ANNALES

DE

L'INSTITUT FOURIER

Hiroshi IRITANI

Quantum Cohomology and Periods

Tome 61, nº 7 (2011), p. 2909-2958.

<http://aif.cedram.org/item?id=AIF_2011__61_7_2909_0>

Ann. Inst. Fourier, Grenoble
61, 7 (2011) 2909-2958

QUANTUM COHOMOLOGY AND PERIODS

by Hiroshi IRITANI (*)

ABSTRACT. — In a previous paper, the author introduced an integral structure in quantum cohomology defined by the K-theory and the Gamma class and showed that it is compatible with mirror symmetry for toric orbifolds. Applying the quantum Lefschetz principle to the previous results, we find an explicit relationship between solutions to the quantum differential equation of toric complete intersections and the periods (or oscillatory integrals) of their mirrors. We describe in detail the mirror isomorphism of variations of integral Hodge structure for a mirror pair of Calabi-Yau hypersurfaces (Batyrev's mirror).

RÉSUMÉ. — Dans un précédent article, l'auteur a défini une structure entière sur la cohomologie quantique à l'aide de la K-théorie et d'une classe Gamma. Cette structure est compatible avec la symétrie miroir pour les orbifolds toriques. Le principe de Lefschetz quantique appliqué aux résultats précédents, nous donne une relation explicite entre les solutions du module différentiel quantique pour une intersection complète torique et les périodes (ou les intégrales oscillantes) de leur miroir. Nous expliquons en détail l'isomorphisme miroir pour une variation de structure de Hodge entière pour une paire miroir (au sens de Batyrev) d'hypersurfaces de Calabi-Yau.

1. Introduction

Hodge theoretic mirror symmetry is concerned with the equivalence of Hodge structures from symplectic geometry (A-model or Gromov-Witten theory) of Y and complex geometry (B-model or Kodaira-Spencer theory) of the mirror $\check{Y}$. In [37], we introduced a $\mathbb{Z}$-structure in the A-model Hodge theory in terms of the K-group and the $\widehat{\Gamma}$-class of Y. When Y is a weak Fano compact toric orbifold, we showed that this $\mathbb{Z}$-structure in the A-side is in fact mirror to the natural $\mathbb{Z}$-structure in the B-side. This was based

Keywords: quantum cohomology, mirror symmetry, Gamma class, K-theory, period, oscillatory integral, variation of Hodge structure, GKZ system, toric variety, orbifold.
Math. classification: 14N35, 14D05, 14D07, 14J33, 32G20, 53D37.
(*) The author thanks Etienne Mann who kindly provided the French translation of the abstract. This research is supported by Grant-in-Aid for Young Scientists (B) 22740042.

on the mirror theorem [15] for toric orbifolds which will be shown in joint work with Coates, Corti and Tseng and a calculation of oscillatory integrals on the B-side. In this paper we extend the previous results in [37] to the case of complete intersections in toric orbifolds.

For simplicity, we explain the case where Y is a Calabi-Yau manifold. The variation of Hodge structure on the A-side is given by the trivial holomorphic vector bundle $\mathscr{H} = H^*(Y) \times H^2(Y) \to H^2(Y)$ endowed with the flat Dubrovin connection

$$\nabla_V = d_V + V\circ_\tau, \qquad V \in H^2(Y)$$

where $V\circ_\tau$ is the quantum multiplication by V at $\tau \in H^2(Y)$. The Hodge filtration and the polarization form are given by $\mathscr{F}^p = H^{\leqslant 2(\dim Y - p)}(Y)$ and $Q(\alpha,\beta) = (2\pi\mathbf{i})^{\dim Y}\int_Y ((-1)^{\frac{\deg}{2}}\alpha)\cup\beta$ respectively. For $\mathcal{E} \in K(Y)$, we have a unique flat section $\mathfrak{s}(\mathcal{E})$ of the Dubrovin connection satisfying

$$\mathfrak{s}(\mathcal{E}) \sim (2\pi\mathbf{i})^{-\dim Y} e^{-\tau}\left(\widehat{\Gamma}_Y \cup (2\pi\mathbf{i})^{\frac{\deg}{2}}\operatorname{ch}(\mathcal{E})\right)$$

in the large radius limit, *i.e.*, as $e^{\langle\tau,d\rangle} \to 0$ for all nonzero effective classes $d \in H_2(Y;\mathbb{Z})$. The Gamma class $\widehat{\Gamma}_Y$ here plays the role of a "square root" of the Todd class (see (3.4)) so that we have $Q(\mathfrak{s}(\mathcal{E}_1),\mathfrak{s}(\mathcal{E}_2)) = \chi(\mathcal{E}_1,\mathcal{E}_2)$ by Hirzebruch-Riemann-Roch. The $\widehat{\Gamma}$-*integral structure* is defined to be the $\mathbb{Z}$-local system consisting of the flat sections $\mathfrak{s}(\mathcal{E})$, $\mathcal{E} \in K(Y)$. We call the pairing

$$\Pi(\phi,\mathcal{E}) := Q(\phi(\tau),\mathfrak{s}(\mathcal{E})(\tau))$$

of any section $\phi(\tau) \in \mathscr{H}$ with the flat section $\mathfrak{s}(\mathcal{E})$ the *A-period* of Y. Our main theorem identifies the A-periods of Y with the usual periods of the mirror $\check{Y}$.

Let $\mathcal{Y}$ be a quasi-smooth Calabi-Yau hypersurface in a weak Fano Gorenstein toric orbifold $\mathcal{X}$. Here we allow $\mathcal{Y}$ to have orbifold singularities. Let $\Delta \subset \mathbf{N}_\mathbb{R}$ be the fan polytope of $\mathcal{X}$. The *Batyrev mirror* of $\mathcal{Y}$ is the hypersurface $\check{Y}_\alpha = \{W_\alpha(t) = 1\}$ in the algebraic torus $\check{\mathbb{T}} = \operatorname{Hom}(\mathbf{N},\mathbb{C}^\times) \cong (\mathbb{C}^\times)^n$ defined by the Laurent polynomial $W_\alpha(t) = \sum_{b\in\Delta\cap\mathbf{N}} \alpha_b t^b$ on $\check{\mathbb{T}}$. The affine hypersurface $\check{Y}_\alpha$ can be compactified to a Calabi-Yau orbifold $\check{\mathcal{Y}}_\alpha$.

Theorem 1.1 (Theorems 5.7, 6.9, 6.10). — *The A-period for $\mathcal{Y}$ associated to $\mathcal{E} \in K(\mathcal{Y})$ can be written as a period of $\check{Y}_\alpha$ for some integral cycle $C_\mathcal{E}$ if either $\mathcal{E}$ is pulled-back from the ambient toric orbifold $\mathcal{X}$ or $\mathcal{E} = \mathcal{O}_{\mathrm{pt}}$:*
(1.1)
$$\Pi(\Upsilon_v,\mathcal{E})(\varsigma(\alpha)) = \int_{C_\mathcal{E}} (-1)^{\operatorname{age}(v)}\operatorname{age}(v)!\operatorname{Res}\left(\frac{\alpha^v t^v \frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_n}{t_n}}{(W_\alpha(t)-1)^{\operatorname{age}(v)+1}}\right).$$

Here Υ_v is a section of $\mathscr{H}$ which (see (4.6)) is asymptotically the same in the large radius limit as the unit class $\mathbf{1}_v$ on the twisted sector associated to $v \in \mathrm{Box}$ and $\varsigma(\alpha)$ is the mirror map.

We calculate the left-hand side of (1.1) as explicit hypergeometric series (Theorem 4.6) by applying the quantum Lefschetz principle [17, 16] to the mirror theorem [15] for toric orbifolds. Theorem 1.1 then follows from the Laplace transformation of the previous results in [37]. Similar results for toric complete intersections are given in Theorem 5.7. We use Theorem 1.1 to establish the mirror isomorphism between the *ambient A-model VHS* of $\mathcal{Y}$ and the *residual B-model VHS* of $\check{\mathcal{Y}}_\alpha$ which preserves certain integral structures (Theorem 6.9).

