Quantum Chaos and $1/f$ Noise

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It is shown that the energy spectrum fluctuations of quantum systems can be formally considered as a discrete time series. The power spectrum behavior of such a signal for different systems suggests the following conjecture: The energy spectra of chaotic quantum systems are characterized by $1/f$ noise.

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The understanding of quantum chaos has greatly advanced during the past two decades. It is well known that there is a clear relationship between the energy level fluctuation properties of a quantum system and the large time scale behavior of its classical analogue. The pioneering work of Berry and Tabor [1] showed that the spectral fluctuations of a quantum system whose classical analogue is fully integrable are well described by Poisson statistics; i.e., the successive energy levels are not correlated. In a seminal paper, Bohigas *et al.* [2] conjectured that the fluctuation properties of generic quantum systems, which in the classical limit are fully chaotic, coincide with those of random matrix theory (RMT). This conjecture is strongly supported by experimental data, many numerical calculations, and analytical work based on semiclassical arguments. A review of later developments can be found in [3,4].

We propose in this Letter a different approach to quantum chaos based on traditional methods of time series analysis. The essential feature of chaotic energy spectra in quantum systems is the existence of level repulsion and correlations. To study these correlations, we can consider the energy spectrum as a discrete signal, and the sequence of energy levels as a time series. For example, the sequence of nearest level spacings has similarities with the diffusion process of a particle. But generally we do not need to specify the nature of the analogue time series. As we shall see, examination of the power spectrum of energy level fluctuations reveals very accurate power laws for completely regular or completely chaotic Hamiltonian quantum systems. It turns out that chaotic systems have $1/f$ noise, in contrast to the Brown noise of regular systems.

The first step, previous to any statistical analysis of the spectral fluctuations, is the unfolding of the energy spectrum. Level fluctuation amplitudes are modulated by the value of the mean level density $\bar{\rho}(E)$, and therefore, to compare the fluctuations of different systems, the level density smooth behavior must be removed. The unfolding consists in locally mapping the real spectrum into another with mean level density equal to one. The actual energy levels E_i are mapped into new dimensionless levels ϵ_i,

$$E_i \rightarrow \epsilon_i = \bar{N}(E_i), \qquad i = 1, \ldots, N, \tag{1}$$

where N is the dimension of the spectrum and $\bar{N}(E)$ is given by

$$\bar{N}(E) = \int_{-\infty}^{E} dE' \bar{\rho}(E'). \tag{2}$$

This function is a smooth approximation to the step function $N(E)$ that gives the true number of levels up to energy E. The form of the function $\bar{\rho}(E)$ can be determined by a best fit of $\bar{N}(E)$ to $N(E)$.

The nearest neighbor spacing sequence is defined by

$$s_i = \epsilon_{i+1} - \epsilon_i, \qquad i = 1, \ldots, N-1. \tag{3}$$

For the unfolded levels, the mean level density is equal to 1 and $\langle s \rangle = 1$. In practical cases, the unfolding procedure can be a difficult task for systems where there is no analytical expression for the mean level density [5].

Generally, two suitable statistics are used to study the fluctuation properties of the unfolded spectrum. The nearest neighbor spacing distribution $P(s)$ gives information on the short range correlations among the energy levels. The $\Delta_3(L)$ statistic makes it possible to study correlations of length L: As we change the L value, we obtain information on the level correlations at all scales. By contrast, in this paper we characterize the spectral fluctuations by the statistic δ_n [6] defined by

$$\delta_n = \sum_{i=1}^{n} (s_i - \langle s \rangle) = \sum_{i=1}^{n} w_i, \tag{4}$$

where the index n runs from 1 to $N-1$. The quantity w_i gives the fluctuation of the ith spacing from its mean value $\langle s \rangle = 1$. The function $\langle \delta_n^2 \rangle$ is closely related to the covariance matrix and thus provides important information on level correlations. Recently, it has been shown [7] that, under certain assumptions, $\langle \delta_n^2 \rangle$ is a logarithmic function of n for the RMT ensembles.

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The subject of the present work is the function δ_n, instead of $\langle \delta_n^2 \rangle$. It represents the deviation of the unfolded excitation energy from its mean value n. From our point of view, the function δ_n has a formal similarity with a time series. For example, we may compare the energy level spectrum with the diffusion process of a particle. The analogy is clear if the index i of the nearest level spacings is considered as a discrete time, and the spacing fluctuation w_i as the analogue of the particle displacement d_i from the collision at time i to the next collision. Certainly, there are some differences. For instance, there is no limitation for a single particle displacement d_i, whereas $w_i > -1$ for a nearest level spacing sequence because there cannot exist negative spacings. We also note that the amplitude and sign of the displacements are given by a certain probability function that may be very complex and depend on the previous particle trajectory, i.e., on the values d_j, $j < i$. On the other hand, in a spectrum the position of each level, and consequently w_i, depends not only on the lower energy levels, but also on those with higher energy. Considered as a time series, there is a strong dependence on the *past* as well as on the *future* history. Nevertheless, in spite of these peculiarities, the analogy exists, and the function δ_n is the analogue of the particle total displacement at time n.

