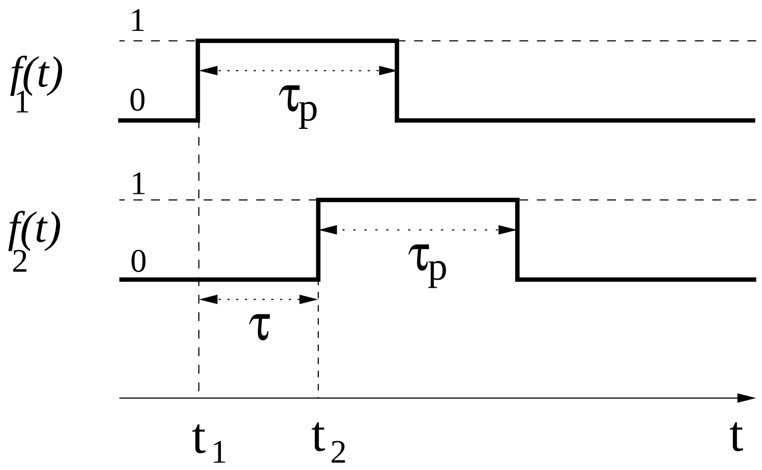
$$\epsilon_n(\mu = 1) = H(p_0)/n$$

$\epsilon_n(\mu) \approx \frac{1}{2\ln 2}\frac{\mu^2}{n}$ for $\mu \ll 1$

Bosonic bath environment

$$H(t) = H_E - \frac{1}{2} X_E F(t) + H_C, \quad H_E = \sum_\alpha \omega_\alpha b_\alpha^\dagger b_\alpha,$$

$$X_E = \sum_\alpha (b_\alpha^\dagger + b_\alpha), \quad F(t) = \lambda \sum_{j=1}^{n} \sigma_z^{(j)} f_j(t), \quad H_C = \sum_\alpha \frac{\lambda^2}{4\omega_\alpha} \sum_{j=1}^{n} \sigma_z^{(j)}$$



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$$\rho(t) = \mathrm{Tr}_E[U(t)(\rho \otimes \rho_E)U^\dagger(t)], \quad U(t) = Te^{-\frac{i}{\hbar}\int_0^t ds H(s)}$$

$U(t|i) = \langle i|U(t)|i\rangle$ conditional evolution operator for the environment alone

$$(\rho')_{ij} = (\rho)_{ij} \sum_\alpha {}_E\langle\alpha|U(t|i)\rho_E U^\dagger(t|j)|\alpha\rangle_E$$

Multimode environment of oscillators initially at thermal equilibrium,
$\rho_E = \exp(-\beta H_E)$:

$$\sum_\alpha {}_E\langle\alpha|U(t|i)\rho_E U^\dagger(t|j)|\alpha\rangle_E = e^{\left\{-\lambda^2 \int_0^\infty \frac{d\omega}{\pi} S(\omega) \frac{1-\cos(\omega\tau_p)}{\omega^2} \left| \sum_{k=1}^n (i_k - j_k) e^{i\omega(k-1)\tau} \right|^2 \right\}}$$

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Assume that the bath correlation function

$$C(t) \equiv \frac{1}{2}\langle X_E(t)X_E(0) + X_E(0)X_E(t)\rangle$$

decays exponentially with time (forgetful channel)

This is the case, e.g., for a Lorentian power spectrum

$$S(\omega) = \frac{2\tau_c}{[1 + (\omega\tau_c)^2]}$$

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Decoherence-protected subspace

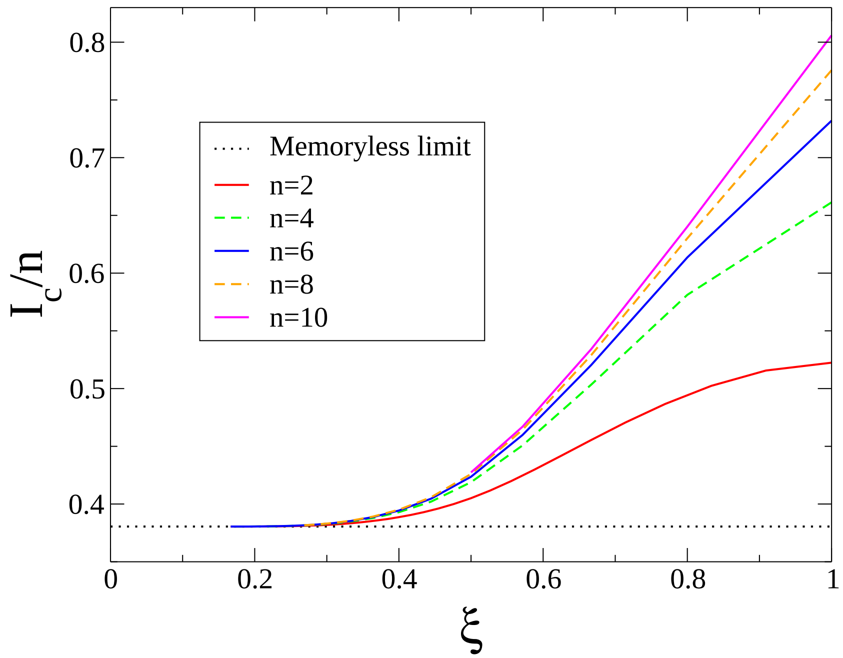
In the limit of perfect memory $(\tau_c \to \infty)$ there exists for any n a decoherence-free subspace corresponding to a qubit train with an equal number of $|0\rangle$ and $|1\rangle$ states

Since the dimension d of this subspace is such that $\log_2 d \approx n - \frac{1}{2}\log_2 n$, then the channel is asymptotically noiseless $(Q = 1)$

If $\bar{n} \gg 1$ qubits can be sent within the memory time scale τ_c and the quantum information is encoded in the decoherence-protected subspace, then

$1 - \log_2 \bar{n}/(2\bar{n})$ lower bound for $Q_{\bar{n}}/\bar{n}$

Lower bound for Q_n/n



maximally mixed input state

Lorentzian power spectrum

$$\xi \equiv \tau_c/(\tau + \tau_c)$$

Numerial data suggest that I_c/n converges for $n \to \infty$: it is possible to increase the transmission rate if quantum information is encoded in long blocks, separated by time intervals larger than τ_c

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Conclusions

The coherent information in a dephasing channel with memory is maximized by separable input states

Computed the quantum capacity Q for a Markov chain noise model and provided numerical evidence of a lower bound for Q in the case of a bosonic bath

• Find realistic coding strategies for few-qubit trains

• The results of this study could be adapted to environments with algebraically decaying memory effects, e.g. low-frequency noise in the solid state?

• Effects of integrable/chaotic environments or of quantum phase transitions on quantum channel capacity