On the quantum, classical and total amount of correlations in a quantum state

Berry Groisman, Sandu Popescu, and Andreas Winter

March 9, 2011

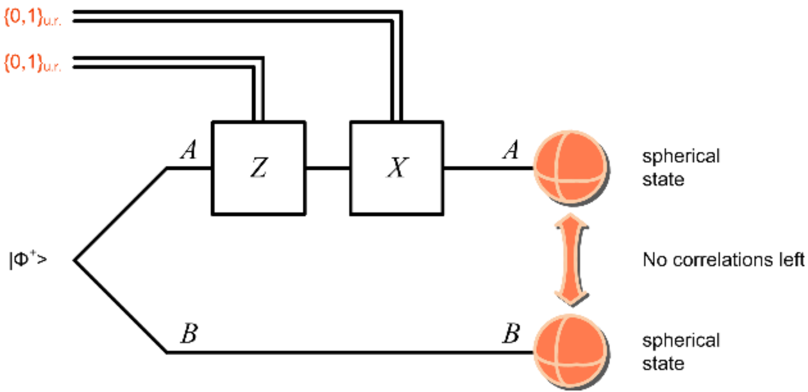
Presentation prepared for COMP598 - Quantum Shannon Theory by Jan Florjanczyk

History (forward and backwards)

- Classical - Landauer's thermodynamic principle: amount of information stored is equal to amount of work necessary to erase it.
- Previous work - Oppenheim and H^3 showed that the work achievable by two parties on a bipartite state is less than work on the whole state.
- Schumacher's "entropy exchange" will be the quantity of choice to measure the work in a quantum state.

- Calculated the amount necessary to erase all correlations in a quantum state, as well as quantum and classical correlations individually.
- Conjecture: quantum correlations $\leq$ classical correlations for any bipartite state.
- Operational interpretation (and straightforward proof) of strong subadditivity of mutual information.

A simple example



A simple example

Begin with the state $\rightarrow$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle^{AB} + |11\rangle^{AB}\right).$$

Apply $\mathbb{I}^{AB}$ or $Z^A \otimes \mathbb{I}^B$ with equal probability $\rightarrow$

$$\rho = \frac{1}{2}|00\rangle\langle 00|^{AB} + \frac{1}{2}|11\rangle\langle 11|^{AB}.$$

Apply $\mathbb{I}^{AB}$ or $X^A \otimes \mathbb{I}^B$ with equal probability $\rightarrow$

$$\rho' = \pi^A \otimes \pi^B.$$

Thus 2 bits are required to erase the total correlations in the state (1 classical, 1 pure entanglement).

Total bipartite correlations

Definition (Randomizing map)

Let R be as follows:

$$R : \rho^{AB} \to \sum_{i=1}^{N} p_i \left(U_i^A \otimes V_i^B \right) \rho \left(U_i^A \otimes V_i^B \right)^\dagger$$

R $\varepsilon -$ *decorrelates* a state ρ^{AB} if there exists $\omega^A \otimes \omega^B$ such that

$$\left\| R(\rho) - \omega^A \otimes \omega^B \right\|_1 \leq \varepsilon$$

We call R a COLUR map. If all $V_i = \mathbb{I}$ then it is a A-LUR map. If all $U_i = \mathbb{I}$ then it is a B-LUR map. The composition of A-LUR and B-LUR is a LUR map.

Definition (Entropy exchange)

For a purification $|\psi\rangle\langle\psi|^{ZAB}$ of ρ^{AB} and the map R, we define

$$S_e(R^{AB}, \rho^{AB}) := S\left(\left(\mathbb{I}^Z \otimes R^{AB} \right) |\psi\rangle\langle\psi| \right)$$

Total bipartite correlations

First note that

$$\log N \geq H(p) \geq S_e(R, \rho).$$

Lemma (Size of COLUR maps)

Any $\varepsilon-$ decorrelating COLUR map R on $A^n B^n$ has the lower bound

$$S_e\left(R, \rho^{\otimes n}\right) \geq n\left(I(A;B) - O(\varepsilon)\right)$$