The present work is motivated by Givental's celebrated paper [25] on mirror symmetry for toric complete intersections, where Givental remarked that each component of the I-function can be written as an oscillatory integral. In terms of a hypergeometric differential system, essentially the same integral structure has been identified in the work of Borisov-Horja [9] and Hosono [33]. The $\widehat{\Gamma}$-structure was also proposed by Katzarkov-Kontsevich-Pantev [40] independently. Our results give a partial affirmative answer to the conjecture of Hosono [33, Conjecture 6.3].

The concept of orbifold has been a rich source of ideas in mirror symmetry. For example, Batyrev's mirror may not admit a full crepant resolution for dimension bigger than 3. By the development of orbifold Gromov-Witten theory [14, 13, 1], we can now work with partial resolutions with orbifold singularities. In this paper, we encounter a phenomenon of *multi-generation*[(1)] of orbifold quantum D-modules. This phenomenon was first observed by Guest-Sakai [29] (in a different language) for a degree 3 Fano hypersurface in $\mathbb{P}(1,1,1,2)$. For an orbifold hypersurface, it can happen that the ambient part[(2)] of the small quantum D-module is not generated by the single unit class $\mathbf{1}$ as an $\mathcal{O}[z]\langle z\partial\rangle$-module[(3)], but is generated by $\mathbf{1}$ and the unit classes $\mathbf{1}_v$ supported on twisted sectors. Here ∂ denotes the derivative in the $H^{\leqslant 2}_{\mathrm{orb}}$-direction and z is an additional variable in the quantum D-module (see Definition 3.1). For the A-model VHS of a Calabi-Yau hypersurface, this means that each Hodge filter $\mathscr{F}^p$ may not be generated by $\leqslant (\dim\mathcal{Y} - p)$ times derivatives of the top filter $\mathscr{F}^{\dim\mathcal{Y}}$. In fact, we

(1) This does *not* mean that the quantum D-module is not cyclic.

(2) The ambient part is the subbundle of the quantum D-module with fiber $\iota^* H^*_{\mathrm{orb}}(\mathcal{X}) \subset H_{\mathrm{orb}}(\mathcal{Y})$ where $\iota\colon \mathcal{Y} \to \mathcal{X}$ is the inclusion of the hypersurface $\mathcal{Y}$ into the ambient toric orbifold $\mathcal{X}$.

(3) On the other hand, when z inverted, it is generated by $\mathbf{1}$ as an $\mathcal{O}[z,z^{-1}]\langle z\partial\rangle$-module under the assumption on the ambient toric orbifold in this paper.

will describe the quantum D-module of toric Calabi-Yau hypersurfaces in terms of the *multi-GKZ system* (Theorem 6.13) — a GKZ system defined by multiple generators. The same generalization of the GKZ system was proposed in a recent work by Borisov-Horja [8] who called it *better behaved GKZ system*. This multi-generation is a reason why we needed to show Theorem 1.1 also for twisted sectors $v \neq 0$.

Acknowledgments. The author has learned a lot from various joint works with Alessandro Chiodo, Tom Coates, Alessio Corti, Sergey Galkin, Vasily Golyshev, Yongbin Ruan and Hsian-Hua Tseng. He would like to thank them all. He also would like to thank Yukiko Konishi and Satoshi Minabe for very helpful discussions concerning their work [44]. He is grateful to Etienne Mann and Thierry Mignon for informing the author of their work [45] and to Martin Guest for very helpful comments on a draft version of this paper.

2. Preliminaries

2.1. Orbifold Gromov-Witten Invariants

Gromov-Witten theory for orbifolds has been developed by Chen-Ruan for symplectic orbifolds and by Abramovich-Graber-Vistoli for smooth Deligne-Mumford stacks. Here we fix notation for orbifold Gromov-Witten invariants. For the details of the subject, we refer the reader to the original articles [14, 13, 1].

Let $\mathcal{X}$ be a proper smooth Deligne-Mumford stack over $\mathbb{C}$ and X be its coarse moduli space. Set $n = \dim_{\mathbb{C}} \mathcal{X}$. We assume that X is projective. Let $\mathcal{IX}$ be the inertia stack, which is the fiber product $\mathcal{X} \times_{\mathcal{X}\times\mathcal{X}} \mathcal{X}$ of the diagonal morphisms $\Delta\colon \mathcal{X} \to \mathcal{X} \times \mathcal{X}$. A $\mathbb{C}$-valued point of $\mathcal{IX}$ is a pair (x, g) of a $\mathbb{C}$-valued point $x \in \mathcal{X}$ and a stabilizer $g \in \mathrm{Aut}(x)$ at x. Let

$$\mathcal{IX} = \bigsqcup_{v\in\mathsf{T}} \mathcal{X}_v = \mathcal{X}_0 \sqcup \bigsqcup_{v\in\mathsf{T}'} \mathcal{X}_v, \quad \mathcal{X}_0 = \mathcal{X}.$$

be the decomposition of $\mathcal{IX}$ into connected components. The index set T contains a special element $0 \in \mathsf{T}$ corresponding to the trivial stabilizer $g = 1$. We set $\mathsf{T}' = \mathsf{T} \setminus \{0\}$. Let $\mathrm{age}(v) \in \mathbb{Q}_{\geqslant 0}$ be the age (or degree shifting number) of the component $\mathcal{X}_v$. The *Chen-Ruan orbifold cohomology group* $H^*_{\mathrm{orb}}(\mathcal{X})$ is the $\mathbb{Q}$-graded vector space given by

$$H^p_{\mathrm{orb}}(\mathcal{X}) := \bigoplus_{\{v\in\mathsf{T}\mid p-2\,\mathrm{age}(v)\in 2\mathbb{Z}\}} H^{p-2\,\mathrm{age}(v)}(\mathcal{X}_v;\mathbb{C}), \quad p \in \mathbb{Q}.$$

Throughout the paper, we ignore odd cohomology classes in Gromov-Witten theory *i.e.*, elements in $H^{p-2\,\mathrm{age}(v)}(\mathcal{X}_v)$ with $p-2\,\mathrm{age}(v)$ odd. ($H^*_{\mathrm{orb}}(\mathcal{X})$ is sometimes denoted by $H^*_{\mathrm{CR}}(\mathcal{X})$ in the literature.) We have an involution $\mathrm{inv}\colon \mathcal{IX} \to \mathcal{IX}$ given by $(x,g) \mapsto (x, g^{-1})$. This induces an involution $\mathrm{inv}^*\colon H^*_{\mathrm{orb}}(\mathcal{X}) \to H^*_{\mathrm{orb}}(\mathcal{X})$. The orbifold Poincaré pairing $(\cdot,\cdot)_{\mathrm{orb}}\colon H^*_{\mathrm{orb}}(\mathcal{X}) \otimes H^*_{\mathrm{orb}}(\mathcal{X}) \to \mathbb{C}$ is defined by

$$(\alpha,\beta)_{\mathrm{orb}} := \int_{\mathcal{IX}} \alpha \cup \mathrm{inv}^* \beta.$$

This is a nondegenerate symmetric bilinear form of degree $-2n$. Let $\mathcal{X}_{0,l,d}$ denote the *moduli stack of stable maps* of genus 0, l-pointed and degree $d \in H_2(X,\mathbb{Z})$. (This is the same as the *stack of twisted stable maps* $\mathcal{K}_{0,l}(\mathcal{X},d)$ in [1].) This is equipped with a virtual fundamental class $[\mathcal{X}_{0,l,d}]^{\mathrm{vir}} \in H_*(\mathcal{X}_{0,l,d};\mathbb{Q})$ and the evaluation maps

$$\mathrm{ev}_i\colon \mathcal{X}_{0,l,d} \to \overline{\mathcal{IX}}, \quad i = 1,\dots,l$$

to the rigidified inertia stack(4) $\overline{\mathcal{IX}}$ (see [1]). Take $\alpha_1,\dots,\alpha_l \in H^*_{\mathrm{orb}}(\mathcal{X})$ and nonnegative integers $k_1,\dots,k_l$. The *orbifold Gromov-Witten invariants* are defined by

$$\left\langle \alpha_1\psi^{k_1}, \alpha_2\psi^{k_2}, \dots, \alpha_l\psi^{k_l} \right\rangle_{0,l,d} := \int_{[\mathcal{X}_{0,l,d}]^{\mathrm{vir}}} \prod_{i=1}^{l} (\mathrm{ev}_i^*(\alpha_i) \cup \psi_i^{k_i}).$$

Because $\overline{\mathcal{IX}}$ and $\mathcal{IX}$ are the same as topological spaces, we can define the pull-back $\mathrm{ev}_i^*(\alpha_i)$ for $\alpha_i \in H^*_{\mathrm{orb}}(\mathcal{X})$. The class ψ_i is the first Chern class of the i-th universal cotangent line bundle $\mathcal{L}_i \to \mathcal{X}_{0,l,d}$ whose fiber at a stable map $f\colon \mathcal{C} \to \mathcal{X}$ is the cotangent space $T^*_{x_i}C$ at the i-th marked point of the *coarse* domain curve C.

