Quantum Channel Capacities With Passive Environment Assistance

Siddharth Karumanchi, Stefano Mancini, Andreas Winter, and Dong Yang

***Abstract*— We initiate the study of passive environment-assisted communication via a quantum channel, modeled as a unitary interaction between the information carrying system and an environment. In this model, the environment is controlled by a benevolent helper, who can set its initial state such as to assist sender and receiver of the communication link (the case of a malicious environment, also known as jammer, or arbitrarily varying channel, is essentially well-understood and comprehensively reviewed). Here, after setting out precise definitions, focusing on the problem of quantum communication, we show that entanglement plays a crucial role in this problem: indeed, the environment-assisted capacity where the helper is restricted to product states between the channel uses is different from the one with unrestricted helper. Furthermore, prior shared entanglement between the helper and the receiver makes a difference, too.**

***Index Terms*— Quantum channels, quantum capacity, super-activation, entanglement.**

I. Introduction

IN quantum Shannon theory it is customary to model communication channels as completely positive and trace preserving (CPTP) maps on states; this notion contains as a special case classical channels [42]. It is a well-known fact that each CPTP map can be decomposed into a unitary interaction with a suitable environment system and the discarding of that environment. This means that the noise of the channel can be entirely attributed to losing information into the environment, which raises the question of how much better one could communicate over the channel if one had access to the environment. Note that "access to the environment" is ambiguous at this point, but that one can distinguish at least two broad directions, one concerned with the exploitation of the information in the environment after the interaction and the other with the control of the state of the environment before the interaction – and of course both.

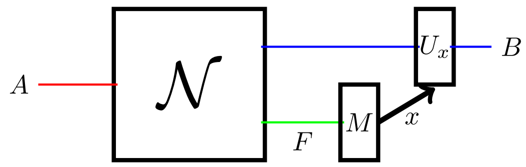

Fig. 1. Diagrammatic view of the three parties involved in the communication setting with an active helper. In this model the helper measures the output state with a POVM (M_x), $\sum_x M_x = \mathbb{1}$, and sends the classical message x to Bob, who applies a corresponding unitary U_x to recover the initial message of Alice.

The first direction has been addressed starting from Gregoratti and Werner's "quantum lost and found" [18], [19] and focusing on the error correction ability of this scheme for random unitary channels [8] as well as for other channel types [31], [32]. The problem was set in an information theoretic vein in [21] and culminated in the determination of the "environment-assisted" quantum capacity of an interaction with fixed initial state of the environment, but arbitrary measurements on the environment output fed forward to the receiver [41] (see Fig. 1). These findings were partially extended to the classical capacity [44], which revealed an interesting connection to data hiding and highlighted the impact of the precise restriction on the measurements being performed on the combined channel-output and environment-output system. Note that, whereas the usual capacity theory for quantum channels treats the environment as completely inaccessible, these results assume full access to the environment and classical communication to the receiver. Thus, whoever controls the environment can be considered as an *active helper*.

In the present paper we are concerned with the second avenue, to be precise, a model where the communicating parties have no access to the environment-output but instead there is a third party controlling the initial state of the environment. The choice of initial environment state effectively is a way of preparing a channel between Alice and Bob. Depending on the aim of that party, we call the model "communication with a *passive helper*" if she is benevolent (because she only chooses the initial state and does not intervene otherwise), or "communication in the presence of a *jammer*" if she is malicious (see Fig. 2).

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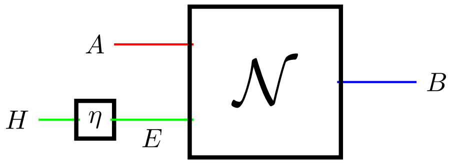

Fig. 2. Diagrammatic view of the three parties involved in the communication with a party controlling the environment input system. Depending on the goal of the party controlling the environment-input, either to assist or to obstruct the communication between the sender Alice and the receiver Bob, we call it passive helper (Helen) or jammer (Jenny), respectively.

In the next Section II we shall define the model rigorously, as well as the different notions of assisted and adversarial codes and associated (quantum) capacities, and make initial general observations. In Section III we then go on to study two-qubit unitaries, which allow for the computation or estimation of capacities. They also show a range of general phenomena, including super-activation of capacities that are discussed in Section IV. These findings put into the focus a variation of the passive helper, where she can use pre-shared entanglement with the receiver, a model which we explore in Section V. We conclude in Section VI with a number of open problems and suggestions for future investigations. Two appendices contain the technical details of the random coding capacity formula of the jammer model (Appendix A), and the analysis of the (anti-)degradability properties of two-qubit unitaries (Appendix B).

II. Assisted and Adversarial Capacities

As mentioned in the introduction we are concerned with the model of communication where there is a third party, other than the sender and receiver, who has access to the environment input system. The party's role is either to assist or hamper the quantum communication from Alice to Bob, which is distinguished in our nomenclature as Helen (helper) and Jenny (jammer), respectively.

Let A, E, B, F, etc. be finite dimensional Hilbert spaces, $\mathcal{L}(X)$ denote the space of linear operators on the Hilbert space X, and $|X|$ denote the dimension of the Hilbert space.

We denote the identity operator in $\mathcal{L}(X)$ as $\mathbb{1}^X$ and the ideal map, $\mathrm{id} : \mathcal{L}(X) \to \mathcal{L}(X)$ is denoted by id^X. For a density operator α^X the *von Neumann entropy* is defined as

$$S(X)_\alpha = S(\alpha) := -\operatorname{Tr}(\alpha \log \alpha). \tag{1}$$

Furthermore, the binary entropy is denoted by

$$H_2(p) = -p \log(p) - (1-p)\log(1-p). \tag{2}$$

Consider an isometry $V : A \otimes E \hookrightarrow B \otimes F$, which defines the channel (CPTP map) $\mathcal{N} : \mathcal{L}(A \otimes E) \to \mathcal{L}(B)$, whose action on the input state is

$$\mathcal{N}^{AE\to B}(\rho) = \operatorname{Tr}_F V\rho V^\dagger.$$

The complementary channel, $\widetilde{\mathcal{N}} : \mathcal{L}(A \otimes E) \to \mathcal{L}(F)$, is given by

$$\widetilde{\mathcal{N}}^{AE\to F}(\rho) = \operatorname{Tr}_B V\rho V^\dagger.$$

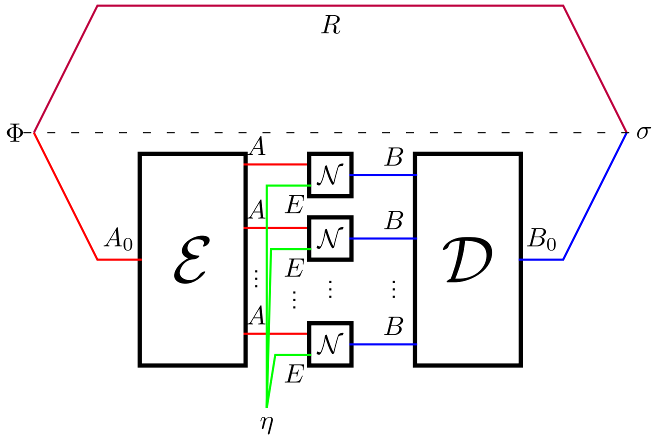

Fig. 3. Schematic of a general protocol to transmit quantum information with passive assistance from the environment; $\mathcal{E}$ and $\mathcal{D}$ are the encoding and decoding maps respectively, the initial state of the environment is η.

By inputting an environment state η on E, an effective channel $\mathcal{N}_\eta : \mathcal{L}(A) \to \mathcal{L}(B)$ is defined, via

$$\mathcal{N}_\eta^{A\to B}(\rho) = \mathcal{N}^{AE\to B}(\rho \otimes \eta).$$

Clearly, for channels $\mathcal{N}_i : \mathcal{L}(A_i E_i) \to \mathcal{L}(B_i)$ and states η_i,

$$(\mathcal{N}_1 \otimes \mathcal{N}_2)_{\eta_1 \otimes \eta_2} = (\mathcal{N}_1)_{\eta_1} \otimes (\mathcal{N}_2)_{\eta_2}.$$

Note that if η is pure, then the complementary channel is given by

$$\widetilde{\mathcal{N}_\eta} = (\widetilde{\mathcal{N}})_\eta,$$

but this is not true in general for mixed states η. In other words taking the effective channel and then the complementary is equivalent to take the complementary channel and then the effective channel provided that the state of the environment is pure.

Referring to Fig. 3, to send information down this channel from Alice to Bob, we furthermore need an encoding CPTP map $\mathcal{E} : \mathcal{L}(A_0) \to \mathcal{L}(A^n)$ and a decoding CPTP map $\mathcal{D} : \mathcal{L}(B^n) \to \mathcal{L}(B_0)$, where the dimension of A_0 is equal to the dimension of B_0. The output after the overall dynamics, when we input a maximally entangled test state Φ^{RA_0}, with R being an inaccessible reference system, is $\sigma^{RB_0} = \mathcal{D}\big(\mathcal{N}^{\otimes n}\big(\mathcal{E}(\Phi^{RA_0}) \otimes \eta^{E^n}\big)\big)$.

Definition 1: A *passive environment-assisted quantum code* of block length n is a triple $(\mathcal{E}^{A_0\to A^n}, \eta^{E^n}, \mathcal{D}^{B^n\to B_0})$. Its *fidelity* is given by $F = \operatorname{Tr}\Phi^{RA_0}\sigma^{RB_0}$, and its *rate* $\frac{1}{n}\log|A_0|$.

A rate R is called *achievable* if there are codes for all block lengths n with fidelity converging to 1 and rate converging to R. The *passive environment-assisted quantum capacity* of V, denoted $Q_H(V)$, or equivalently $Q_H(\mathcal{N})$, is the supremum of all achievable rates.

If the helper is restricted to fully separable states η^{E^n}, i.e. convex combinations of tensor products $\eta^{E^n} = \eta_1^{E_1} \otimes \cdots \otimes \eta_n^{E_n}$, the supremum of all achievable rates is denoted $Q_{H\otimes}(V) = Q_{H\otimes}(\mathcal{N})$.

A very similar model, however with the aim of maximizing the "transfer fidelity" (averaged over all pure states of A), was considered recently by Liu *et al.* [27]. Although the figure of merit is different, the objective of that paper is, like ours, a quantitative index for the transmission power of a bipartite unitary, assisted by a benevolent helper.

As the fidelity is linear in the environment state η, without loss of generality η may be assumed to be pure, both for the unrestricted and separable helper. We shall assume this from now on always in the helper scenario, without necessarily specifying it each time.

Remark 2: Our model, since it allows for an isometry V, includes the plain Stinespring dilation $V : A \hookrightarrow B \otimes F$ of a quantum channel (CPTP map) $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$, for trivial (1-dimensional) $E = \mathbb{C}$ so that the helper doesn't really have any choice of initial state. In this case the quantum capacity is well-understood thanks to the works of Schumacher, Lloyd, Shor and Devetak. The fundamental quantity is the *coherent information* [2], [35], [36], see also [42]

$$I(A'\rangle B)_\sigma := S(\sigma^B) - S(\sigma^{A'B}) = -S(A'|B)_\sigma,$$

which needs to be evaluated for states $\sigma^{A'B} = (\mathrm{id} \otimes \mathcal{N}^{A\to B})\phi^{A'A}$, where $\phi^{A'A}$ is a purification of a generic density matrix ρ^A:

$$I_c(\rho; \mathcal{N}) := I(A'\rangle B)_{(\mathrm{id}\otimes\mathcal{N})\phi} = S(\mathcal{N}(\rho)) - S(\widetilde{\mathcal{N}}(\rho)).$$

Then [2], [12], [30], [35], [36], [38],

$$Q(\mathcal{N}) = \sup_n \max_{\rho^{(n)}} \frac{1}{n} I_c\big(\rho^{(n)}; \mathcal{N}^{\otimes n}\big),$$

where the maximum is over all states $\rho^{(n)}$ on A^n. It is known that the supremum over n (the "regularization") is necessary [10], [14], [15], [37], except for some special channels – see below.

On the other hand, the helper has the largest range of options to assist if V is a *unitary*. This will be the case that shall occupy us most in the sequel. However, in any case, we assume that the input to V is a product state between Alice and Helen, since they have to act independently, albeit in coordination.

