QUANTUM COADJOINT ACTION

C. DE CONCINI, V. G. KAC, AND C. PROCESI

0. INTRODUCTION AND NOTATIONS

0.1. This paper is a continuation of the paper [DCK] on representations of quantum groups at roots of 1. We give a solution to most of the conjectures stated in [DCK, §5] on the center and on the quantum coadjoint action (some of the conjectures needed modification to be correct).

As in the case of Lie groups, "simply connected" quantum groups are nicer than the more popular "adjoint" quantum groups. The primary object of the present paper is the simply connected quantum group, the adjoint quantum group being the subalgebra of invariants of the center of the corresponding simply connected Lie group.

The classical orbit method relates representations of a Lie group to the orbits of the coadjoint action of this group in the dual of the Lie algebra. The basic observation of the present paper is that representations of a quantum group at roots of 1 are closely related to the orbits of the action of the corresponding group on itself by conjugation.

0.2. Let us first introduce the necessary notations. Fix an $n \times n$ indecomposable matrix (a_{ij}) with integer entries such that $a_{ii} = 2$ and $a_{ij} \leq 0$ for $i \neq j$ and a vector $(d_1, \ldots, d_n)$ with relatively prime positive integral entries d_i such that the matrix $(d_i a_{ij})$ is symmetric and positive definite.

Let P be a free abelian group with basis ω_i, $i = 1, \ldots, n$, and let

$$\rho = \sum_{i=1}^{n} \omega_i, \qquad \alpha_j = \sum_{i=1}^{n} a_{ij} \omega_i \qquad (j = 1, \ldots, n).$$

Let $Q = \sum_i \mathbb{Z} \alpha_i$, $Q_+ = \sum_i \mathbb{Z}_+ \alpha_i$. For $\beta = \sum_i k_i \alpha_i \in Q$ let $\mathrm{ht}\, \beta = \sum_i k_i$ be the height of β.

Define a bilinear pairing $P \times Q \to \mathbb{Z}$ by $(\omega_i \,|\, \alpha_j) = \delta_{ij} d_j$. Then $(\alpha_i \,|\, \alpha_j) = d_i a_{ij}$, giving a symmetric $\mathbb{Z}$-valued bilinear form on Q such that $(\alpha \,|\, \alpha) \in 2\mathbb{Z}$. Note also that $2(\omega_i \,|\, \rho) \in \mathbb{Z}$ since $2\rho \in Q$.

Define automorphisms s_i of P by $s_i(\omega_j) = \omega_j - \delta_{ij} \alpha_i$ $(i, j = 1, \ldots, n)$. Then $s_i(\alpha_j) = \alpha_j - a_{ij} \alpha_i$. Let W be the (finite) subgroup of $\mathrm{GL}(P)$ generated

151

by $s_1, \ldots, s_n$. Then Q is W-invariant and the pairing $P \times Q \to \mathbb{Z}$ is W-invariant. Let

$$\Pi = \{\alpha_1, \ldots, \alpha_n\}, \qquad R = W\Pi, \quad R^+ = R \cap Q_+.$$

Then, of course, R is the root system corresponding to the Cartan matrix (a_{ij}), Q is the root lattice, W is the Weyl group, R^+ is a set of positive roots, Π the corresponding set of simple roots, etc.

Given a lattice M, we denote as usual by $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ the dual lattice. For example, P^* may be identified, using the bilinear form $(\cdot|\cdot)$, with the coroot lattice $Q^\vee = \sum_i \mathbb{Z}\alpha_i^\vee$, where $\alpha_i^\vee = d_i^{-1}\alpha_i$.

0.3. Let q be an indeterminate and let $q_i = q^{d_i}$. The *simply connected quantum group* is the $\mathbb{C}(q)$-algebra U_P on generators E_i, F_i, L_i, L_i^{-1} $(1 \le i \le n)$ and the following defining relations:

(1) $L_i L_j = L_j L_i$, $L_i L_i^{-1} = L_i^{-1} L_i = 1$;

(2) $L_i E_j L_i^{-1} = q_i^{\delta_{ij}} E_j$, $L_i F_j L_i^{-1} = q_i^{-\delta_{ij}} F_j$;

(3) $E_i F_j - F_j E_i = \delta_{ij}(K_i - K_i^{-1})/(q_i - q_i^{-1})$,

where for $\beta = \sum m_i \omega_i \in P$ we let $K_\beta = \prod_j L_j^{m_j}$, and let $K_i = K_{\alpha_i}$;

(4) certain Chevalley type relations between the E_i and between the F_i (see, e.g., [DCK, (1.2.4 and 5)]).

The quantum group of Drinfeld-Jimbo is the subalgebra of U_P over $\mathbb{C}(q)$ generated by the E_i, F_i, K_i, K_i^{-1} $(i = 1, \ldots, n)$. More generally, for any lattice M between P and Q one may consider the intermediate quantum group U_M generated by the E_i, F_i $(i = 1, \ldots, n)$ and the K_β with $\beta \in M$. In this paper by a *quantum group* we mean one of these algebras. We denote by U^+, U^-, and U^0 the $\mathbb{C}(q)$-subalgebra of U_M generated by the E_i, the F_i, and the K_β respectively, and by $\overline{U}^+$ and $\overline{U}^-$ the two sided ideals of U^+ and U^- generated by the E_i and the F_i respectively. We shall sometimes add the subscript M to emphasize the dependence on M, like U_M^0, etc.

0.4. As usual, for $n \in \mathbb{Z}$ and $d \in \mathbb{N}$ we let

$$[n]_d = (q^{dn} - q^{-dn})/(q^d - q^{-d}), \qquad [n]_d! = [1]_d[2]_d \cdots [n]_d.$$

Given $s \in \mathbb{N}$ we shall write $E_i^{(s)}$ and $F_i^{(s)}$ for $E_i^s/[s]_{d_i}!$ and $F_i^s/[s]_{d_i}!$ respectively.

Due to [L] we have an action of the braid group $\mathscr{B}$ with generators $T_1, \ldots, T_n$ (see §2.1 for its definition) by automorphisms of U_M defined as follows:

$$T_i E_i = -F_i K_i, \qquad T_i E_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-s} E_i^{(-a_{ij}-s)} E_j E_i^{(s)} \quad \text{if } i \neq j,$$

$$T_i F_i = -K_i^{-1} E_i, \qquad T_i F_j = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-s} F_i^{(s)} F_j F_i^{(-a_{ij}-s)} \quad \text{if } i \neq j,$$

$$T_i K_\beta = K_{s_i \beta}.$$

0.5. Now let l be an odd integer greater than 1 and relatively prime to all the d_i (this condition, imposed throughout the paper, comes from the use of formulas in [L, DCK]), and let ε be a primitive lth root of 1. We denote by $U_{M,\varepsilon}$ the algebra over $\mathbb{C}$ obtained from U_M by specializing q to ε, and by U_ε^+, U_ε^-, etc. the specializations of the subalgebras U^+, U^-, etc. More precisely, we let $\mathscr{A} = \mathbb{C}[q, q^{-1}]$ and denote by $U_{M,\mathscr{A}}$ the $\mathscr{A}$-subalgebra of U_M generated by the E_i, F_i, K_β, and $(K_i - K_i^{-1})/(q_i - q_i^{-1})$; then $U_{M,\varepsilon} = U_{M,\mathscr{A}}/(q - \varepsilon)$, etc.

Let $c_i = (\varepsilon^{d_i} - \varepsilon^{-d_i})^l$ and consider the elements

$$y_i = c_i F_i^l, \qquad z_\beta = K_\beta^l \quad (\beta \in M), \qquad z_i = z_{\alpha_i}.$$

These elements lie in the center Z_ε of $U_{M,\varepsilon}$ [DCK, §3.1]. Denote by Z_0 the smallest $\mathscr{B}$-invariant subalgebra of Z_ε containing all these elements.

Our first main result is a description of the $\mathscr{B}$-algebra Z_0 by generators and relations (§3.5). In the simply laced case, i.e., when (a_{ij}) is symmetric, the result is:

Let Z_0^0 be the subalgebra of Z_0 spanned by the z_β $(\beta \in M)$ with the $\mathscr{B}$-action given by $T_i z_\beta = z_{s_i\beta}$. Then Z_0 is a commutative $\mathscr{B}$-algebra over the $\mathscr{B}$-algebra Z_0^0 on generators $T y_i$ $(T \in \mathscr{B}, i = 1, \dots, n)$ and defining relations

  (1) $T_i^2 y_i = z_i^2 y_i$,
  (2) $T_i y_j = y_j$ if $a_{ij} = 0$,
  (3a) $T_i T_j y_i = y_j$ if $a_{ij} = -1$,
  (3b) $T_i y_j + T_j y_i = y_i y_j$ if $a_{ij} = -1$.

It is interesting to note a close connection of this with the well-known action of $\mathscr{B}$ on the simple Lie algebra $\mathfrak{g}$ associated to the matrix (a_{ij}) (Proposition 4.2):

Let D be the smallest $\mathscr{B}$-invariant subalgebra in $S(\mathfrak{g})$ (the symmetric algebra over $\mathfrak{g}$) containing the Chevalley generators $f_1, \dots, f_n$ (root vectors attached to negative simple roots). Then D is a commutative $\mathscr{B}$-algebra over $\mathbb{C}$ on generators $T f_i$ $(T \in \mathscr{B}, i = 1, \dots, n)$ and the "homogenized" defining relations of Z_0

$$T_i^2 f_i = f_i, \qquad T_i f_j = f_j \quad \text{if } a_{ij} = 0,$$
$$T_i T_j f_i = f_j \quad \text{and} \quad T_i f_j + T_j f_i = 0 \quad \text{if } a_{ij} = -1.$$

0.6. Let G be the connected complex Lie group whose Lie algebra is $\mathfrak{g}$ and such that any maximal complex torus of G has character group M (so that Center $G = M/Q$). Let T be a maximal torus of G, and let U_+ (resp. U_-) be the maximal unipotent subgroup of G corresponding to R^+ (resp. to $-R^+$). Note that $T = \operatorname{Spec} Z_0^0$ with the usual W-action.

Our main construction is the map π of $\Omega_M := \operatorname{Spec} Z_0$ to the "big cell" $G^0 := U_- T U_+$ constructed as follows. Fix a reduced expression J of the longest element of W: $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$. Let $T_{k-1}^J = T_{i_1} \cdots T_{i_{k-1}}$ $(k = 1, \dots, N)$,

$T_0 = T_N^J$, and consider the elements

$$y_k^J = T_{k-1}^J y_{i_k} \in Z_0, \qquad f_k^J = T_{k-1}^J f_{i_k} \in \mathfrak{g}.$$

Let Z_0^- (resp. Z_0^+) be the subalgebra generated by the y_k^J (resp. $T_0 y_k^J$), $k = 1, \ldots, N$. These are polynomial algebras (independent of the choice of J; see §3.3) and we have

$$Z_0 = Z_0^- \otimes Z_0^0 \otimes Z_0^+,$$

so that $\Omega_M \simeq \mathbb{C}^N \times T \times \mathbb{C}^N$ (noncanonically). Define the maps

$$Y : \operatorname{Spec} Z_0^- \to U_-, \qquad X : \operatorname{Spec} Z_0^+ \to U_+, \qquad Z : T \to T$$

as

$$Y = (\exp y_N^J f_N^J)(\exp y_{N-1}^J f_{N-1}^J) \cdots (\exp y_1^J f_1^J), \qquad X = T_0(Y),$$
$$Z(t) = t^2, \qquad t \in T.$$

Here $\mathscr{B}$ acts in the obvious way: $T_i(\cdots \exp y f \cdots) = \cdots \exp((T_i y)(T_i f)) \cdots$, $y \in Z_0, f \in \mathfrak{g}$, where its action on $\mathfrak{g}$ is the usual one (see §4.1). We show that the maps Y and X are independent of the choice of J, and we let

$$\pi = YZX: \Omega_M \to G^0.$$

This is an unramified cover of degree 2^n.

In §4 we study the interplay between the map π and the action of the braid group $\mathscr{B}$ on Ω_M and on G.

0.7. The primary object of our study is the quantum coadjoint action defined as follows [DCK]. We have derivations $\underline{e}_i$ and $\underline{f}_i$ of $U_{M,\varepsilon}$ ($i = 1, \ldots, n$) defined by

$$\underline{e}_i(u) = \lim_{q \mapsto \varepsilon} [E_i^{(l)}, u], \qquad \underline{f}_i = T_i \underline{e}_i T_i^{-1}.$$

We denote by $\widetilde{G}$ the (infinite-dimensional) group of analytic automorphisms of the variety Ω_M generated by the 1-parameter groups $\exp t \underline{e}_i$ and $\exp t \underline{f}_i$ ($t \in \mathbb{C}$, $i = 1, \ldots, n$).

The key calculations of the paper are the following formulas relating the Chevalley generators e_i and f_i to the derivations $\underline{e}_i$ and $\underline{f}_i$ (Theorems 5.4 and 5.5)

$$(!) \qquad\qquad\qquad \underline{e}_i = z_i f_i, \qquad \underline{f}_i = -z_i e_i.$$

Here by e_i and f_i we understand the pullback via the covering π of the Killing vector fields on G defined by the $e_i, f_i \in \mathfrak{g}$.

The proof of the formulas (!) is rather straightforward, but requires heavy computations. For simplicity of the exposition, we present all details of the calculation in the simply laced case, leaving out the details in other cases.

We use these formulas to describe the orbits of the group $\widetilde{G}$ on Ω_M (§6). In order to state the result we need one more construction.

Let G_c be the simply connected cover of G, so that $G = G_c/C$, where $C \simeq P/M$. We let $G' = G_c/C^2$. This is an unramified C/C^2-cover $\phi: G' \to G$

of G. The map π factors through G', i.e., there exists a unique unramified cover $\pi': \Omega_M \to G'^0$ such that $\pi = \phi \circ \pi'$.

Let $\mathscr{O}$ be a conjugacy class in G' of a noncentral element and let $\mathscr{O}^0 = \mathscr{O} \cap G'^0$. We show that $(\pi')^{-1}\mathscr{O}^0$ is a $\widetilde{G}$-orbit in Ω_M and these are all orbits of nonfixed points of $\widetilde{G}$; all the fixed points are points of the fibers of π' over central elements of G' (Theorem 6.6).

The study of the orbits of $\widetilde{G}$ on $\operatorname{Spec} Z_\varepsilon$ and $\operatorname{Spec} Z_0$ is important for the following reason [DCK]. Let $\operatorname{Spec} U_{M,\varepsilon}$ denote the set of all equivalence classes of finite-dimensional irreducible representations of $U_{M,\varepsilon}$. The group $\widetilde{G}$ acts on $\operatorname{Spec} U_{M,\varepsilon}$ in a natural way. Associating to an irreducible representation its central character gives a surjective $\widetilde{G}$-equivariant map $\chi: \operatorname{Spec} U_{M,\varepsilon} \to \operatorname{Spec} Z_\varepsilon$ that is generically bijective. The inclusion $Z_0 \subset Z_\varepsilon$ induces a finite (hence surjective) map $\tau: \operatorname{Spec} Z_\varepsilon \to \operatorname{Spec} Z_0$. Thus, we obtain a sequence of canonical surjective maps

$$\operatorname{Spec} U_{M,\varepsilon} \xrightarrow{\chi} \operatorname{Spec} Z_\varepsilon \xrightarrow{\tau} \operatorname{Spec} Z_0.$$

Representations from the same $\widetilde{G}$-orbit in $\operatorname{Spec} U_{M,\varepsilon}$ are practically the same. Thus in order to describe $\operatorname{Spec} U_{M,\varepsilon}$, it suffices for each orbit to describe the fiber of χ. The structure of representations from the fiber should be intimately related to geometric properties of the orbit (see, e.g., Conjecture 6.8).

The first important application of the description of the orbits of $\widetilde{G}$ in Ω_M is the "triangulizability" of any $\pi \in \operatorname{Spec} U_{M,\varepsilon}$: there exists an automorphism σ and a nonzero vector v in the representation space of π such that $\pi(\sigma(E_i))v = 0$ for all i.

0.8. Denote by $Q_2^\vee$ the subgroup of T consisting of elements $\exp \pi i \alpha$, $\alpha \in Q^\vee \subset M^*$, and let $\widetilde{W} = W \ltimes Q_2^\vee$. It follows from §0.7 that two nonfixed points of $T \subset \Omega_M$ lie in the same orbit of $\widetilde{G}$ if and only if they lie in the same orbit of $\widetilde{W}$. This leads to a Chevalley type restriction theorem (Theorem 6.7):

The restriction homomorphism induced by inclusion $T \subset \Omega_M$ gives an isomorphism of algebras of invariants: $Z_0^{\widetilde{G}} \xrightarrow{\sim} Z_0^{0\widetilde{W}}$. (One can actually show that this extends to an isomorphism $Z_\varepsilon^{\widetilde{G}} \xrightarrow{\sim} U_\varepsilon^{0\widetilde{W}}$.)

0.9. Let Z be the center of the quantum group U_M. In [DCK] an explicit isomorphism $\overline{h}: U^{0\widetilde{W}} \xrightarrow{\sim} Z$ was constructed (only the case $M = Q$ was considered there, but the construction and the proof extend to arbitrary M). We show that $\overline{h}$ can be specialized to $q = \varepsilon$ to give an injective homomorphism $\overline{h}_\varepsilon: U_\varepsilon^{0\widetilde{W}} \to Z_\varepsilon$ (Proposition 6.2). We denote the image of this homomorphism by Z_1. One can show in fact that $Z_1 = Z_\varepsilon^{\widetilde{G}}$ (Theorem 6.7(c)).

Our next important result (Theorem 6.4(a)) states that the subalgebras $Z_{P,0}$ and $Z_{P,1}$ generate the center $Z_{P,\varepsilon}$ (for arbitrary M this may be false). The proof of this theorem is roughly as follows. Consider the subalgebra $\widetilde{Z}_{M,\varepsilon}$ generated by Z_0 and Z_1. It is easy to see for arbitrary M that the quotient

fields of $\widetilde{Z}_{M,\varepsilon}$ and $Z_{M,\varepsilon}$ coincide. Also it was shown in [DCK] that $Z_{M,\varepsilon}$ is integrally closed (this again holds for arbitrary M). Thus, we have to show that $\widetilde{Z}_{P,\varepsilon}$ is integrally closed. We show that $\widetilde{Z}_{P,\varepsilon}$ is a complete intersection ring that is smooth in codimension one and hence (by Serre's theorem) is integrally closed. We therefore deduce that $\operatorname{Spec} Z_{P,\varepsilon}$ is a complete intersection (Theorem 6.4(b)).