2.2. Twisted Invariants

Following [17, 55, 16], we introduce the orbifold Gromov-Witten invariants *twisted* by a vector bundle $\mathcal{V}$ on $\mathcal{X}$ and a characteristic class $\mathbf{c}$. We use these invariants to calculate the Gromov-Witten invariants of a complete intersection in $\mathcal{X}$. Let $\mathbf{c}(\cdot) = \exp(\sum_{k=0}^{\infty} s_k \,\mathrm{ch}_k(\cdot))$ be a universal invertible multiplicative characteristic class with parameters $\boldsymbol{s} = (s_0, s_1, s_2, \dots)$. Let

(4) The rigidified inertia stack $\overline{\mathcal{IX}}$ is obtained from $\mathcal{IX}$ by taking the quotient of the automorphism group at $(x,g) \in \mathcal{IX}$ by the cyclic group generated by g.

$\mathcal{IV}$ be the vector bundle on $\mathcal{IX}$ whose fiber at (x,g) is the g-fixed subspace of $\mathcal{V}_x$. In the twisted theory, the pairing $(\cdot,\cdot)_{\mathrm{orb}}$ is replaced with the following twisted Poincaré pairing:

$$(\alpha,\beta)^{\mathbf{c}}_{\mathrm{orb}} = \int_{\mathcal{IX}} \alpha \cup \mathrm{inv}^*(\beta) \cup \mathbf{c}(\mathcal{IV}).$$

Using the universal family $u\colon \mathcal{C}_{0,l,d} \to \mathcal{X}$ over $\mathcal{X}_{0,l,d}$, we define a K-group element $\mathcal{V}_{0,l,d} \in K^0(\mathcal{X}_{0,l,d})$ by $\mathcal{V}_{0,l,d} = R\pi_* u^*\mathcal{V}$.

$$\begin{array}{ccc} \mathcal{C}_{0,l,d} & \xrightarrow{u} & \mathcal{X} \\ {\scriptstyle\pi}\big\downarrow & & \\ \mathcal{X}_{0,l,d} & & \end{array}$$

Define the *twisted Gromov-Witten invariants* by

(2.1)
$$\left\langle \alpha_1\psi^{k_1}, \alpha_2\psi^{k_2}, \dots, \alpha_l\psi^{k_l} \right\rangle^{\mathbf{c}}_{0,l,d} := \int_{[\mathcal{X}_{0,l,d}]^{\mathrm{vir}}} \mathbf{c}(\mathcal{V}_{0,l,d}) \cup \prod_{i=1}^{l} \mathrm{ev}_i^*(\alpha_i)\psi_i^{k_i}.$$

Note that the twisted invariants equal the original ones when $\mathbf{c}$ is trivial (*i.e.*, $\mathbf{c} \equiv 1$).

2.3. Twisted Quantum Cohomology

We can define both untwisted and twisted quantum cohomology, but we begin with the twisted version because the untwisted version is obtained from it by the specialization $\mathbf{c} = 1$. Let $\mathrm{Eff}_{\mathcal{X}} \subset H_2(X;\mathbb{Z})$ denote the semigroup generated by effective curves. The Novikov ring Λ is defined to be the completion of the group ring $\mathbb{C}[\mathrm{Eff}_{\mathcal{X}}]$. For a curve class $d \in \mathrm{Eff}_{\mathcal{X}}$, let Q^d be the corresponding element in Λ. Define $\Lambda_{\boldsymbol{s}}$ to be the completion of $\mathbb{C}[\mathrm{Eff}_{\mathcal{X}}][s_0,s_1,s_2,\dots]$ with respect to the additive valuation v given by

$$v(Q^d) = \int_d \omega, \quad v(s_k) = k+1.$$

where ω is a Kähler class of $\mathcal{X}$. Let $\{\phi_1,\dots,\phi_N\} \subset H^*_{\mathrm{orb}}(\mathcal{X})$ be a homogeneous $\mathbb{C}$-basis, $\{\tau^1,\dots,\tau^N\}$ be the dual co-ordinates on $H^*_{\mathrm{orb}}(\mathcal{X})$ and $\tau = \sum_{i=1}^N \tau^i\phi_i$ be a general point on $H^*_{\mathrm{orb}}(\mathcal{X})$. The *twisted quantum product* $\bullet^{\mathbf{c}}_\tau$ is defined by the formula:

$$(\alpha \bullet^{\mathbf{c}}_\tau \beta, \gamma)^{\mathbf{c}}_{\mathrm{orb}} = \sum_{l\geqslant 0}\sum_{d\in\mathrm{Eff}_{\mathcal{X}}} \langle \alpha,\beta,\gamma,\tau,\dots,\tau\rangle^{\mathbf{c}}_{0,l+3,d}\frac{Q^d}{l!} \tag{2.2}$$

where $\alpha,\beta,\gamma\in H^*_{\rm orb}(\mathcal{X})$. This defines a unique element $\alpha\bullet^{\mathbf{c}}_\tau\beta$ in $H^*_{\rm orb}(\mathcal{X})\otimes\Lambda_{\boldsymbol{s}}[\![\tau]\!]$. Here $\Lambda_{\boldsymbol{s}}[\![\tau]\!]:=\Lambda_{\boldsymbol{s}}[\![\tau^1,\dots,\tau^N]\!]$. The product $\bullet^{\mathbf{c}}_\tau$ is extended bilinearly over $\Lambda_{\boldsymbol{s}}[\![\tau]\!]$ and defines a ring structure on $H^*_{\rm orb}(\mathcal{X})\otimes\Lambda_{\boldsymbol{s}}[\![\tau]\!]$. We call the ring $(H^*_{\rm orb}(\mathcal{X})\otimes\Lambda_{\boldsymbol{s}}[\![\tau]\!],\bullet^{\mathbf{c}}_\tau)$ the *twisted quantum cohomology*. For a topological ring R with an additive valuation $v\colon R\to\mathbb{R}\cup\{\infty\}$, we define $R\{z,z^{-1}\}$ to be the space of all power series $\sum_{k\in\mathbb{Z}}a_kz^k$ with $a_k\in R$ such that $\lim_{|k|\to\infty}v(a_k)=\infty$. Let $R\{z\}$ (resp. $R\{z^{-1}\}$) denote the subspace of $R\{z,z^{-1}\}$ consisting of nonnegative (resp. nonpositive) power series in z. These are rings when R is complete. We define the *Dubrovin connection* $\boldsymbol{\nabla}^{\mathbf{c}}_i\colon H^*_{\rm orb}(\mathcal{X})\otimes\Lambda_{\boldsymbol{s}}\{z\}[\![\tau]\!]\to z^{-1}H^*_{\rm orb}(\mathcal{X})\otimes\Lambda_{\boldsymbol{s}}\{z\}[\![\tau]\!]$ by

$$\boldsymbol{\nabla}^{\mathbf{c}}_i=\frac{\partial}{\partial\tau^i}+\frac{1}{z}\phi_i\bullet^{\mathbf{c}}_\tau .$$

The differential equation $\boldsymbol{\nabla}^{\mathbf{c}}_i s(\tau,z)=0$ for a cohomology-valued function s is called the *quantum differential equation*. Define $\mathbf{L}^{\mathbf{c}}(\tau,z)\in\operatorname{End}(H^*_{\rm orb}(\mathcal{X}))\otimes\Lambda_{\boldsymbol{s}}\{z^{-1}\}[\![\tau]\!]$ by

(2.3)
$$(\mathbf{L}^{\mathbf{c}}(\tau,z)\alpha,\beta)^{\mathbf{c}}_{\rm orb}=(\alpha,\beta)^{\mathbf{c}}_{\rm orb}+\sum_{\substack{(d,l)\neq(0,0)\\ d\in\mathrm{Eff}_{\mathcal{X}},\,l\geqslant0}}\left\langle\frac{\alpha}{-z-\psi},\tau,\dots,\tau,\beta\right\rangle^{\mathbf{c}}_{0,l+2,d}\frac{Q^d}{l!}.$$

Here $1/(-z-\psi)$ in the correlator should be expanded in the series $\sum_{k\geqslant0}(-z)^{-k-1}\psi^k$.