Our aim is to study the δ_n signal of chaotic quantum systems. We can analyze their spectral statistics with numerical techniques normally used in the study of complex systems and try to relate the emerging properties with some universal features that appear in many other branches of physics. One of those techniques is the calculation of the power spectrum $S(k)$ of a discrete and finite series δ_n given by

$$S(k) = |\hat{\delta}_k|^2, \tag{5}$$

where $\hat{\delta}_k$ is the Fourier transform of δ_n,

$$\hat{\delta}_k = \frac{1}{\sqrt{N}} \sum_n \delta_n \exp\left(\frac{-2\pi i k n}{N}\right), \tag{6}$$

and N is the size of the series.

As an example of a very chaotic system, we take the atomic nucleus at high excitation energy, where the level density is very large. To obtain the energy spectrum, shell-model calculations for selected nuclei are performed, using realistic interactions that reproduce well experimental data of nuclei in a mass region. The Hamiltonian matrices for different angular momenta, parity, and isospin are fully diagonalized, and careful global unfolding is performed. Then, sets of 256 consecutive levels of the same $J^\pi T$, from the high level density region, are selected. To characterize the statistical properties of the δ_n signal, we calculate an ensemble average of its power spectrum, in order to reduce statistical fluctuations and clarify its main trend. The average $\langle S(k) \rangle$ is calculated with 25 sets.

Figure 1 shows the results for a typical stable *sd* shell nucleus, ^{24}Mg, with matrix dimensionalities up to about 2000; and for a very exotic nucleus, ^{34}Na, with dimensions up to about 5000, in the *sd* proton and *pf* neutron shells. Clearly, the power spectrum of δ_n follows closely a power law. We may assume the simple functional form

$$\langle S(k) \rangle \sim \frac{1}{k^\alpha}. \tag{7}$$

A least squares fit to the data of Fig. 1 gives $\alpha = 1.11 \pm 0.03$ for ^{34}Na, and $\alpha = 1.06 \pm 0.05$ for ^{24}Mg. These results raise the question of whether there is a general relationship between quantum chaos and the power spectrum of the δ_n fluctuations of the system.

Probably, the simplest and most reliable way to clarify this issue is to compare δ_n and $\langle S(k) \rangle$ for Poisson energy levels and random matrix spectra. Random matrix theory plays a predominant role in the description of chaotic quantum systems [3,4]. It deals with three basic Hamiltonian matrix ensembles: the Gaussian orthogonal ensemble (GOE) of N-dimensional matrices, applicable for time-reversal invariant chaotic systems with rotational symmetry or integer spin and broken rotational symmetry; the Gaussian unitary ensemble (GUE), applicable for chaotic systems in which the time-reversal invariance is violated; and the Gaussian symplectic ensemble (GSE), applicable for time-reversal invariant chaotic systems with half-integer spin and broken rotational symmetry. There are, of course, other more complex ensembles such as deformed ensembles, band matrix ensembles, etc., but they will not be considered in this work. Instead, we include an ensemble of diagonal matrices whose elements are random Gaussian variables. We call it the *Gaussian diagonal ensemble* (GDE).

Figure 2 shows the signal δ_n for a GDE (Poisson) and a GOE spectrum of dimension 1000. Clearly, the two

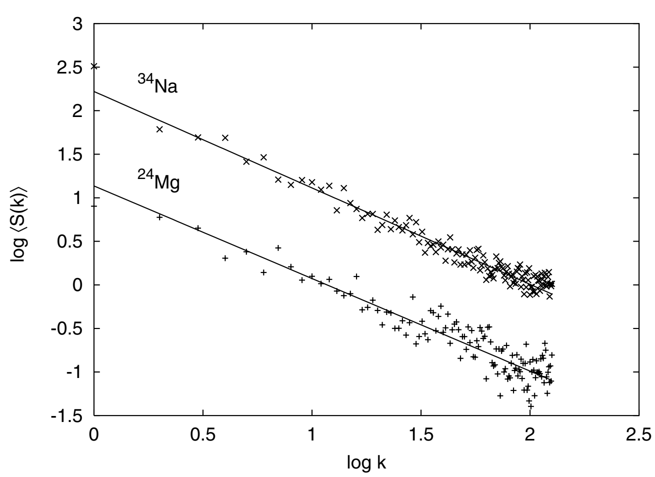

FIG. 1. Average power spectrum of the δ_n function for ^{24}Mg and ^{34}Na, using 25 sets of 256 levels from the high level density region. The plots are displaced to avoid overlapping.