At the same time, we obtain a simple geometric construction of the center $Z_{P,\varepsilon}$. Let G be simply connected. Denote by ρ_l the map $g \mapsto g^l$ of G into itself. Let $\mathbb{C}[G]^G$ be the algebra of invariant polynomials on G under conjugation, let $G//G$ denote the corresponding affine variety, and let $\sigma: G \to G//G$ be the quotient map. Then $\operatorname{Spec} Z_{P,\varepsilon}$ is an unramified cover with the Galois group $\frac{1}{2}Q^\vee/Q^\vee$ of the fiber product

$$(G//G) \times_{G//G} G^0,$$

where the first map is induced by ρ_l and the second map is the restriction of σ to the big cell. We show that this fiber product may be obtained using the Stein factorization of the lth power map in G (§6.5). Note that this is the only instance where l enters in the geometric picture. We derive from this result a description of the action of $\widetilde{G}$ on $\operatorname{Spec} Z_{P,\varepsilon}$ and of the fibers of the map τ (Corollary 6.4).

0.10. In §7 we interpret our results in the language of Poisson algebraic groups. Explicitly, we show that Ω_M has a canonical structure of a Poisson algebraic group, which turns out to be isomorphic to the dual of the celebrated Sklyanin-Drinfeld Poisson Lie group. The $\widetilde{G}$-orbits are precisely the symplectic leaves of this Poisson structure. The map π and the action of $\mathscr{B}$ have a simple description in this framework.

It is worth mentioning here that the Hopf and Poisson structures of the $\mathscr{B}$-algebra Z_0 (defined by generators and relations in §0.5) can be easily calculated using that Δ is a Poisson map and that the Poisson structure is $\mathscr{B}$-invariant. For example in the case when (a_{ij}) is symmetric, it suffices to use the following formulas (where $x_i = T_i y_i$):

$$\Delta y_i = y_i \otimes z_i^{-1} + 1 \otimes y_i, \qquad \Delta x_i = x_i \otimes z_i^{-1} + 1 \otimes x_i,$$
$$\Delta z_\alpha = z_\alpha \otimes z_\alpha;$$
$$\{z_\alpha, z_\beta\} = 0, \qquad \{y_i, z_\alpha\} = \tfrac{1}{2}(\alpha \,|\, \alpha_i) y_i z_\alpha,$$
$$\{y_i, y_j\} = 0 \quad \text{if } a_{ij} = 0,$$
$$\{y_i, y_j\} = \tfrac{1}{2} y_i y_j - T_i y_j \quad \text{if } a_{ij} = -1,$$
$$\{z_i x_i, y_j\} = \delta_{ij}(z_i - z_i^{-1}).$$

0.11. We would like to thank the referee whose severe criticism considerably improved the exposition.

1. THE LONGEST ELEMENT OF THE WEYL GROUP W

1.1. Recall that W is a *Coxeter group* on generators s_i ($i = 1, \ldots, n$) and defining relations

$$s_i^2 = 1 \quad \text{and} \quad (s_i s_j)^{m_{ij}} = 1 \quad \text{when } i \neq j,$$

where $m_{ij} = 2, 3, 4$, or 6 for $a_{ij}a_{ji} = 0, 1, 2$, or 3 respectively $(i \neq j)$.

Recall that any element of W has a length $l(w)$ that can be defined as the length of a shortest expression of w as a product of s_i and equals the cardinality of $R_w := \{\beta \in R^+ \mid w(\beta) < 0\}$; such an expression is called a *reduced expression* of w. Recall that

$$l(ws_i) = l(w) + 1 \text{ if } w(\alpha_i) > 0 \quad \text{and} \quad l(ws_i) = l(w) - 1 \text{ if } w(\alpha_i) < 0.$$

1.2. Since W acts transitively on the bases there exists a unique element w_0 of longest length N such that $w_0(R^+) = -R^+$. Of course $w_0 = w_0^{-1}$ and $\Pi = -w_0(\Pi)$. Let us denote by $j \mapsto \bar{j}$ the permutation of $1, 2, \ldots, n$ such that $\alpha_{\bar{j}} = -w_0(\alpha_j)$.

We have that $s_j w_0 = w_0 s_{\bar{j}}$. More precisely, writing $w_0 = s_j s_{i_1} s_{i_2} \cdots s_{i_{N-1}} = s_{i_1} s_{i_2} \cdots s_{i_{N-1}} s_{\bar{j}}$ we deduce

Lemma. $s_{i_1} s_{i_2} \cdots s_{i_{N-1}}(\alpha_{\bar{j}}) = \alpha_j$.

Proof. $s_{i_1} s_{i_2} \cdots s_{i_{N-1}}(\alpha_{\bar{j}}) = s_j w_0(\alpha_{\bar{j}}) = s_j(-\alpha_j)$. $\square$

If $w = ab \in W$ is such that $l(w) = l(a) + l(b)$ we will say that this is a *reduced* decomposition.

Given an element $w \in W$ we set $\overline{w} := w_0 w w_0^{-1}$ (so that $\bar{s}_j = s_{\bar{j}}$).

1.3. If w is not the longest element there must exist a simple root α_i such that $w(\alpha_i) > 0$; thus any reduced expression $s_{i_1} s_{i_2} \cdots s_{i_m}$ can be completed to a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_N}$ of w_0.

In particular given an element $a \in W$ there exist $b, a' \in W$ such that $w_0 = ab = ba'$ are reduced decompositions. In this case we clearly have

Lemma. $a' = \bar{a}$.

2. THE BRAID GROUP $\mathscr{B}$

2.1. Recall that the braid group is an infinite group $\mathscr{B}$ on generators T_i, $i = 1, \ldots, n$, and the *braid relations*: for $i \neq j$ we take the word of (even) length $(T_i T_j)^{m_{ij}}$, split it in half and impose that the first half be equal to the second written in reverse order. Of course the Weyl group W is the quotient of $\mathscr{B}$ under the further relations $T_i^2 = 1$.

It will be convenient to use the abbreviated notation,

$$T_{ij}^{(m)} = T_i T_j T_i \cdots \qquad (m \text{ factors}).$$

For example, the braid relations read: $T_{ij}^{(m_{ij})} = T_{ji}^{(m_{ji})}$ if $i \neq j$.

2.2. The main tool of the computations to follow is a result of Matsumoto and Steinberg.

Theorem. *If $s_{i_1} s_{i_2} \cdots s_{i_m}$ is a reduced expression of an element $w \in W$, then the element $T_w := T_{i_1} T_{i_2} \cdots T_{i_m}$ in $\mathscr{B}$ depends only on w (and not on its reduced expression). Moreover, the different reduced expressions of w can be transformed into one another by the braid relations.*

This means that we define a canonical section $w \to T_w$ of W in $\mathscr{B}$; of course, this section is not multiplicative but $T_a T_b = T_{ab}$ if $l(ab) = l(a) + l(b)$. Going back to the longest element we set $T_0 := T_{w_0}$ and will often use the following

Lemma. *If $a \in W$ we have $T_a T_0 = T_0 T_{\overline{a}}$.*

Proof. From §1.3 there exists b such that $w_0 = ab = b\overline{a}$ are reduced decompositions. We thus have $T_0 = T_a T_b = T_b T_{\overline{a}}$. So $T_a T_0 = T_a T_b T_{\overline{a}} = T_0 T_{\overline{a}}$. $\square$

Recall another well-known useful fact.

Proposition. *If $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ is a reduced expression of w_0, then one gets the following ordering of the set of positive roots*:

$$R^+ = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), s_{i_1} s_{i_2}(\alpha_{i_3}), \ldots, s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})\}.$$

2.3. One of the difficulties in working with quantum groups occurs due to the existence of nontrivial $w \in W$ such that $w(\alpha_i) = \alpha_h$. To deal with this difficulty, we need a fact on root systems. Consider a simple root α_h and the set A_h formed by all the pairs (w, i) such that $w \in W$ and $w(\alpha_i) = \alpha_h$. We make this set into a graph by joining (w, i) with (ws_j, i) if $m_{ij} = 2$, with $(ws_j s_i, j)$ if $m_{ij} = 3$, with $(ws_j s_i s_j, i)$ if $m_{ij} = 4$, and with $(ws_j s_i s_j s_i s_j, i)$ if $m_{ij} = 6$. We have

Lemma. *The graph A_h is connected.*

Proof. We connect any given element (w, i) with $(1, h)$ as follows. Fix a reduced expression of w. Then, by §1.1, ws_i is also a reduced expression and by §1.3 we can complete it to a reduced expression $w_0 = ws_i w_1$. By Proposition 2.2 we have: $R^+ = \{\ldots, w(\alpha_i) = \alpha_h, \ldots\}$. On the other hand, taking a reduced expression of w_0 that starts with s_h, we get an ordering of R^+ that starts with α_h. Now we pass from the first reduced expression to the second one using braid relations. Then at each step either the braid relation lies entirely in w or w_1 and then the presentation of α_h does not change, or α_h becomes an element of the form $\alpha_h = w'(\alpha_j)$ where (w', j) is joined with (w, i) in A_h. At the last step the presentation of α_h becomes $\alpha_h = 1(\alpha_h)$. $\square$

Theorem. *Let $\mathscr{B}$ act on a set $\mathscr{S}$, and let $u_1, \ldots, u_n \in \mathscr{S}$ be elements with the properties*:

$$T_{ij}^{(2)} u_i = u_j \ \ if \ m_{ij} = 3 \quad and \quad T_{ij}^{(m_{ij}-1)} u_j = u_j \ \ if \ m_{ij} = 2, 4, \ or \ 6.$$

Then

    (a) *For a pair (w, i) in A_h we have $T_w(u_i) = u_h$.*

    (b) *$T_i(u_i) = T_0(u_{\overline{i}})$ and $T_0^2(u_i) = T_i^2(u_i)$.*

Proof. (a) is just a restatement of the lemma using the relations of the theorem. As for (b) we have $T_0 = T_i T_w = T_w T_{\bar{i}}$ where $w = s_i w_0$ and so $w(\alpha_{\bar{i}}) = \alpha_i$. Thus $T_i(u_i) = T_i T_w(u_{\bar{i}}) = T_0(u_{\bar{i}})$, so $T_i^2(u_i) = T_i T_0(u_{\bar{i}}) = T_0 T_{\bar{i}}(u_{\bar{i}}) = T_0^2(u_i)$. □

In particular we can apply the theorem to the action of the braid group on a quantum group and the elements $u_i = E_i$ or $u_i = F_i$. In this case the relations of the theorem are easily verified (see [L, 5.1]). Thus, we have: if α_i, α_j are two simple roots and $w \in W$ is such that $w(\alpha_i) = \alpha_j$ then $T_w(E_i) = E_j$. We also have

$$T_0(E_{\bar{i}}) = T_i(E_i) = -F_i K_i, \qquad T_0(F_{\bar{i}}) = T_i(F_i) = -K_i^{-1} E_i.$$

3. A UNIVERSAL CONSTRUCTION ASSOCIATED TO THE BRAID GROUP

3.1. Let $U_{M,\varepsilon}$ be a quantum group at a primitive lth root of unity ε, l odd. In order to explain our next general construction let us introduce some notations and prove some identities in $U_{M,\varepsilon}$. Let $c_i := (\varepsilon^{d_i} - \varepsilon^{-d_i})^l$ and consider the following central elements of $U_{M,\varepsilon}$:

$$z_\beta := K_\beta^l \qquad (\beta \in M),$$

$$z_i = z_{\alpha_i}, \qquad y_i := c_i F_i^l, \qquad x_i := T_0(y_{\bar{i}}) \qquad (i = 1, \ldots, n)$$

(one should remark on the difference in notation used in [DCK]).

By Theorem 2.3 we have $T_i(y_i) = T_0(y_{\bar{i}})$, which implies

(0) $\quad x_i = -c_i E_i^l z_i^{-1}$.

The following formulas are now immediate by [DCK, (3.3.3 and 4)]:

(1) $\quad T_i(x_i) = z_i^2 y_i$, $\ T_i(y_i) = x_i$, $\ T_i(z_\beta) = z_{s_i \beta}$.

In order to write down further relations, we introduce one more notation:

$$y_{ij}^{(m)} = T_{ij}^{(m)} y_j \ \text{ if } m \text{ is odd} \quad \text{and} \quad y_{ij}^{(m)} = T_{ij}^{(m)} y_i \ \text{ if } m \text{ is even}.$$

We have the following formulas $(i \neq j)$:

(2) $\quad y_{ij}^{(m_{ij}-1)} = y_j$.

(3) $\quad y_{ij}^{(m_{ij}-2)} + y_{ji}^{(1)} = y_i y_j$ if $m_{ij} \geq 3$ and $a_{ij} = -1$.

(4) $\quad y_{ij}^{(m_{ij}/2-1)} - y_{ji}^{(m_{ij}/2)} = y_i^2 y_j - 2 y_{ji}^{(1)} y_i$ if $m_{ij} \geq 4$ and $a_{ji} = -1$.

(5) $\quad y_{ij}^{(1)} + y_{ji}^{(4)} = y_i^3 y_j - 3 y_i^2 y_{ji}^{(1)} + 3 y_i y_{ji}^{(3)}$, $\ y_{ij}^{(3)} - y_{ji}^{(2)} = y_i^3 y_j^2 - 3 y_i^2 y_j y_{ji}^{(1)} + 3 y_i y_{ji}^{(1)2} - 3 y_{ji}^{(1)} y_{ji}^{(3)} + y_j y_{ji}^{(3)}$ if $m_{ij} = 6$ and $a_{ji} = -1$.

Formula (2) follows from 2.3. The remaining formulas are deduced from [L, 5.3–5.5; DCK, (3.4.14)], §5.1, and the identity

$$\sum_{j=1}^{l-1} (-1)^j \frac{\varepsilon^{-j}}{[j]! [l-j]!} = \frac{c}{2}(1 - l^{-1}), \quad \text{where } c = (\varepsilon - \varepsilon^{-1})^l$$

(which follows easily from the Gauss binomial formula: see e.g., [DCK, (1.1.1)]).

Recall that we also have an involution ω, defined by $\omega E_i = F_i$, $\omega K_\beta = K_{-\beta}$. It commutes with the braid group and we have $\omega(z_\beta) = z_{-\beta}$ and $\omega(x_i) = -z_i y_i$.

In view of the previous identities we start the following construction.

3.2. Denote by Z_0^0 the algebra with basis z_β ($\beta \in M$) and multiplication $z_\alpha z_\beta = z_{\alpha+\beta}$. The Weyl group, and hence the braid group, acts on Z_0^0 by $s_i(z_\beta) = z_{s_i \beta}$. Consider the polynomial algebra over Z_0^0 in the indeterminates $T y_i$, as $i = 1, 2, \ldots, n$ and $T \in \mathscr{B}$; we write $y_i := 1 y_i$. We extend the action of $\mathscr{B}$ from Z_0^0 to this polynomial algebra by

$$T_1(T_2 y_i) := (T_1 T_2) y_i.$$

We define a quotient algebra of this polynomial algebra by the smallest ideal of relations, stable under $\mathscr{B}$ and containing the following relations:

(1) $T_i^2 y_i = z_i^2 y_i$,
(2) $y_{ij}^{(m_{ij}-1)} = y_j$ if $i \neq j$,
(3) relations (3)–(5) from §3.1.

Denote by Z_0 the resulting algebra. Remark that relations (1) and (3) are not homogeneous, and we can also consider the associated homogeneous relations:

(1′) $T_i^2(y_i) = y_i$,
(3′) the left-hand sides of (3)–(5) are zero.

In this case we can ignore the variables z_i and consider the algebra D over $\mathbb{C}$ generated by the $T y_i$ ($T \in \mathscr{B}$, $i = 1, \ldots, n$) and the relations (1′), (2), and (3′).

3.3. Next given an element $w \in W$ we need to consider the set of all possible reduced expressions J of w. If $J: w = s_{i_1} s_{i_2} \cdots s_{i_t}$ is such a reduced expression we define $J^{-1}: w^{-1} = s_{i_t} s_{i_{t-1}} \cdots s_{i_1}$ a reduced expression of w^{-1}. For $k = 1, \ldots, t$ we also set

$$w_k^J := s_{i_1} s_{i_2} \cdots s_{i_k}, \qquad \beta_k^J := s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) = w_{k-1}^J(\alpha_{i_k}).$$

By Proposition 2.2, the elements β_k^J are distinct positive roots. In fact we have $w^{-1}(\beta_k^J) = s_{i_t} s_{i_{t-1}} \cdots s_{i_1} s_{i_1} s_{i_2} \cdots s_{i_{k-1}}(\alpha_{i_k}) = -\beta_{t-k+1}^{J^{-1}}$. Thus we get a total ordering of the set $R_{w^{-1}}$.

We can apply this analysis to the element w_0 and the set of all positive roots. In this case for a given reduced expression J we can construct another reduced expression $\overline{J}: w_0 = s_{\overline{i}_1} s_{\overline{i}_2} \cdots s_{\overline{i}_N}$. It is clear that $\beta_k^{\overline{J}} = -w_0(\beta_k^J)$.

We now define in $\mathscr{B}$ and in Z_0 the elements

$$T_{k-1}^J := T_{i_1} T_{i_2} \cdots T_{i_{k-1}}, \qquad y_k^J := T_{i_1} T_{i_2} \cdots T_{i_{k-1}}(y_{i_k}) = T_{k-1}^J(y_{i_k}).$$

Let us choose a reduced expression $J: w = s_{i_1} s_{i_2} \cdots s_{i_t}$ of an element w. Denote by Z^J the Z_0^0-subalgebra of Z_0 generated by the elements y_k^J, and let $\overline{Z}^J$ be its augmentation ideal.

Proposition. *One has*: $y_k^J = \pm y_k^{J'} \mod(\overline{Z}^J)^2$. *In particular*, Z^J *and* $\overline{Z}^J$ *are independent of the choice of the reduced expression* J *of* w.

Proof. Since one can pass from one expression to another by the use of braid relations we may by induction restrict to the case where J' is obtained from J by a single braid relation. We give a proof in the cases $m_{ij} = 2$ and $m_{ij} = 3$. (In the remaining cases the proof is similar.)

(1) $w = as_i s_j b = as_j s_i b$; $s_i s_j = s_j s_i$ $(m_{ij} = 2)$,
(2) $w = as_i s_j s_i b = as_j s_i s_j b$; $s_i s_j s_i = s_j s_i s_j$ $(m_{ij} = 3)$.

Let $k - 1 = l(a)$. In case (1) we remark that $y_h^J = y_h^{J'}$ if $h \neq k, k+1$, $y_k^J = T_a(y_i) = T_a T_j(y_i) = y_{k+1}^{J'}$, and similarly $y_{k+1}^J = y_k^{J'}$. In case (2) we get $y_h^J = y_h^{J'}$ if $h \neq k, k+1, k+2$, $y_k^J = T_a(y_i) = T_a T_j T_i(y_j) = y_{k+2}^{J'}$, and similarly $y_k^{J'} = y_{k+2}^J$; $y_{k+1}^J = T_a T_i(y_j) = T_a(-T_j(y_i) - y_i y_j) = -y_{k+1}^{J'} - y_k^J y_k^{J'} = -y_{k+1}^{J'} - y_{k+2}^{J'} y_k^{J'}$. $\square$

3.4. The previous proof and Theorem 2.3 have an important

Corollary. (a) *If for a given* J *and* k, $\beta_k^J = \alpha_i$ *is a simple root then* $y_k^J = y_i$.