PROPOSITION 2.1. — *The* $\operatorname{End}(H^*_{\rm orb}(\mathcal{X}))$*-valued function* $\mathbf{L}^{\mathbf{c}}(\tau,z)$ *gives a fundamental solution to the quantum differential equation: It satisfies*

$$\boldsymbol{\nabla}^{\mathbf{c}}_i(\mathbf{L}^{\mathbf{c}}(\tau,z)\alpha)=0,\quad 1\leqslant i\leqslant N,\quad \forall\alpha\in H^*_{\rm orb}(\mathcal{X})$$

and $\mathbf{L}^{\mathbf{c}}(\tau,z)=\mathrm{id}+O(Q,\tau)$. *We also have*

$$(\mathbf{L}^{\mathbf{c}}(\tau,-z)\alpha,\mathbf{L}^{\mathbf{c}}(\tau,z)\beta)^{\mathbf{c}}_{\rm orb}=(\alpha,\beta)^{\mathbf{c}}_{\rm orb}. \tag{2.4}$$

Proof. — See [36, Proposition 2.3] and [45] when $\mathcal{X}$ is a smooth variety. In this proof, we will freely use the language of Givental's Lagrangian cone for which we refer the reader to [27, 16]. From Tseng's orbifold Quantum Riemann-Roch (QRR) [55], it follows that the twisted Gromov-Witten invariants (2.1) satisfy the String Equation (SE), the Dilaton Equation (DE) and the Topological Recursion Relation (TRR) listed *e.g.*, in [49, Section 1]. (In the TRR, we need to use the twisted Poincaré pairing.) This is because these equations correspond to certain special geometric properties of Givental's Lagrangian cone (see [27]) and the symplectic operator in Tseng's QRR preserves such properties. The differential equation for $\mathbf{L}^{\mathbf{c}}(\tau,z)$ has been proved for the untwisted theory for manifolds in [49, Proposition 2] using TRR and the same proof applies to our case. It is easy to see that $\mathbf{L}^{\mathbf{c}}(\tau,z)^\dagger\beta$ is a tangent vector of Givental's Lagrangian cone for the twisted theory. Here $\mathbf{L}^{\mathbf{c}}(\tau,z)^\dagger$ denotes the adjoint of $\mathbf{L}^{\mathbf{c}}(\tau,z)$, *i.e.*, $(\alpha,\mathbf{L}^{\mathbf{c}}(\tau,z)^\dagger\beta)^{\mathbf{c}}_{\mathrm{orb}} = (\mathbf{L}^{\mathbf{c}}(\tau,z)\alpha,\beta)^{\mathbf{c}}_{\mathrm{orb}}$. By the Lagrangian property of the cone, we know that $(\mathbf{L}^{\mathbf{c}}(\tau,-z)^\dagger\alpha,\mathbf{L}^{\mathbf{c}}(\tau,z)^\dagger\beta)^{\mathbf{c}}_{\mathrm{orb}}$ contains only nonnegative powers in z. On the other hand $\mathbf{L}^{\mathbf{c}}(\tau,z)^\dagger\beta = \beta + O(z^{-1})$. Therefore we have $(\mathbf{L}^{\mathbf{c}}(\tau,-z)^\dagger\alpha,\mathbf{L}^{\mathbf{c}}(\tau,z)^\dagger\beta)^{\mathbf{c}}_{\mathrm{orb}} = (\alpha,\beta)^{\mathbf{c}}_{\mathrm{orb}}$ and so $\mathbf{L}^{\mathbf{c}}(\tau,-z)^\dagger$ is inverse to $\mathbf{L}^{\mathbf{c}}(\tau,z)$. This proves (2.4). □

Remark 2.2. — The existence of a fundamental solution implies that the Dubrovin connection $\boldsymbol{\nabla}^{\mathbf{c}}$ is flat, *i.e.*, $[\boldsymbol{\nabla}^{\mathbf{c}}_i,\boldsymbol{\nabla}^{\mathbf{c}}_j]=0$. This in turn shows that the twisted quantum product $\bullet^{\mathbf{c}}_\tau$ is associative.

Definition 2.3 ([25, 16]). — *We define the J-function of the twisted theory by*

$$\mathbf{J}^{\mathbf{c}}(\tau,z) := \mathbf{L}^{\mathbf{c}}(\tau,z)^{-1}\,\mathbf{1} = \mathbf{L}^{\mathbf{c}}(\tau,-z)^\dagger\,\mathbf{1}\,. \tag{2.5}$$

2.4. Equivariant Euler Twist

We consider the case where $\mathbf{c}$ is the S^1-equivariant Euler class $\mathbf{e}_\lambda$. Here S^1 acts on vector bundles by scaling the fibers and $\lambda\in H^2_{S^1}(\mathrm{pt})$ denotes a generator. We have $\mathbf{e}_\lambda(\mathcal{E}) = \sum_{i=0}^r \lambda^i c_{r-i}(\mathcal{E})$ for a rank r vector bundle $\mathcal{E}$. Then $\mathbf{e}_\lambda$ corresponds to the choice of parameters

$$s_0 = \log\lambda,\quad s_i = (-1)^{i-1}(i-1)!\lambda^{-i}\ (i\geqslant 1).$$

If $\mathcal{V}_{0,l,d}$ is not represented by a vector bundle, the $\mathbf{e}_\lambda$-twisted invariants take values in $\mathbb{C}[\lambda,\lambda^{-1}]$. In this paper, we only consider the case where $\mathcal{V}_{0,n,d}$ is a vector bundle and no negative powers of λ appear. Then we can take the ground ring to be (instead of $\Lambda_{\boldsymbol{s}}$) the completion Λ_λ of $\mathbb{C}[\mathrm{Eff}][\lambda]$ with respect to the valuation $v(Q^d)=\int_d\omega$, $v(\lambda)=0$.

We assume that $\mathcal{V}$ is the sum $\mathcal{L}_1\oplus\cdots\oplus\mathcal{L}_c$ of line bundles such that $c_1(\mathcal{L}_j)$ is nef and $\mathcal{L}_j$ is a pull-back from the coarse moduli space X for all $1\leqslant j\leqslant c$. Let $\mathcal{Y}$ be a quasi-smooth complete intersection in $\mathcal{X}$ with respect to a regular section of $\mathcal{V}$. Let $\iota\colon\mathcal{Y}\subset\mathcal{X}$ denote the inclusion. The pull-back $\iota^*\colon H^*_{\mathrm{orb}}(\mathcal{X})\to H^*_{\mathrm{orb}}(\mathcal{Y})$ and the push-forward $\iota_*\colon H^*_{\mathrm{orb}}(\mathcal{Y})\to H^*_{\mathrm{orb}}(\mathcal{X})$ are defined by the inclusion $\mathcal{IY}\subset\mathcal{IX}$. We also write $\mathbf{L}_{\mathcal{Y}}(\tau,z)$, $\mathbf{J}_{\mathcal{Y}}(\tau,z)$ for the fundamental solution and the J-function of the *untwisted* theory of $\mathcal{Y}$.

Proposition 2.4. — *Under the above assumption, $\mathbf{L}^{\mathbf{e}_\lambda}(\tau,z)$ and $\mathbf{J}^{\mathbf{e}_\lambda}(\tau,z)$ contain no negative powers in λ. So we can set $\mathbf{L}^{\mathbf{e}}(\tau,z) := \mathbf{L}^{\mathbf{e}_\lambda}(\tau,z)|_{\lambda=0}$,*

$\mathbf{J}^{\mathbf{e}}(\tau,z) := \mathbf{J}^{\mathbf{e}_\lambda}(\tau,z)|_{\lambda=0}$. *Moreover, we have*

$$\iota^*\mathbf{L}^{\mathbf{e}}(\tau,z)\alpha = \mathbf{L}_{\mathcal{Y}}(\iota^*\tau,z)\iota^*\alpha\Big|_{H_2(Y;\mathbb{Z})\to H_2(X;\mathbb{Z})}.$$

Here $\alpha,\beta\in H^*_{\mathrm{orb}}(\mathcal{X})$. *The notation* $H_2(Y;\mathbb{Z})\to H_2(X;\mathbb{Z})$ *means to replace* Q^d *with* $Q^{\iota_*(d)}$ *for* $d\in H_2(Y;\mathbb{Z})$.