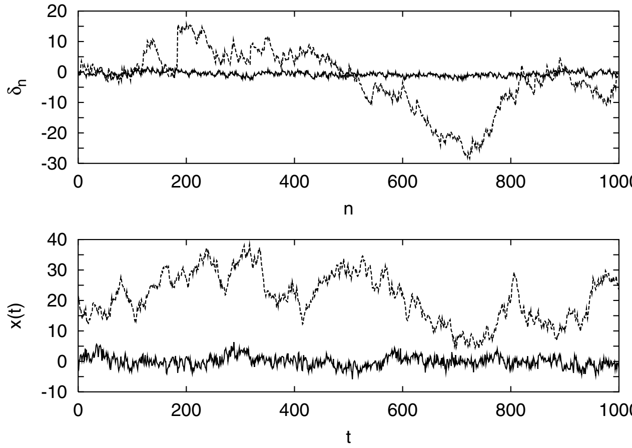

FIG. 2. Comparison of the δ_n function for Poisson (dashed line) and GOE spectra (solid line), with a standard time series $x(t)$ with $1/k^\alpha$ power spectrum, for $\alpha = 2$ (dashed line) and $\alpha = 1$ (solid line).

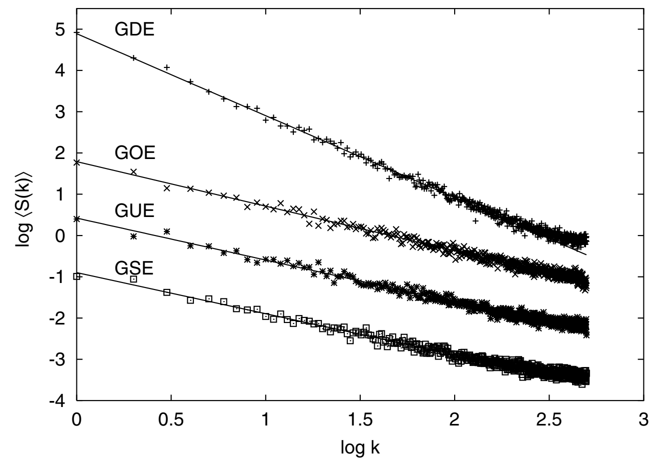

FIG. 3. Power spectrum of the δ_n function for GDE (Poisson) energy levels, compared to GOE, GUE, and GSE. The plots are displaced to avoid overlapping.

signals are very different. It is also illustrative to compare those signals with a discrete time series $x(t)$, with $1/k$ and $1/k^2$ power laws, generated with the random-phase approximation procedure used in [8]. The similarity of the $\alpha = 2$ time series with the Poisson spectrum and the $\alpha = 1$ time series with the GOE spectrum is obvious.

To compute the average $\langle S(k)\rangle$, we generate 30 different matrices of dimension 1000 for each type of random matrix ensemble. Figure 3 shows the results of these calculations in a decimal log-log scale. In all the cases, the main trend is essentially linear, except for very high frequencies, where some deviation is observed, probably due to finite size effects.

Ignoring frequencies greater than $\log k = 2.2$, the fit to (7) gives $\alpha_{\mathrm{GDE}} = 1.99$ with an uncertainty near 2%. The spectrum of any matrix pertaining to GDE consists of N uncorrelated levels. This is due to the diagonal character of the matrix and to the fact that its matrix elements are independent random variables. Consequently, the nearest level spacings are also uncorrelated, and δ_n is just a sum of $N - 1$ independent random variables. The power spectrum of such a signal is well known to present $1/k^2$ behavior, and that is in full agreement with our numerical value for α in the Poisson spectrum. Furthermore, Berry and Tabor [1] showed that, in a semiclassical integrable system, the spacings s_i are random independent variables for $i \gg 1$. As a consequence, their δ_n power spectrum behaves as $1/k^2$. However, this behavior may be modified by the levels of the ground state region.

By contrast, the spectrum of any GOE member of large dimension is generally considered the paradigm of chaotic quantum spectra. It presents level correlations at all scales. The same applies to GUE and GSE, in increasing order of level repulsion. As is well known, the nearest neighbor spacing distribution for these three ensembles behaves as $P(s) \sim s^\beta$ for small s, where β is known as the level repulsion parameter. For our diagonal ensemble with Poisson statistics, $\beta = 0$, while $\beta = 1$, 2, and 4 for GOE, GUE, and GSE, respectively [3,4].