(b) $T_i(x_i) = z_i^2 y_i$.

(c) *If* $\beta_k^J = \beta_h^{J'}$ *then* $y_k^J = \pm y_h^{J'} + P$ *where* P *is a polynomial in the elements* $y_i^{J'}$ *involving only the indices* i *for which* $\beta_i^{J'}$ *has height strictly less than that of* $\beta_h^{J'}$.

Proof. (a) This is a special case of Theorem 2.3(a).
(b) This follows directly from Theorem 2.3 and the defining relations 3.2.
(c) This follows from the proof of Proposition 3.3. $\square$

Remark. The construction and statements of §§3.2–3.4, as well as the first statement of Theorem 3.5 below, hold over $\mathbb{Z}$.

3.5. We will now denote $Z_w^- = Z^J$ and $Z_w^+ := T_0(Z_w^-)$, and let $x_k^J := T_0(y_k^J)$. If $w = w_0$ is the longest element we will simply write Z_0^-, Z_0^+.

Theorem. *The algebra* Z_0 *is the tensor product* $Z_0^- \otimes Z_0^0 \otimes Z_0^+$ *and, given a reduced expression* J *of* w_0, *it is the polynomial ring*

$$Z_0^0[y_k^J, x_k^J; \quad k = 1, \ldots, N].$$

Thus this algebra coincides with the subalgebra Z_0 *of the center of the quantum group* $U_{M,\varepsilon}$ *studied in* [DCK].

Proof. First of all we will show that the subring $S := Z_0^0[y_k^J, x_k^J; k = 1, \ldots, N]$ is stable under the action of $\mathcal{B}$, which will prove that it is the entire algebra;

then we will see that the $2N$ given elements are algebraically independent by using the quantum groups. Let us begin with the first claim. We need to show that, for a given j, $T_j(S) \subset S$.

From the independence of S from J we can choose a reduced decomposition J of w_0 so that s_j appears on the extreme left. Then we have $T_j(y_k^J) = y_{k+1}^J$ unless $k = N$; in the latter case $T_j(y_N^J) = T_0(y_{i_N}) = x_{\bar{i}_N}$. Now $T_j(x_k^J) = T_j T_0(y_k^J) = T_0 T_{\bar{j}}(y_k^J)$. Again $T_{\bar{j}}(y_k^J)$ is in Z_0^- and so $T_j(x_k^J)$ is in Z_0^+ unless $y_k^J = y_{\bar{j}}$, in which case we use 3.2(1) and get $T_j(x_{\bar{j}}) = z_j^2 y_j$ as desired.

Next we have to verify the algebraic independence. This follows from the fact that the same statement is true for the subalgebra Z_0 of the center of the quantum group, and this is a consequence of the existence and the form of a PBW basis (cf. [DCK]). □

4. The big cell

4.1. In this section we want to make the link between our formal constructions and the actual Lie algebras and Lie groups. Fix once and for all a sublattice M of P containing Q.

Consider the simple Lie algebra $\mathfrak{g}$ corresponding to the Cartan matrix (a_{ij}), let $\mathfrak{h}$ be a Cartan subalgebra, and let $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in R} \mathfrak{g}_\alpha)$ be the root space decomposition. We denote by e_i, f_i, h_i the usual Chevalley generators. Let G be the corresponding connected algebraic group, and let T be the maximal complex torus of G corresponding to $\mathfrak{h}$ whose group of characters is the lattice M. Let U_+ and U_- denote the unipotent subgroups of G corresponding to positive and negative roots, and let $G^0 = U_- T U_+$ be the *big cell* of G (this is a Zariski open dense subset of G). We set

$$t_i := \exp(f_i) \exp(-e_i) \exp(f_i) \in G.$$

One knows the following facts that go back to Tits (see, e.g., [KP]):

(1) The mapping $T_i \mapsto t_i$ extends to a homomorphism of $\mathscr{B}$ to G, hence an action of $\mathscr{B}$ on $\mathfrak{g}$.

(2) If J is a reduced expression, setting $f_k^J = T_{k-1}^J(f_{i_k})$, we have that f_k^J is a root vector relative to the negative root $-\beta_k^J$.

4.2. Let us now consider the action of $\mathscr{B}$ on $\mathfrak{g}$. We verify directly the following relations:

(1') $T_i^2(f_i) = f_i$.

(2') $T_{ij}^{(m_{ij}-1)}(f_j) = f_j$ if $m_{ij} = 2, 4$, or 6.

(3') $T_j T_i(f_j) = f_i$, $T_i(f_j) + T_j(f_i) = 0$ if $m_{ij} = 3$.

(4') $T_i(f_j) = T_j T_i(f_j)$ if $m_{ij} = 4$ and $a_{ji} = -1$.

(5') $T_{ij}^{(m_{ij}-2)}(f_i) + T_j(f_i) = 0$ if $m_{ij} \geq 4$ and $/ a_{ji} = -1$.

(6') $T_{ij}^{(2)}(f_i) = T_{ji}^{(3)}(f_i)$, $T_i(f_j) + T_{ji}^{(4)}(f_j) = 0$, $T_{ij}^{(3)}(f_j) = T_{ji}^{(2)}(f_j)$ if $m_{ij} = 6$ and $a_{ji} = 1$.

We can take these relations to define a commutative algebra D with a $\mathscr{B}$-action.

Proposition. *The subalgebra generated in* $S(\mathfrak{g})$ *by the* $\mathscr{B}$-*translates of the* f_i *is* $\mathscr{B}$-*isomorphic to the algebra* D.

Proof. The $\mathscr{B}$-equivariant mapping sending y_i to f_i is well defined since the relations defining D hold in our ring. D is generated by the elements y_k^J and x_k^J that map to the corresponding vectors f_k^J and e_k^J. These vectors are linearly independent (in $\mathfrak{g}$) and this finishes the proof. $\square$

4.3. We return now to the algebra $Z_0 = Z_0^- \otimes Z_0^0 \otimes Z_0^+$, and let $\Omega_M = \mathrm{Spec}\, Z_0$ be the algebraic variety of its $\mathbb{C}$-valued points. Thus Ω_M is a product of the N-dimensional affine space $\mathrm{Spec}\, Z_0^-$, the complex torus $T = \mathrm{Spec}\, Z_0^0$ (we identify Z_0^0 with $U_\varepsilon^0 = \mathbb{C}[T]$ via the map $z_\alpha \mapsto K_\alpha$), and the N-dimensional affine space $\mathrm{Spec}\, Z_0^+$. The group $\mathscr{B}$ acts on Ω_M algebraically. We act with $\mathscr{B}$ on G by inner conjugation by the elements $t_i \colon T_i(g) := t_i g t_i^{-1}$. We consider the set $\mathscr{F}$ of (regular) maps $F \colon \Omega_M \to G$ and act on $\mathscr{F}$ with $\mathscr{B}$ in the obvious way: $(T_i F)(p) := T_i(F(T_i^{-1}(p)))$. We construct now some special maps $\Omega_M \to G^0$,

$$Y_k = \exp(y_k f_k) \colon p \mapsto \exp(y_k(p) f_k);$$

for a reduced expression J,

$$Y_k^J := \exp(y_k^J f_k^J);$$

and finally the map Z that is trivial on the first and the third factors and is induced on the second factor by the map $\beta \mapsto 2\beta$, $\beta \in M$.

Since $y_k^J = T_{k-1}^J(y_{i_k})$ and $f_k^J = t_{k-1}^J(f_{i_k})$ we get from the definition of the braid group action on $\mathscr{F}$ that

$$Y_k^J = T_{k-1}^J(Y_{i_k}).$$

Lemma. *If* $\beta_k^J = \alpha_i$ *is a simple root we have* $Y_k^J = Y_i$.

Proof. This follows from Corollary 3.4 and the analogous statement for the action of $\mathscr{B}$ on the root vectors. $\square$

4.4. **Lemma.** (1) $Y_i T_i(Y_j) = Y_j T_j(Y_i)$ *if* $m_{ij} = 2$.

(2) $Y_i T_j(Y_i) Y_j = Y_j T_i(Y_j) Y_i$ *if* $m_{ij} = 3$.

(3) $Y_i T_j T_i(Y_j) T_j(Y_i) Y_j = Y_j T_i T_j(Y_i) T_i(Y_j) Y_i$ *if* $m_{ij} = 4$.

(4) $Y_i (T_{ji}^{(4)} Y_j)(T_{ji}^{(3)} Y_i)(T_{ji}^{(2)} Y_j)(T_j Y_i) Y_j = Y_j (T_{ij}^{(4)} Y_i)(T_{ij}^{(3)} Y_j)(T_{ij}^{(2)} Y_i)(T_i Y_j) Y_i$ *if* $m_{ij} = 6$.

Proof. By direct calculation. (1) is clear and as for (2) we can compute in SL_3. With the usual notation of elementary matrices we have

$$Y_1 = 1 + y_1 e_{21}, \qquad Y_2 = 1 + y_2 e_{32},$$
$$t_1 = -e_{12} + e_{21} + e_{33}, \qquad t_2 = e_{11} - e_{23} + e_{32}.$$

Hence we get

$$T_1(Y_2) = t_1(1 + T_1(y_2)e_{32})t_1^{-1} = 1 - T_1(y_2)e_{31}$$

and

$$T_2(Y_1) = t_2(1 + T_2(y_1)e_{21})t_2^{-1} = 1 + T_2(y_1)e_{31},$$

so

$$Y_1 T_2(Y_1)Y_2 = 1 + y_1 e_{21} + T_2(y_1)e_{31} + y_2 e_{32},$$

while

$$Y_2 T_1(Y_2)Y_1 = 1 + y_1 e_{21} - T_1(y_2)e_{31} + y_2 e_{32} + y_1 y_2 e_{31};$$

thus the relation follows from 3.1(3). The proofs of (3) and (4) is similar. □

4.5. We define now, for $J : w = s_{i_1} s_{i_2} \cdots s_{i_t}$ the maps

$$Y^J := Y_t^J Y_{t-1}^J \cdots Y_1^J, \qquad X^J := T_0(Y^J).$$

Proposition. (1) Y^J and X^J are independent of J; we denote them: Y_w and $X_{\overline{w}}$.

(2) If $w = ab$ is a reduced decomposition then

$$Y_w = T_a(Y_b)Y_a \quad and \quad X_w = T_a(X_b)X_a.$$

(3) Y_w can be thought of as an algebraic isomorphism between the affine space with coordinates y_k^J and the unipotent group $w^{-1}(U_+) \cap U_-$.

Proof. (1) It suffices to do it for Y^J. Again it is enough to do it for two reduced expressions which differ by a single braid relation. In this case we see that the factors in the two products coincide except for two or three, etc. consecutive ones. Then Lemmas 4.3 and 4.4 finish the proof.

(2) This is clear from the definitions for the Y_w and follows from Lemma 2.2 for the X_w.

(3) Y_w is the product of the root subgroups relative to the roots $-\beta_k^J$ and these are the negative roots that w maps to positive roots. □

We shall refer to Y and X instead of Y_{w_0} and X_{w_0}. We shall also write X_i in place of $X_{s_i} = \exp(-x_i e_i)$.

4.6. **Proposition.** $T_i(Y) = T_i(Y_i)YY_i^{-1}$, $T_i(X) = T_i(X_i)XX_i^{-1}$, and $T_i(Z) = Z$.

Proof. Let $w_0 = s_i w = w s_{\overline{i}}$ so that, from Proposition 4.5(2), we have $Y = T_w(Y_{\overline{i}})Y_w = T_i(Y_w)Y_i$. But, since $w(\alpha_{\overline{i}}) = \alpha_i$, from Lemma 4.3 we have $T_w(Y_{\overline{i}}) = Y_i$ and so

$$Y = Y_i Y_w, \qquad T_i(Y) = T_i(Y_i)T_i(Y_w) = T_i(Y_i)YY_i^{-1}.$$

Furthermore,

$$T_i(X) = T_i T_0(Y) = T_0 T_{\overline{i}}(Y) = T_0(T_{\overline{i}}(Y_{\overline{i}})YY_{\overline{i}}^{-1})$$
$$= T_0(T_{\overline{i}}(Y_{\overline{i}}))XT_0(Y_{\overline{i}}^{-1}) = T_i(T_0(Y_{\overline{i}}))XT_0(Y_{\overline{i}}^{-1}) = T_i(X_i)XX_i^{-1}.$$

Finally, the statement for Z is clear from the definitions. $\square$

4.7. Theorem. $T_i(YZX) = X_i YZX X_i^{-1}$.

Proof. From Proposition 4.6 we have $T_i(YZX) = X_i YY_i^{-1} ZT_i(X_i)XX_i^{-1}$, so we need to show that $Z = Y_i^{-1}ZT_i(X_i)$ or $Z^{-1}Y_iZ = T_i(X_i)$. We write $T_i(X_i) = \exp(-T_i(x_i)t_i(e_i)) = \exp(z_i^2 y_i f_i)$. But by the definition of the map Z we have that $Z^{-1}\exp(y_i f_i)Z = \exp(z_i^2 y_i f_i)$. $\square$

Corollary. *The map YZX of Ω_M onto the big cell $U_- \times T \times U_+$ is of degree 2^n, and the functions on the group G invariant under conjugation pullback to functions on Ω_M invariant under $\mathscr{B}$.*

Proof. The first part is clear from the definitions. If we restrict a G-invariant function f from G to the open cell we have

$$T_i(f(YZX(p))) = f(YZX(T_i^{-1}(p))) = f(t_i(YZX(T_i^{-1}(p)))t_i^{-1})$$
$$= f(T_i(YZX)(p)) = f(X_i YZX X_i^{-1}(p)) = f(YZX(p)). \quad \square$$

Corollary. *For $w \in W$ we have $T_w(YZX) = X_w YZX X_w^{-1}$.*

Proof. This is an immediate consequence of the theorem and Proposition 4.5(2) using induction on the length of w. $\square$

   In particular $T_0(YZX) = XYZ$, and since $X = T_0(Y)$, we have $T_0(X) = Z^{-1}YZ$.

Remark. One can connect the formulas with the Cartan involution σ in G and the transformation ω. In fact $\sigma\omega(Y_k^J) = \exp(z_{\beta_k^J} x_k^J e_k^J)$ so $\sigma\omega(Y) = Z^{1/2}XZ^{-1/2}$.

5. The quantum coadjoint action

5.1. As in [DCK, §3.4] suppose one has an element $b \in U_{M,\mathscr{A}}$ with the property that $[b, a] \in [l]U_{M,\mathscr{A}}$ for all $a \in U_{M,\mathscr{A}}$. Then when we set $q = \varepsilon$, we have that b is central in $U_{M,\varepsilon}$ but one can also define the derivation D_b of $U_{M,\varepsilon}$ by $D_b(a) := [b/[l], a]_{q=\varepsilon}$. In particular for $b = E_i^l$ we have the derivations $\underline{e}_i$ of $U_{M,\varepsilon}$ given by $\underline{e}_i(a) := [E_i^{(l)}, a]_{q=\varepsilon}$. The following formulas can be derived from [L, 5.3–5.5; DCK, §3.4] (with our change of notations):

$$\underline{e}_i(z_\alpha) = (\alpha \mid \alpha_i)z_{\alpha+\alpha_i}x_i/(\alpha_i \mid \alpha_i);$$
$$\underline{e}_i(x_i) = -z_i x_i^2;$$
$$\underline{e}_i(x_j) = z_i x_{ji}^{(m_{ji}-2)} \text{ if } i \neq j \text{ and } |a_{ij}| \leq |a_{ji}|;$$
$$\underline{e}_i(x_j) = -a_{ij}z_i T_j(x_i) \text{ if } a_{ji} = -1;$$
$$\underline{e}_i(T_j(x_i)) = 0 \text{ if } m_{ij} = 3;$$
$$\underline{e}_i(T_j(x_i)) = z_i T_j T_i(x_j), \; \underline{e}_i(T_j T_i(x_j)) = z_i T_j(x_i) \text{ if } m_{ij}=4 \text{ and } a_{ji}= -1;$$
$$\underline{e}_i(T_j(x_i)) = z_i(T_j T_i(x_j))^2, \; \underline{e}_i(T_j T_i(x_j)) = 0 \text{ if } m_{ij} = 4 \text{ and } a_{ij} = -1;$$
$$\underline{e}_i(T_j(x_i)) = 2z_i x_{ji}^{(3)}, \; \underline{e}_i(x_{ji}^{(3)}) = z_i x_{ji}^{(4)};$$

$$\underline{e}_i(x_{ji}^{(2)}) = 3z_i x_{ji}^{(3)2} \text{ if } m_{ij} = 6 \text{ and } a_{ji} = -1;$$

$$\underline{e}_i(x_j^{(4)}) = 0 \text{ if } m_{ij} = 6;$$

$$\underline{e}_i(x_{ji}^{(3)}) = z_i x_{ji}^{(4)3}, \quad \underline{e}_i(x_{ji}^{(2)}) = z_i x_{ji}^2;$$

$$\underline{e}_i(T_j(x_i)) = 3z_i x_{ji}^{(4)} x_{ji}^{(2)} - z_i x_{ji}^{(3)} \text{ if } m_{ij} = 6 \text{ and } a_{ij} = -1;$$

$$\underline{e}_i(y_j) = \delta_{ij}(z_i - z_i^{-1});$$

$$\underline{e}_i(T_i(y_j)) = z_i y_j \text{ if } a_{ij} = -1;$$

$$\underline{e}_i(T_i(y_j)) = -a_{ij} z_i T_i T_j(y_i) \text{ if } a_{ji} = -1;$$

$$\underline{e}_i(T_i T_j(y_i)) = z_i y_j^2 \text{ if } m_{ij} = 4 \text{ and } a_{ij} = -1;$$

$$\underline{e}_i(y_{ij}^{(3)}) = 3z_i y_{ij}^{(4)2}, \quad \underline{e}_i(y_{ij}^{(2)}) = 2z_i y_{ij}^{(4)};$$

$$\underline{e}_i(y_{ij}^{(4)}) = z_i y_j \text{ if } m_{ij} = 6 \text{ and } a_{ji} = -1;$$

$$\underline{e}_i(y_{ij}^{(3)}) = z_i y_j^2, \quad e_i(y_{ij}^{(4)}) = z_i y_j^3;$$

$$\underline{e}_i(y_{ij}^{(2)}) = -z_i y_{ij}^{(4)} + 3z_i y_j y_{ij}^{(3)} \text{ if } m_{ij} = 6 \text{ and } a_{ij} = -1.$$

Finally, if ϕ is any automorphism of $U_{M,\varepsilon}$ we obviously have $\phi D_b \phi^{-1}(a) = D_{\phi(b)}(a)$ or in other form (substituting $\phi^{-1}(a)$ for a), $\phi(D_b(a)) = D_{\phi(b)}(\phi(a))$. An important case is when α_i, α_j are two simple roots and $w \in W$ is such that $w(\alpha_i) = \alpha_j$ so that $T_w(E_i) = E_j$. Thus we have, taking $\phi = T_w$,

$$T_w(\underline{e}_i(a)) = \underline{e}_j(T_w(a)).$$

In order to perform our computations we need some preliminary steps.