Before we continue with our development of the theory of passive environment-assisted capacities, we pause for a moment to reflect on the role of the environment. While our above definitions model a benevolent agent controlling the environment input, one may ask what results if instead she is *malevolent*, i.e. trying to jam the communication between Alice and Bob. This is captured by the following definition:

Definition 3: A *quantum code* of block length n for the jammer channel $\mathcal{N}^{AE\to B}$ is a pair $(\mathcal{E}^{A_0\to A^n}, \mathcal{D}^{B^n\to B_0})$, with two spaces A_0 and B_0 of the same dimension. Its rate is, as before, $\frac{1}{n}\log|A_0|$, while the fidelity is given by

$$F := \min_{\eta^{E^n}} \mathrm{Tr}\, \Phi^{RA_0}\sigma^{RB_0},$$

where η^{E^n} ranges over all states on E^n, and $\sigma^{RB_0} = \mathcal{D}\big(\mathcal{N}^{\otimes n}\big(\mathcal{E}(\Phi^{RA_0}) \otimes \eta^{E^n}\big)\big)$, with a maximally entangled state Φ^{RA_0}.

A *random quantum code* is given by an ensemble of codes $(\mathcal{E}_\lambda^{A_0\to A^n}, \mathcal{D}_\lambda^{B^n\to B_0})$ with a random variable λ. The rate is as before, and the fidelity

$$\overline{F} := \min_{\eta^{E^n}} \mathbb{E}_\lambda \,\mathrm{Tr}\, \Phi^{RA_0}\sigma_\lambda^{RB_0},$$

where now $\sigma_\lambda^{RB_0} = \mathcal{D}_\lambda\big(\mathcal{N}^{\otimes n}\big(\mathcal{E}_\lambda(\Phi^{RA_0}) \otimes \eta^{E^n}\big)\big)$.

The corresponding *adversarial quantum capacities*, to emphasize the presence of the jammer, are denoted $Q_J(\mathcal{N})$ and $Q_{J,r}(\mathcal{N})$, respectively.

Remark 4: The special case where the jammer controls a classical input E, i.e. there is an orthonormal basis $\{|s\rangle\}$ of E such that

$$\mathcal{N}(\rho \otimes |s\rangle\langle t|) = \delta_{st}\, \mathcal{N}_s(\rho),$$

has been introduced and studied in-depth by Ahlswede *et al.* [1] under the name of *arbitrarily varying quantum channel (AVQC)*. In other words, there the communicating parties are controlling genuine quantum systems (naturally, as they are supposed to transmit quantum information), whereas the jammer effectively only has a classical choice s.

Our model here lifts this restriction and generalizes the AVQC to a fully quantum jammer channel. This has the very important consequence that the jammer now can choose to prepare channels for Alice and Bob that are not tensor products of n single-system channels, or convex combinations thereof, but have other, more subtle noise correlations between the n systems.

It turns out that the worst behaviour of the jammer, at least in the random code case where the shared randomness is secret from the jammer, is to choose one, pessimal, environment input to $\mathcal{N}$ and use it in all n instances. The following theorem is proved in Appendix A.

Theorem 5: For any jammer channel $\mathcal{N}^{AE\to B}$,

$$Q_{J,r}(\mathcal{N}) = \sup_n \max_{\rho^{(n)}} \min_\eta \frac{1}{n} I_c\big(\rho^{(n)}; (\mathcal{N}_\eta)^{\otimes n}\big),$$

where the maximization is over states $\rho^{(n)}$ on A^n, and the minimization is over arbitrary (mixed) states η on E.

See [1], [5] for a detailed discussion of the role of shared randomness in the theory of the AVQC model; these authors suggest that $Q_J = Q_{J,r}$ for all jammer channels, at least for all AVQCs, which however should be contrasted with the findings of [6] that there are AVQCs for which the *classical* capacity assisted by shared randomness is positive while without that resource it is zero.

Let us now resume our discussion of environment-assisted quantum capacity, deriving capacity theorems analogous to the one above for the jammer model. For the latter we saw that (mixed) product states are asymptotically optimal for the jammer. It will turn out that restricting the helper to product (separable) states can be to severe disadvantage; while from the definitions, for any isometry V we have $Q_{H\otimes}(V) \leq Q_H(V)$, the inequality can be strict.

Theorem 6: For an isometry $V : AE \longrightarrow BF$, the passive environment-assisted quantum capacity is given by

$$Q_H(V) = \sup_n \max_{\eta^{(n)}} \frac{1}{n} Q(\mathcal{N}_{\eta^{(n)}}^{\otimes n}) = \sup_n \max_{\rho^{(n)},\eta^{(n)}} \frac{1}{n} I_c\big(\rho^{(n)}; \mathcal{N}_{\eta^{(n)}}^{\otimes n}\big), \quad (3)$$

where the maximization is over states $\rho^{(n)}$ on A^n and *pure* environment input states $\eta^{(n)}$ on E^n.

Similarly, the capacity with separable helper is given by the same formula,

$$Q_{H\otimes}(V) = \sup_n \max_{\eta^{(n)}=\eta_1\otimes\cdots\otimes\eta_n} \frac{1}{n} Q(\mathcal{N}_{\eta_1} \otimes \cdots \otimes \mathcal{N}_{\eta_n}) = \sup_n \max_{\rho^{(n)},\eta^{(n)}=\eta_1\otimes\cdots\otimes\eta_n} \frac{1}{n} I_c\big(\rho^{(n)}; \mathcal{N}_{\eta^{(n)}}^{\otimes n}\big), \quad (4)$$

but now varying only over (pure) product states, i.e. $\eta^{(n)} = \eta_1 \otimes \cdots \otimes \eta_n$.

As a consequence, $Q_H(V) = \lim_{n\to\infty} \frac{1}{n} Q_{H\otimes}(V^{\otimes n})$.

Proof: The direct parts, i.e. the "$\geq$" inequality, follows directly from the Lloyd-Shor-Devetak (LSD) theorem [12], [30], [38], applied to the channel $(\mathcal{N}^{\otimes n})_{\eta^{(n)}}$, to be precise asymptotically many copies of this block-channel, so that the i.i.d. theorems apply (cf. [42]).

For the converse (i.e. "$\leq$"), we apply directly the argument of Barnum *et al.* [2], Schumacher [35], and Schumacher and Nielsen [36]: Consider a code of block length n and fidelity F, where the helper uses an environment state $\eta^{(n)}$; otherwise we use notation as in Fig. 3. Then, first of all, $\frac{1}{2}\|\sigma - \Phi\|_1 \leq \sqrt{1-F} =: \epsilon$, cf. [17]. Now, Fannes' inequality [16] can be applied, at least once $2\epsilon \leq \frac{1}{e}$ (i.e. when F is large enough), yielding

$$\begin{aligned} I(R\rangle B_0)_\sigma &= S(\sigma^{B_0}) - S(\sigma^{RB_0}) \\ &\geq S(\sigma^{B_0}) \\ &\geq S(\Phi^{A_0}) - 2\epsilon \log|B_0| - H_2(2\epsilon) \\ &\geq (1-2\epsilon)\log|A_0| - 1. \end{aligned}$$

On the other hand, with $\omega = (\mathrm{id} \otimes \mathcal{E})\Phi$,

$$\begin{aligned} I(R\rangle B_0)_\sigma &\leq I(R\rangle B^n)_{(\mathrm{id}\otimes\mathcal{N}_{\eta^{(n)}}^{\otimes n})\omega} \\ &\leq \max_{|\phi\rangle^{RA^n}} I(R\rangle B^n)_{(\mathrm{id}\otimes\mathcal{N}_{\eta^{(n)}}^{\otimes n})\phi} \\ &= \max_{\rho^{(n)}} I_c\big(\rho^{(n)}; (\mathcal{N}^{\otimes n})_{\eta^{(n)}}\big), \end{aligned}$$

using first data processing of the coherent information and then its convexity in the state [36]. As $n \to \infty$ and $F \to 1$, the upper bound on the rate follows – depending on Q_H or $Q_{H\otimes}$, without or with restrictions on $\eta^{(n)}$. ■

Remark 7: The channels $\mathcal{N} : \mathcal{L}(A \otimes E) \longrightarrow \mathcal{L}(B)$ can equivalently be seen as (two-sender-one-receiver) *quantum multi-access channels*. These channels were introduced and studied in [43] and [46] under the aspect of characterizing their *capacity region* of all pairs or rates (R_A, R_E) at which the users, Alice and Helen, controlling the two input registers can communicate with Bob. In fact, while in [43] only special channels and classical communication were considered, Ref. [46] extended this to general CPTP maps and the consideration of quantum communication.

Clearly, knowing the capacity region for some $\mathcal{N}^{AE\to B}$ implies the environment-assisted capacity:

$$Q_H(\mathcal{N}) = \max\{R : (R, 0) \in \text{ capacity region}\}.$$

Unfortunately, however, in general only a regularized capacity formula is available, much like our Theorem 6. Thus, the general multi-access viewpoint does not seem to help particularly with the computation of Q_H or $Q_{H\otimes}$. Furthermore, our aim is to study the impact of the helper, its entanglement, etc. which is a different question to MAC.

Proposition 8: The capacities Q_H, $Q_{H\otimes}$ and $Q_{J,r}$ are continuous in the channel, with respect to the diamond (or completely bounded) norm. Concretely, if $\|\mathcal{N} - \mathcal{M}\|_\diamond \leq \epsilon$, then

$$\begin{aligned} \big|Q_{H\otimes}(\mathcal{N}) - Q_{H\otimes}(\mathcal{M})\big| &\leq 8\epsilon \log|B| + 4H_2(\epsilon), \\ \big|Q_H(\mathcal{N}) - Q_H(\mathcal{M})\big| &\leq 8\epsilon \log|B| + 4H_2(\epsilon), \\ \big|Q_{J,r}(\mathcal{N}) - Q_{J,r}(\mathcal{M})\big| &\leq 8\epsilon \log|B| + 4H_2(\epsilon). \end{aligned}$$

Proof: This is essentially the argument of Leung and Smith [29, Th. 6, Lemma 1, Corollary 2]. We can apply this because we have the formulas for these capacities in terms of coherent informations $\frac{1}{n} I_c\big(\rho^{(n)}; \mathcal{N}_{\eta^{(n)}}^{\otimes n}\big)$, according to Theorem 6. The only new ingredient is that now the parameter is the joint input state $\rho^{(n)} \otimes \eta^{(n)}$, but fixing that the proof via the "hybrid argument" in [29] goes through. ■

We remark here that it is not known at the time of writing, whether Q_J is continuous in the channel, a problem that is in fact closely tied to the question whether $Q_J = Q_{J,r}$ for all channels.

Given that in our formulation of the environment-assisted quantum capacity, the ordinary quantum channel capacity is contained as a special case, it is clear that we cannot make many general statements about either Q_H or $Q_{H\otimes}$. However, focusing henceforth on unitaries $V : AE \longrightarrow BF$, we will in the sequel explore the environment-assisted capacities by looking at specific classes of interactions which exhibit interesting or even unexpected behaviour.

To start, what are the unitaries $V : AE \longrightarrow BF$, say with equal dimensions of A and B, with maximal capacity $\log|B|$? For $Q_H(V)$ this seems a non-trivial question, but for $Q_{H\otimes}(V)$, invoking the result of [7], we find that $Q_{H\otimes}(V) = \log|B|$ if and only if there exist states $|\eta\rangle \in E$, $|\phi\rangle \in F$, and a unitary $U : A \longrightarrow B$ such that

$$V(|\psi\rangle^A|\eta\rangle^E) = (U|\psi\rangle)^B|\phi\rangle^F,$$

which in principle can be checked algebraically. In other words, in this case, one of the channels $\mathcal{N}_\eta$ induced by choosing an environment input state is the conjugation by a unitary. In the search for non-trivial channels, we find the following result.