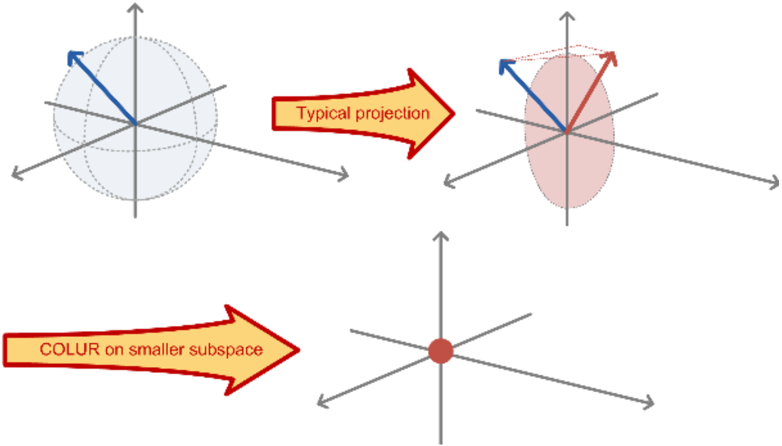
The above can be proved entirely via concavity of entropy and Fannes inequality

Lemma (Size of A-LUR maps)

There exists an $\varepsilon-$ decorrelating A-LUR map R on $A^n B^n$ with the upper bound

$$\log N \leq n\left(I(A;B) + O(\varepsilon)\right)$$

Total bipartite correlations



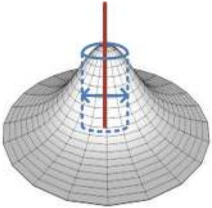
Chernoff bound

Lemma (Chernoff bound)

Le X_1, ..., X_N be i.i.d. random variables taking values in the operator interval $[0;\mathbb{I}]$ and with expectation $\mathbb{E}X \geq \mu\mathbb{I}$. Then for $0 \leq \varepsilon \leq 1$,

$$\Pr\left\{\frac{1}{N}\sum_i X_i \notin [(1-\varepsilon)\mathbb{E}X;(1-\varepsilon)\mathbb{E}X]\right\} \leq \exp\left(-N\frac{\mu\varepsilon^2}{2}\right).$$

Our random variable is U_i and $N = 2^{n(I(A;B)+4\varepsilon)}$ is sufficient to make this bound ≤ 1.



Total bipartite correlations

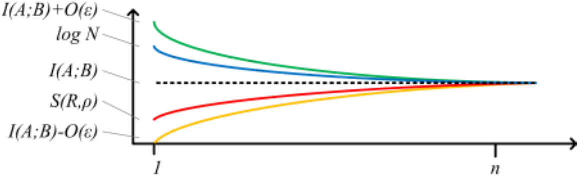
Together, the two lemmas give an extremely robust statement

Theorem

The amount of local noise needed to turn ρ^{AB} into a product state is measured by

$$\begin{aligned} &\sup_{\varepsilon} \lim_{n\to\infty} \inf \frac{1}{n} \min \left\{ S_e(R, \rho^{\otimes n}) : R,\ \varepsilon - COLUR \right\} \\ &= \quad \sup_{\varepsilon} \lim_{n\to\infty} \inf \frac{1}{n} \min \left\{ \log N : R,\ \varepsilon - A - LUR \right\} \\ &= \quad I(A;B) \end{aligned}$$

"Smallest R for worst ε in the asymptotic limit"

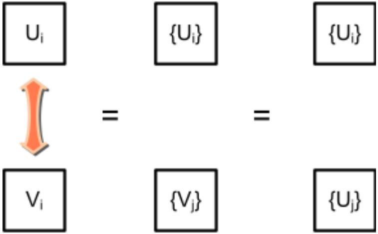


Moving freely between COLURs and LURs

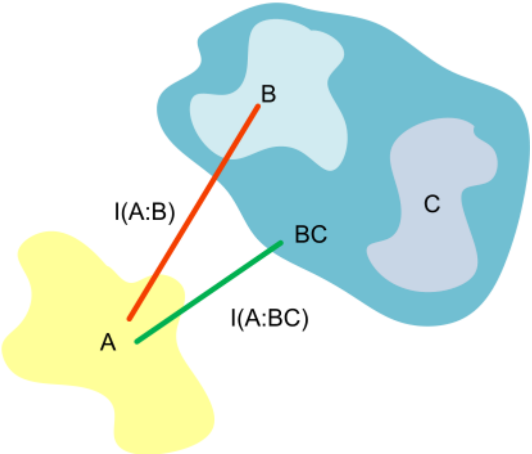
Implementing the correlated unitary randomizing map requires providing i to Alice and Bob via the state