5.2. For SL_2 we have the elements

$$Y = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \qquad Z = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}, \qquad X = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$$

and their product

$$A := \begin{pmatrix} z & -zx \\ zy & -zxy + z^{-1} \end{pmatrix}.$$

We apply the operator $\underline{e}$ to A and get by the formulas in §5.1,

$$\underline{e}(A) = \begin{pmatrix} z^2 x & 0 \\ z^2 xy + z^2 - 1 & -z^2 x \end{pmatrix} = [ze_{21}, A].$$

(We have dropped the subscript 1 in all formulas.)

5.3. Next, we show that for $G = SL_3$, again if $A = YZX$ then

$$\underline{e}_1(A) = [z_1 e_{21}, A].$$

We have by Proposition 4.5(2), $Y = \overline{Y}Y_1$, where $\overline{Y} = T_1(Y_{s_2 s_1})$ and $X = T_{s_2 s_1}(X_2)X_{s_2 s_1} = X_1\overline{X}$, where $\overline{X} = X_{s_2 s_1}$, since the longest element of W is $s_1 s_2 s_1 = s_2 s_1 s_2$. We also decompose $Z = Z_1\overline{Z}$, where $\overline{Z}$ is the kernel of α_1

in T. Explicitly,

$$Y_1 = 1 + y_1 e_{21}, \qquad Y_2 = 1 + y_2 e_{32};$$
$$t_1 = -e_{12} + e_{21} + e_{33}, \qquad t_2 = e_{11} - e_{23} + e_{32}, \qquad t_0 := t_1 t_2 t_1 = e_{31} - e_{22} + e_{13};$$
$$Y_{s_2 s_1} = T_2(Y_1)Y_2 = 1 + y_2 e_{32} + T_2(y_1)e_{31};$$
$$\overline{Y} = 1 - T_1(y_2)e_{31} + y_2 e_{32};$$
$$Z_1 = \mathrm{diag}(z_{\omega_1}^2 z_{\omega_2}^{-2}, z_{\omega_1}^{-2}, z_{\omega_2}^2), \qquad \overline{Z} = \mathrm{diag}(z_{\omega_2}^2, z_{\omega_2}^2, z_{\omega_2}^{-4});$$
$$X_1 = t_0 Y_2 t_0 = 1 - x_1 e_{12}, \qquad X_2 = t_0 Y_1 t_0 = 1 - x_2 e_{23};$$
$$\overline{X} = T_2(X_1)X_2 = 1 - T_2(x_1)e_{13} - x_2 e_{23}.$$

Using §5.1, we obtain

$$\underline{e}_1(\overline{Y}) = -z_1 y_2 e_{31} = [z_1 e_{21}, \overline{Y}];$$
$$\underline{e}_1(\overline{X}) = -z_1 T_2(x_1)e_{23} = [z_1 e_{21}, \overline{X}];$$
$$\underline{e}_1(\overline{Z}) = 0 = [z_1 e_{21}, \overline{Z}].$$

Finally since $A = \overline{Y}(Y_1 Z_1 X_1)\overline{Z}\,\overline{X}$ and the derivations $\underline{e}_1$ and $z_1 ade_{21}$ coincide on all four factors by the above formulas and (a version of) §5.2, we deduce that they coincide on A.

5.4. Consider again the big cell $G^0 = U_- T U_+$ and the mapping YZX that is an unramified covering: $\Omega_M \to G^0$. Thus given a vector field on G we can pull it back to Ω_M. We will do this with the elements of the Lie algebra $\mathfrak{g}$, in particular with the Chevalley generators e_i, h_i, and f_i. We consider a representation ρ of G and its Lie algebra $\mathfrak{g}$. In a given basis the entries of the matrix $\rho(g)$ are functions on G and if $a \in \mathfrak{g}$ the entries of $[a, \rho(g)]$ are the derivatives of the entries of $\rho(g)$ according to the vector field a. A mapping f of Ω_M in G can be composed with ρ to get a map to matrices. We will drop the symbol ρ if there is no ambiguity. For the pull back a^* of a we have of course the same formula $[a, \rho(f(g))] = a^*(\rho(f(g)))$. We wish to show that

Theorem. $\underline{e}_i = z_i f_i$.

Proof. We have made the computation for SL_2 and SL_3. Similar but lengthier calculations for the four-dimensional representation of C_2 and the seven-dimensional representation of G_2 show that the formula hold for all rank 2 groups as well. We will reduce the general case to these cases.

For each $\alpha \in R^+$ pick $w_\alpha \in W$ such that $\alpha = w_\alpha(\alpha_j)$, $\alpha_j \in \Pi$, and let

$$y_\alpha = T_{w_\alpha} y_j, \qquad x_\alpha = T_{w_\alpha} x_j.$$

Since, by Corollary 3.4(c), the elements x_α and y_α generate Z_0 over Z_0^0 and since the derivatives $\underline{e}_i$ and $z_i f_i$ coincide on Z_0^0 by the rank 2 calculations, it suffices to show that

$$(1) \qquad \underline{e}_i(y_\alpha) = z_i f_i(y_\alpha), \qquad \underline{e}_i(x_\alpha) = z_i f_i(x_\alpha), \qquad \alpha \in R^+.$$

Given two nonproportional roots α and β, we denote by $R_{\alpha,\beta}$ the intersection of the $\mathbb{Z}$-span of α and β with R and let $R^+_{\alpha,\beta} = R_{\alpha,\beta} \cap R^+$. Then $R_{\alpha,\beta}$ is a rank 2 root system with $R^+_{\alpha,\beta}$ being a subset of positive roots.

There exist two simple roots, say, $\alpha_1, \alpha_2 \in \Pi$, and $w \in W$ such that

$$wR^+_{\alpha_1,\alpha_2} = R^+_{\alpha_i,\alpha} \quad \text{and} \quad w\alpha_1 = \alpha_i \ (\alpha \text{ need not be equal to } w(\alpha_2)).$$

Fix a reduced expression $w = s_{i_1} \cdots s_{i_k}$. Let $w'_0 = s_1 s_2 s_1 \cdots s_\varepsilon$, where $\varepsilon = 1$ or 2, be the reduced expression J' of the longest element of the Weyl group of R_{α_1,α_2}, and let $m = l(w'_0)$. Then the expression $ww'_0 = s_{i_1} \cdots s_{i_k} s_1 s_2 \cdots s_\varepsilon$ is reduced; this is easily checked by using §1.1. Complete to the right this expression to a reduced expression J of w_0. Let $\{\beta_1, \ldots, \beta_N\}$ be the corresponding ordering of R^+ associated to this reduced expression (see Proposition 2.1). Then R^+ breaks into four pieces:

$$R^1 := \{\beta_1, \ldots, \beta_k\} = R_{w^{-1}}, \qquad \beta_{k+1} = \alpha_i,$$
$$R^2 := \{\beta_{k+2}, \ldots, \beta_{k+m}\} = R^+_{\alpha_i,\alpha} \backslash \{\alpha_i\}, \qquad R^3 = \{\beta_{k+m+1}, \ldots, \beta_N\}.$$

Furthermore, let $\mathfrak{g}^i_\pm = \bigoplus_{\gamma \in R^i} \mathbb{C}e_{\pm\gamma}$, $i = 1, 2, 3$. It is easy to see that these are subalgebras of the Lie algebra $\mathfrak{g}$ normalized by the three-dimensional subalgebra $\mathbb{C}e_i + \mathbb{C}h_i + \mathbb{C}f_i$. Indeed, we have to show that for $s = 1$ and 3, the R^s are additively closed subsets of R^+ such that $R \cap (R^s \pm \alpha_i) \subset R^s$. Since $R^1 = R_{w^{-1}}$, it is additively closed. If $\beta \in R^1$ and $\beta \pm \alpha_i \in R$, then $\beta \pm \alpha_i \in R^+$ since α_i is simple and $w^{-1}(\beta \pm \alpha_i) = w^{-1}(\beta) \pm \alpha_1 \in -R^+$ since α_1 is simple. Furthermore, $R^3 = R^+ \backslash R_{w'_0 w^{-1}}$ and hence is additively closed. If $\beta \in R^3$ and $\beta \pm \alpha_i \in R$, then, as above, $\beta \pm \alpha_i \in R^+$ and $w^{-1}(\beta \pm \alpha_i) = w^{-1}(\beta) \pm \alpha_1 \in R^+$; hence $w'_0 w^{-1}(\beta \pm \alpha_i) = w'_0 w^{-1}(\beta) \pm w'_0(\alpha_1) \in R^+$ since $-w'_0(\alpha_1)$ is simple. Let $U^i_\pm$ be the subgroups of $U_\pm$ corresponding to the $\mathfrak{g}^i_\pm$.

We turn now to the map YZX. We have the decompositions of Y and X according to the above decomposition of R^+

$$YZX = Y_3 Y_2 (\exp y_i f_i) Y_1 Z X_1 (\exp -x_i e_i) X_2 X_3$$
$$= Y_3 Y_2 Y'_1 ((\exp y_i f_i) Z (\exp -x_i e_i)) X'_1 X_2 X_3,$$

where $Y'_1 = (\exp y_i f_i) Y_1 (\exp -y_i f_i) \subset U^1_-$ and $X'_1 = (\exp -x_i e_i) X_1 (\exp x_i e_i) \in U^1_+$.

Consider the subalgebra $Z^{1,2}_0$ of Z_0 generated over Z^0_0 by all x_γ and y_γ with $\gamma \in R^+_{\alpha_1,\alpha_2}$. We want to prove

$$(2) \qquad (z_i f_i)(T_w(a)) = T_w(z_1 f_1(a)) \quad \text{for } a \in Z^{1,2}_0.$$

This formula implies (1). Indeed using the formula at the end of §5.1 and the calculation in the rank 2 case, we have for $a \in Z^{1,2}_0$, $\underline{e}_i(T_w(a)) = T_w \underline{e}_1(a) = T_w(z_1 f_1(a))$.

In order to prove (2) note that the action of $z_i f_i$ on Z_0 may be calculated as follows. Write for $t \in \mathbb{C}$

$$(\exp t z_i f_i) YZX(\exp -t z_i f_i) = \prod_{s=N}^{1} (\exp y_s^J(t) f_s^J) Z(t) \prod_{s=1}^{N} (\exp x_s^J(t) e_s^J).$$

Then $z_i f_i(x_s^J) = \frac{d}{dt} x_s^J(t)|_{t=0}$, and similarly for y.

But x_α (resp. y_α) occurs only in X_2 (resp. Y_2) and all other factors of YZX lie in the subgroups normalized by $\exp t z_i f_i$ and having trivial intersection with U_+^2 (resp. U_-^2). Thus, it suffices to perform the calculation in U_+^2 (resp. U_-^2). We have

$$\prod_{s=k+2}^{k+m} \exp x_s^J(t) e_s^J = (\exp t z_i f_i) \prod_{s=2}^{m} \exp T_w(x_s^{J'} e_s^{J'})(\exp -t z_i f_i)$$

$$= T_w\left((\exp t z_1 f_1)\left(\prod_{s=2}^{m} \exp x_s^{J'} e_s^{J'}\right)(\exp -t z_1 f_1)\right),$$

and we can use again the calculation in the rank 2 case. This proves (2). □

In the next section we shall need the following result.

Lemma. *Let* $\beta \in R^+$ *and let* $p \in \mathbb{Z}_+$ *be such that* $\beta + p\alpha_i \in R^+$ *but* $\beta + (p+1)\alpha_i \notin R^+$. *Then for suitable choices of* x_β *and* $x_{\beta+\alpha_i}$ *one has*

$$\underline{e}_i(x_\beta) = p z_i x_{\beta + \alpha_i}.$$

Proof. If $\beta \in R_{\alpha_1, \alpha_2}^+$ for some simple roots α_1 and α_2, and $i = 1$, the lemma holds due to §5.1. This case implies the general one if we take $w \in W$ such that $w R_{\alpha_1, \alpha_2}^+ = R_{\alpha_i, \beta}^+$, and $w\alpha_1 = \alpha_i$, and apply T_w. □

5.5. Let

$$\underline{f}_i = T_0 \underline{e}_{\bar{i}} T_0^{-1}.$$

As in §2.3, we have

$$\underline{f}_i = T_i \underline{e}_i T_i^{-1}.$$

As in §5.4, it suffices to verify the following formula in the rank 1 and 2 cases where we verify it as in §5.3.

Theorem. $\underline{f}_i = -z_i e_i$.

Recall that given functions f, g and vector fields U, V, one has

$$[fU, gV] = fg[U, V] + fU(g)V - gV(f)U.$$

Using this, we immediately deduce the

Lemma. (a) $\underline{h}_i := [\underline{e}_i, \underline{f}_i] = z_i^2 h_i + z_i^2 x_i e_i + y_i f_i$.

(b) $[\underline{e}_{i_1}, [\underline{e}_{i_2}, [\dots, [\underline{e}_{i_{k-1}}, \underline{e}_{i_k}] \dots]] = z_\beta [f_{i_1}, [f_{i_2}, [\dots, [f_{i_{k-1}}, f_{i_k}] \dots]] + f$, *where* $\beta = \alpha_{i_1} + \cdots + \alpha_{i_k}$ *and* f *is a linear combination of vector fields (with*

coefficients functions) corresponding to root vectors attached to roots $-\alpha$ *such that* $\alpha \in R^+$ *and* $\text{ht}\,\alpha < \text{ht}\,\beta$.

Let $\tilde{\mathfrak{g}}$ be the Lie algebra of vector fields on Ω_M generated by the vector fields $\underline{e}_i$ and $\underline{f}_i$, $i = 1, \ldots, n$. Recall that using the map YZX, the Lie algebra $\mathfrak{g}$ also can be viewed as a Lie algebra of vector fields on Ω_M, generated by the vector fields e_i and f_i, $i = 1, \ldots, n$. Using Lemma 5.5, we obtain the following important corollary of Theorems 5.4 and 5.5.

Corollary. *At every point of* Ω_M *the spaces consisting of vectors of all vector fields from* $\tilde{\mathfrak{g}}$ *and from* $\mathfrak{g}$ *coincide.*

Remark. (a) As in [DCK,§3.4] introduce the derivations $\underline{k}_i$ of U_ε by

$$\underline{k}_i(u) = c_i^{-1}[K_i^{(l)}, u]_{q=\varepsilon}.$$

Then one easily checks the following formula:

$$\underline{k}_i = z_i h_i/2.$$

(b) The derivation $\underline{f}_i$ is different from the derivation f_i introduced in [DCK, §3.4], which (in order to avoid confusion) we denote here by f'_i. It follows from [DCK, (3.4.7)] that $\underline{f}_i = -z_i f'_i - y_i \underline{k}_i$, and hence $f'_i = -\frac{1}{2}y_i h_i + e_i$.

5.6. Recall that the quantum group U_M and its specialization $U_{M,\varepsilon}$ are Hopf algebras with comultiplication Δ, antipode S, and counit η defined by

$$\Delta E_i = E_i \otimes 1 + K_i \otimes E_i, \qquad \Delta F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i, \qquad \Delta K_\alpha = K_\alpha \otimes K_\alpha,$$

$$SE_i = -K_i^{-1}E_i, \qquad SF_i = -F_iK_i, \qquad SK_\alpha = K_{-\alpha},$$

$$\eta E_i = 0, \qquad \eta F_i = 0, \qquad \eta K_\alpha = 1.$$

Lemma. *Let* $x \in U_{M,\varepsilon}$ *be such that* $\Delta(x) \in Z_\varepsilon \otimes Z_\varepsilon$. *Then for* $s = 1, \ldots, n$ *one has*

$$\Delta(\underline{e}_s(x)) = (\underline{e}_s \otimes 1 + z_s \otimes \underline{e}_s - \underline{k}_s \otimes z_s x_s)\Delta(x).$$

Proof. We compute in U_M,

$$\Delta([E_s^{(l)}, x]) = [E_s^{(l)} \otimes 1 + K_s^l \otimes E_s^{(l)} + \cdots, \Delta(x)],$$

where dots refer to elements that are regular at $q = \varepsilon$ and hence can be ignored. Let $\Delta(x) = \sum_i a_i \otimes b_i$; we have

$$\Delta(\underline{e}_s(x)) = \sum_i ([E_s^{(l)}, a_i] \otimes b_i + [K_s^{(l)}, a_i] \otimes E_s^l b_i + a_i K_s^l \otimes [E_s^{(l)}, b_i]) + \cdots.$$

Specializing at $q = \varepsilon$ completes the proof. $\square$

Let $Z_{\geq 0} = Z_0^0 Z_0^+$ and $Z_{\leq 0} = Z_0^0 Z_0^-$; these are Q_+-graded subalgebras of Z_0. For $\alpha \in R^+$ let Z_α (resp. Z_α^+) $= Z_0^0[x_\gamma | \gamma \in R^+, \gamma \leq (\text{resp.} <) \alpha]$; we let $Z_\alpha^+ = 0$ if $\alpha \in \Pi$. These are Q_+-graded subalgebras of $Z_{\geq 0}$, whose γ-component coincides with that of $Z_{\geq 0}$ if $\gamma \leq (\text{resp.} <) \alpha$. Similarly, we define subalgebras $Z_{-\alpha}$ and $Z_{-\alpha}^+$ of $Z_{\leq 0}$. Let $I = \text{Ker}\,\eta$ be the augmentation ideal of $Z_{\geq 0}$.

Proposition. (a) $\Delta(x_\alpha) = x_\alpha \otimes z_{-\alpha} + 1 \otimes x_\alpha + P_\alpha$, where $P_\alpha \in Z_\alpha^+ \otimes Z_\alpha^+$.

(b) *Let* $\alpha \in R_+$, $j \in \{1, \ldots, n\}$, *and let* $r_{\alpha, j} = \max\{k \mid \alpha - k\alpha_j \in R_+\}$. *Then for a suitable choice of* x_α *we have* $\mod I^2 \otimes Z_{\geq 0} + Z_{\geq 0} \otimes I^2$,

$$\Delta x_\alpha = x_\alpha \otimes z_{-\alpha} + 1 \otimes x_\alpha + x_{\alpha-\alpha_j} \otimes x_j \quad \text{if } \alpha = \alpha_i + \alpha_j \,;$$

$$\Delta x_\alpha = x_\alpha \otimes z_{-\alpha} + 1 \otimes x_\alpha + \sum_i r_{\alpha, i} x_{\alpha-\alpha_i} \otimes x_i$$

$$+ \sum_{\substack{\beta, \gamma \in R_+ \backslash \Pi \\ \beta+\gamma=\alpha}} c_{\beta, \gamma} x_\beta \otimes x_\gamma \quad \text{if } \operatorname{ht} \alpha > 2 \,,$$

where $c_{\beta, \alpha} \in \mathbb{C}$.