Proof. — The proof parallels the argument in [49, Section 2.1]. By the assumption, for every stable map $u\colon\mathcal{C}\to\mathcal{X}$ in $\mathcal{X}_{0,l+2,d}$, the convexity $H^1(\mathcal{C},u^*\mathcal{V})=0$ holds and the natural map $H^0(\mathcal{C},u^*\mathcal{V})\to(u^*\mathcal{V})_{x_{l+2}}$ is surjective. Here x_{l+2} is the last marked point on $\mathcal{C}$. Therefore $\mathcal{V}_{0,l+2,d}$ is a vector bundle and we can define the subbundle $\mathcal{V}'_{0,l+2,d}$ by the following exact sequence:

$$0\longrightarrow\mathcal{V}'_{0,l+2,d}\longrightarrow\mathcal{V}_{0,l+2,d}\longrightarrow\mathrm{ev}^*_{l+2}\mathcal{I}\mathcal{V}\longrightarrow 0. \tag{2.6}$$

Here note that $\mathcal{I}\mathcal{V}$ defines a vector bundle on the rigidified inertia stack $\overline{\mathcal{I}\mathcal{X}}$ whose fiber at $(x,g)\in\overline{\mathcal{I}\mathcal{X}}$ is $\mathcal{V}_x$. Using $\mathbf{e}_\lambda(\mathcal{V}_{0,l+2,d})=\mathbf{e}_\lambda(\mathcal{V}'_{0,l+2,d})\cup\mathrm{ev}^*_{l+2}\mathbf{e}_\lambda(\mathcal{I}\mathcal{V})$, we find that $\mathbf{L}^{\mathbf{e}_\lambda}(\tau,z)\alpha$ equals

$$\alpha+\sum_{\substack{(d,l)\neq(0,0)\\ d\in\mathrm{Eff}_{\mathcal{X}},\ l\geqslant 0}}\frac{Q^d}{l!}\mathrm{inv}^*\mathrm{ev}_{l+2*}\left[\frac{\mathrm{ev}_1^*\alpha}{-z-\psi_1}\left(\prod_{j=2}^{l+1}\mathrm{ev}_j^*(\tau)\right)\mathbf{e}_\lambda(\mathcal{V}'_{0,l+2,d})\cap[\mathcal{X}_{0,l+2,d}]^{\mathrm{vir}}\right].$$

This shows that $\mathbf{L}^{\mathbf{e}_\lambda}$ does not contain negative powers of λ. Since $\mathbf{L}^{\mathbf{e}_\lambda}=\mathrm{id}+O(Q,\tau)$, $(\mathbf{L}^{\mathbf{e}_\lambda})^{-1}$ and $\mathbf{J}^{\mathbf{e}_\lambda}=(\mathbf{L}^{\mathbf{e}_\lambda})^{-1}\mathbf{1}$ do not contain negative powers of λ either. We denote by $\mathrm{ev}^{\mathcal{X}}\colon\mathcal{X}_{0,l+2,d}\to(\overline{\mathcal{I}\mathcal{X}})^{l+2}$ and $\mathrm{ev}^{\mathcal{Y}}\colon\mathcal{Y}_{0,l+2,d}\to(\overline{\mathcal{I}\mathcal{Y}})^{l+2}$ the collection $(\mathrm{ev}_1,\dots,\mathrm{ev}_{l+2})$ of the evaluation maps. For the second statement, it suffices to show that

$$f^*\mathrm{ev}^{\mathcal{X}}_*\left(\psi_1^k\mathbf{e}(\mathcal{V}'_{0,l+2,d})\cap[\mathcal{X}_{0,l+2,d}]^{\mathrm{vir}}\right)=\sum_{d':\iota_*(d')=d}g_*\mathrm{ev}^{\mathcal{Y}}_*\left(\psi_1^k\cap[\mathcal{Y}_{0,l+2,d'}]^{\mathrm{vir}}\right)$$

where f and g are the inclusions:

$$(\overline{\mathcal{I}\mathcal{Y}})^{l+1}\times\overline{\mathcal{I}\mathcal{Y}}\xrightarrow{\ g\ }(\overline{\mathcal{I}\mathcal{X}})^{l+1}\times\overline{\mathcal{I}\mathcal{Y}}\xrightarrow{\ f\ }(\overline{\mathcal{I}\mathcal{X}})^{l+1}\times\overline{\mathcal{I}\mathcal{X}}.$$

We consider the fiber diagram

$$\begin{CD}
\mathcal{Z} @>{i}>> \mathcal{X}_{0,l+2,d}\\
@V{\mathrm{ev}^{\mathcal{Z}}}VV @VV{\mathrm{ev}^{\mathcal{X}}}V\\
\overline{\mathcal{I}\mathcal{X}}^{l+1}\times\overline{\mathcal{I}\mathcal{Y}} @>{f}>> (\overline{\mathcal{I}\mathcal{X}})^{l+2}
\end{CD}$$

When $\mathcal{Y}$ is the zero locus of a regular section $s \in H^0(\mathcal{X}, \mathcal{V})$, $\mathcal{Z}$ is defined to be the zero locus of $\mathrm{ev}_{l+2}^*(s) \in H^0(\mathcal{X}_{0,l+2,d}, \mathrm{ev}_{l+2}^* \mathcal{I}\mathcal{V})$. Using the *refined Gysin map* $f^!$ in [23, 56], we have

$$f^* \mathrm{ev}_*^{\mathcal{X}} \left(\psi_1^k \mathbf{e}(\mathcal{V}'_{0,l+2,d}) \cap [\mathcal{X}_{0,l+2,d}]^{\mathrm{vir}}\right) = \mathrm{ev}_*^{\mathcal{Z}} f^! \left(\psi_1^k \mathbf{e}(\mathcal{V}'_{0,l+2,d}) \cap [\mathcal{X}_{0,l+2,d}]^{\mathrm{vir}}\right).$$

Let $j \colon \mathcal{Y}_{0,l+2,d} \to \mathcal{Z}$ be the inclusion. It now suffices to show the equality of classes on $\mathcal{Z}$:

$$\sum_{d' : \iota_*(d') = d} j_* \left(\psi_1^k \cap [\mathcal{Y}_{0,l+2,d'}]^{\mathrm{vir}}\right) = f^! \left(\psi_1^k \mathbf{e}(\mathcal{V}'_{0,l+2,d}) \cap [\mathcal{X}_{0,l+2,d}]^{\mathrm{vir}}\right).$$