Theorem 9: Let $|A| = |B| = 2$, $|E| = |F| = d \leq 4$ and consider d linearly independent unitaries $U_k^{A\to B} \in \mathrm{U}(2)$. If the unitary $V : AE \longrightarrow BF$ is such that it induces a mixture of conjugation by U_k's for any state $|\eta\rangle \in E$, then V is a controlled-unitary gate:

$$V^{AE\to BF} = \sum_k U_k^{A\to B} \otimes |f_k\rangle^F \langle e_k|^E,$$

with suitable orthonormal bases $\{|e_k\rangle\}_k$ and $\{|f_k\rangle\}_k$ of E and F, respectively.

Proof: Let us start from the requirement that V gives rise to a mixture of conjugation by U_k's in the states $\{|j\rangle\}_j$ of a basis of the environment E. W.l.o.g. we can write the action of V as follows

$$V|\psi\rangle^A|j\rangle^E = \sum_k U_k|\psi\rangle|v_{jk}\rangle, \tag{5}$$

where $|v_{jk}\rangle$ are non-normalized states of E. Then, let us consider a standard maximally entangled state $|\Psi\rangle^{RA}$ between a reference system R and the input system A. We have

$$(I\otimes V)|\Psi\rangle^{RA}|j\rangle^E = \sum_k (I\otimes U_k)|\Psi\rangle|v_{jk}\rangle =: \sum_k |\Psi_k\rangle|v_{jk}\rangle,$$

with all the $|\Psi_k\rangle^{RA}$ maximally entangled states. The trace over E gives the Choi-Jamiolkowski state of the channel which in turn must represent a mixture of conjugations by U_k's, hence the following equality must hold true:

$$\sum_{kk'} \langle v_{jk'}|v_{jk}\rangle |\Psi_k\rangle\langle\Psi_{k'}| = \sum_k p_k|\Psi_k\rangle\langle\Psi_k|,$$

for some probability distribution $\{p_k\}_k$. Since the $|\Psi_k\rangle$ are linearly independent (as a consequence of the linear independence of the unitaries U_k), we necessarily must have vanishing scalar products $\langle v_{jk}|v_{jk'}\rangle = 0$ for all j and all $k \neq k'$.

For a generic environment state $|\eta\rangle = \sum_j \eta_j|j\rangle$ it is

$$V|\psi\rangle \sum_j \eta_j|j\rangle = \sum_k U_k|\psi\rangle \sum_j \eta_j|v_{jk}\rangle, \tag{6}$$

and using the same argument as above we end up with the requirement that the states $\{\sum_j \eta_j|v_{jk}\rangle\}_k$ have to be orthogonal (for different values of k). Actually this must be true for any value of the η_js, hence the only possibility is that the vectors $|v_{jk}\rangle$ result as $|v_{jk}\rangle = c_{jk}|f_k\rangle$ with $\{|f_k\rangle\}_k$ orthonormal.

This can be proved by considering the scalar product between

$$\sum_j \eta_j|v_{jr}\rangle \text{ and } \sum_j \eta_j|v_{js}\rangle,$$

(for arbitrary values $r \neq s$) with all $\eta_j = 0$ except η_m and η_n (for any values $m \neq n$), which yield the following conditions:

$$\big(\overline{\eta}_m\langle v_{mr}| + \overline{\eta}_n\langle v_{nr}|\big)(\eta_m|v_{ms}\rangle + \eta_n|v_{ns}\rangle) = 0.$$

Then, we may notice that

$$\eta_m = \eta_n = 1 \;\Rightarrow\; \langle v_{mr}|v_{ns}\rangle = -\langle v_{nr}|v_{ms}\rangle,$$
$$\eta_m = \eta_n = i \;\Rightarrow\; \langle v_{mr}|v_{ns}\rangle = \langle v_{nr}|v_{ms}\rangle.$$

To simultaneously satisfy these conditions it must hold that $\langle v_{mr}|v_{ns}\rangle = \langle v_{nr}|v_{ms}\rangle = 0$. Due to the arbitrariness of r, s, m, n we can conclude that $\langle v_{jk}|v_{j'k'}\rangle = 0$ for $k \neq k'$ and for any j, j', i.e. $|v_{jk}\rangle = c_{jk}|f_k\rangle$ with $\{|f_k\rangle\}_k$ orthonormal.

Thus, the action (6) of V in the environment basis states $\{|j\rangle\}_j$ will result as

$$V|\psi\rangle|j\rangle = \sum_k U_k|\psi\rangle c_{jk}|f_k\rangle.$$

Therefore, in the basis $\{|j\rangle\}_j$ the unitary V can be written as

$$V = \sum_{j,k} U_k \otimes c_{jk}|f_k\rangle\langle j| = \sum_k U_k \otimes |f_k\rangle\langle e_k|,$$

where we have defined the vectors

$$|e_k\rangle := \sum_j \overline{c}_{kj}|j\rangle.$$

Finally using the condition

$$\sum_k \langle v_{jk}|v_{j'k}\rangle = \delta_{jj'}$$

coming from the unitarity of V, we have

$$\sum_j c_{jk}\overline{c}_{k'j} = \delta_{kk'},$$

expressing the orthonormality of $\{|e_k\rangle\}_k$. ■

We conjecture furthermore that for $|A| = |B| = 2$ and $|E| = |F| = d$ arbitrary, if $V : AE \longrightarrow BF$ is such that it induces random-unitary (equivalently: unital [25]) channels $\mathcal{N}_\eta$ for all states $|\eta\rangle \in E$, then V is essentially a controlled-unitary gate:

$$V^{AE\to BF} = \sum_j U_j^{A\to B} \otimes |f_j\rangle^F \langle e_j|^E,$$

with qubit unitaries U_j and with suitable orthonormal bases $\{|e_j\rangle\}$ and $\{|f_j\rangle\}$ of E and F, respectively.

To turn the other way, what are the useless unitary interactions, i.e. those with $Q_H(V) = 0$, or at least $Q_{H\otimes}(V) = 0$? In the next section we will encounter some families of two-qubit V with the latter property. On the other hand, unitaries with $Q_H(V) = 0$ do not seem to be so obvious, except for the example of SWAP, which swaps two isomorphic systems A and E, i.e. $\text{SWAP}(|\psi\rangle^A|\varphi\rangle^E) = |\varphi\rangle^B|\psi\rangle^F$, because it results in channels with constant output.

III. Two-Qubit Unitaries

In this section we will look at two-qubit unitary interactions, hence in principle study all qubit channels which can be described by a single qubit environment. This is motivated by quantum channels derived from such unitaries having nice properties, which allow us to characterize their environment-assisted capacities.

A general two-qubit unitary interaction can be described by 15 real parameters. For the analysis of quantum capacity under consideration we follow the arguments used in [26] to reduce the parameters to 3 by the action of local unitaries. According to the definition of the capacities, the local unitaries on A, B, E and F do not affect the environment-assisted quantum capacity, as they could be incorporated into the encoding and decoding maps, respectively, or can be reflected in a different choice of environment state.

Lemma 10 (Kraus/Cirac [26]): Any two-qubit unitary interaction V^{AE} is equivalent, up to local unitaries before and after the V^{AE}, to one of the form

$$\begin{aligned} U^{AE} &= \sum_k e^{-i\lambda_k} |\Phi_k\rangle\langle\Phi_k| \\ &= \exp -\frac{i}{2}\big(\alpha_x \sigma_x \otimes \sigma_x + \alpha_y \sigma_y \otimes \sigma_y + \alpha_z \sigma_z \otimes \sigma_z\big), \\ &=: U(\alpha_x, \alpha_y, \alpha_z), \end{aligned}$$

where σ_x, σ_y and σ_z are the Pauli operators and $\frac{\pi}{2} \geq \alpha_x \geq \alpha_y \geq |\alpha_z| \geq 0$. Furthermore the λ_ks are

$$\begin{aligned} \lambda_1 &= \frac{\alpha_x - \alpha_y + \alpha_z}{2}, \\ \lambda_2 &= \frac{-\alpha_x + \alpha_y + \alpha_z}{2}, \\ \lambda_3 &= \frac{-\alpha_x - \alpha_y - \alpha_z}{2}, \\ \lambda_4 &= \frac{\alpha_x + \alpha_y - \alpha_z}{2}, \end{aligned}$$

and $|\Phi_k\rangle$ the so-called "magic basis" [22],

$$\begin{aligned} |\Phi_1\rangle &= \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \\ |\Phi_2\rangle &= \frac{-i(|00\rangle - |11\rangle)}{\sqrt{2}}, \\ |\Phi_3\rangle &= \frac{|01\rangle - |10\rangle}{\sqrt{2}}, \\ |\Phi_4\rangle &= \frac{-i(|01\rangle + |10\rangle)}{\sqrt{2}}. \end{aligned}$$

This is of course the familiar Bell basis, but note the peculiar phases. ■

Hence the parameter space given by

$$\mathfrak{T}_{total} = \Big\{(\alpha_x, \alpha_y, \alpha_z) : \frac{\pi}{2} \geq \alpha_x \geq \alpha_y \geq |\alpha_z| \geq 0\Big\}, \quad (7)$$

describes all two-qubit unitaries up to local basis choice. This forms a tetrahedron with vertices $(0,0,0)$, $(\frac{\pi}{2},0,0)$, $(\frac{\pi}{2},\frac{\pi}{2},-\frac{\pi}{2})$ and $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$.

As we are interested in evaluating the capacities of unitaries, we use,

$$U\Big(\alpha_x, \alpha_y, \frac{\pi}{2}+\alpha_z\Big) = -i(\sigma_z \otimes \mathbb{1})U^*\Big(\alpha_x, \alpha_y, \frac{\pi}{2}-\alpha_z\Big)(\mathbb{1} \otimes \sigma_z), \quad (8)$$

where U^* is the complex conjugate of U. Note that the latter has the same environment-assisted classical capacities; indeed, any code for U is transformed into one for U^* by taking complex conjugates. The reduced parameter space given by

$$\mathfrak{T} = \Big\{(\alpha_x, \alpha_y, \alpha_z) : \frac{\pi}{2} \geq \alpha_x \geq \alpha_y \geq \alpha_z \geq 0\Big\}, \quad (9)$$

describes all two-qubit unitaries up to local basis choice and complex conjugation.

This forms a tetrahedron with vertices $(0,0,0)$, $(\frac{\pi}{2},0,0)$, $(\frac{\pi}{2},\frac{\pi}{2},0)$ and $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$, see Fig. 4. Familiar two-qubit gates can easily be identified within this parameter space: for instance, $(0,0,0)$ represents the identity $\mathbb{1}$, $(\frac{\pi}{2},0,0)$ the CNOT, $(\frac{\pi}{2},\frac{\pi}{2},0)$ the DCNOT (double controlled not), and $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ the SWAP gate, respectively.

Remark 11: $\mathfrak{T}$ is closely related to the Weyl-chamber as discussed in [47] which has the parametric region given by tetrahedron with vertices $(0,0,0)$, $(\pi,0,0)$, $(\frac{\pi}{2},\frac{\pi}{2},0)$ and $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$. It has been shown that each non-local two-qubit unitary can be identified with a unique point in the Weyl-chamber except on the base, where the triangles formed by the vertices $(0,0,0)$, $(\frac{\pi}{2},0,0)$ and $(\frac{\pi}{2},\frac{\pi}{2},0)$ is equivalent to the triangle formed with the vertices $(\pi,0,0)$, $(\frac{\pi}{2},0,0)$ and $(\frac{\pi}{2},\frac{\pi}{2},0)$.

Example 12: To illustrate this parametrization, let us look at a controlled-unitary V (cf. Theorem 9) of the form $V = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes U_1$, where $U_i \in \mathrm{SU}(2)$. One can work out that this has parametric representation $(t,0,0)$, i.e. in the parameter tetrahedron $\mathfrak{T}_{total}$, these unitaries are on the edge joining the identity $\mathbb{1}$ and CNOT.