(c) $Z_{\pm\alpha}$ *and* $Z_{\pm\alpha}^+$ *are Hopf subalgebras of* $U_{M, \varepsilon}$.

(d) Z_0, $Z_{\geq 0}$, *and* $Z_{\leq 0}$ *are Hopf subalgebras of* $U_{M, \varepsilon}$.

Proof. (c) and (d) follow from (a).

We prove (a) and (b) by induction on $\alpha \in Q_+$.

For $\alpha \in \Pi$, (a) is clear. Otherwise, by Lemma 5.4 we have for suitable choices of x_α and x_β, provided that $\beta = \alpha - \alpha_i \in R^+ : x_\alpha = p^{-1} z_i^{-1} \underline{e}_i(x_\beta)$. Applying Lemma 5.6, we obtain

$$\Delta(x_\alpha) = p^{-1}(z_i^{-1} \otimes z_i^{-1})(\underline{e}_i \otimes 1 + z_i \otimes \underline{e}_i - \underline{k}_i \otimes z_i x_i)\Delta(x_\beta).$$

(a) follows by induction. Substituting $\Delta(x_\beta)$ given by the inductive assumption in (b) and using that $\underline{e}_i(z_{-\beta}) = -(\beta \mid \alpha_i)z_{-\beta+\alpha_i} x_i/(\alpha_i \mid \alpha_i)$ and $\underline{k}_i(x_\beta) = (\beta \mid \alpha_i)x_\beta z_i/(\alpha_i \mid \alpha_i)$ proves (b). $\square$

It follows from Proposition 5.6 that $\operatorname{Spec} Z_0$ is a connected algebraic group and that $\operatorname{Spec} Z_{\geq 0}$ and $\operatorname{Spec} Z_{\leq 0}$ are its connected normal subgroups whose intersection is the torus $\operatorname{Spec} Z_0^0$ and whose product is $\operatorname{Spec} Z_0$. Let L, L^+, L^-, and L^0 be the Lie algebras of these groups respectively. We have $L = L^+ + L^-$ (sum of ideals) and $L^0 = L^- \cap L^+$.

We shall calculate below the structure of Lie algebras L^+ and L^-. We may assume that $M = Q$. Recall that $L^+ = (I/I^2)^*$, and that, denoting by $\overline{x}$ the class in I/I^2 of $x \in I$, the bracket in L^+ can be calculated as follows $(\varphi_1, \varphi_2 \in L^+, x \in I)$:

$$[\varphi_1, \varphi_2](\overline{x}) = \sum_i (\varphi_1(\overline{a_i - \eta(a_i)})\varphi_2(\overline{b_i - \eta(b_i)}) - \varphi_1(\overline{b_i - \eta(b_i)})\varphi_2(\overline{a_i - \eta(a_i)}).$$

Here $\Delta(x) = \sum_i a_i \otimes b_i$, where $a_i, b_i \in Z_{\geq 0}$.

Note that by Corollary 3.4, $\overline{x}_\alpha$ up to a sign is independent of the choice of the x_α. The elements $\overline{z_i - 1}$ $(i = 1, \ldots, n)$, $\overline{x}_\alpha$ $(\alpha \in R^+)$ form a basis of I/I^2; let h_i $(i = 1, \ldots, n)$, S_α $(\alpha \in R^+)$ be the dual basis of L^+. In order to calculate brackets of these elements, we use Proposition 5.6(b).

A straightforward calculation shows that

$$[h_i, S_\beta] = \frac{2(\omega_i \mid \beta)}{(\alpha_i \mid \alpha_i)} S_\beta \,.$$

It follows that

$$[S_\alpha, S_\beta] = 0 \qquad \text{if } \alpha + \beta \notin R^+,$$

$$[S_\alpha, S_\beta] = m_{\alpha\beta} S_{\alpha+\beta} \quad \text{if } \alpha + \beta \in R^+.$$

Again, it is straightforward to show that

$$m_{\alpha_i, \beta} = r_{\beta, i} + 1 \quad \text{if } \alpha_i + \beta \in R^+.$$

Similar formulas hold for L^-. It follows that L^0 and the S_{α_i} (resp. $S_{-\alpha_i}$) generate L^+ (resp. L^-) and that the Chevalley-Serre relations hold for them. Hence we obtain the following

Theorem. L^+ (resp. L^-) *is isomorphic to a Borel subalgebra* $\mathfrak{b}$ *of the Lie algebra* $\mathfrak{g}$.

This result has been announced recently by N. Reshetikhin (MIT seminar). In §7 we calculate the group $\operatorname{Spec} Z_0$ more explicitly, along with its Poisson structure.

Remark. Let $\pi_i \in \operatorname{Spec} U_{M,\varepsilon}$ $(i = 1, 2)$ be two irreducible representations, and let g_i $(i = 1, 2)$ be the corresponding elements of $\operatorname{Spec} Z_0$. It is clear that if $\pi_1 \otimes \pi_2$ and $\pi_2 \otimes \pi_1$ are equivalent, then $g_1 g_2 = g_2 g_1$ in the algebraic group $\operatorname{Spec} Z_0$. To what extent is this condition sufficient?

6. The geometry of the quantum coadjoint action

6.1. It is proved in [DCK, §3.5] that the derivations $\underline{e}_i$ (and hence $\underline{f}_i$) of the algebra Z_0 integrate to global 1-parameter groups of analytic automorphisms $\exp s\underline{e}_i$ (resp. $\exp s\underline{f}_i$) of the algebraic variety $\Omega_M = \operatorname{Spec} Z_0$. Denote by $\widetilde{G}$ the (infinite-dimensional) group generated by the groups $\exp s\underline{e}_i$ and $\exp s\underline{f}_i$, $i = 1, \ldots, n$. The action of $\widetilde{G}$ on Ω_M is called the quantum coadjoint action [DCK].

Let, as before, $G^0 = U_- T U_+$ be the big cell of G, and let π denote the map $YZX: \Omega_M \to G^0$. Recall that $\Omega_M = \operatorname{Spec} Z_0^- \times T \times \operatorname{Spec} Z_0^+$ is an unramified (algebraic) Galois covering with the group $\frac{1}{2}M/M$. Let $F = \pi^{-1}(\operatorname{Center} G) = \{\chi \in T \subset \Omega_M \mid \chi(z_\beta) = \pm 1 \text{ for all } \beta \in Q\}$.

Given a conjugacy class $\mathcal{O}$ in G, it always intersects G^0 in a smooth connected variety $\mathcal{O}^0$. Theorems 5.4 and 5.5 and Corollary 5.5 immediately give us

Proposition. (a) *The connected components of the variety* $\pi^{-1}(\mathcal{O}^0)$ *are orbits of the group* $\widetilde{G}$.

(b) *The set* F *is the fixed point set of* $\widetilde{G}$ *in* Ω_M. *The action of* $\widetilde{G}$ *in the tangent space to each point* $p \in F$ *induces the coadjoint action of* G *on* $\mathfrak{g}$ *(after dividing by the kernel).*

Proof. (b) is clear. To see (a) we use the following simple geometric fact: Let φ_i $(i = 1, \ldots, m)$ be global 1-parameter groups of diffeomorphisms of a connected

manifold M and let X_i be the corresponding vector fields. Assume that the X_i span the tangent space to M at every point of M. Then M is an orbit of the group K generated by the φ_i. (Indeed, our assumption implies that every K-orbit in M is open. Since M is connected, the claim follows.) $\square$

Proposition 6.1(a) has a number of important consequences, some of which are solutions to some of the conjectures of [DCK]. In order to state these results, introduce some notation and terminology.

Denote by I (resp. I^+) the ideal in Z_0 generated by all elements x_k^J and y_k^J (resp. x_k^J). Then $T = \{\chi \in \Omega_M \mid \chi(I) = 0\}$, and we let $T_{\text{reg}} = \{\chi \in T \mid \chi(z_\beta)^2 \neq 1$ for all $\beta \in R\}$. A $\widetilde{G}$ orbit in Ω_M is called *unipotent* if its Zariski closure contains a point from F. A $\widetilde{G}$-orbit in Ω_M is called *semisimple* (resp. *regular semisimple*) if it intersects T (resp. T_{reg}).

Theorem. (a) *Every $\widetilde{G}$-orbit in Ω_M contains an element χ such that $\chi(I^+) = 0$.*

(b) *Every $\widetilde{G}$-orbit $\mathcal{O}$ in Ω_M is Zariski open in its Zariski closure $\overline{\mathcal{O}}$.*

(c) *A $\widetilde{G}$-orbit is closed if and only if it is semisimple.*

(d) *The union of all regular semisimple orbits $\widetilde{G} \cdot T_{\text{reg}}$ is Zariski open and dense in Ω_M.*

(e) *There is only a finite number of unipotent orbits.*

Proof. Follows from Proposition 6.1 and the well-known results on conjugacy classes in simple Lie groups [K, S]. $\square$

Corollary. *Every finite-dimensional irreducible representation ρ of $U_{M,\varepsilon}$ in a vector space V is triangularizable; i.e., there exists $\sigma \in \widetilde{G}$ and a nonzero vector $v \in V$ such that*

$$\rho(\sigma(\overline{U}_\varepsilon^+))v = 0 \quad and \quad \rho(\sigma(K_\beta))v = \lambda_\beta v, \qquad \lambda_\beta \in \mathbb{C}, \ \beta \in M.$$

Proof. By Theorem 6.1(a) there exists $\sigma \in \widetilde{G}$ such that $\rho(\sigma(I^+)) = 0$. It follows that $\rho(\sigma(E_\alpha))^l = 0$ for all $\alpha \in R^+$. By the PBW theorem [L], it follows that $\rho(\sigma(\overline{U}_\varepsilon^+))$ consists of nilpotent endomorphisms; hence the subspace $V_0 = \{v \in V \mid \rho(\sigma(\overline{U}_\varepsilon^+))v = 0\}$ is nonzero. Since the K_β normalize U_ε^+, V_0 is K_β-invariant, and hence the K_β have in V_0 a common eigenvector. $\square$

One can deduce from [BC, Theorem 7.1] a more precise statement than Theorem 6.1(a): Every $\widetilde{G}$-orbit in Ω_M contains an element χ such that

(i) $\chi(I^+) = 0$,

(ii) the set $R_\chi^+ := \{\beta_k^J \in R^+ \mid \chi(y_k^J) \neq 0\}$ is linearly independent (and then it is independent of J),

(iii) if $\chi(z_\alpha)^2 \neq 1$ then $\alpha \notin R_\chi^+$.

6.2. We turn now to the center Z of U_M and the center Z_ε of $U_{M,\varepsilon}$. Recall that $U_M = U^0 \oplus (\overline{U}^- U_M + U_M \overline{U}^+)$ and denote by h' the projection of U_M on U^0. Similarly, define the projection $h_\varepsilon' : U_{M,\varepsilon} \to U^0$.

Lemma. *Let* $z \in Z_\varepsilon$ *be such that*

$$\widetilde{G} \cdot z = z \quad and \quad h'_\varepsilon(z) = 0.$$

Then $z = 0$.

Proof. Since the map $\operatorname{Spec} Z_\varepsilon \to \operatorname{Spec} Z_0$ (induced by inclusion) is surjective, [DCK, §3.6] due to Theorem 6.1(d), it suffices to check that the eigenvalue α of z on each diagonal module $\overline{M}_\varepsilon(\lambda)$ (see [DCK, §3.2]) is 0. We have

$$z v_\lambda = \alpha v_\lambda = u_- v_\lambda, \quad \text{where } u_- \in \overline{U}_\varepsilon^-.$$

Since u_- is a nilpotent endomorphism, we see that $\alpha = 0$. □

Recall that the maximal torus T of G is the affine variety with coordinate ring $\mathbb{C}[M]$, the group ring of M. We identify $\mathbb{C}[M]$ with U_M^0 via $\beta \mapsto K_\beta$, $\beta \in M$. It is convenient to look at T as the algebraic group $\mathbb{C}^\times \otimes_\mathbb{Z} M^*$. The action by left translations of T on functions on T can then be written as

$$(\lambda \otimes m) K_\beta = \lambda^{\langle m, \beta \rangle} K_\beta, \qquad \lambda \in \mathbb{C}^\times, \ m \in M^*.$$

Consider the map of $\frac{1}{2} Q^\vee$ in T defined by $\frac{1}{2} \gamma \mapsto (-1) \otimes \gamma$ and let $Q_2^\vee$ denote its image. One should note that $Q^\vee \subset M^*$ since $Q^\vee = P^*$ and that, for the same reason, $Q_2^\vee$ is the set of all elements of T of period 2 if G is simply connected (otherwise $Q_2^\vee$ may be smaller). Thus we have an action of the group $\widetilde{W} := W \ltimes Q_2^\vee$ on U_M^0.

Now we can recall the construction of Z. Denote by γ the automorphism of $U^{0 Q_2^\vee}$ defined by $\gamma(K_\beta) = q^{(\rho \mid \beta)} K_\beta$, and let $h = \gamma^{-1} \circ h'$. At this point we use that $2(\rho \mid P) \subset \mathbb{Z}$. Then we have an isomorphism of algebras [DCK, Proposition 2.2(b)]

$$h : Z \xrightarrow{\sim} U^{0 \widetilde{W}}.$$

More explicitly, for each $\varphi \in U^{0 \widetilde{W}}$ there exists a unique central element (see [DCK] for notation) of the form

$$z_\varphi = \gamma(\varphi) + \sum_{\eta > 0} \sum_{k, r \in \operatorname{Par}(\eta)} F^k \varphi_{k, r} E^r,$$

where the $\varphi_{k, r} \in U^0$ can be computed from a recurrent formula and $Z = \{ z_\varphi \mid \varphi \in U^{0 \widetilde{W}} \}$. The recurrent formula shows that the only poles of the $\varphi_{k, r}$ are $q =$ roots of 1 and the poles of φ.

Proposition. *All the* $\varphi_{k, r}$ *do not have a pole at* $q = \varepsilon$ *if* φ *does not.*

Proof. Let $P_l(q)$ be the lth cyclotomic polynomial and let m be the maximum of orders of poles of the $\varphi_{k, r}$. Let $z' = P_l(q)^m z$. Then the element $z'|_{q=\varepsilon}$ satisfies all conditions of the lemma and is nonzero, a contradiction. □

Thus, we have a well-defined injective homomorphism $\overline{h}_\varepsilon : U_\varepsilon^{0 \widetilde{W}} \to Z_\varepsilon$ given by $\varphi \mapsto z_\varphi$. Denote by Z_1 its image. (Note that it is different from that introduced in [DCK, §5]. As we shall see, all conjectures stated there are true for the Z_1, introduced here.)

Corollary. (a) *The subalgebra* Z_1 *of* $Z_{M,\varepsilon}$ *is pointwise fixed under the action of* $\widetilde{G}$ *and the action of* $\mathscr{B}$.

(b) *The center* Z_M *of* U_M *is pointwise fixed under the action of* $\mathscr{B}$.

Proof. The first part of (a) is clear and the second part follows from (b). Now let $z \in Z_M$ and let $z' := T_i(z) - z \in Z_M$. If $z' \neq 0$, then by Lemma 6.2 one can arrange for z' to be defined and yet $h'_\varepsilon(z') \neq 0$. This contradicts [DCK, (5.4.2)]. □

6.3. We return to the study of the quantum coadjoint action. Recall that $G = G_c/C_G$, where G_c is the simply connected cover of G and $C_G = M^*/Q^\vee$ is its fundamental group. Note that, given $m \in \mathbb{N}$, we have

$$T_{(m)} := \{\sigma \in T \mid \sigma^m = 1\} = m^{-1}M^*/M^* ,$$

and that $Q_2^\vee \subset T_{(2)}$ since $Q^\vee \subset M^*$. Denote by S the subgroup of $T_{(2l)}$ generated by $T_{(l)}$ and $Q_2^\vee$. We have canonical isomorphisms

$$T_{(2l)}/S \simeq \frac{1}{2l}M^* \Big/ \left(\frac{1}{2}Q^\vee + \frac{1}{l}M^*\right) \simeq M^*/(Q^\vee + 2M^*) \simeq T_{(2)}/Q_2^\vee .$$

Thus

$$T_{(2l)}/S \simeq C_G/C_G^2 .$$

Let $G' = G_c/C_G^2$, so that $\mathrm{Center}\, G' = C_G/C_G^2$. The coroot lattice of the group G' is $Q^\vee + 2M^*$ and the weight lattice is $P \cap \frac{1}{2}M$. We list in Table 1, in the case of the adjoint group, $G = G_{\mathrm{ad}}$ (i.e., when C_G is the largest) the group G' , its center, the groups C_G , and C_G/C_G^2 .

TABLE 1.

(a_{ij})	G'	Center G'	C_G	C_G^2
A_{2k}	G_{ad}	1	$\mathbb{Z}_{2k+1}$	$\mathbb{Z}_{2k+1}$
A_{2k-1}	$\mathrm{SL}_{2k}/\mathbb{Z}_k$	$\mathbb{Z}_2$	$\mathbb{Z}_{2k}$	$\mathbb{Z}_k$
B_n, C_n	G_c	$\mathbb{Z}_2$	$\mathbb{Z}_2$	1
D_{2k}	G_c	$\mathbb{Z}_2 + \mathbb{Z}_2$	$\mathbb{Z}_2 + \mathbb{Z}_2$	1
D_{2k-1}	SO_{4k-2}	$\mathbb{Z}_2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$
E_6	G_{ad}	1	$\mathbb{Z}_3$	$\mathbb{Z}_3$
E_7	G_c	$\mathbb{Z}_2$	$\mathbb{Z}_2$	1
E_8, F_4, G_2	G_c	1	1	1

Note that $Z_0^0 = U_\varepsilon^{0\,T_{(l)}}$; hence U_ε^{0S} is a subalgebra of Z_0^0 . Consider the following subalgebra of Z_0 (which is independent of the choice of the reduced

expression J of w_0),

$$Z_0' = U_\varepsilon^{0S}[x_k^J, y_k^J],$$

and let $\Omega_M' = \operatorname{Spec} Z_0'$. The embedding $Z_0' \subset Z_0$ induces an unramified cover $\Omega_M \to \Omega_M'$ with the Galois group $Q_2^\vee$. Recall that the map $\Omega_M \to G^0$ is an unramified cover with the Galois group $T_{(2)}$. Also the homomorphism $G' \to G$ is an unramified cover with the Galois group $C_G/C_G^2 \simeq T_{(2)}/Q_2^\vee$. It follows that there exists a unique isomorphism of algebraic varieties $\tau: \Omega_M' \overset{\sim}{\to} G'^0$, the big cell of G', which makes the diagram

$$
\begin{array}{ccc}
\Omega_M & \xrightarrow{\;Q_2^\vee\;} & \Omega_M' \\
{\scriptstyle T_{(2)}}\Big\downarrow & & \Big\downarrow \\
G^0 & \xleftarrow{\;T_{(2)}/Q_2^\vee\;} & G'^0
\end{array}
$$

commutative. The map τ induces an embedding $\mathbb{C}[G'] \subset Z_0'$.