The power spectrum of δ_n for these three ensembles is displayed in Fig. 3. The fit of $\langle S(k)\rangle$ to the power law (7) is excellent. For the exponents, we obtain $\alpha_{\mathrm{GOE}} = 1.08$, $\alpha_{\mathrm{GUE}} = 1.02$, and $\alpha_{\mathrm{GSE}} = 1.00$. In all the cases, the error of the linear regression is about 2%. The three ensembles yield the same power law, with $\alpha \simeq 1$. This is in agreement with preliminary results [9] showing that a logarithmic behavior of $\langle \delta_n^2 \rangle$, as for the GOE [7], is related to a $1/k^\alpha$ behavior of the δ_n power spectrum, with $\alpha = 1$. The small deviations from $\alpha = 1$ observed in our numerical results are probably simple finite size effects of the matrix dimensions, which introduce additional inaccuracies through statistical fluctuations and the unfolding procedure. As Table I shows, α_{GOE} approaches 1 as the matrix dimensionality increases.

Clearly, the power spectrum $\langle S(k)\rangle$ behaves as $1/k^\alpha$ both in regular and chaotic energy spectra, but level correlations decrease the exponent from the $\alpha = 2$ limit for uncorrelated spectra to apparently a minimum value $\alpha = 1$ for chaotic quantum systems.

The concept of quantum chaos has no precise definition as yet. Quantum systems with classical analogues are considered chaotic when their classical analogues are chaotic. Quantum systems without classical analogues may be called chaotic if they show the same kind of fluctuations as chaotic quantum systems with classical analogues. In practice, the Bohigas-Gianoni-Schmit conjecture is generally used as a criterion.

TABLE I. Dependence of the power spectrum exponent α_{GOE} on the ensemble matrix dimensionality N.

N	128	512	2048
α_{GOE}	1.10	1.09	1.06

The results obtained above for the power spectrum of the δ_n statistic suggest the following conjecture.

The energy spectra of chaotic quantum systems are characterized by $1/f$ *noise.*

This conjecture has several appealing features. It is a property characterizing the chaotic spectrum by itself, without any reference to the properties of other systems such as GOE. It is universal for all kinds of chaotic quantum systems, either time-reversal invariant or not, either of integer or half-integer spin. Furthermore, the $1/f$ noise characterization of quantum chaos includes these physical systems into a widely spread kind of systems appearing in many fields of science, which display $1/f$ fluctuations. Thus, the energy spectrum of chaotic quantum systems exhibits the same kind of fluctuations as many other complex systems. However, there is no indication that $1/f$ spectral fluctuations in a quantum system imply $1/f$ noise in its classical analogue. Neither have we found any relationship with $1/f$ noise in classical chaotic phenomena such as intermittency [10].

Chaotic quantum energy spectra are characterized by strong level repulsion and strong spectral rigidity. In terms of the conventional $\Delta_3(L)$ statistic, strong rigidity corresponds to small values of $\Delta_3(L)$ and slow, logarithmic dependence on L. We can try to interpret what spectral rigidity means in terms of the δ_n function.

Rigidity of the energy spectrum means that the deviations of the energy spacings s_i from their mean value $\langle s \rangle = 1$ are generally small, and that the spectrum is organized in such a way that a deviation of a spacing from the mean tends to be balanced by neighboring spacings. Therefore it is unlikely to find a long series of consecutive spacings all above or below the mean spacing.

In a time series, antipersistence means that an increasing or decreasing trend in the past makes the opposite trend in the future probable. In the present approach, where a quantum energy spectrum is considered as a time series, the spectral rigidity is analogous to antipersistence. We have seen that a rigid energy spectrum gives rise to a δ_n power spectrum of $1/k^\alpha$ type with $\alpha = 1$. On the other hand, as is well known, a time series with a $1/k^\alpha$ power spectrum where $\alpha \simeq 1$ is very antipersistent. Therefore, the interpretation of spectral rigidity as the analogue of antipersistence is consistent with the behavior of the power spectrum.

In summary, we have seen that for quantum systems the δ_n function can be considered as a time series, where the level order index n plays the role of a discrete time. The power spectrum $\langle S(k) \rangle$ of δ_n has been studied for representative energy spectra of regular and chaotic quantum systems. Neat power laws $\langle S(k) \rangle \sim 1/k^\alpha$ have been found in all cases. For Poisson spectra, we get $\alpha = 2$, as expected for independent random variables. For spectra of atomic nuclei at higher energies, in regions of high level density, and for the GOE, GUE, and GSE ensembles, we obtain $\alpha = 1$.

These results suggest the conjecture that chaotic quantum systems are characterized by $1/f$ noise in the energy spectrum fluctuations. This property is not a mere statistic to measure the chaoticity of the system. It provides an intrinsic characterization of quantum chaotic systems without any reference to the properties of RMT ensembles. As is well known, $1/f$ noise is quite ubiquitous. It characterizes sunspot activity, the flow of the Nile river, music, and chronic illness [11]. And we believe that it characterizes quantum chaos as well.

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