To see this, we use the argument described in Appendix A of [20]: Observe that the spectrum of $V^T V$ is $(e^{-2i\lambda_1}, e^{-2i\lambda_2}, e^{-2i\lambda_3}, e^{-2i\lambda_4})$, where the transpose operator is with respect to the magic basis. In this way, $(|0\rangle\langle 0| \otimes \mathbb{1})^T = |1\rangle\langle 1| \otimes \mathbb{1}$ and $(\mathbb{1} \otimes U)^T = (\mathbb{1} \otimes U^\dagger)$, thus $V^T = |1\rangle\langle 1| \otimes U_0^\dagger + |0\rangle\langle 0| \otimes U_1^\dagger$ and $V^T V = |1\rangle\langle 1| \otimes U_0^\dagger U_1 + |0\rangle\langle 0| \otimes U_1^\dagger U_0$. The eigenvalues of $U = U_0^\dagger U_1$ are e^{id} and e^{-id}, where $2\cos d = \operatorname{Tr} U$. The spectrum of $V^T V$ is thus $\big(e^{id}, e^{id}, e^{-id}, e^{-id}\big)$. Using the order property $\frac{3\pi}{4} \geq \lambda_4 \geq \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq -\frac{3\pi}{4}$ (condition (9) written in terms of λ_k) and solving the linear equations in α_x, α_y and α_z, we get the parametric point as $(t,0,0)$ where $t = d$ when $d \leq \frac{\pi}{2}$ and $t = \pi - d$ when $d \geq \frac{\pi}{2}$.

Now we come to the main reason why we investigate this class of unitaries, apart from obviously furnishing the smallest possible examples: Recall that a quantum channel $\mathcal{N} : \mathcal{L}(A) \to \mathcal{L}(B)$ is called *degradable* [13] if there exists a degrading CPTP map $\mathcal{M} : \mathcal{L}(B) \to \mathcal{L}(F)$ such that for any input ρ^A, $\widetilde{\mathcal{N}}(\rho) = \mathcal{M}(\mathcal{N}(\rho))$. That is, Bob can simulate the environment output by applying a CPTP map on his system. It means that the complementary channel is noisier than the channel itself, in an operationally precise sense.

A quantum channel is *anti-degradable* if its complementary channel is degradable, i.e. if there exists a CPTP map $\mathcal{M} : \mathcal{L}(F) \to \mathcal{L}(B)$ such that for any input ρ^A, $\mathcal{N}(\rho^{AE}) = \mathcal{M}(\widetilde{\mathcal{N}}(\rho^A))$.

It is well-known that the quantum capacity of anti-degradable channels is zero, by the familiar *cloning argument*: Namely, if an anti-degradable channel were to have positive quantum capacity, F can apply the degrading map followed by the same decoder as B and thus A would be transmitting the same quantum information to B and F. This is in contradiction to the no-cloning theorem as observed in [3]. Furthermore, for each anti-degradable channel $\mathcal{N}$ we can identify a zero-capacity degradable extension $\mathcal{T}$ i.e. there exists a channel $\mathcal{R}$ such that $\mathcal{N} = \mathcal{R} \circ \mathcal{T}$ and $\mathcal{T}$ is degradable [39]. On the other hand, if a channel is degradable, Devetak and Shor [13] showed that the quantum capacity can be characterized very concisely. Namely, they proved that for degradable $\mathcal{N}_i : \mathcal{L}(A_i) \longrightarrow \mathcal{L}(B_i)$,

$$\max_{\rho^{(n)}} I_c\big(\rho^{(n)}; \mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n\big) = \max_{\rho_1\otimes\cdots\otimes\rho_n} I_c\big(\rho_1 \otimes \cdots \otimes \rho_n; \mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n\big) = \sum_{i=1}^{n} \max_{\rho_i} I_c(\rho_i; \mathcal{N}_i),$$

which implies for degradable channel $\mathcal{N}$ that

$$Q(\mathcal{N}) = \max_{\rho} I_c(\rho; \mathcal{N}).$$

Furthermore, the coherent information in this case is a concave function of ρ, so the maximum can be found efficiently.

Notice that by interchanging the registers in B and F we go from degradable channels to anti-degradable ones, and vice versa. But many channels are neither degradable nor anti-degradable. However, in [45] it was shown that qubit channels with one-qubit environment are either degradable or anti-degradable or both. Hence, for any initial state of the environment, all the two-qubit unitary interactions give rise to qubit channels that are either degradable or anti-degradable or both. Ref. [45] also provided an analytical criterion for determining whether a channel is degradable or anti-degradable (or both, becoming *symmetric* in such a case). The criterion is revisited here for our purposes.

Lemma 13 (Wolf/Perez-García [45]): Given an isometry $V : A \otimes E \to B \otimes F$ and an initial input to environment $|\eta\rangle \in E$, let $\{K_i\}$ be the Kraus operators in normal form (i.e. $\mathrm{Tr}\, K_i^\dagger K_j = 0$ for $i \neq j$) of the qubit channel $\mathcal{N}_\eta(\rho) = \mathrm{Tr}_F\, V(\rho^A \otimes \eta^E)V^\dagger$.

Then, the condition for degradability is given by the sign of $\det(2K_0^\dagger K_0 - \mathbb{1})$. The channel is degradable when $\det(2K_0^\dagger K_0 - \mathbb{1}) \geq 0$, anti-degradable when $\det(2K_0^\dagger K_0 - \mathbb{1}) \leq 0$, and symmetric when $\det(2K_0^\dagger K_0 - \mathbb{1}) = 0$. ■

This characterization has the consequence that the separable environment-assisted quantum capacity of two-qubit unitaries can be calculated fairly easily:

Theorem 14: For a two-qubit unitary $V : AE \longrightarrow BF$,

$$Q_{H\otimes}(V) = \max_{\eta^E} \max_{\rho^A} I_c(\rho^A; \mathcal{N}_\eta).$$

In addition, the maximization over helper states η may be restricted to pure states such that $\mathcal{N}_\eta$ is degradable, and for each such fixed η, the inner maximization over ρ is a convex optimization problem (concave function on a convex domain).

Proof: The capacity in general is given by Theorem 6, Eq. (4):

$$Q_{H\otimes}(V) = \sup_n \max_{\eta_1\otimes\cdots\otimes\eta_n} \max_{\rho^{(n)}} \frac{1}{n} I_c\big(\rho^{(n)}; \mathcal{N}_{\eta_1} \otimes \cdots \otimes \mathcal{N}_{\eta_n}\big).$$

By Wolf and Perez-Garcia's Lemma 13, each of the $\mathcal{N}_{\eta_i}$ is degradable or anti-degradable, so by Devetak and Shor [13], Smith and Smolin [39], the coherent information is additive:

$$\max_{\rho^{(n)}} I_c\big(\rho^{(n)}; \mathcal{N}_{\eta_1} \otimes \cdots \mathcal{N}_{\eta_n}\big) = \sum_{i=1}^{n} \max_{\rho_i} I_c(\rho_i; \mathcal{N}_{\eta_i}),$$

hence $Q_{H\otimes}(V) = \max_\eta \max_\rho I_c(\rho; \mathcal{N}_\eta)$ as advertised.

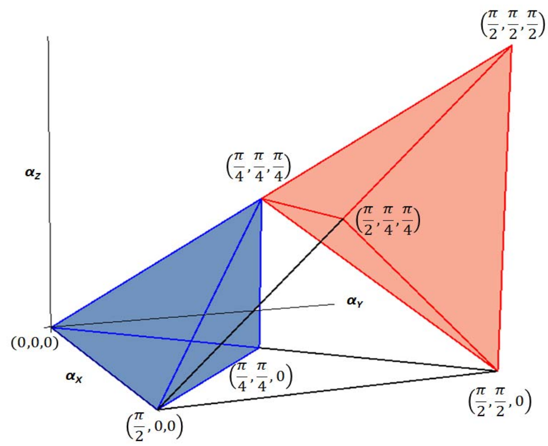

Fig. 4. Universally anti-degradable and degradable regions inside the parameter space $\mathfrak{T}$. The upper (red) tetrahedron corresponds to $\mathfrak{A}$, the lower (blue) one corresponds to $\mathfrak{D}$. The universally degradable (anti-degradable) region in $\mathfrak{T}_{total}$ is obtained by union of $\mathfrak{D}$ ($\mathfrak{A}$) and its reflection about the α_x and α_y plane.

Clearly, for those η such that $\mathcal{N}_\eta$ is anti-degradable, we know that the r.h.s. is 0, so we may discount them in the optimization. ■

Definition 15: We say that a unitary operator U is *universally degradable* (resp. *anti-degradable*), if for every $|\eta\rangle \in E$, the qubit channel $\mathcal{N}_\eta : \mathcal{L}(A) \to \mathcal{L}(B)$ is degradable (resp. anti-degradable). The set of universally degradable (anti-degradable) unitaries in the parameter space $\mathfrak{T}$ is denoted $\mathfrak{D}$ ($\mathfrak{A}$) and in the parameter space $\mathfrak{T}_{total}$ is denoted by $\mathfrak{D}_{total}$ ($\mathfrak{A}_{total}$).

Clearly, SWAP $\in \mathfrak{A}$ and id $\in \mathfrak{D}$, hence both $\mathfrak{A}$ and $\mathfrak{D}$ are non-empty. Furthermore, $U \in \mathfrak{D}$ if and only if SWAP $\cdot U \in \mathfrak{A}$. Indeed, the set $\{(\alpha_x, \alpha_y, \alpha_z) \in \mathfrak{T} : U(\alpha_x, \alpha_y, \alpha_z) \in \mathfrak{A}\}$ is a tetrahedron with vertices $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$, $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$, $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$, shown in Fig. 4. Similarly, the set $\mathfrak{D}$ corresponds to the tetrahedron with vertices $(0,0,0)$, $(\frac{\pi}{2}, 0, 0)$, $(\frac{\pi}{4}, \frac{\pi}{4}, 0)$ and $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$. For a detailed analysis of the sets $\mathfrak{A}$ and $\mathfrak{D}$ and their parameter regions we refer to Appendix B.

Let us first consider the unique edge of the tetrahedron $\mathfrak{T}$ which contains points either belonging to $\mathfrak{A}$ or $\mathfrak{D}$. This is the line segment joining the identity $\mathbb{1}$ $(0,0,0)$ with SWAP $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$. Each unitary on that line is a γ-th root of SWAP with a parameter $\gamma \in (0,1)$, i.e.

$$\mathrm{SWAP}^\gamma = \frac{1+e^{i\pi\gamma}}{2}\mathbb{1} + \frac{1-e^{i\pi\gamma}}{2}\mathrm{SWAP} \equiv U\Big(\frac{\gamma\pi}{2}, \frac{\gamma\pi}{2}, \frac{\gamma\pi}{2}\Big). \qquad (10)$$

It is actually elementary to evaluate the universally anti-degradable region of this line segment. Due to the invariance of SWAP under conjugation with unitaries of the form $u \otimes u$, it is enough to examine the anti-degradability of the channel that arise when the initial state of the environment is $|0\rangle$: either all $\mathcal{N}_\eta$ are anti-degradable or none. The Kraus operators are

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{1+e^{i\pi\gamma}}{2} \end{bmatrix}, \quad \begin{bmatrix} 0 & \frac{1-e^{i\pi\gamma}}{2} \\ 0 & 0 \end{bmatrix},$$

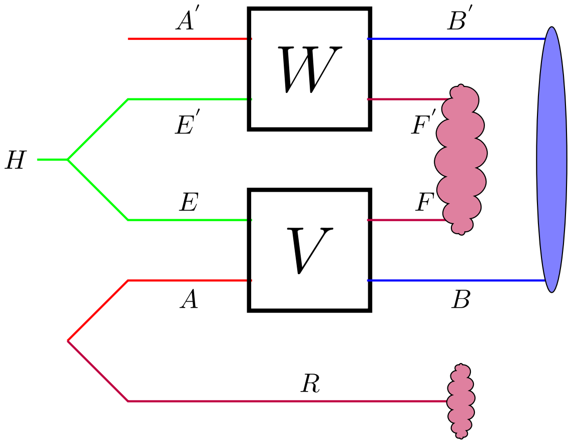

Fig. 5. The inputs controlled by Alice are A' and A, R is the purification of A. Helen controls E' and E, Bob's systems are labelled as B' and B. The inaccessible output-environment systems are labelled as F' and F. Alice inputs $|0\rangle$ in A' and $|\Phi\rangle$ in AR. Helen inputs a Bell state $|\Phi\rangle$ in $E'E$.

making it a generalized amplitude damping channel with damping parameter $\frac{1+e^{i\pi\gamma}}{2}$.