Proposition. (a) $Z_0 \cap Z_1 = \mathbb{C}[G']^{G'}$, where G' acts on itself by conjugation.

   (b) $Z_0 \cap Z_1 \subset Z_0'$.

   (c) $\overline{h}_\varepsilon(U_\varepsilon^{0\widetilde{W}_\varepsilon \ltimes T_{(l)}}) = Z_0 \cap Z_1$.

   (d) *The map* $\operatorname{Spec} Z_0' \to \operatorname{Spec} Z_0 \cap Z_1$ *induced by inclusion is the restriction to the big cell of the quotient morphism* $G' \to \operatorname{Spec} \mathbb{C}[G']^{G'}$.

Proof. In order to prove (a) note that by the construction of Z_1, $h_\varepsilon(Z_0 \cap Z_1) \subset Z_0'$, and also $h_\varepsilon(Z_1) = U_\varepsilon^{0\widetilde{W}}$. So, if $a \in Z_0 \cap Z_1$ then $h_\varepsilon(a) \in Z_0' \cap U_\varepsilon^{0\widetilde{W}}$, hence there exists a unique $\overline{a} \in \mathbb{C}[G']^{G'} \subset Z_0'$ such that $h_\varepsilon(\overline{a}) = h_\varepsilon(a)$. (Here we use the well-known restriction isomorphism $\mathbb{C}[G']^{G'} \overset{\sim}{\to} \mathbb{C}[T']^W$, see [S].) Since, by Proposition 6.1(a), both a and $\overline{a}$ are $\widetilde{G}$-invariant, applying Lemma 6.2, we get $a = \overline{a}$. So, $Z_0 \cap Z_1 \subset \mathbb{C}[G']^{G'}$. Conversely, if $a \in \mathbb{C}[G']^{G'}$ then $h_\varepsilon(a) \in (Z_0^0 \cap Z_0')^W = Z_0^{0\widetilde{W}}$. But again, there exists $\overline{a} \in Z_1$ such that $h_\varepsilon(\overline{a}) = h_\varepsilon(a)$, and by Lemma 6.2 we get $a = \overline{a} \in Z_1$, hence $\mathbb{C}[G']^{G'} \subset Z_0' \cap Z_1$. (b) is clear since $Z_0 \cap Z_1 = \mathbb{C}[G']^{G'} \subset Z_0'$. (c) and (d) follow immediately from (b) and the definition of Z_0'. $\square$

Corollary. *The map* $\operatorname{Spec} Z_0' \to \operatorname{Spec} Z_0 \cap Z_1$ *induced by the inclusion is smooth in codimension* 1.

Proof. Due to Proposition 6.3(d), it suffices to show that the map $\varphi': G' \to \operatorname{Spec} \mathbb{C}[G']^G$ is smooth in codimension 1. For this recall that the map $\varphi: G_c \to \operatorname{Spec} \mathbb{C}[G_c]^{G_c} \simeq \mathbb{C}^n$, given by $g \mapsto (\chi_1(g), \dots, \chi_n(g))$, where the χ_i are the characters of fundamental representations of G_c, is smooth in codimension 1 [S, §§5, 8]. Since φ' is obtained from φ by dividing by C_G^2 it suffices to

show that the quotient map of the action of C_G^2 on $\operatorname{Spec}\mathbb{C}[G_c]^{G_c}$ is smooth in codimension 1. Note that C_G^2 is a cyclic group (see Table 1). Let s denote its order and let g be its generator. Let η be a primitive sth root of 1; then $g\chi_j = \eta^{s_j}\chi_j$. If $s = 1$, there is nothing to prove. Otherwise, by a case-wise inspection we see that there are at least two s_j which are relatively prime to s. This completes the proof. $\square$

6.4. Our next objective is to prove that $Z_{P,\varepsilon}$ is generated by its subalgebras Z_0 and Z_1. (Remark 6.4(a) below shows that this may be false if $M \neq P$.) For this we shall prove the

Proposition. *The ring* $A := \mathbb{C}[P]^W$ *is a complete intersection over its subring* $A_l := \mathbb{C}[lP]^W$.

First, we prove a lemma. Let $P_+ = \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+,\ i = 1, \ldots, n\}$, $P_+^l = \{\lambda \in P_+ \mid \langle \lambda, \alpha_i^\vee \rangle < l,\ i = 1, \ldots, n\}$. For $\lambda \in P_+$ let $\chi_\lambda = \sum_{\mu \in W(\lambda)} e^\mu$. Then $\{\chi_\lambda\}_{\lambda \in P_+}$ (resp. $\{\chi_{l\lambda}\}_{\lambda \in P_+}$) form a $\mathbb{C}$-basis of A (resp. A_l) and $\{\chi_\lambda\}_{\lambda \in P_+^l}$ form a basis of the A_l-module A. For $\lambda = \sum n_i \omega_i \in P_+$ let $M_\lambda = \prod_i \chi_{\omega_i}^{n_i}$. Finally, define a partial ordering on P_+ by letting $\lambda \geq \mu$ if $\lambda - \mu = \sum_i a_i \alpha_i$ with $a_i \in \mathbb{Q}$, $a_i \geq 0$.

Lemma. *Let* $\lambda \in P_+$. (a)*We can write* $\chi_\lambda = \sum_{\mu \in P_+^l} a_{\lambda\mu}\chi_\mu$, *where* $a_{\lambda\mu} \in A_l$ *and* $a_{\lambda\mu} \neq 0$ *only if* $\lambda \geq \mu$.

(b) *We can write* $M_\lambda = \sum_{\mu \in P_+^l} b_{\lambda\mu}\chi_\mu$, *where* $b_{\lambda\mu} \in A_l$ *and* $b_{\lambda\mu} \neq 0$ *only if* $\lambda \geq \mu$. *Also,* $b_{\lambda\lambda} = 1$ *if* $\lambda \in P_+^l$.

(c) *We can write* $M_\lambda = \sum_{\mu \in P_+^l} c_{\lambda\mu}M_\mu$, *where* $c_{\lambda\mu} \in A_l$ *and* $c_{\lambda\mu} \neq 0$ *only if* $\lambda \geq \mu$.

Proof. Write $\lambda = l\lambda' + \lambda''$, where $\lambda' \in P_+$, $\lambda'' \in P_+^l$. Then $\lambda \geq \lambda''$ and

$$\chi_\lambda = \chi_{l\lambda'}\chi_{\lambda''} + \sum_{\mu < \lambda} a'_{\lambda\mu}\chi_\mu, \quad \text{where } a'_{\lambda\mu} \in \mathbb{Z}.$$

Thus (a) follows by induction on the ordering.
Since

$$M_\lambda = \chi_\lambda + \sum_{\mu < \lambda} b'_{\lambda\mu}\chi_\mu, \qquad b'_{\lambda\mu} \in \mathbb{Z},$$

the first part of (b) follows by induction on the ordering and (a). The second part of (b) is clear.

Thus, the matrix $B := (b_{\lambda\mu})$ in (b) (where $\lambda, \mu \in P_+^l$) is invertible over A_l and $B^{-1} = (c'_{\lambda\mu})$ has the same properties, i.e., we have for $\lambda \in P_+^l$

$$\chi_\lambda = \sum_{\mu \leq \lambda} c''_{\lambda\mu}M_\mu, \qquad c''_{\lambda\mu} \in A_l,\ c''_{\lambda\lambda} = 1.$$

Substituting this in (b) we get (c). $\square$

Proof of Proposition 6.4. By Lemma 6.4(c) we have

$$M_{\omega_i}^l = \sum_{\mu \in P_+', \; \mu < l\omega_i} d_{i\mu} M_\mu, \qquad d_{i\mu} \in A_l.$$

Consider the polynomial algebra $A_l[x_1, \ldots, x_n]$ let $x_\lambda = x_1^{k_1} \cdots x_n^{k_n}$ for $\lambda = \sum k_i \omega_i$. Let $P_i = x_i^l - \sum_\mu d_{i\mu} x_\mu \in A_l[x_1, \ldots, x_n]$, let $I = (P_1, \ldots, P_n)$ and let $B = A_l[x_1, \ldots, x_n]/I$. We have a surjective homomorphism $B \to A$ defined by $x_i \mapsto \chi_{\omega_i}$. We claim that it is injective. This will prove the proposition.

In order to prove the injectivity, it suffices to show that the x_λ, $\lambda \in P_+^l$, span B. Take $\lambda \in P_+$ and consider the monomial $x_\lambda = x_1^{k_1} \cdots x_n^{k_n}$. If $k_i < l$ for all i, we are done; if not, then $k_j > l$ for some j and we have

$$x_\lambda = x_j^l x_{\lambda - l\omega_i} = \sum_{\mu < l\omega_j, \; \mu \in P_+^l} d_{j\mu} x_{\lambda - l\omega_j + \mu}.$$

Since $\lambda - l\omega_j + \mu < \lambda$, the proof is completed by induction on the ordering. $\square$

Theorem. (a) $Z_{P,\varepsilon}$ *is generated by* Z_0 *and* Z_1.

(b) $Z_{P,\varepsilon}$ *is an integrally closed complete intersection ring (over* Z_0).

Proof. Let $\widetilde{Z}_{P,\varepsilon} = Z_0 \otimes_{Z_0 \cap Z_1} Z_1$. We have natural morphisms $Z_0 \hookrightarrow \widetilde{Z}_{P,\varepsilon} \to Z_{P,\varepsilon}$. In order to prove (a), it suffices to show that $\widetilde{Z}_{P,\varepsilon}$ is normal. Indeed, since $Z_{P,\varepsilon}$ is finite over Z_0 [DCK], the normality of $\widetilde{Z}_{P,\varepsilon}$ implies that the map $\widetilde{Z}_{P,\varepsilon} \to Z_{P,\varepsilon}$ is injective. From [DCK, §5.3] it follows that this is a birational inclusion, hence an isomorphism since $Z_{P,\varepsilon}$ is normal [DCK].

Recall that

$$(1) \qquad Z_1 \simeq \mathbb{C}[P]^{\widetilde{W}} = \mathbb{C}[2P]^W, \qquad Z_0 \cap Z_1 = \mathbb{C}[lP]^{\widetilde{W}} = \mathbb{C}[2lP]^{\widetilde{W}}.$$

Hence by Proposition 6.4, $\widetilde{Z}_{P,\varepsilon}$ is a complete intersection ring over Z_0.

According to Serre's theorem [Se, Chapter 4], a complete intersection variety, which is smooth outside of a subvariety of codimension 2, is normal. Thus, in order to establish normality of $\widetilde{Z}_{P,\varepsilon}$, it suffices to show that $\mathrm{Spec}\,\widetilde{Z}_{P,\varepsilon}$ is smooth outside of a subvariety of codimension 2.

It follows from (1) that $\mathrm{Spec}\,\widetilde{Z}_{P,\varepsilon}$ is an open subset in an unramified covering of the fiber product

$$X_G := T/W \times_{T/W} G,$$

where the first map $\rho_l : T/W \to T/W$ is induced by the lth power map $(t \mapsto t^l)$ and the second map $\sigma : G \to T/W$ is the quotient map.

By [S, §§5, 8], $G = G_1 \cup G_2$, where G_1 is open, G_2 is a closed subvariety of codimension ≥ 2, and $\sigma : G_1 \to T/W$ is a smooth map (here we use that G is simply connected). Hence $T/W \times_{T/W} G_1 \to T/W$ is a smooth map, and since T/W is smooth, we obtain that $T/W \times_{T/W} G_1$ is smooth. Since the map

$T/W \to T/W$ is finite, we obtain that $T/W \times_{T/W} G_2$ has codimension ≥ 2 in $T/W \times_{T/W} G$, completing the proof.

(b) is clear from the previous discussion. □

Remarks. (a) If $M \neq P$, the conclusion of Theorem 6.4(a) is false. For example, if $U_{Q,\varepsilon}$ is the (adjoint) quantum group of type A_2 and ε is a $3d$ root of 1, then the element $K_1^2 K_2$ is central but does not lie in $\widetilde{Z}_{Q,\varepsilon}$. Note also that the proof of a result similar to Theorem 6.4(a) in [KW] on the center of the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of a simple algebraic group G in characteristic p contains a gap. Our argument proves this result as well provided that the quotient map $\mathfrak{g}^* \to \mathfrak{g}^*/G$ is smooth in codimension 1.

(b) We have shown that $\mathrm{Spec}\, Z_{P,\varepsilon}$ is an unramified cover with the Galois group $\frac{1}{2} Q^\vee / Q^\vee$ of the fiber product $G//G \times_{G//G} G^0$, where the first map is the lth power map and the second is the restriction of the quotient map.

(c) The proof of normality of $\mathrm{Spec}\, Z_\varepsilon$ given in [DCK] may be simplified. Namely, it is easy to prove the following fact (cf. [DCK, Proposition 1.8]): Let C be an S-filtered algebra (where S is an ordered set such that any decreasing sequence stabilizes) such that $Gr\, C$ has no zero divisors and is integrally closed. Then C has these properties as well. (Indeed, it is standard that C has no zero divisors. Furthermore, if $C \subset B \subset z^{-1}C$, where B is a subring and z is a central element of C, then $\bar{z}^{-1} Gr\, zB$ is a subring between $Gr\, C$ and $\bar{z}^{-1} Gr\, C$, hence coincides with $Gr\, C$. Take $x \in B \backslash C$ and let $y = zx$. Since $\bar{y} = \overline{zx_1}$ for some $x_1 \in C$, we obtain an element $x - x_1 \in B \backslash C$ such that $\deg z(x - x_1) < \deg y$.) Also the fact that any quasi-polynomial algebra is integrally closed is proved by the argument at the end of the proof of Proposition 1.8 of [DCK].

Corollary. (a) *The action of $\widetilde{G}$ on*

$$\mathrm{Spec}\, Z_{P,\varepsilon} = \mathrm{Spec}\, Z_1 \times_{\mathrm{Spec}\, Z_0 \cap Z_1} \mathrm{Spec}\, Z_0$$

extends from $\mathrm{Spec}\, Z_0$ by a trivial action on $\mathrm{Spec}\, Z_1$. Orbits of the action of $\widetilde{G}$ on $\mathrm{Spec}\, Z_{P,\varepsilon}$ are connected components of the preimages of orbits of $\widetilde{G}$ on Ω_P under the map $\tau \colon \mathrm{Spec}\, Z_{P,\varepsilon} \to \Omega_P$ induced by inclusion of coordinate rings.

(b) *Let $x \in \Omega_P$ and let $\overline{x} = \sigma \circ \pi(x) \in T/W$. Recall the map $\rho_l \colon T/W \to T/W$ induced by the lth power map. Then*

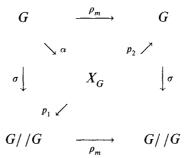$$|\tau^{-1}(x)| = |\rho_l^{-1}(\overline{x})|.$$

6.5. We shall give here another construction of the fiber product $G//G \times_{G//G} G$ (cf. Remark 6.4(b)). Let B be a finitely generated algebra without zero divisors, and let A be a finitely generated subalgebra of B. Denote by $\overline{A}$ the integral closure of A in B. This gives us a factorization of the map $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$ (induced by the inclusion),

$$\mathrm{Spec}\, B \to \mathrm{Spec}\, \overline{A} \to \mathrm{Spec}\, A,$$

such that $\mathrm{Spec}\, \overline{A}$ is normal, the first map has connected (possibly empty) fibers, and the second map is finite, called the *Stein factorization.*

Now let G be a semisimple connected affine algebraic group. Given a positive integer m, denote by $\rho_m: G \to G$ the map defined by $g \mapsto g^m$, $g \in G$. Let $G \to X'_G \to G$ be the Stein factorization of this map.

More explicitly this factorization can be constructed as follows (cf. §6.4). Denote by $G//G$ the affine variety with coordinate ring the regular functions on G invariant under conjugation, and let $\sigma: G \to G//G$ be the quotient map. The map ρ_m induces a map $\rho_m: G//G \to G//G$, so that we have a commutative diagram:

$$
\begin{array}{ccc}
G & \xrightarrow{\ \rho_m\ } & G \\
 & \searrow \alpha \qquad p_2 \nearrow & \\
\sigma \downarrow & X_G & \downarrow \sigma \\
 & p_1 \nearrow & \\
G//G & \xrightarrow[\rho_m]{} & G//G
\end{array}
$$

Here X_G denotes the fiber product $G//G \times_{G//G} G$ and p_i is the projection on the ith factor. By the universality property of the fiber product, there exists a morphism $\alpha: G \to X_G$ making the diagram commutative.

Proposition. $G \xrightarrow{\alpha} X_G \xrightarrow{p_2} G$ is the Stein factorization of $\rho_m: G \to G$, so that $X'_G \simeq X_G$.

Proof. Let T be a maximal torus; then we have the canonical isomorphism $G//G \xrightarrow{\sim} T/W$, where W is the Weyl group. Since the map $\rho_m: T \to T$ is finite, it follows that the map $\rho_m: G//G \to G//G$ is finite, hence the map $p_2: X_G \to G$ is finite. Furthermore, it is clear that the map $\alpha: G \to X_G$ is birational. It remains to show that X_G is normal. As shown in §6.4, this is the case if G is simply connected. Hence it is true for arbitrary G. $\square$

Remark. Note that if G is an arbitrary connected simply connected affine algebraic group, we denote by $\overline{G}$ its quotient by the unipotent radical, and then X_G is the fiber product of G and $X_{\overline{G}}$ over $\overline{G}$.

6.6. The orbits of the action of $\widetilde{G}$ on Ω_M are described by the following

Theorem. *Consider the unramified cover* $\pi': \Omega_M \to G'^0$, *with the Galois group* $Q_2^\vee = Q^\vee/(Q^\vee \cap 2M^*)$. *Let* $\mathscr{O}$ *be a conjugacy class of a noncentral element of* G' *and let* $\mathscr{O}_0 = \mathscr{O} \cap G'^0$. *Then* $(\pi')^{-1}(\mathscr{O}_0)$ *is an orbit of* $\widetilde{G}$ *in* Ω_M.

Proof. We know (Proposition 6.1(a)) that connected components of $(\pi')^{-1}(\mathscr{O}_0)$ are orbits. Hence it suffices to show that $(\pi')^{-1}(\mathscr{O}_0)$ is connected. For that we have to show that the composite homomorphism $\varphi: \pi_1(\mathscr{O}_0) \to Q_2^\vee$ defined by

$$
\pi_1(\mathscr{O}_0) \xrightarrow{\ i_*\ } \pi_1(G'^0) = 2M^* + Q^\vee \to \frac{2M^* + Q^\vee}{2Q^*} = Q_2^\vee
$$

is surjective. We need the following lemma.