$$\gamma = \sum_i p_i |i\rangle\langle i|^A \otimes |i\rangle\langle i|^B$$

We consider the least cost of erasing $\rho \otimes \gamma$ minus the cost of erasing γ. We know this to be $I(A;B)$ by the theorem. However, this implies **catalysis does not increase the cost**.

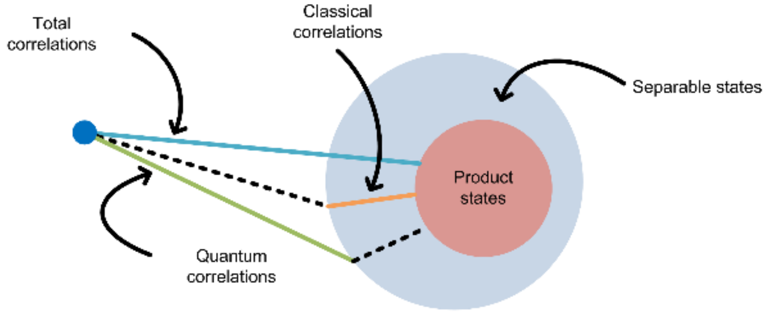


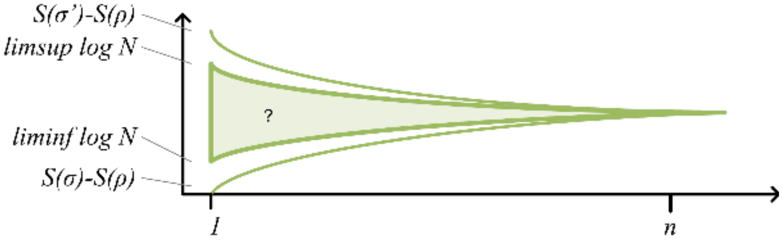
Subadditivity



Clearly the green process simulates the red process $\Longrightarrow I(A:BC) \geq I(A:B)$

Quantum correlations





We define two quantities, the first is well-motivated, the second is desirable,

$$
\begin{aligned}
Cl_{\mathrm{er}}(\rho) &:= \sup_{\varepsilon>0} \lim_{n\to\infty} \sup \sup_{\|\sigma - R(\rho^{\otimes n}\|_1 \leq \varepsilon} \frac{1}{n} I(A;B)_\sigma \\
Cl^\star_{\mathrm{er}}(\rho) &:= \sup_{\varepsilon>0} \lim_{n\to\infty} \sup \sup_{\|\sigma - T(\rho^{\otimes n}\|_1 \leq \varepsilon} \frac{1}{n} I(A;B)_\sigma
\end{aligned}
$$

where T is any local CPTP map.

Remark For pure states it is shown that $E_{\mathrm{er}}(\psi) = Cl^\star_{\mathrm{er}}(\psi) = \frac{1}{2} C_{\mathrm{er}}(\psi) = E(\psi)$. However for mixed states the optimal paths of erasure do not necessarily coincide.

Open questions

- Do the cost of entanglement erasure ($E_{\rm er}$) and optimistic cost ($\underline{E}_{\rm er}$) coincide asymptotically?

$$\sup_{\varepsilon>0} \lim_{n\to\infty} \sup \inf_{\|\sigma - R(\rho^{\otimes n}\|_1 \leq \varepsilon} \frac{1}{n} S(\sigma) - S(\rho) = \lim_{n\to\infty} \sup \inf_{\sigma = R(\rho^{\otimes n}} \frac{1}{n} S(\sigma) - S(\rho)$$

- Is $E_{\rm er}$ monotonic under local operations and classical communication? Is it convex?
- Is $E_{\rm er}$ a bound on the distillable entanglement and is $CI_{\rm er}$ a bound on the distillable secret key?