Hence we can invoke the criterion of Lemma 13, as these Kraus operators are in normal form. It results that $\mathcal{N}_{|0\rangle\langle 0|}$ is anti-degradable for $\gamma \in [\frac{1}{2}, 1]$, i.e. $U\left(\frac{\gamma\pi}{2}, \frac{\gamma\pi}{2}, \frac{\gamma\pi}{2}\right) \in \mathfrak{A}$.

From the above arguments it follows that $Q_{H\otimes}\left(U\left(\frac{\gamma\pi}{2}, \frac{\gamma\pi}{2}, \frac{\gamma\pi}{2}\right)\right) = 0$ for $\gamma \in [\frac{1}{2}, 1]$. We do not know whether it is even true that $Q_H(\text{SWAP}^\gamma) = 0$ for these values of γ, which would require to show that $Q_{H\otimes}\left((\text{SWAP}^\gamma)^{\otimes n}\right) = 0$ for all integers n.

IV. Super-Activation

The significance of $U \in \mathfrak{A}$ is that a Helen restricted to n-separable environment states cannot help Alice to communicate quantum information to Bob, $Q_{H\otimes}(U) = 0$, in accordance with Theorem 14. The natural question now arising is whether an unrestricted Helen can perform any better. In this section we show that this can indeed be the case.

A. Two Different Unitaries

The edges of the universally anti-degradable tetrahedron (Fig. 4) provide examples of super-activation ($Q_{H\otimes}(W) = Q_{H\otimes}(V) = 0$ and $Q_{H\otimes}(W \otimes V) > 0$). These are discussed below by referring to the setting and notation of Fig. 5. The input state we will consider below in all the further analysis, unless mentioned otherwise, shall be $|\Psi\rangle = |0\rangle^{A'} \otimes |\Phi\rangle^{E'E} \otimes |\Phi\rangle^{AR}$, where $|\Phi\rangle$ is the two-qubit maximally entangled state.

The global unitary G is given by $W \otimes V \otimes \mathbb{1}_R$, so that the coherent information is given by $S(\rho^{B'B}) - S(\rho^{F'F})$, where $\rho^{B'B} = \text{Tr}_{F'FR}\, G|\Psi\rangle\langle\Psi|G^\dagger$ and $\rho^{F'F} = \text{Tr}_{B'BR}\, G|\Psi\rangle\langle\Psi|G^\dagger$ are the output states of Bob and Eve, respectively.

A-1 Let W be a unitary on the edge joining SWAP and DCNOT, i.e. $W = U(\frac{\pi}{2}, \frac{\pi}{2}, \frac{t\pi}{2})$ with a parameter $t \in [0, 1]$; $V = \text{SWAP}^\gamma$ with $\gamma \in [0.5, 1]$. Then W has $\lambda_1 = \frac{t\pi}{4}$, $\lambda_2 = \frac{t\pi}{4}$, $\lambda_3 = -\frac{\pi}{4}(t+2)$ and $\lambda_4 = -\frac{\pi}{4}(t-2)$.

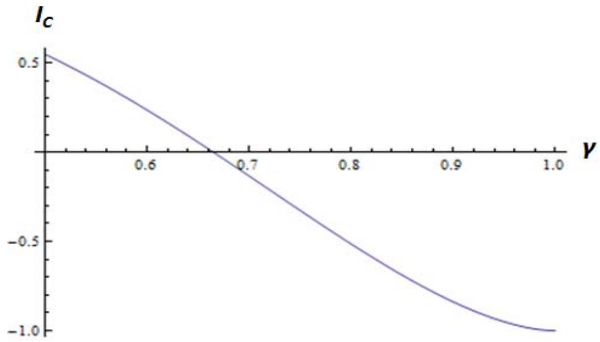

Fig. 6. Example A-1: Plot of the coherent information $I_c = S(B'B) - S(F'F)$ when $W = U(\frac{\pi}{2}, \frac{\pi}{2}, \frac{t\pi}{2})$ and $V = \text{SWAP}^\gamma$, over $\gamma \in [0.5, 1]$.

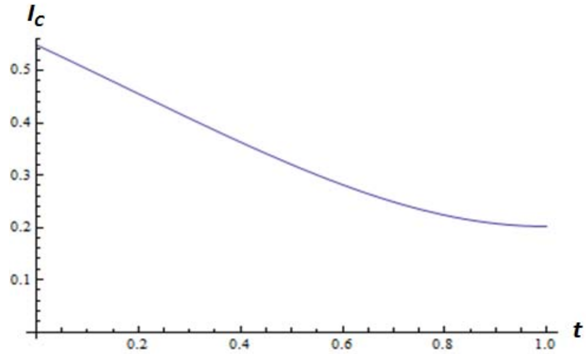

Fig. 7. Example A-2: Plot of the coherent information $I_c = S(B'B) - S(F'F)$ when $W = \text{SWAP}$ and $V = (\frac{\pi}{4} + \frac{t\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$, over $t \in [0, 1]$.

Hence, $W = e^{\frac{it\pi}{4}}\tilde{U}$ where

$$\tilde{U} = \begin{bmatrix} e^{-\frac{it\pi}{2}} & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & e^{-\frac{it\pi}{2}} \end{bmatrix},$$

written in the computational basis. Bob's output state is then given by

$$\begin{aligned} \rho^{B'B} = \frac{1}{4}\Bigg[& \frac{3 - \cos\pi\gamma}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|) \\ & + ie^{-\frac{it\pi}{2}}(1 - \cos\pi\gamma)|00\rangle\langle 11| \\ & - ie^{\frac{it\pi}{2}}(1 - \cos\pi\gamma)|11\rangle\langle 00| \\ & + \frac{1 + \cos\pi\gamma}{2}(|01\rangle\langle 01| + |10\rangle\langle 10|)\Bigg], \end{aligned}$$

whose eigenvalues are $\frac{5-3\cos\pi\gamma}{8}$ (single) and $\frac{1+\cos\pi\gamma}{8}$ (triple), while $\rho^{F'F} = |0\rangle\langle 0|^{F'} \otimes \frac{1}{2}\mathbb{1}^F$. The coherent information vanishes at $\gamma^* \approx 0.6649$, see Fig. 6. Hence each unitary $U(\frac{\pi}{2}, \frac{\pi}{2}, \frac{t\pi}{2})$ with $t \in [0, 1]$ super-activates SWAP^γ for $\gamma \in [0.5, \gamma^*)$.

A-2 Let $W = \text{SWAP}$ and $V = (\frac{\pi}{4} + \frac{t\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ with $t \in [0, 1]$. Here, V sits on the edge joining $\sqrt{\text{SWAP}}$ to $U(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$. The coherent information is positive for $t \in [0, 1]$ as depicted in Fig. 7.

A-3 $\sqrt{\text{SWAP}}$ activates $U(\frac{\pi}{2}, \frac{\pi}{2}, \frac{t\pi}{2})$ for $t \in [0, 1]$ as shown in example A-1. The coherent information is given by

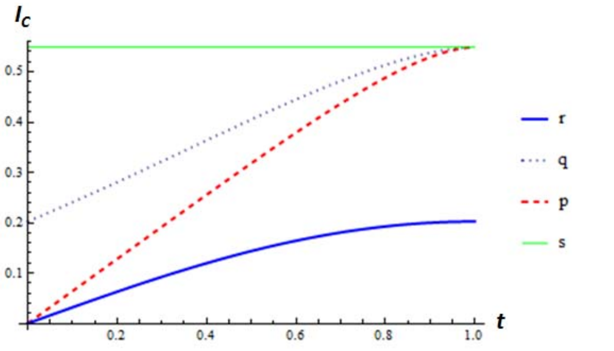

Fig. 8. Plots of the coherent information when $V = \sqrt{\text{SWAP}}$ and W is on one of the edges of the tetrahedron $\mathfrak{A}$, examples A-3a through A-3e.

the curve s in Fig. 8. Here let us evaluate the coherent information for the setting described in Fig. 5, when we have $V = \sqrt{\text{SWAP}}$ and W is a unitary on the edges of the tetrahedron corresponding to $\mathfrak{A}$. By varying the parameter t from $[0, 1]$ we move along one of the edges of $\mathfrak{A}$.

a) The edge joining $\sqrt{SWAP}$ to DCNOT: $W = U(\frac{\pi}{4} + \frac{t\pi}{4}, \frac{\pi}{4} + \frac{t\pi}{4}, \frac{\pi}{4} - \frac{t\pi}{4})$. The coherent information is given by the curve p in Fig. 8, which is positive for $t \in (0, 1]$.
b) The edge joining $\sqrt{\text{SWAP}}$ to SWAP (SWAP^{γ}): $W = U(\frac{\pi}{4} + \frac{t\pi}{4}, \frac{\pi}{4} + \frac{t\pi}{4}, \frac{\pi}{4} + \frac{t\pi}{4})$. The coherent information is given by the curve p in Fig. 8, which is positive for $t \in (0, 1]$. Here $t = 2\gamma - 1$, and the coherent information is positive for $\gamma \in (\frac{1}{2}, 1]$.
c) The edge joining $U(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ to SWAP: $W = U(\frac{\pi}{2}, \frac{\pi}{4} + \frac{t\pi}{4}, \frac{\pi}{4} + \frac{t\pi}{4})$. The coherent information is given by the curve q in Fig. 8, which is positive for $t \in [0, 1]$.
d) The edge joining $U(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$ to DCNOT: $W = U(\frac{\pi}{2}, \frac{\pi}{4} + \frac{t\pi}{4}, \frac{\pi}{4} - \frac{t\pi}{4})$. The coherent information is given by the curve q in Fig. 8, which is positive for $t \in [0, 1]$.
e) The edge joining $\sqrt{\text{SWAP}}$ to $U(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$: $W = U(\frac{\pi}{4} + \frac{t\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$. The coherent information is given by the curve r in Fig. 8, which is positive for $t \in (0, 1]$.

It results that each unitary corresponding to a point on the edge of the tetrahedron $\mathfrak{A}$ is super-activated by some another $V \in \mathfrak{A}$. Actually a single unitary, $V = \sqrt{\text{SWAP}}$, super-activates every other unitary on the edges of the universally anti-degradable tetrahedron $\mathfrak{A}$ (except itself). Furthermore, from the numerical analysis we have that $V = \sqrt{\text{SWAP}}$ super-activates every $W \in \mathfrak{A}$ (except itself).

Thus, in all the above cases,

$$Q_{H\otimes}(W \otimes V) > Q_{H\otimes}(W) + Q_{H\otimes}(V) = 0.$$

In other words, two seemingly useless unitaries can transfer a positive rate of quantum information when used in conjunction and the input environments are entangled. All the above W and V show super-activation of $Q_{H\otimes}$. In addition, in the examples A-1 and A-2, we have $W = \text{SWAP}$, hence in fact even $Q_H(W) = 0$. In particular the roots SWAP^{γ} of the SWAP gate are interesting. When $\sqrt{\text{SWAP}}$ is used in conjunction with a different $W \in \mathfrak{A}$ and the input environments are entangled, then they could transfer positive quantum information i.e. $Q_{H\otimes}(\sqrt{\text{SWAP}} \otimes W) > 0$.

Remark 16: We should note that in general $U \otimes V$ and $U \otimes V^*$ have different environment-assisted capacities and in such cases we should consider $\mathfrak{T}_{total}$, say for example to provide a complete characterization of super-activation. So, $U(\frac{\pi}{4}, \frac{\pi}{4}, \frac{-\pi}{4})$ activates every other unitary on the edges of the bottom half of universally anti-degradable region $\mathfrak{A}_{total}$ (the tetrahedron with the vertices $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{-\pi}{4}), (\frac{\pi}{2}, \frac{\pi}{4}, \frac{-\pi}{4}), (\frac{\pi}{2}, \frac{\pi}{2}, \frac{-\pi}{2})$ and $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$).

B. Self-Super-Activation

So far we have considered two different unitaries. The question is if two copies of the same unitary R ($R \in \mathfrak{A}$) can yield positive capacity when the initial states environments are entangled? In other words, can $Q_{H\otimes}$ be *self-super-activated*? The answer to this question is affirmative as we shall show now.