Lemma. *Consider the big cell in the group* $SL_2(\mathbb{C})$

$$V = \left\{ \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x, y \in \mathbb{C}, \ z \in \mathbb{C}^\times \right\},$$

and let $a \in \mathbb{C}$. *Let* V_a *denote the intersection with* V *of the conjugacy class of noncentral elements of* $SL_2(\mathbb{C})$ *with trace* a. *Then the inclusion map induces a surjective homomorphism of fundamental groups* $\pi_1(V_a) \to \pi_1(V)$.

Proof. The set V_a is a hypersurface in V given by the equation $z + z^{-1} + zxy = a$, with the point $(0, 0, \pm 1)$ deleted if $a = \pm 2$. Consider the mth cover $\sigma_m \colon \operatorname{Spec} \mathbb{C}[x, y, t, t^{-1}] \to V = \operatorname{Spec} \mathbb{C}[x, y, z, z^{-1}]$ given by $(x, y, t) \mapsto (x, y, t^m)$. Note that $\sigma_m^{-1}(V_a)$ has equation $t^m + t^{-m} + t^m xy = a$, which is irreducible in $\mathbb{C}[x, y, t, t^{-1}]$. Since $\pi_1(V) = \mathbb{Z}$, this proves the lemma. $\square$

Now we complete the proof of Theorem 6.6. Since our claim holds for $\mathscr{O}$, if it holds for an orbit in its closure, it suffices to consider two cases (since the closure of $\mathscr{O}$ contains a semisimple conjugacy class):

(a) $\mathscr{O}$ is a conjugacy class of a noncentral semisimple element $\exp 2\pi i h$, $h \in \mathfrak{h}$.

(b) $\mathscr{O}$ is a conjugacy class of a nontrivial unipotent element.

In case (a) there exists a long root α such that $(\alpha^\vee | h) \notin \mathbb{Z}$, where $\alpha^\vee = 2\alpha/(\alpha | \alpha)$. Let $\gamma_\alpha \colon SL_2(\mathbb{C}) \to G'$ be the homomorphism corresponding to α. It is well known that γ_α is injective (see, e.g., [KW, Proposition 2.1]). Then $\mathscr{O} \cap \gamma_2(SL_2(\mathbb{C}))$ is a noncentral conjugacy class in $SL_2(\mathbb{C})$. Due to the lemma, this implies that the image of i^* contains $\alpha^\vee$. Using the Weyl group, we see that the image of i^* contains $\alpha^\vee$ for each long root α, hence contains $Q^\vee$, and hence the map φ is surjective. In case (b), it suffices to look at the conjugacy class $\mathscr{O}$ of $\exp e_\alpha$, where e_α is a root vector attached to a long root α, since it is well known that the closure of any nontrivial unipotent conjugacy class contains this one (since the projectivization of the orbit of e_α in $\mathfrak{g}$ is the only closed G-orbit in $\mathbb{P}\mathfrak{g}$). The same argument as that in case (a) now completes the proof. $\square$

Corollary. (a) *Two elements of* $T \backslash F$ *lie in the same orbit of* $\widetilde{G}$ *if and only if they lie in the same orbit of* $\widetilde{W}$.

(b) *The closure of a* $\widetilde{G}$-*orbit* $\mathscr{O}$ *of a nonunipotent point in* Ω_M *contains a unique closed* $\widetilde{G}$-*orbit, which we denote by* $\mathscr{O}_s$.

6.7. In this section we study the invariants of the action of $\widetilde{G}$ on Ω_M.

Theorem. (a) *The restriction homomorphism induced by inclusion* $i \colon T \to \Omega_M$ *gives an isomorphism of the algebras of invariants*

$$i^* \colon Z_0^{\widetilde{G}} \xrightarrow{\sim} Z_0^{0\widetilde{W}}.$$

(b) *The map* h_ε *induces an isomorphism* $Z_{M,\varepsilon}^{\widetilde{G}} \simeq U_\varepsilon^{0\widetilde{W}}$.

(c) $Z_1 = Z_{M,\varepsilon}^{\widetilde{G}}$.

Proof. First of all, the homomorphism $i^*: Z_0^{\widetilde{G}} \to U_\varepsilon^0$ is injective by Lemma 6.2. Furthermore, by Corollary 6.6(a), $i^*: Z_0^{\widetilde{G}} \to Z_0^{0\widetilde{W}}$. This homomorphism induces an isomorphism of the fields of fractions. Indeed, consider the composite map $\operatorname{Spec} Z_0 \to G^0 \to T/W$. Using Proposition 6.1(a) and the Stein factorization we see that there exists a Zariski open $\widetilde{G}$-invariant subset $C' \subset \operatorname{Spec} Z_0$ and a morphism of C' to some algebraic variety whose fibers consist of $\widetilde{G}$-orbits. It follows from Corollary 6.5(a) that the latter variety is birationally isomorphic to $T/\widetilde{W}$, proving the claim.

Finally, $i^*(Z_0^{\widetilde{G}}) = Z_0^{0\widetilde{W}}$. Indeed, if $p \in Z_0^{0\widetilde{W}}$, by the above, there exists Q in the field of fractions of $Z_0^{\widetilde{G}}$ such that $i^*(Q) = p$. Suppose that $Q \notin Z_0^{\widetilde{G}}$. Since Z_0 is normal, we can write $Q = p_1/p_2$ in such a way that there exists $a \in \operatorname{Spec} Z_0$ such that $p_1(a) \neq 0$, but $p_2(a) = 0$. Also, clearly, $g \cdot p_i = \varphi(g)p_i$, $g \in \widetilde{G}$, where $\varphi(g) \in Z_0^{0\times}$. Hence the set of zeros of p_2 in $\operatorname{Spec} Z_0$ is $\widetilde{G}$-invariant; let $\mathscr{O}$ be a closed $\widetilde{G}$-orbit in this set (it exists by Proposition 6.1(a) by taking an orbit of minimal dimension). By Theorem 6.1(c), $\mathscr{O} \cap T \neq \varnothing$, hence p is not regular on T, a contradiction. This proves (a).

(b) follows from (a) by the same arguments as in [KW, §5]. Finally since $Z_1 \subset Z_{M,\varepsilon}^{\widetilde{G}}$ (by Corollary 6.2(a)) and $h_\varepsilon(Z_1) = U_\varepsilon^{0\widetilde{W}}$, by Lemma 6.2 we conclude that $Z_1 = Z_{M,\varepsilon}^{\widetilde{G}}$, proving (c). $\square$

Remark. (a) This theorem may be used to write down explicit equations for the z_φ. Consider the homomorphism $f_l: U_\varepsilon^0 \to U_\varepsilon^0$ given by $f_l(K_\beta) = K_{l\beta}$ and recall the identification of T with $\operatorname{Spec} U_\varepsilon^0$. It is easy to see that for each $\varphi \in U_\varepsilon^0$ there exists a unique $\tilde{\varphi}_1 \in U_\varepsilon^0$ such that

$$\tilde{\varphi} := \prod_{g \in T_l} (\varphi - \varphi(g)) = \tilde{\varphi}_1 \circ f_l.$$

If $\varphi \in U_\varepsilon^{0\widetilde{W}}$, it follows from the theorem that $z_{\tilde{\varphi}_1} \in Z_0$ and that z_φ satisfies

$$\prod_{g \in T_l} (z_\varphi - \gamma(\varphi)(g)) = z_{\tilde{\varphi}_1}.$$

Note that in the case of $U_q(sl_2)$ this equation appears in [DCK, §4].

(b) It follows from Theorem 6.7 that $Z_1 \cap Z_{P,0}$ is a polynomial algebra on n generators $Q_1, \dots, Q_n$ constructed as follows. Let σ_j be the jth fundamental representation of the group G_P in a vector space V_j. Then $Q_j(a) = \operatorname{tr}_{V_j} \sigma_j(\pi(a))$, $a \in \operatorname{Spec} Z_{P,0}$.

6.8. In conclusion of this section, we present a conjecture (similar to that in [WK] on representations of Lie algebras in characteristic p) and state a result (similar to Theorem 2 from [WK]) that confirms this conjecture.

Let σ be a finite-dimensional irreducible representation of the quantum group $U_{M,\varepsilon}$ in a vector space V. Let $\chi_\sigma: Z_0 \to \mathbb{C}$ be the corresponding point of Ω_M, and let $\mathscr{O}$ be the $\widetilde{G}$-orbit of χ_σ.

Conjecture. dim V *is divisible by* $l^{(\dim \mathscr{O})/2}$. *In particular,* $\dim V = l^N$ *if* $\mathscr{O}$ *is an orbit of maximal dimension.*

Let $g \in \widetilde{G}$ be such that $\chi := g\chi_\sigma$ has the property that $\chi(I^+) = 0$ (see Theorem 6.1(a)). Define $\chi_s \in \Omega_M$ by $\chi_s(z_\beta) = \chi(z_\beta)$, $\beta \in M$, and $\chi_s(I) = 0$, so that $\pi(\chi_s) \in T$ (and $\mathscr{O}_s = \widetilde{G} \cdot \chi_s$). Let R^σ be the set of $\alpha \in R$ that vanishes on the Lie algebra of the center of the centralizer of $\pi(\chi_s)$. We call σ a representation of *parabolic type* if $R^\sigma \neq R$ (i.e., the center of the centralizer of $\pi(\chi_s)$ in G is infinite). In this case we may assume that $g \in \widetilde{G}$ is such that $R^\sigma := \mathbb{Z}\Pi' \cap R$ where Π' is a subset of Π different from Π. Let $M' = \mathbb{Z}\Pi'$ and let $U'_{M', \varepsilon}$ denote the subalgebra of $U_{M, \varepsilon}$ generated by U^0_ε and the E_i and F_i such that $\alpha_i \in \Pi'$, and let $U^\sigma = U'_{M', \varepsilon} U^+_\varepsilon$ be the corresponding "parabolic" subalgebra. The proof of the following result is essentially the same as that of Theorem 2 from [WK]:

Theorem [DCK1]. *Let σ be an irreducible representation of parabolic type of $U_{M, \varepsilon}$ in a finite-dimensional vector space V. Then V contains a unique irreducible U^σ-submodule V' (which is in fact a U'_ε-module), and the $U_{M, \varepsilon}$-module V is induced from the U^σ-module V'. In particular, $\dim V = l^{t/2} \dim V'$, where $t = |R \backslash R^\sigma|$.*

Remarks. (a) The irreducible $U'_{M, \varepsilon}$-module V' is a representation of nonparabolic type. Thus, Theorem 6.8 reduces the representation theory of $U_{M, \varepsilon}$ to the study of representations of nonparabolic type.

(b) If σ is not of parabolic type, then $\pi(\chi_s)$ is conjugate in G to an element of the form $\sigma_m u$, where u is a unipotent element of G commuting with σ_m and σ_m is a finite order element of G such that $(\mathrm{Ad}\,\sigma_m)e_j = e_j$ and $(\mathrm{Ad}\,\sigma_m)f_j = f_j$ for $m \neq j$, $(\mathrm{Ad}\,\sigma_m)e_m = (\exp(2\pi i/a_m))e_m$, $(\mathrm{Ad}\,\sigma_m)f_m = (\exp(-2\pi i/a_m))f_m$. Here $\sum_{j=1}^n a_j\alpha_j$ is the decomposition of the highest root and $m = 0, 1, \ldots, n$.

(c) In the A_n case (and only in that case) all nonparabolic representations are unipotent (i.e., χ_s is unipotent).

(d) If $\chi(\sigma) \notin F$, then $\dim V$ is divisible by l^n. Indeed, using $\widetilde{G}$, we can make all $x_i(\chi(\sigma))$ nonzero and hence all the E_i invertible. Then all weight spaces with respect to U^0 have the same dimension.

7. Ω_M AS A POISSON ALGEBRAIC GROUP

7.1. Recall that a *Poisson bracket* on a commutative algebra A is a bilinear map

$$\{\ ,\ \}: A \wedge A \to A$$

that satisfies the Jacobi identity and for which $\{ab, c\} = a\{b, c\} + \{a, c\}b$.

Given two commutative algebras with a Poisson bracket, A and B, one defines a Poisson bracket on $A \otimes B$ by

$$\{a_1 \otimes b_1, a_2 \otimes b_2\} = \{a_1, a_2\} \otimes b_1 b_2 + a_1 a_2 \otimes \{b_1, b_2\}.$$

A commutative Hopf algebra A with comultiplication Δ, antipode S, and counit η is called a *Poisson Hopf algebra* if $\Delta \colon A \to A \otimes A$ and $\eta \colon A \to k$ are homomorphisms of Poisson algebras.

Let $\Delta' = p \circ \Delta$, where $p \colon A \otimes A \to A \otimes A$ is the permutation map, and let

$$\delta = \Delta - \Delta' \colon A \to A \wedge A.$$

Let $\mathfrak{m} = \operatorname{Ker} \eta$. Then $\delta(\mathfrak{m}) \subset \mathfrak{m} \wedge \mathfrak{m}$ since

$$\delta(a) = \sum_i ((a_i - \eta(a_i)) \otimes (b_i - \eta(b_i))) - ((b_i - \eta(b_i)) \otimes (a_i - \eta(a_i))),$$

where $\Delta(a) = \sum_i a_i \otimes b_i$. It follows that $\delta(\mathfrak{m}^2) \subset \mathfrak{m}^2 \wedge \mathfrak{m}^2$, hence δ induces a map $\mathfrak{m}/\mathfrak{m}^2 \to (\mathfrak{m}/\mathfrak{m}^2) \wedge (\mathfrak{m}/\mathfrak{m}^2)$, which is again denoted by δ. This is a Lie comultiplication map, i.e., the contragredient map defines a Lie algebra structure on $(\mathfrak{m}/\mathfrak{m}^2)^*$.

On the other hand, $\{\mathfrak{m}, \mathfrak{m}\} \subset \mathfrak{m}$, hence $\{\mathfrak{m}^2, \mathfrak{m}^2\} \subset \mathfrak{m}^2$. Thus the Poisson bracket on A induces a Lie bracket on $\mathfrak{m}/\mathfrak{m}^2$. The property that Δ is a homomorphism of Poisson algebras implies that δ is a 1-cocycle for the Lie algebra $\mathfrak{m}/\mathfrak{m}^2$ acting in the usual way on $(\mathfrak{m}/\mathfrak{m}^2) \wedge (\mathfrak{m}/\mathfrak{m}^2)$.

Thus we obtain a Lie bialgebra $L(A) = (\mathfrak{m}/\mathfrak{m}^2)^*$ associated to the Poisson Hopf algebra A (cf. [D]).

If A is a finitely generated commutative Poisson Hopf algebra, then $\operatorname{Spec} A$ is an algebraic group called a *Poisson algebraic group*. Then $L(A)$ is called the Lie bialgebra of $\operatorname{Spec} A$ (these notions were introduced by Drinfeld [D] in the category of Lie groups).

7.2. We shall need the following simple

Lemma. *Let A and B be two commutative Poisson Hopf algebras, and let $\varphi \colon A \to B$ be an algebra isomorphism. Let $a_1, \ldots, a_s$ be a set of Poisson generators of A (i.e., A equals the smallest Poisson subalgebra containing all the a_i). Suppose that*

  (i) $\varphi(\eta_A(a))k = \eta_B(\varphi(a))$, $a \in A$;
  (ii) $\varphi \otimes \varphi(\Delta_A(a_i)) = \Delta_B \varphi(a_i)$, $i = 1, \ldots, s$;
  (iii) $\{\varphi(a_i), \varphi(a)\} = \varphi(\{a_i, a\})$, $i = 1, \ldots, s$, $a \in A$.

Then φ is an isomorphism of Poisson Hopf algebras.

Proof. Due to the Jacobi identity, (iii) implies that φ is a Poisson algebra isomorphism. Now, due to the compatibility of Δ and η with $\{\,,\,\}$, conditions (i) and (ii) imply that φ is a Hopf algebra isomorphism. $\square$

7.3. Recall (see §5) that $Z_{0,M}$ is a finitely generated commutative Hopf algebra. We give it a structure of a Poisson Hopf algebra by a usual formula (the choice of its normalization will become clear later)

$$\{a, b\} = (\hat{a}\hat{b} - \hat{b}\hat{a})/l(q^l - q^{-l}) \pmod{q - \varepsilon}.$$

Here $\hat{a}$ stands for a preimage of $a \in Z_0$ in $U_{M,\mathscr{A}}$ under the canonical homomorphism $\varphi_\varepsilon \colon U_{M,\mathscr{A}} \to U_{M,\varepsilon}$ (Δ and η are Poisson algebra homomorphisms

since $\Delta\colon U_{M,\mathscr{A}} \to U_{M,\mathscr{A}} \otimes U_{M,\mathscr{A}}$ and $\eta\colon U_{M,\mathscr{A}} \to \mathscr{A}$ are algebra homomorphisms).

Remark. (a) Let $N_\varepsilon = \varphi_\varepsilon^{-1}(Z_\varepsilon)$. Given $u \in N_\varepsilon$ and $v \in U_\varepsilon$, let

$$P_u(v) = (u\hat{v} - \hat{v}u)/(q - \varepsilon) \quad \mathrm{mod}(q - \varepsilon).$$

The map $u \to P_u$ defines a representation of $\overline{N}_\varepsilon := N_\varepsilon/(q - \varepsilon)^2 U_\varepsilon$ (viewed as a Lie algebra) by derivations of U_ε. (Restricting to Z_ε we recover, up to a constant factor, the Poisson bracket.) We have the following exact sequence of Lie algebras

$$0 \to U_\varepsilon \xrightarrow{q-\varepsilon} \overline{N}_\varepsilon \xrightarrow{\varphi_\varepsilon} Z_\varepsilon \to 0.$$

(b) Denote by $\widetilde{\widetilde{G}}$ the group generated by all convergent series $\exp D$, $D \in \overline{N}_\varepsilon$. Since the orbits of $\widetilde{\widetilde{G}}$ in Ω_M are connected, it follows from Proposition 6.1(a) that the orbit decompositions for $\widetilde{G}$ and $\widetilde{\widetilde{G}}$ in Ω_M are the same.

7.4. Here we recall (in a convenient to us form) an important example of a Poisson Hopf algebra (cf. [STS, D, LuR] and references therein). In the next sections we will show that this Poisson Hopf algebra is isomorphic to the Poisson Hopf algebra $Z_{0,M}$ considered in §7.3.