Note that we only need to consider the case $k = 0$ since ψ_1^k factors out. By the functoriality [43] of virtual classes we have

$$\sum_{d' : \iota_*(d') = d} [\mathcal{Y}_{0,l+2,d'}]^{\mathrm{vir}} = 0_{\mathcal{X}}^! [\mathcal{X}_{0,l+2,d}]^{\mathrm{vir}}$$

where $0_{\mathcal{X}} \colon \mathcal{X}_{0,l+2,d} \to \mathcal{V}_{0,l+2,d}$ is the zero section (which is the bottom row of the diagram below). We can make the following fiber diagram:

$$\begin{tikzcd}
\mathcal{Y}_{0,l+2,d} \arrow[r, "j"] \arrow[d, "j"'] & \mathcal{Z} \arrow[r, "i"] \arrow[d, "\tilde{s}_{\mathcal{Z}}"'] & \mathcal{X}_{0,l+2,d} \arrow[d, equal] \\
\mathcal{Z} \arrow[r, "0_{\mathcal{Z}}"] \arrow[d, "i"'] & \mathcal{V}'_{0,l+2,d}|_{\mathcal{Z}} \arrow[d, "h|_{\mathcal{Z}}"'] & \mathcal{X}_{0,l+2,d} \arrow[d, "\tilde{s}"'] \\
\mathcal{X}_{0,l+2,d} \arrow[r, "0'_{\mathcal{X}}"] & \mathcal{V}'_{0,l+2,d} \arrow[r, "h"] & \mathcal{V}_{0,l+2,d}
\end{tikzcd}$$

where $\tilde{s}$ and $\tilde{s}_{\mathcal{Z}}$ are the sections of $\mathcal{V}_{0,l+2,d}$ and $\mathcal{V}'_{0,l+2,d}|_{\mathcal{Z}}$ induced from $s \in H^0(\mathcal{X}, \mathcal{V})$, $0'_{\mathcal{X}}$ and $0_{\mathcal{Z}}$ are the zero sections and h is the natural inclusion. We have $0_{\mathcal{X}} = h \circ 0'_{\mathcal{X}}$. Using the properties of the Gysin maps, we have

$$\begin{aligned} j_* 0_{\mathcal{X}}^! [\mathcal{X}_{0,l+2,d}]^{\mathrm{vir}} &= j_* 0'^!_{\mathcal{X}} h^! [\mathcal{X}_{0,l+2,d}]^{\mathrm{vir}} = 0_{\mathcal{Z}}^* (\tilde{s}_{\mathcal{Z}})_* h^! [\mathcal{X}_{0,l+2,d}]^{\mathrm{vir}} \\ &= \mathbf{e}(\mathcal{V}'_{0,l+2,d}) h^! [\mathcal{X}_{0,l+2,d}]^{\mathrm{vir}} = f^! \left(\mathbf{e}(\mathcal{V}'_{0,l+2,d}) \cap [\mathcal{X}_{0,l+2,d}]^{\mathrm{vir}}\right). \end{aligned}$$

In the last step we used the exact sequence (2.6). The conclusion follows. □

Using $\phi_i \bullet_\tau^{\mathbf{e}_\lambda} = -(z\partial_{\tau^i} \mathbf{L}^{\mathbf{e}_\lambda}(\tau, z))(\mathbf{L}^{\mathbf{e}_\lambda}(\tau, z))^{-1}$, we obtain the following corollary.

Corollary 2.5. — *Under the same assumption, the equivariant Euler twisted quantum product $\bullet_\tau^{\mathbf{e}_\lambda}$ has the non-equivariant limit $\bullet_\tau^{\mathbf{e}}$ and we have*

$$\iota^*(\alpha \bullet_\tau^{\mathbf{e}} \beta) = (\iota^* \alpha) \bullet_{\iota^* \tau} (\iota^* \beta)\Big|_{H_2(Y;\mathbb{Z}) \to H_2(\mathcal{X};\mathbb{Z})}$$

where $\alpha, \beta \in H^*_{\mathrm{orb}}(\mathcal{X})$ and $\bullet_{\iota^*\tau}$ in the right-hand side denotes the untwisted quantum product of $\mathcal{Y}$.

2.5. The Specialization at $Q = 1$

The divisor equation ([1, Theorem 8.3.1]) shows that the Novikov parameter Q is actually redundant in the product $\bullet^{\mathbf{c}}_\tau$ (2.2). Writing

$$\tau = \tau_{0,2} + \tau', \quad \tau_{0,2} \in H^2(\mathcal{X}_0), \quad \tau' \in \bigoplus_{p \neq 2} H^p(\mathcal{X}) \oplus \bigoplus_{v \in \mathsf{T}'} H^*(\mathcal{X}_v), \tag{2.7}$$

we have

$$(\alpha \bullet^{\mathbf{c}}_\tau \beta, \gamma)^{\mathbf{c}}_{\mathrm{orb}} = \sum_{l \geqslant 0} \sum_{d \in \mathrm{Eff}_{\mathcal{X}}} \langle \alpha, \beta, \gamma, \tau', \dots, \tau' \rangle^{\mathbf{c}}_{0,l+3,d} \frac{e^{\langle \tau_{0,2}, d\rangle} Q^d}{l!}.$$

Therefore the parameter Q plays the same role as $e^{\tau_{0,2}}$. We define

$$\circ^{\mathbf{c}}_\tau := \bullet^{\mathbf{c}}_\tau|_{Q=1}.$$

The new product $\circ^{\mathbf{c}}_\tau$ is a formal power series in τ' and a formal *Fourier* series in $\tau_{0,2}$. Similarly, by the divisor equation, the fundamental solution (2.3) can be specialized to $Q = 1$. Writing $L^{\mathbf{c}}(\tau, z) := \mathbf{L}^{\mathbf{c}}(\tau, z)|_{Q=1}$, we have

$$\begin{aligned}(L^{\mathbf{c}}(\tau, z)\alpha, \beta)^{\mathbf{c}}_{\mathrm{orb}} &= (e^{-\tau_{0,2}/z}\alpha, \beta)^{\mathbf{c}}_{\mathrm{orb}} \\ &\quad + \sum_{\substack{(d,l) \neq (0,0) \\ d \in \mathrm{Eff}_{\mathcal{X}}, l \geqslant 0}} \left\langle \frac{e^{-\tau_{0,2}/z}\alpha}{-z - \psi}, \tau', \dots, \tau', \beta \right\rangle^{\mathbf{c}}_{0,l+2,d} \frac{e^{\langle \tau_{0,2}, d\rangle}}{l!}.\end{aligned} \tag{2.8}$$

Here the action of $\tau_{0,2}$ on $H^*_{\mathrm{orb}}(\mathcal{X})$ is defined by $\tau_{0,2} \cdot \alpha = \mathrm{pr}^*(\tau_{0,2}) \cup \alpha$ where $\mathrm{pr}\colon \mathcal{IX} \to \mathcal{X}$ is the natural projection. The classical limit $Q = \tau = 0$ corresponds, after the specialization $Q = 1$, to the limit $\tau' = 0$ and $e^{\langle \tau_{0,2}, d\rangle} \to 0$ for all nonzero $d \in \mathrm{Eff}_{\mathcal{X}}$. This is called the *large radius limit.*

3. $\widehat{\Gamma}$-Integral Structure in Quantum Cohomology

In this section we review the quantum D-module for stacks and its $\widehat{\Gamma}$-integral structure following [37]. See also [38] for a review.

3.1. Untwisted Quantum D-Module with $Q=1$

We denote by $\circ_\tau := \circ^{\mathbf{c}}_\tau|_{\boldsymbol{s}=0}$ the quantum product of the untwisted theory of $\mathcal{X}$ specialized to $Q=1$. In all the examples we treat in our paper, it turns out a posteriori that the quantum product $\circ_\tau$ is convergent in τ. So henceforth we assume that $\circ_\tau$ is convergent over the region $U \subset H^*_{\mathrm{orb}}(\mathcal{X})$ containing the set

$$\left\{\tau \in H^*_{\mathrm{orb}}(\mathcal{X}) \;\middle|\; \|\tau'\| \leqslant e^{-M},\ \Re(\langle \tau_{0,2}, d\rangle) \leqslant -M \ \forall d \in \mathrm{Eff}_{\mathcal{X}} \setminus \{0\}\right\}$$

for some $M>0$. Here $\|\cdot\|$ is a certain norm on $H^*_{\mathrm{orb}}(\mathcal{X})$ and we used the decomposition (2.7). The region U is considered as a neighborhood of the large radius limit point.

Let (τ,z) denote a general point on $U\times\mathbb{C}$ and $(-)\colon U\times\mathbb{C}\to U\times\mathbb{C}$ be the map sending (τ,z) to $(\tau,-z)$. In the untwisted theory we can extend the Dubrovin connection in the z-direction.