Remark 17: From the super-activation of a unitary W with another unitary V, such that both W and V are universally anti-degradable, we can get a self-super-activating unitary by doubling the size of the environment. More precisely, we can construct the new unitary $R : A \otimes E \otimes E' \to B \otimes F \otimes F'$, with $E' = F' = \mathbb{C}^2$: $R = W^{AE} \otimes |0\rangle\langle 0|^{E'} + V^{AE} \otimes |1\rangle\langle 1|^{E'}$.

To see that this works, clearly if Helen inputs $|0\rangle$ into E', she determines that the unitary on AE is W, if she inputs $|1\rangle$ into E', the unitary is V; hence from two uses, $R \otimes R$, she can get $W \otimes V$, which has positive environment-assisted capacity by assumption. On the other hand $Q_{H\otimes}(R) = 0$, because in fact R is itself universally anti-degradable. Namely, observe that if the channels induced by W and V for environment input states ψ and φ are denoted by $\mathcal{N}_\psi$ and $\mathcal{M}_\varphi$, respectively, then a generic input state $\sqrt{p}|\psi\rangle|0\rangle + \sqrt{1-p}|\varphi\rangle|1\rangle$ to the EE' registers of W results in the channel $p\mathcal{N}_\psi + (1-p)\mathcal{M}_\varphi$. As both components are anti-degradable, so is their convex combination.

However, by looking at our two-qubit classification more carefully, we can also find self-super-activation in this simplest possible setting.

B-1 Let us consider the unitaries $U\left(\frac{\pi}{4} + t\frac{\pi}{4}, \frac{\pi}{4} + t\frac{\pi}{4}, \frac{\pi}{4} - t\frac{\pi}{4}\right)$ with $t \in [0, 1]$. We have seen in example A-3a that these unitaries are activated by $\sqrt{\text{SWAP}}$ in $t \in (0, 1]$. Now we shall explore the case when $W = V = U\left(\frac{\pi}{4} + t\frac{\pi}{4}, \frac{\pi}{4} + t\frac{\pi}{4}, \frac{\pi}{4} - t\frac{\pi}{4}\right)$. The coherent information $S(BB') - S(FF')$ is positive for $t \in (0, 1)$ as shown by curve m in Fig. 9.

When Helen can create quantum correlation between the environment inputs we see that a seemingly "useless" unitary can transmit quantum information. That is, the unrestricted Helen can super-activate the interaction $U\left(\frac{\pi}{4} + t\frac{\pi}{4}, \frac{\pi}{4} + t\frac{\pi}{4}, \frac{\pi}{4} - t\frac{\pi}{4}\right)$, with $t \in (0, 1)$ which translates to

$$\begin{aligned} &Q_H\Big(U\Big(\frac{\pi}{4} + t\frac{\pi}{4}, \frac{\pi}{4} + t\frac{\pi}{4}, \frac{\pi}{4} - t\frac{\pi}{4}\Big)\Big) \\ &\quad > Q_{H\otimes}\Big(U\Big(\frac{\pi}{4} + t\frac{\pi}{4}, \frac{\pi}{4} + t\frac{\pi}{4}, \frac{\pi}{4} - t\frac{\pi}{4}\Big)\Big) = 0. \end{aligned} \quad (11)$$

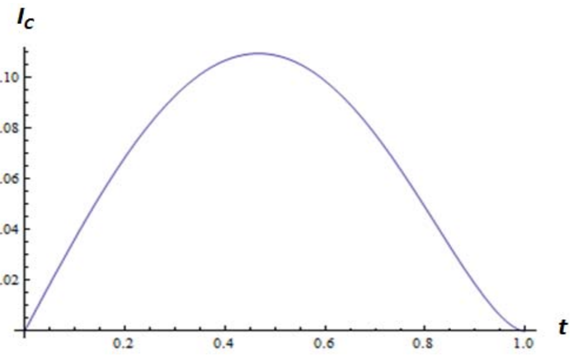

Fig. 9. Example B-1: Plot of the coherent information for the family $W = V = U\left(\frac{\pi}{4} + \frac{t\pi}{4}, \frac{\pi}{4} + \frac{t\pi}{4}, \frac{\pi}{4} - \frac{t\pi}{4}\right)$; it is positive for $t \in (0, 1)$.

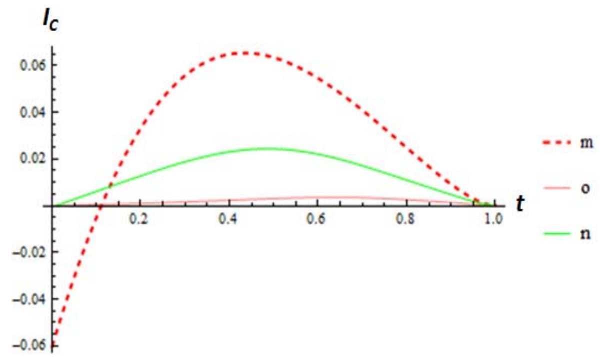

Fig. 10. Example B-2: Plots of the coherent information for the family $W = V = U\left(\frac{\pi}{2}, \frac{\pi}{4} + t\frac{\pi}{4}, \frac{\pi}{4} - t\frac{\pi}{4}\right)$. Curves m, n, o correspond to input states $|\Psi\rangle_\theta$ with $\theta = \frac{1}{2}, 2^{-6}, 2^{-10}$, respectively.

B-2 We can provide another family of unitaries which exhibit self-super-activation by the unitaries $W = V = U\left(\frac{\pi}{2}, \frac{\pi}{4} + t\frac{\pi}{4}, \frac{\pi}{4} - t\frac{\pi}{4}\right)$. The environment input state is $|\Psi\rangle_\theta = |1\rangle^{A'} \otimes |\Phi\rangle^{E'E} \otimes \left(\sqrt{\theta}|00\rangle + \sqrt{1-\theta}|11\rangle\right)^{AR}$. By optimizing over θ, we numerically find positive coherent information for $t \in (0.0004, 0.9999)$. The plots in Fig. 10 show $\theta = \frac{1}{2}$ (curve m), $\theta = 2^{-6}$ (curve n) and $\theta = 2^{-10}$ (curve o). The coherent information achievable seems to get smaller and smaller as t approaches 0.

For all the above U, $Q_{H\otimes}(U) = 0$ but $Q_H(U) > 0$, showing that to unlock the full potential of an interaction U, the helper may need to entangle the environments of different instances of U.

Remark 18: The phenomenon of self-super-activation taking place thanks to entanglement across environments resembles the super-additivity of the capacity in quantum channels with memory [11], [28].

V. Entanglement-Environment-Assisted Helper

Entanglement played a pivotal role in the instances of super-activation exhibited above; when Helen could create correlation between the environment input registers, she could enhance quantum communications from Alice to Bob. In this section we consider the model when there is pre-shared entanglement between Helen and Bob. This model is motivated by the equivalence of the two schemes presented in Fig. 11.

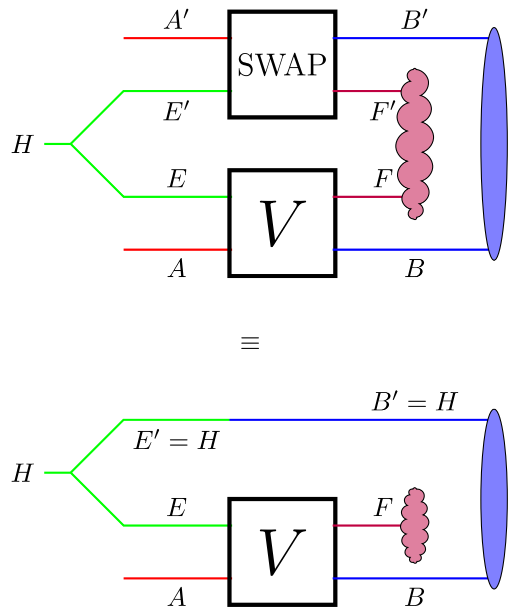

Fig. 11. When inputting an entangled state across $E'E$ and an arbitrary state in A' (top), the SWAP acts like a "dummy" channel but helps to establish entanglement between the receiver BB' and the environment E. This is equivalent to sharing an entangled state between Helen and Bob (bottom).

SWAP merely exchanges the input and environment registers, which could be used to correlate the environment on the input side with the receiver when the initial environment states are entangled. Indeed, this was behind several of the examples of super-activation in the previous section (A-1 and A-2).

Extending the notation of $\mathcal{N}_\eta = \mathcal{N}(\cdot \otimes \eta)$ introduced in Section II, we let, for a state κ on EH,

$$\mathcal{N}_\kappa^{A\to BH}(\rho) := (\mathcal{N}^{AE\to B} \otimes \mathrm{id}_H)(\rho^A \otimes \kappa^{EH}).$$

Referring to Fig. 12, we can further define the following CPTP maps. An encoding map $\mathcal{E} : \mathcal{L}(A_0) \to \mathcal{L}(A^n)$, and the decoding map $\mathcal{D} : \mathcal{L}(B^n \otimes H) \to \mathcal{L}(B_0)$. The output after the overall dynamics when we input a maximally entangled state Φ^{RA_0}, with the inaccessible reference system R, is given by $\sigma^{RB_0} = \mathcal{D}\left(\mathcal{N}^{\otimes n} \otimes \mathrm{id}_H\left(\mathcal{E}(\Phi^{RA_0}) \otimes \kappa^{E^n}\right)\right)$.

Definition 19: An *entanglement-environment-assisted quantum code* of block length n is a triple $(\mathcal{E}^{A_0\to A^n}, \kappa^{E^nH}, \mathcal{D}^{B^nH\to B_0})$. Its *fidelity* is given by $F = \mathrm{Tr}\, \Phi^{RA_0}\sigma^{RB_0}$, and its *rate* defined as $\frac{1}{n}\log|A_0|$.

A rate R is called *achievable* if there are codes of all block lengths n with fidelity converging to 1 and rate converging to R. The *entanglement-environment-assisted quantum capacity* of V, denoted $Q_{EH}(V)$, or equivalently $Q_{EH}(\mathcal{N})$, is the supremum of all achievable rates.

Theorem 20: The entanglement-environment-assisted quantum capacity of an interaction $V : AE \longrightarrow BF$ is characterized by following regularization.

$$\begin{aligned} Q_{EH}(V) &= \sup_n \max_{|\kappa^{(n)}\rangle\in E^nH} \frac{1}{n} Q\left((\mathcal{N}^{\otimes n})_{\kappa^{(n)}}\right) \\ &= \sup_n \max_{|\kappa\rangle\in E^nH} \max_{\rho^{(n)}} \frac{1}{n} I_c\left(\rho^{(n)}; (\mathcal{N}^{\otimes n})_{\kappa^{(n)}}\right). \end{aligned} \quad (12)$$

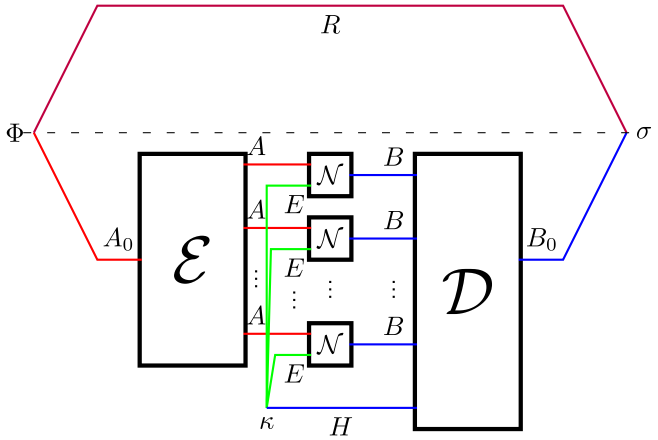

Fig. 12. The general form of a protocol to transmit quantum information when the helper and the receiver pre-share entanglement; $\mathcal{E}$ and $\mathcal{D}$ are the encoding and decoding maps respectively, κ is the initial state of the environments and system H.

The maximization is over (w.l.o.g. pure) states $\kappa^{(n)}$ on E^nH and input states $\rho^{(n)}$ on A^n.

Proof: The direct part, i.e. the "$\geq$" inequality, follows directly from the LSD theorem [12], [30], [38], applied to the channel $(\mathcal{N}^{\otimes n})_{\kappa^{(n)}}$, to be precise asymptotically many copies of this block-channel, so that the i.i.d. theorems apply [42].