We keep the notation of §4.1. In particular, $\mathfrak{g}$ is a simple Lie algebra corresponding to the Cartan matrix (a_{ij}), G is a connected algebraic group with Lie algebra $\mathfrak{g}$ and such that the group of characters of a maximal torus T is M, U_+, and U_- are unipotent subgroups of G corresponding to positive and negative roots, so that $G^0 = U_- T U_+$ is the big cell of G, etc. We normalize an invariant symmetric bilinear form $(\cdot | \cdot)$ on $\mathfrak{g}$ by the condition that it induces the bilinear form $(\cdot | \cdot)$ on Q defined in §0.2 (i.e., the square length of a short root in 2). When restricted to a subalgebra $\mathbb{C}e_i + \mathbb{C}f_i + \mathbb{C}h_i$, this becomes the usual trace form on $\mathrm{sl}_2(\mathbb{C})$ multiplied by d_i^{-1}.

Consider the group $G \times G$ and the following two subgroups

$$\widetilde{\Omega} = \{(t^{-1}u_-, tu_+) \,|\, t \in T, \; u_\pm \in U_\pm\}, \qquad K = \{(g, g) \,|\, g \in G\}.$$

Consider the (nondegenerate invariant) bilinear form on $\mathfrak{g} \oplus \mathfrak{g} = \mathrm{Lie}(G \times G)$

$$((x_1, x_2) \,|\, (y_1, y_2)) = -(x_1 \,|\, y_1) + (x_2 \,|\, y_2).$$

With respect to this form, $\mathfrak{g} \oplus \mathfrak{g}$, $\mathrm{Lie}\,\widetilde{\Omega}$, and $\mathrm{Lie}\,K$ form a Manin triple, i.e., $\mathrm{Lie}\,\widetilde{\Omega}$ and $\mathrm{Lie}\,K$ are isotropic subalgebras whose sum is $\mathfrak{g} \oplus \mathfrak{g}$. This allows us to identify $(\mathrm{Lie}\,\widetilde{\Omega})^*$ with $\mathrm{Lie}\,K$.

Consider the map $\tilde{\pi}\colon \widetilde{\Omega} \to G$ defined by $\tilde{\pi}(a, b) = a^{-1}b$. Note that $\tilde{\pi}$ is an unramified cover of the big cell G^0 of G.

Given an affine algebraic manifold X we denote by $\mathbb{C}[X]$ (resp. $\mathrm{Vect}\,X$, resp. $\mathscr{D}X$) the space of regular functions (resp. vector fields, resp. differential 1-forms) on X. To define a Poisson bracket $\{\ ,\ \}$ on $\mathbb{C}[X]$ is equivalent to defining a homomorphism of $\mathbb{C}[X]$-modules $\tau\colon \mathscr{D}X \to \mathrm{Vect}\,X$ (satisfying certain conditions), so that $\{f, g\} = \langle \tau(df), dg \rangle$.

We define the map $\tau\colon \mathscr{D}\tilde{\Omega} \to \mathrm{Vect}\,\tilde{\Omega}$ as follows. We identify $(\mathrm{Lie}\,\tilde{\Omega})^*$ with the space of left-invariant 1-forms on $\tilde{\Omega}$. On the other hand, the adjoint action of G on itself gives an embedding $\mathrm{Lie}\,G \subset \mathrm{Vect}\,G$. This gives a linear map $\tau_0\colon (\mathrm{Lie}\,\tilde{\Omega})^* \to \mathrm{Vect}\,G$. Since $\tilde{\pi}$ is unramified we have a map $\tilde{\pi}^*\colon \mathrm{Vect}\,G \to \mathrm{Vect}\,\tilde{\Omega}$. Then τ is the homomorphism of $\mathbb{C}[\tilde{\Omega}]$-modules defined by the linear map $\tilde{\pi}^* \circ \tau_0$.

One checks as in [LuR] that τ defines a Poisson bracket on $\mathbb{C}[\tilde{\Omega}]$ making $\mathbb{C}[\tilde{\Omega}]$ a Poisson Hopf algebra. (In fact our setup is a complexification of that of [LuR].)

7.5. We keep using the notation of 4.1. Let

$$n_{\pm} = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}, \qquad n_{\pm}^{(i)} = \bigoplus_{\alpha \in R^+\setminus\{\alpha_i\}} \mathfrak{g}_{\pm\alpha},$$

so that $n_+ = n_+^{(i)} + \mathbb{C}e_i$ and $n_- = n_-^{(i)} + \mathbb{C}f_i$ $(i = 1, \ldots, n)$. We have the corresponding semidirect product of groups

$$U_+ = U_+^{(i)} \ltimes \exp\mathbb{C}e_i, \qquad U_- = U_-^{(i)} \ltimes \exp\mathbb{C}f_i.$$

This allows us to define regular functions $\tilde{x}_i$ and $\tilde{y}_i$ on U_+ and U_- respectively by letting

$$u_+ = u_+^{(i)}\exp-\tilde{x}_ie_i, \qquad u_- = u_-^{(i)}\exp\tilde{y}_if_i, \quad \text{where } u_{\pm}^{(i)} \in U_{\pm}^{(i)}.$$

Recall that $\mathbb{C}[T] = M$, i.e., any $\alpha \in M$ defines a regular function on T, which we denote by $\tilde{z}_{\alpha}$. We extend these functions to the whole $\tilde{\Omega}$ by letting $\tilde{x}_i$, $\tilde{y}_i$, and $\tilde{z}_{\alpha}$ equal on $(t^{-1}u_-, tu_+)$ to $\tilde{x}_i(u_+)$, $\tilde{y}_i(u_-)$, and $\alpha(t)$ respectively.

Define the regular automorphisms $\tilde{T}_i$ of the algebraic variety $\tilde{\Omega}$,

$$\tilde{T}_i(t^{-1}u_-^{-1}, tu_+) = (t_it^{-1}(u_-^{(i)})^{-1}(\exp\tilde{x}_ie_i)t_i^{-1}, \; t_it^{-1}(\exp\tilde{y}_if_i)t^2u_+^{(i)}t_i^{-1}),$$

where the t_i are defined in §4.1.

Finally, define a map $\varphi\colon \Omega_M \to \tilde{\Omega}_M$ as follows. Recall that $\Omega_M = \mathrm{Spec}\,Z_0 = \mathrm{Spec}\,Z_0^- \times \mathrm{Spec}\,Z_0^0 \times \mathrm{Spec}\,Z_0^+$ and that $\mathrm{Spec}\,Z_0^0 = T$. Then $\varphi = \varphi_- \times \varphi_0 \times \varphi_+$, where

$$\varphi_-(u_-) = (Y(u_-)^{-1}, 1), \qquad \varphi_+(u_+) = (1, X(u_+)), \qquad \varphi_0(t) = (t^{-1}, t).$$

Here X and Y are maps defined in §4.4. By the definitions, we have $\pi = \tilde{\pi} \circ \varphi$.

7.6. **Theorem.** (a) φ *is an isomorphism of Poisson algebraic groups.*

(b) *The* $\tilde{T}_i$ *satisfy the braid relations and the map* φ^* *is a* $\mathscr{B}$-*algebra isomorphism such that* $\varphi^*(\tilde{y}) = y_i$, $\varphi^*(\tilde{x}_i) = x_i$, $\varphi^*(\tilde{z}_{\alpha}) = z_{\alpha}$.

Proof. We apply Lemma 7.2 to the map $\varphi^*\colon \mathbb{C}[\tilde{\Omega}] \to \mathbb{C}[\Omega_M]$ and the elements $\tilde{x}_i$, $\tilde{y}_i$ $(i = 1, \ldots, n)$, $\tilde{z}_{\alpha}$ $(\alpha \in M)$. It is clear that $\Delta\tilde{z}_{\alpha} = \tilde{z}_{\alpha} \otimes \tilde{z}_{\alpha}$. Since $\exp\mathbb{C}e_i$ and T normalize $U_+^{(i)}$, it suffices to calculate Δ in the subgroup $\{(t^{-1}, t\exp-\tilde{x}_ie_i)\}$ of $\tilde{\Omega}$, which gives

$$\Delta\tilde{x}_i = 1 \otimes \tilde{x}_i + \tilde{x}_i \otimes \tilde{z}_{-\alpha_i}.$$

Similarly we have

$$\Delta \tilde{y}_i = 1 \otimes \tilde{y}_i + \tilde{y}_i \otimes \tilde{z}_{-\alpha_i} \,.$$

Thus the assumption (ii) of Lemma 7.2 holds. Since assumption (i) holds trivially, it remains to show that assumption (iii) holds. For $\tilde{a} \in \tilde{\Omega}$ (resp. $a \in \Omega_M$) let $\tilde{P}_{\tilde{a}}$ (resp. P_a) be the operators on $\mathbb{C}[\tilde{\Omega}]$ (resp. $\mathbb{C}[\Omega_M]$) defined by $\tilde{P}_{\tilde{a}}(u) = \{\tilde{a}, u\}$ (resp. $P_a(u) = \{a, u\}$). We have to show that these operators correspond to each other, i.e., that $\varphi^* \circ \tilde{P}_{\tilde{a}} = P_a \circ \varphi^*$, for $\tilde{a} = \tilde{x}_i$, $\tilde{y}_i$, and $\tilde{z}_a$.

Let $\tilde{m} = \operatorname{Ker} \eta \subset \mathbb{C}[\tilde{\Omega}]$ and let $\overline{x}_i$ be the image of $\tilde{x}_i$ in $\tilde{m}/\tilde{m}^2 = \operatorname{Lie} K$. As above, it is clear that $\overline{x}_i = c_i(f_i, f_i)$, for some $c_i \in \mathbb{C}$. Since

$$\overline{x}_i(e_i) = -1 \quad \text{and} \quad ((f_i, f_i) \mid (0, e_i)) = d_i^{-1} \,,$$

we deduce that $\overline{x}_i = -d_i(f_i, f_i)$.

Now we apply the left translation L_g to $\overline{x}_i$, $g \in \tilde{\Omega}$. Let $m_g = \{u \in \mathbb{C}[\tilde{\Omega}] \mid u(g) = 0\}$. Recall that if $u \in m$ and $\Delta u = \sum_i a_i \otimes b_i$, then $L_g \overline{u} = \sum a_i(g^{-1}) b_i \mod m_g^2$. Hence

$$L_g \overline{x}_i = \tilde{x}_i + \tilde{x}_i(g^{-1}) \tilde{z}_i^{-1} \in m_g \,,$$

so that $\tilde{x}_i(g) + \tilde{x}_i(g^{-1}) \tilde{z}_i(g)^{-1} = 0$, and substituting in the above formula, we obtain

$$L_g \overline{x}_i = \tilde{x}_i - \tilde{z}_i^{-1}(\tilde{x}_i \tilde{z}_i)(g) = \tilde{z}_i^{-1} d(\tilde{z}_i \tilde{x}_i) \quad \mod m_g^2 \,,$$

(since $du = u - u(g) \mod m_g^2$). We conclude that

$$\tilde{P}_{\tilde{z}_i \tilde{x}_i} = -d_i z_i(f_i, f_i) \,.$$

On the other hand, Theorem 5.4 can be rewritten as

$$P_{z_i x_i} = -d_i z_i f_i$$

(here the normalization of §7.3 is essential). This proves that $\varphi^* \circ \tilde{P}_{\tilde{z}_i \tilde{x}_i} = P_{z_i x_i} \circ \varphi$. Similarly, we prove that $\varphi^* \circ \tilde{P}_{\tilde{y}_i} = P_{y_i} \circ \varphi$ and $\varphi^* \circ \tilde{P}_{\tilde{z}_\alpha} = P_{z_\alpha} \circ \varphi$ using Theorem 5.5 and Remark 5.5(a) respectively.

In order to complete the proof of (a), it remains to show that the elements $\tilde{x}_i$, $\tilde{y}_i$, $\tilde{z}_\alpha$ generate $\mathbb{C}[\tilde{\Omega}]$ as a Poisson algebra. Note that $\mathbb{C}[\tilde{\Omega}] = \mathbb{C}[U_-] \otimes \mathbb{C}[T] \otimes \mathbb{C}[U_+]$, where $U_\pm$ and T are embedded in $\tilde{\Omega}$ in a natural way as Poisson algebraic subgroups. Hence it suffices to show that the $\tilde{x}_i$ (resp. $\tilde{y}_i$) generate $\mathbb{C}[U_+]$ (resp. $\mathbb{C}[U_-]$) as a Poisson subalgebra. For this note that $\mathbb{C}[U_+]$ is $\mathbb{Z}_+$-graded by $\deg x_i = 1$. Let $\tilde{m}_+$ be its augmentation ideal. In order to show that the $\tilde{x}_i$ generate $\mathbb{C}[U_+]$ as a Poisson subalgebra, it suffices to show that $\tilde{m}_+/\tilde{m}_+^2$ is generated as a Lie algebra by the $\overline{x}_i$. But this is clear since $\tilde{m}_+/\tilde{m}_+^2$ is the subalgebra n_+ in the Lie algebra $\tilde{m}/\tilde{m}_+^2 \simeq \mathfrak{g}$, the $\overline{x}_i$ corresponding to the e_i. This completes the proof of (a).

(b) follows from (a) and Theorem 4.7. $\square$

7.7. *Remarks.* (a) The orbits of $\widetilde{G}$ on Ω_M are precisely the symplectic leaves of the Poisson structure on Ω_M.

(b) Recall that, by Theorem 6.7(a) we have canonical isomorphisms $T/\widetilde{W} \simeq \mathrm{Spec}(Z_0^{0\widetilde{W}}) \simeq \mathrm{Spec}(Z_0^{\widetilde{G}})$. Here $Z_0^{\widetilde{G}}$ is the ring of $\widetilde{G}$-invariant regular functions on $\Omega_M = $ the ring of regular functions on Ω_M constant along the symplectic leaves. Thus, the inclusion $Z_0^{\widetilde{G}} \hookrightarrow Z_0$ gives a map $\sigma\colon \Omega_M \to T/\widetilde{W}$. On the other hand, we have the *l*th power map $\rho_l : T/\widetilde{W} \to T/\widetilde{W}$. Consider the fiber product $Y_M := T/\widetilde{W} \times_{T/\widetilde{W}} \Omega_M$. We extend the Poisson structure from Ω_M to Y_M trivially. We have the canonical map $\mathrm{Spec}\, Z_{M,\varepsilon} \to Y_M$ and due to §6.4 this map is an isomorphism if $M = P$. Thus, according to Theorem 7.6, $\mathrm{Spec}\, Z_{P,\varepsilon}$ as a Poisson variety can be constructed entirely in terms of the Poisson algebraic group $\widetilde{\Omega}_P$.

(c) For an n-tuple $\beta = (\beta_1, \ldots, \beta_n)$ of elements of M one can associate a comultiplication Δ_β of U_M by the formula

$$\Delta_\beta E_i = E_i \otimes K_{\beta_i} + K_{\alpha_i - \gamma_i} \otimes E_i,$$
$$\Delta_\beta F_i = F_i \otimes K_{-\alpha_i - \beta_i} + K_{\gamma_i} \otimes F_i,$$
$$\Delta_\beta K_\alpha = K_\alpha \otimes K_\alpha,$$

where the $\gamma_i \in M$ are defined by equations $(\gamma_i | \alpha_j) = -(\beta_j | \alpha_i)$, $i, j = 1, \ldots, n$. The Lie comultiplication on $\mathfrak{m}/\mathfrak{m}^2$ is then

$$\delta\alpha_i = 0, \qquad \delta E_i = (\alpha_i - \beta_i - \gamma_i) \wedge E_i, \qquad \delta F_i = (\alpha_i + \beta_i + \gamma_i) \wedge E_i.$$

The corresponding Manin triple is $(\mathfrak{g} \oplus \mathfrak{g}, \mathrm{Lie}\,\widetilde{\Omega}_\beta, \mathrm{Lie}\,K)$, where $\mathrm{Lie}\,\widetilde{\Omega}_\beta = (\mathfrak{n}_-, \mathfrak{n}_+) + \sum_i \mathbb{C}(-\alpha_i + \beta_i + \gamma_i, \alpha_i + \beta_i + \gamma_i)$. Theorem 7.6 holds in this more general situation.

NOTE ADDED IN PROOF

1. One may replace throughout the paper the condition $(l, d_i) = 1$ for all i by $l > d_i$ for all i (l odd). This takes care of (a_{ij}) of type G_2 and (odd) l divisible by 3. Then all results of the present paper still hold (with a little modification of their proofs). The same is true for [DCK] except that Theorem 3.2 and Corollary 3.2 should be modified. Consequently, the maximal dimension of an irreducible representation of U_ε of type G_2 in the case when 3 divides l is $l^6/27$ (instead of l^6 when 3 does not divide l; see [DCK, Theorem 3.8]).

2. The case of l divisible by 4 has been worked out recently by Jonathan Beck in his MIT thesis.

3. We can prove now the second part of Conjecture 6.8: if $\mathcal{O}$ is an orbit of maximal dimension, then $\dim V = l^N$.

REFERENCES

[BC]   P. Bala and R. W. Carter, *Classes of unipotent elements in simple algebraic groups*, II, Math. Proc. Cambridge Philos. Soc. **80** (1976), 1–18.

[D]   V. G. Drinfeld, *Quantum groups*, Proc. ICM Berkeley **1** (1986), 789–820.

[DCK]   C. De Concini and V. G. Kac, *Representations of quantum groups at roots of* 1 , Colloque Dixmier 1989, pp. 471–506; Progress in Math., vol. 92, Birkhäuser, Boston, MA, 1990.

[DCK1] ____, *Representations of quantum groups at roots of* 1: *reduction to the exceptional case*, RIMS 792 preprint, 1991.

[KP]   V. G. Kac and D. H. Peterson, *Defining relations of certain infinite-dimensional groups*, Astérisque, hors série (1985), 165–208.

[KW]   V. G. Kac and B. Yu. Weisfeiler, *Coadjoint action of a semi-simple algebraic group and the center of the enveloping algebra in characteristic* p , Indag. Math. **38** (1976), 136–151.

[K]   B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. **86** (1963), 327–402.

[L]   G. Lusztig, *Quantum groups at roots of* 1 , Geom. Dedicata **35** (1990), 89–114.

[LuR]   J.-H. Lu and T. Ratiu, *On the nonlinear convexity theorem of Kostant*, J. Amer. Math. Soc. **4** (1991), 349–363.

[STS]   M. A. Semenov-Tian-Shansky, *Dressing transformations and Poisson group actions*, Publ. RIMS **21** (1985), 1237–1260.

[Se]   J.-P. Serre, *Algebre locale. Multiplicites*, Lecture Notes in Math., vol. 11, Springer-Verlag, Berlin-New York, 1975.

[S]   R. Steinberg, *Regular elements of semi-simple algebraic groups*, Publ. Math. IHES **25** (1965), 49–80.

[WK]   B. Yu. Weisfeiler and V.G. Kac, *On irreducible representations of Lie p-algebras*, Funct. Anal. Appl. **5** (1971), 28–36.

SCUOLA NORMALE SUPERIORE, PISA, ITALY

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

DIPARTMENTO DI MATEMATICA, UNIVERSITA DEGLI STUDI DI ROMA "LA SAPIENZA", ROME, ITALY