Definition 3.1 ([37, Definition 2.2]). — *The quantum D-module $QDM(\mathcal{X})$ is the triple $(F,\nabla,(\cdot,\cdot)_F)$ consisting of the trivial holomorphic vector bundle $F := H^*_{\mathrm{orb}}(\mathcal{X})\times(U\times\mathbb{C})\to(U\times\mathbb{C})$, the meromorphic flat connection of F*

$$\nabla := d + \frac{1}{z}\sum_{i=1}^N (\phi_i \circ_\tau) d\tau^i + \left(-\frac{1}{z}(E\circ_\tau) + \frac{\deg}{2}\right)\frac{dz}{z}$$

and the pairing $(\cdot,\cdot)_F\colon (-)^*\mathcal{O}(F)\otimes\mathcal{O}(F)\to z^n\mathcal{O}_{U\times\mathbb{C}}$ defined by

$$(\alpha,\beta)_F := (2\pi\mathtt{i}z)^n(\alpha,\beta)_{\mathrm{orb}} \qquad \text{for } \alpha\in F_{(\tau,-z)},\ \beta\in F_{(\tau,z)}.$$

Here $E\in\mathcal{O}(F)$ is the Euler vector field

$$E := c_1(T\mathcal{X}) + \sum_i \left(1-\frac{1}{2}\deg\phi_i\right)\tau^i\phi_i$$

and deg *denotes the degree as a class in $H^*_{\mathrm{orb}}(\mathcal{X})$. (In the definition of ∇, $\frac{\deg}{2}$ should be understood as an element of $\mathrm{End}(H^*_{\mathrm{orb}}(\mathcal{X}))$.) The connection ∇ is called the (extended) Dubrovin connection. It has poles of order $\leqslant 2$ along $z=0$. The pairing $(\cdot,\cdot)_F$ is flat with respect to ∇. When we refer to $QDM(\mathcal{X})$ as a D-module, we consider the action of the ring $\mathcal{O}_U[z]\langle z\partial_1,\dots,z\partial_N\rangle$ of differential operators on $\mathcal{O}(F)$ given by $z\partial_i\mapsto z\nabla_i$.*

Remark 3.2. — We work with the different conventions for ∇ and $(\cdot,\cdot)_F$ from [37] to get a better match with the B-side. For the flat connection ∇^{old} and the pairing $(\cdot,\cdot)^{\mathrm{old}}_F$ in [37], we have $\nabla=\nabla^{\mathrm{old}}+\frac{n}{2}\frac{dz}{z}$ and $(\cdot,\cdot)_F = (2\pi\mathtt{i}z)^n(\cdot,\cdot)^{\mathrm{old}}_F$, where $n=\dim_{\mathbb{C}}\mathcal{X}$. In what follows, we will translate the contents in [37] in this new convention, but we will not remark the difference every time.

Remark 3.3. — The quantum D-module can be considered as a variation of *generalized* Hodge structure. Generalizations of Hodge structure have been studied by many people and referred to in various ways: *semi-infinite Hodge structure* [2, 37], *TERP structure* [31] and *non-commutative Hodge structure* [40] etc.

The quantum D-module has a certain symmetry which we called the *Galois action* in [37]. This comes from the divisor equation and the monodromy constraints for orbifold stable maps. Let $H^2(\mathcal{X};\mathbb{Z})$ denote the sheaf cohomology on the topological stack $\mathcal{X}$ which classifies topological orbifold line bundles. For $\xi \in H^2(\mathcal{X};\mathbb{Z})$, let $\mathcal{L}_\xi$ be the corresponding orbifold line bundle, $\xi_0 \in H^2(\mathcal{X};\mathbb{Q})$ denote the image of ξ and $f_v(\xi) \in [0,1) \cap \mathbb{Q}$ be the rational number such that the stabilizer along $\mathcal{X}_v$ acts on fibers of $\mathcal{L}_\xi$ by $\exp(2\pi\mathtt{i} f_v(\xi))$. (The number $f_v(\xi)$ is called the *age* of $\mathcal{L}_\xi$ along $\mathcal{X}_v$.) Define the map $G(\xi)\colon H^*_{\mathrm{orb}}(\mathcal{X}) \to H^*_{\mathrm{orb}}(\mathcal{X})$ by

$$G(\xi)(\tau_0 \oplus \bigoplus_{v\in\mathsf{T}'} \tau_v) = (\tau_0 - 2\pi\mathtt{i}\xi_0) \oplus \bigoplus_{v\in\mathsf{T}'} e^{2\pi\mathtt{i} f_v(\xi)}\tau_v \tag{3.1}$$

where $\tau_v \in H^*(\mathcal{X}_v)$. Consider the following bundle isomorphism of F

$$\begin{aligned} G^F(\xi)\colon H_{\mathrm{orb}}(\mathcal{X}) \times (U\times\mathbb{C}) &\longrightarrow H_{\mathrm{orb}}(\mathcal{X}) \times (U\times\mathbb{C}), \\ (\alpha,(\tau,z)) &\longmapsto (dG(\xi)\alpha, (G(\xi)\tau, z)) \end{aligned} \tag{3.2}$$

where $dG(\xi) \in \mathrm{End}(H^*_{\mathrm{orb}}(\mathcal{X}))$ is the differential of $G(\xi)$.

Proposition 3.4 ([37, Proposition 2.3]). — *The bundle isomorphism $G^F(\xi)$ preserves the connection ∇ and the pairing $(\cdot,\cdot)_F$. This defines the $H^2(\mathcal{X},\mathbb{Z})$-action on $QDM(\mathcal{X})$ and $QDM(\mathcal{X})$ descends*[5] *to the quotient $(U/H^2(\mathcal{X},\mathbb{Z}))\times\mathbb{C}$.*

The solution to the extended quantum differential equation $\nabla s = 0$ is given by the fundamental solution $L(\tau,z) := L^{\mathbf{c}}(\tau,z)|_{\boldsymbol{s}=0}$ in (2.8) multiplied by $z^{-\frac{\deg}{2}}z^\rho$.

Proposition 3.5 ([37, Proposition 2.4]). — *Set $\rho := c_1(T\mathcal{X})$ and define*

$$z^{-\frac{\deg}{2}}z^\rho := \exp\left(-\frac{\deg}{2}\log z\right)\exp(\rho\log z).$$

(5) We can assume that U is invariant under the action of $H^2(\mathcal{X};\mathbb{Z})$.

Then $s_i(\tau, z) = L(\tau, z) z^{-\frac{\deg}{2}} z^{\rho} \phi_i$, $i = 1, \ldots, N$ form a basis of (multi-valued) ∇-flat sections. Each s_i is characterized by the asymptotic initial condition $s_i(\tau, z) \sim z^{-\frac{\deg}{2}} z^{\rho} e^{-\tau_{0,2}} \phi_i$ in the large radius limit.

Note that $L(\tau, z)$ is convergent on $U \times \mathbb{C}^*$ so far as the quantum product $\circ_\tau$ is analytic on U since it is a solution to the quantum differential equation.

3.2. $\widehat{\Gamma}$-Integral Structure

Let $\mathcal{S}(\mathcal{X})$ denote the space of multi-valued flat sections for ∇. By Proposition 3.5, it is a $\mathbb{C}$-vector space spanned by $L(\tau, z) z^{-\frac{\deg}{2}} z^{\rho} \phi_i$, $1 \leqslant i \leqslant N$. We will introduce a $\mathbb{Z}$-lattice $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ in the space $\mathcal{S}(\mathcal{X})$ using the K-group. A similar rational structure was introduced also by Katzarkov-Kontsevich-Pantev [40]. Define a pairing $(\cdot, \cdot)_{\mathcal{S}} \colon \mathcal{S}(\mathcal{X}) \otimes \mathcal{S}(\mathcal{X}) \to \mathbb{C}$ by

$$(s_1, s_2)_{\mathcal{S}} := (s_1(\tau, e^{\pi \mathbf{i}} z), s_2(\tau, z))_{\mathrm{orb}}.$$

Here $s_1(\tau, e^{\pi \mathbf{i}} z)$ denotes the analytic continuation of $s_1(\tau, z)$ along the path $[0, 1] \ni \theta \mapsto e^{\pi \mathbf{i} \theta} z$. Since s_1, s_2 are flat sections, the right-hand side of the above formula does not depend on τ and z. Note that $(\cdot, \cdot)_{\mathcal{S}}$ is neither symmetric nor anti-symmetric in general. It is symmetric (resp. anti-symmetric) when $\mathcal{X}$ is an even (resp. odd) dimensional Calabi-Yau orbifold. The Galois action on $QDM(\mathcal{X})$ induces the following automorphism $G^{\mathcal{S}}(\xi)$ of $\mathcal{S}(X)$ for $\xi \in H^2(\mathcal{X}; \mathbb{Z})$: $(G^{\mathcal{S}}(\xi) s)(\tau, z) := dG(\xi) s(G(\xi)^{-1} \tau, z)$ for $s \in \mathcal{S}(\mathcal{X})$.