The converse ("$\leq$"), works as before in Theorem 6, following Barnum *et al.* [2], Schumacher [35], and Schumacher and Nielsen [36]: Consider a code of block length n and fidelity F, where the helper uses an environment state $\kappa^{(n)}$; otherwise we use notation as in Fig. 12. We have $\frac{1}{2}\|\sigma - \Phi\|_1 \leq \sqrt{1-F} =: \epsilon$, cf. [17]. Now, Fannes' inequality [16] can be applied, at least once $2\epsilon \leq \frac{1}{e}$ (i.e. when F is large enough), yielding

$$\begin{aligned} I(R\rangle B_0)_\sigma &= S(\sigma^{B_0}) - S(\sigma^{RB_0}) \\ &\geq S(\sigma^{B_0}) \\ &\geq S(\Phi^{A_0}) - 2\epsilon \log|B_0| - H_2(2\epsilon) \\ &\geq (1-2\epsilon)\log|A_0| - 1. \end{aligned}$$

On the other hand, with $\omega = (\mathrm{id} \otimes \mathcal{E})\Phi$,

$$\begin{aligned} I(R\rangle B_0)_\sigma &\leq I(R\rangle B^n)_{(\mathrm{id}\otimes\mathcal{N}^{\otimes n}_{\kappa^{(n)}})\omega} \\ &\leq \max_{|\phi\rangle^{RA^n}} I(R\rangle B^n)_{(\mathrm{id}\otimes\mathcal{N}^{\otimes n}_{\kappa^{(n)}})\phi} \\ &= \max_{\rho^{(n)}} I_c\big(\rho^{(n)}; (\mathcal{N}^{\otimes n})_{\kappa^{(n)}}\big), \end{aligned}$$

using first data processing of the coherent information and then its convexity in the state [36]. As $n \to \infty$ and $F \to 1$, the upper bound on the rate follows. ■

Proposition 21: The entanglement-environment-assisted quantum capacity is continuous. The statement and proof are analogous to the ones in Proposition 8, following [29]. ■

The super-activation of $U \in \mathfrak{A}$ with SWAP depicted in Fig. 5 translates to positive capacity with an entangled helper. We discuss two concrete examples of two-qubit unitaries:

E-1 $Q_{EH}(\mathrm{SWAP}^\gamma) > 0$ for $\gamma \in [0.5, 0.6649)$, cf. Section IV, example A-1.

E-2 Consider U corresponding to a point on the line segment joining $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$ and $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$. These points are vertices of $\mathfrak{A}$ (see Fig. 4), and hence the line segment is an edge of the universally anti-degradable tetrahedron. As we saw in Section IV, example A-2, this is super-activated by SWAP.

We now show how to evaluate the single-copy coherent information in the entanglement-environment-assisted capacity of SWAP^γ, with $\gamma \in [0, 1]$, as per Theorem 20, Eq. (12); the setting is as in the lower part of Fig. 11. To proceed, we need the following lemma.

Lemma 22: If an isometry $U : AE \longrightarrow BF$ is universally degradable, then for every $|\kappa\rangle \in EH$, the channel $\mathcal{N}_\kappa : \mathcal{L}(A) \longrightarrow \mathcal{L}(BH)$ is degradable.

Proof: Recall $\mathcal{N}_\kappa(\rho) = \mathrm{Tr}_F(\mathcal{N} \otimes \mathrm{id}_H)(\rho^A \otimes \kappa^{EH})$, with Stinespring dilation $V|\varphi\rangle = (U \otimes \mathbb{1})(|\varphi\rangle|\kappa\rangle)$, mapping A to $BH \otimes F$. Hence, the complementary channel is given by

$$\widetilde{\mathcal{N}_\kappa}(\rho) = \mathrm{Tr}_B\, \mathcal{N}(\rho^A \otimes \kappa^E),$$

with the reduced state $\kappa^E = \mathrm{Tr}_H\, \kappa$.

Let $|\kappa\rangle = \sum_i \sqrt{p_i}|\eta_i\rangle^E|i\rangle^H$ be the Schmidt decomposition. Then, on the one hand,

$$\widetilde{\mathcal{N}_\kappa} = \sum_i p_i \widetilde{\mathcal{N}_{\eta_i}} = \sum_i p_i \mathcal{D}_i \circ \mathcal{N}_{\eta_i},$$

with degrading CPTP maps $\mathcal{D}_i^{B\to F}$ by assumption.

As i is accessible in the output of $\mathcal{N}_\kappa$ by measuring H in the computational basis, we obtain the degrading map $\mathcal{D}^{BH\to F}$ such that $\widetilde{\mathcal{N}_\kappa} = \mathcal{D} \circ \mathcal{N}_\kappa$, via $\mathcal{D}(\sigma \otimes |i\rangle\langle j|) = \delta_{ij}\mathcal{D}_i(\sigma)$. ■

Returning to SWAP^γ, the combined channel and environment input is $\rho^A \otimes \kappa^{EH}$. Because of the $u \otimes u$-symmetry of the gate, we may without loss of generality choose the bases of E and H such that $|\kappa\rangle^{EH} = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle$.

Now, κ is invariant under the action of $Z^E \otimes Z^{\dagger H}$, hence we obtain a covariance property of the channel:

$$\mathcal{N}_\kappa(Z\rho Z^\dagger) = (Z \otimes Z^\dagger)\mathcal{N}_\kappa(\rho)(Z^\dagger \otimes Z).$$

By Lemma 22, $\mathcal{N}_\kappa$ is degradable, hence the coherent information is concave in ρ^A [13] and so the coherent information is maximized on an input density ρ^A that commutes with Z. I.e. we may assume that $\rho^A = \mu|0\rangle\langle 0| + (1-\mu)|1\rangle\langle 1|$.

We then find for the output states of Bob ($B'B = HB$) and the environment (F) that

$$\begin{aligned} \rho^{B'B} &= \lambda\left(\mu + (1-\mu)\left|\frac{1-e^{i\pi\gamma}}{2}\right|^2\right)|00\rangle\langle 00| \\ &+ (1-\lambda)\left((1-\mu) + \mu\left|\frac{1-e^{i\pi\gamma}}{2}\right|^2\right)|11\rangle\langle 11| \\ &+ \sqrt{\lambda(1-\lambda)}\left(\frac{1}{2} - \frac{\mu}{2}e^{-i\pi\gamma} - \frac{1-\mu}{2}e^{i\pi\gamma}\right)|00\rangle\langle 11| \\ &+ \sqrt{\lambda(1-\lambda)}\left(\frac{1}{2} - \frac{\mu}{2}e^{i\pi\gamma} - \frac{1-\mu}{2}e^{-i\pi\gamma}\right)|11\rangle\langle 00| \\ &+ \lambda(1-\mu)\left|\frac{1+e^{i\pi\gamma}}{2}\right|^2\big(|01\rangle\langle 01| + |10\rangle\langle 10|\big), \end{aligned}$$

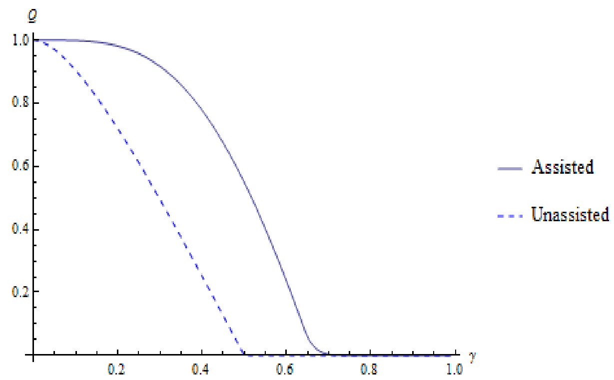

Fig. 13. The unbroken curve in the plot is $Q_{EH\otimes}$, the single-copy coherent information in the formula for the entanglement-environment-assisted quantum capacity of SWAP$^\gamma$, Eq. (12), i.e. the maximum of $I_c(\rho; \mathcal{N}_\kappa)$ over states ρ^A and κ^{EH}. The dashed line is the restricted environment-assisted quantum capacity $Q_{H\otimes}(\text{SWAP}^\gamma)$.

and ρ^F is diagonal in the computational basis:

$$\rho^F = \left(\lambda\mu+\lambda(1-\mu)\left|\frac{1+e^{i\pi\gamma}}{2}\right|^2+\mu(1-\lambda)\left|\frac{1-e^{i\pi\gamma}}{2}\right|^2\right)|0\rangle\langle 0| + \left((1-\lambda)(1-\mu)+\lambda(1-\mu)\left|\frac{1-e^{i\pi\gamma}}{2}\right|^2 + \mu(1-\lambda)\left|\frac{1+e^{i\pi\gamma}}{2}\right|^2\right)|1\rangle\langle 1|.$$

In Fig. 13 we plot the single-copy coherent information assisted by an entangled environment, maximized over λ and μ, and compare it with the same quantity without pre-shared entanglement. This is actually the quantum capacity assisted by entangled states of the form $\kappa^{E^nH^n} = \kappa^{E_1H_1}\otimes\cdots\otimes\kappa^{E_nH_n}$ in Definition 19, which we might denote $Q_{EH\otimes}(U)$ in analogy with $Q_{H\otimes}(U)$. As shown in the plot, the entanglement between Helen and Bob increases the quantum capacity of SWAP$^\gamma$ to a positive quantity for a large interval of γ values, up to $\gamma^{**}\approx 0.7662$.

Remark 23: It follows that we could achieve super-activations of SWAP$^\gamma$ with SWAP for larger interval of $\gamma\in[0.5, 0.7662)$, when optimizing over the input of SWAP$^\gamma$ and the initial environment state, in Section IV, example A-1.

Remark 24: We could even contemplate a fully entanglement-assisted model, where both Alice and Helen share prior entanglement with Bob. This is a special case of Hsieh *et al.*'s entanglement-assisted multi-access channel [23]: Indeed, if the achievable rate region of pairs of rates (R_A, R_E) for quantum communication via $\mathcal{N}^{AE\to B}$ assisted by arbitrary pre-shared entanglement is known, then the entanglement- and helper-assisted quantum capacity is given by the largest R such that the pair $(R, 0)$ is achievable.

VI. Conclusion

We have laid the foundations of a theory of quantum communication with passive environment-assistance, where a helper is able to select the initial environment state of the channel, modelled as a unitary interaction. The general, multi-letter, capacity formulas we gave for the quantum capacity assisted by an unrestricted, and by a separable helper resemble the analogous formula for the unassisted capacity. Like the latter, which is contained as a special case, the environment-assisted capacities are continuous in the channel, but in general seem to be hard to characterize in simple ways. As noted in Remarks 7 and 24, the passive environment-assisted models (resp. passive entanglement-environment-assisted models) come out as a special case of the quantum MAC (resp. entanglement-assisted MAC). However it is only the model that is a special case, but not the results per se as evident in the case of universally degradable unitaries, where the separable helper capacity is characterized by a single-letter, but we can not say it has a capacity region characterized by single-letter in the multi-access viewpoint. Furthermore, the assisted model viewpoint the transmission capabilities are charactrized by a single rate as opposed to the rates-region which characterizes MAC.

In our development we have then focused on two-qubit unitaries, giving rise to very simple-looking qubit channels for which the environment-assisted quantum capacity with separable helper can be evaluated. Interestingly, there are unitaries giving rise to anti-degradable channels for every input state, hence the capacity with separable helper vanishes; yet, some of these "universally anti-degradable" unitaries could be super-activated by unitaries from the same class, in some cases by themselves. In fact, there is a single unitary $\sqrt{\text{SWAP}}$ that activates all universally anti-degradable unitaries $U\in\mathfrak{A}$ (except itself, according to numerics). In particular, the quantum capacity Q_H with unrestricted helper can be strictly larger than the one with separable helper, $Q_{H\otimes}$, and the computation of the former remains a major open problem.

Some other interesting open questions include the following:

- How to characterize the set of unitaries U such that $Q_H(U) = 0$? Note that in the two-qubit case we only the example $U = \text{SWAP}$, but it seems that $\sqrt{\text{SWAP}}$ is another one, but we lack a proof. In [24] we have identified a class of unitaries with vanishing Q_H of which SWAP is a special case. These unitaries $U_c : A\otimes E\longrightarrow B\otimes F$ with $|A| = |B| = |E| = |F| = d$ are of the form
$$U_c := \sum_i |i\rangle^F\langle i|^A\otimes U_i^{E\to F}$$
Furthermore, for two unitaries U_{c1} and U_{c2} which are of the above form, $Q_H(U_{c1}\otimes U_{c2}) = 0$.
- Can Q_H be super-activated, i.e. are there U, V with $Q_H(U) = Q_H(V) = 0$ but $Q_H(U\otimes V) > 0$? From the above analysis, $U = \text{SWAP}$ and $V = \sqrt{\text{SWAP}}$ seem good candidates.

Finally, we only just started the issue of entanglement-environment-assistance, motivated by the distinguished role of the SWAP gate in many of our examples. But for the moment we do not even have an understanding of super-activations of the entanglement-environment-assisted capacities Q_{EH} and $Q_{EH\otimes}$.

Our model and approach can evidently be adapted to other communication capacities, say for instance the private capacity P and classical capacity C of a channel. Regarding the former, our examples of super-activation and self-super-activation apply directly because private and quantum capacity coincide for degradable and anti-degradable channels. On the classical capacity we have preliminary results which will be reported on in forthcoming work [24].

Looking further afield, one can conceive a very powerful helper having access to both the input and the output of the environment, thus providing a unifying picture between present study and those originated from Refs. [18], [19], [41]. In such a scenario it would be particularly interesting to determine channels that do not trivialize, i.e. that, notwithstanding the power of the helper, lead to non optimal capacities. For example, we can identify SWAP as the unitary which has zero quantum communication capabilities even with the all powerful helper.

Appendix A
Communication in the Presence of a Jammer (QAVC)

The purpose of this appendix is to prove the adversarial channel capacity theorem, which we restate here:

Theorem 5: For any jammer channel $\mathcal{N} : AE \to B$,

$$Q_{J,r}(\mathcal{N}) = \sup_n \max_{\rho^{(n)}} \min_\eta \frac{1}{n} I_c\big(\rho^{(n)}; (\mathcal{N}_\eta)^{\otimes n}\big),$$

where the maximization is over states $\rho^{(n)}$ on A^n, and the minimization is over arbitrary states η on E.

Proof: The converse part, i.e. the "$\leq$" inequality, follows from [1, Th. 27], because in the proof it is enough to consider tensor product strategies $\eta^{(n)} = \eta_1 \otimes \cdots \otimes \eta_n$ of the jammer, hence $\mathcal{N}_{\eta^{(n)}} = \mathcal{N}_{\eta_1} \otimes \cdots \otimes \mathcal{N}_{\eta_n}$ is a tensor product map as in the AVQC model. Thus the proof of [1] applies unchanged.

For the direct part ("$\geq$"), consider input states $\rho^{(n)}$ on A^n and a rate

$$R \leq \min_\eta I_c\big(\rho^{(n)}; (\mathcal{N}_\eta)^{\otimes n}\big) - \delta,$$

for $\delta > 0$ and all integers n. We invoke a result of Bjelaković *et al.* [4] on the so-called *compound channel* $\big((\mathcal{N}_\eta)^{\otimes n}\big)_{\eta \in \mathcal{S}(E)}$, to the effect that there exist codes $(\mathcal{D}_n, \mathcal{E}_n)$ for all block lengths n and with rate R that perform universally well for all the i.i.d. channels $(\mathcal{N}_\eta)^{\otimes n}$:

$$F_n := \min_\eta F\big(\Phi^{RB_0}, (\mathcal{D}_n \circ \mathcal{N}_\eta^{\otimes n} \circ \mathcal{E}_n)\Phi^{RA_0}\big) \geq 1 - c^n,$$

with some $c < 1$. For later use, let us rephrase this condition as a property of $\eta^{(n)} = \eta^{\otimes n}$:

$$\begin{aligned} c^n &\geq 1 - F \\ &= \operatorname{Tr}(\mathbb{1} - \Phi)\big(\mathcal{D}_n \circ \mathcal{N}^{\otimes n}(\mathcal{E}_n(\Phi) \otimes \eta^{(n)})\big) \\ &= \operatorname{Tr}\big((\mathcal{N}^{\dagger \otimes n} \circ \mathcal{D}_n^\dagger)(\mathbb{1} - \Phi)\big)\big(\mathcal{E}_n(\Phi) \otimes \eta^{(n)}\big) \\ &= \operatorname{Tr} X_n \eta^{(n)}, \end{aligned} \tag{13}$$

where $0 \leq X_n \leq \mathbb{1}$ is a constant operator depending only on the code.

We claim that, using a shared uniformly random permutation $\pi \in S_n$ to permute the n input/output systems, the same code is good against the jammer. Concretely, let $\mathcal{U}^\pi$ be the conjugation by the permutation unitary on an n-party system, and define, for a given n,

$$\begin{aligned} \mathcal{E}_\pi &:= \mathcal{U}^\pi \circ \mathcal{E}_n, \\ \mathcal{D}_\pi &:= \mathcal{D}_n \circ \mathcal{U}^{\pi^{-1}}. \end{aligned}$$

Then, for any jammer strategy $\eta^{(n)} \in \mathcal{S}(E^n)$,

$$\begin{aligned} 1 - \overline{F}\Big(\eta^{(n)}\Big) &= \frac{1}{n!} \sum_{\pi \in S_n} 1 - F\big(\Phi^{RB_0}, \big(\mathcal{D}_\pi \circ (\mathcal{N}^{\otimes n})_{\eta^{(n)}} \circ \mathcal{E}_\pi\big)\Phi^{RA_0}\big) \\ &= \operatorname{Tr}\left(X_n \frac{1}{n!} \sum_{\pi \in S_n} \mathcal{U}^\pi(\eta^{(n)})\right) \\ &= \operatorname{Tr} X_n \overline{\eta}^{(n)}, \end{aligned} \tag{14}$$

using Eq. (13), and where $\overline{\eta}^{(n)} = \frac{1}{n!}\sum_{\pi \in S_n} \mathcal{U}^\pi(\eta^{(n)})$ is permutation symmetric.

At this point, we can apply the postselection technique of [9], which relies on the matrix inequality

$$\overline{\eta}^{(n)} \leq (n+1)^{|E|^2} \int_\sigma \mathrm{d}\sigma\, \sigma^{\otimes n},$$

with a certain universal probability measure $\mathrm{d}\sigma$ over states on E. Thus, according to the assumption and the above Eq. (14), we find that for the permutation-symmetrized compound channel code,

$$1 - \overline{F}(\eta^{(n)}) \leq (n+1)^{|E|^2} c^n$$

for all jammer strategies $\eta^{(n)}$, and the right hand side of course still goes to zero exponentially fast, concluding the proof. ■

Appendix B
Parametrization of Two-Qubit Unitaries and Degradability Regions

For the further analysis we require another analytical criterion for anti-degradability:

Lemma 25 (Myhr/Lütkenhaus [33]): A qubit channel with qubit environment is anti-degradable if and only if $\lambda_{\max}(\rho_{RB}) \leq \lambda_{\max}(\rho_B)$, where $\lambda_{\max}(X)$ is the maximum eigenvalue of a Hermitian matrix X. Here ρ_{RB} is the Choi matrix of the given qubit channel and ρ_B is the reduced state after tracing out the reference system R. ■

Following the analysis in Section III, we restrict our attention to the parameter space $\mathfrak{T}$ of $(\alpha_x, \alpha_y, \alpha_z)$ satisfying $\frac{\pi}{2} \geq \alpha_x \geq \alpha_y \geq \alpha_z \geq 0$, which forms a tetrahedron with vertices $(0,0,0)$, $(\frac{\pi}{2},0,0)$, $(\frac{\pi}{2},\frac{\pi}{2},0)$ and $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$.

Given a unitary $U(\alpha_x,\alpha_y,\alpha_z)$ and an initial state of the environment, $|\xi\rangle = \cos(\frac{\theta}{2})|0\rangle + e^{i\varphi}\sin(\frac{\theta}{2})|1\rangle$, where $\theta \in [0,\pi]$, $\varphi \in [0,2\pi)$, we evaluate the Choi matrix by inputting a maximally entangled state $|\Phi\rangle = \frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big)$. Thus the output state is $|\Psi\rangle^{RBF} = (\mathbb{1}^R \otimes U^{AE})\big(|\Phi\rangle^{RA} \otimes |\xi\rangle^E\big)$. From the Schmidt decomposition, the maximum eigenvalue of ρ_{RB} is equal to the maximum eigenvalue of $\rho^F = \operatorname{Tr}_{RB}|\Psi\rangle\langle\Psi|$, which can be written in matrix form as

$$\frac{1}{2}\begin{bmatrix} 1 + a_F & b_F - ic_F \\ b_F + ic_F & 1 - a_F \end{bmatrix}, \tag{15}$$

with the Bloch vector components given by

$$\begin{aligned} a_F &= \cos(\theta)\cos(\alpha_x)\cos(\alpha_y), \\ b_F &= \sin(\theta)\cos(\varphi)\cos(\alpha_z)\cos(\alpha_y), \\ c_F &= \sin(\theta)\sin(\varphi)\cos(\alpha_z)\cos(\alpha_x). \end{aligned}$$

Similarly, $\rho^B = \mathrm{Tr}_{RF}|\Psi\rangle\langle\Psi|$ has Bloch vector components given by

$$\begin{aligned} a_B &= \cos(\theta)\sin(\alpha_x)\sin(\alpha_y), \\ b_B &= \sin(\theta)\cos(\varphi)\sin(\alpha_z)\sin(\alpha_y), \\ c_B &= \sin(\theta)\sin(\varphi)\sin(\alpha_z)\sin(\alpha_x). \end{aligned}$$

The largest eigenvalue of a qubit density matrix ρ with Bloch vector components a, b, c is $\frac{1+\sqrt{a^2+b^2+c^2}}{2}$. When we impose the condition for anti-degradability from Lemma 25 we get the following inequality:

$$\begin{aligned} 0 \geq &\cos^2(\theta)\cos(\alpha_x+\alpha_y)\cos(\alpha_x-\alpha_y) \\ &+ \sin^2(\theta)\cos^2(\varphi)\cos(\alpha_z+\alpha_y)\cos(\alpha_z-\alpha_y) \\ &+ \sin^2(\theta)\sin^2(\varphi)\cos(\alpha_z+\alpha_x)\cos(\alpha_z-\alpha_x). \end{aligned}$$

This must be true for all input states of environment, hence for all $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi)$. Thus we arrive at

$$\alpha_x + \alpha_y, \quad \alpha_y + \alpha_z, \quad \alpha_z + \alpha_x \geq \frac{\pi}{2}, \tag{16}$$

for the universally anti-degradable region. This forms another tetrahedron with vertices $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$, $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4})$, $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2})$, which is depicted in Fig. 4.

By swapping the outputs of unitary $U \in \mathfrak{A}$ we get another unitary $V = \mathrm{SWAP} \cdot U \in \mathfrak{D}$. By applying this transformation to the vertices of the parameter region of $\mathfrak{A}$, we get the vertices of the parameter region $\mathfrak{D}$ given by $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$, $(\frac{\pi}{4}, \frac{\pi}{4}, 0)$, $(\frac{\pi}{2}, 0, 0)$ and $(0, 0, 0)$. The unitary $\sqrt{\mathrm{SWAP}}$, with the parameters $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4})$, is the unitary which lies in the intersection of $\mathfrak{A}$ and $\mathfrak{D}$. This gives rise to symmetric qubit channels for every initial state of the environment.

The universally anti-degradable region in $\mathfrak{T}_{total}$ is obtained by union of $\mathfrak{A}$ and its reflection about the α_x and α_y plane which is denoted by $\mathfrak{A}_{total}$. Thus $\mathfrak{A}_{total}$ is the union of $\mathfrak{A}$ and the tetrahedron with the vertices $(\frac{\pi}{4}, \frac{\pi}{4}, \frac{-\pi}{4})$, $(\frac{\pi}{2}, \frac{\pi}{4}, \frac{-\pi}{4})$, $(\frac{\pi}{2}, \frac{\pi}{2}, \frac{-\pi}{2})$ and $(\frac{\pi}{2}, \frac{\pi}{2}, 0)$.

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