Let $K(\mathcal{X})$ be the Grothendieck group of topological orbifold vector bundles on $\mathcal{X}$. In the following, we could also use the Grothendieck group $K_{\mathrm{alg}}(\mathcal{X})$ of *algebraic* vector bundles. Our integral structure depends only on the Chern character image of the K-group, so the algebraic K-group defines a subgroup of $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$. For an orbifold vector bundle $\mathcal{E}$, take its pull-back $\mathrm{pr}^* \mathcal{E}$ to $\mathcal{I}\mathcal{X}$ ($\mathrm{pr} \colon \mathcal{I}\mathcal{X} \to \mathcal{X}$ is the natural map) and consider the eigenbundle decomposition of $\mathrm{pr}^* \mathcal{E}|_{\mathcal{X}_v}$ with respect to the stabilizer action:

$$\mathrm{pr}^* \mathcal{E}|_{\mathcal{X}_v} = \bigoplus_{0 \leqslant f < 1} (\mathrm{pr}^* \mathcal{E})_{v,f}$$

where $(\mathrm{pr}^* \mathcal{E})_{v,f}$ is the piece on which the stabilizer of $\mathcal{X}_v$ acts by $\exp(2\pi \mathbf{i} f)$. The Chern character map $\widetilde{\mathrm{ch}} \colon K(\mathcal{X}) \to H^*(\mathcal{I}\mathcal{X})$ is defined by

$$\widetilde{\mathrm{ch}}(\mathcal{E}) := \bigoplus_{v \in \mathsf{T}} \sum_{0 \leqslant f < 1} e^{2\pi \mathbf{i} f} \, \mathrm{ch}((\mathrm{pr}^* \mathcal{E})_{v,f}).$$

Let $\delta_{v,f,i}$, $i=1,\dots,l_{v,f}$ be the Chern roots of $(\mathrm{pr}^*\mathcal{E})_{v,f}$, where $l_{v,f} = \mathrm{rank}((\mathrm{pr}^*\mathcal{E})_{v,f})$. The $\widehat{\Gamma}$-class of $\mathcal{E}$ is defined to be

$$\widehat{\Gamma}(\mathcal{E}) := \bigoplus_{v\in\mathsf{T}}\prod_{0\leqslant f<1}\prod_{i=1}^{l_{v,f}}\Gamma(1-f+\delta_{v,f,i}) \in H^*(\mathcal{IX}).$$

Here the Γ-function in the right-hand side should be expanded in Taylor series at $1-f>0$. This is a multiplicative *transcendental* characteristic class. We write $\widehat{\Gamma}_{\mathcal{X}} := \widehat{\Gamma}(T\mathcal{X})$. For simplicity we assume that $\mathcal{X}$ has no generic stabilizers, as this is true for our later examples.

Definition 3.6 ([37, Definition 2.9, Proposition 2.10, Remark 2.11], [40, Definition 3.2]). — *Define the K-group framing*[(6)] $\mathfrak{s}\colon K(\mathcal{X})\to\mathcal{S}(\mathcal{X})$ *of the space $\mathcal{S}(\mathcal{X})$ by*

$$\begin{aligned}\mathfrak{s}(\mathcal{E})(\tau,z) &:= (2\pi\mathbf{i})^{-n}L(\tau,z)z^{-\frac{\deg}{2}}z^{\rho}\Psi(\mathcal{E})\\ &\qquad \text{where } \Psi(\mathcal{E}) := \widehat{\Gamma}_{\mathcal{X}}\cup(2\pi\mathbf{i})^{\frac{\deg_0}{2}}\,\mathrm{inv}^*\,\widetilde{\mathrm{ch}}(\mathcal{E}).\end{aligned} \tag{3.3}$$

Here $\deg_0$ denotes the degree without the age shift, i.e., we define $(2\pi\mathbf{i})^{\frac{\deg_0}{2}}|_{H^{2k}(\mathcal{IX})} := (2\pi\mathbf{i})^k$ and $\widehat{\Gamma}_{\mathcal{X}}\cup$ is the cup product in $H^*(\mathcal{IX})$. The $\widehat{\Gamma}$-integral structure $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}\subset\mathcal{S}(\mathcal{X})$ is defined to be the image of $\mathfrak{s}$. This satisfies the following properties.

(i) *$\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$ is a lattice in $\mathcal{S}(\mathcal{X})$, i.e., $\mathcal{S}(\mathcal{X}) = \mathcal{S}(\mathcal{X})_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{C}$.*

(ii) *We have $G^{\mathcal{S}}(\xi)(\mathfrak{s}(\mathcal{E})) = \mathfrak{s}(\mathcal{E}\otimes\mathcal{L}_\xi^\vee)$ for $\xi\in H^2(\mathcal{X};\mathbb{Z})$. In particular the Galois action preserves the lattice $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$.*

(iii) *The pairing $(\cdot,\cdot)_{\mathcal{S}}$ takes values in $\mathbb{Z}$ on $\mathcal{S}(\mathcal{X})_{\mathbb{Z}}$. For holomorphic vector bundles $\mathcal{E}_1,\mathcal{E}_2$, one has $(\mathfrak{s}(\mathcal{E}_1),\mathfrak{s}(\mathcal{E}_2))_{\mathcal{S}} = (-1)^n\chi(\mathcal{E}_2,\mathcal{E}_1) := \sum_{i=0}^n(-1)^{i+n}\dim\mathrm{Ext}^i(\mathcal{E}_2,\mathcal{E}_1)$.*

The last part (iii) of the properties follows from Kawasaki-Riemann-Roch [42, 54] and the fact that the $\widehat{\Gamma}$-class is roughly the half of the Todd class. In fact, for a smooth variety X, the $\widehat{\Gamma}$-class and the Todd class are related by

$$((-1)^{\frac{\deg_0}{2}}\widehat{\Gamma}_X)\cdot\widehat{\Gamma}_X\cdot e^{\pi\mathbf{i}c_1(X)} = (2\pi\mathbf{i})^{\frac{\deg_0}{2}}\,\mathrm{Td}(TX) \tag{3.4}$$

thanks to the functional equality $\Gamma(1-z)\Gamma(1+z) = \pi z/\sin(\pi z)$. (For an orbifold the relationship is more complicated. See [38, p.124].)

Definition 3.7. — *For $\mathcal{E}\in K(\mathcal{X})$ and a section $\phi(\tau,z)\in\mathcal{O}(F)$ of the quantum D-module of $\mathcal{X}$, we define the A-period $\Pi(\phi,\mathcal{E})$ to be the*

[(6)] The convention here is different from [37].

multi-valued function on $U \times \mathbb{C}^\times$

$$\Pi(\phi,\mathcal{E})(\tau,z) := (\phi(\tau,-z), \mathfrak{s}(\mathcal{E})(\tau,z))_F. \tag{3.5}$$

The special case $Z(\mathcal{E}) := (2\pi\mathtt{i})^{-n}\Pi(\mathbf{1},\mathcal{E})$, $n = \dim_{\mathbb{C}}\mathcal{X}$ *is the quantum cohomology central charge of* $\mathcal{E}$ *defined in* [37].

Under mirror symmetry the flat section $\mathfrak{s}(\mathcal{E})$ should correspond to a Gauss-Manin constant cycle $C_{\mathcal{E}}$ and the above pairing $\Pi(\phi,\mathcal{E})$ to the integration of the de Rham form mirror to ϕ over $C_{\mathcal{E}}$. The unit section $\mathbf{1}$ should correspond to a holomorphic (oscillatory) volume form. Using $L(\tau,z)^\dagger = L(\tau,-z)^{-1}$ (2.4), we can rewrite the A-periods in terms of the inverse fundamental solution.

$$\Pi(\phi,\mathcal{E})(\tau,z) = \left(L(\tau,-z)^{-1}\phi(\tau,-z),\ z^{n-\frac{\deg}{2}}z^{\rho}\Psi(\mathcal{E})\right)_{\mathrm{orb}}. \tag{3.6}$$

In particular, $Z(\mathcal{E})$ is a component of the J-function:

$$Z(\mathcal{E}) = \frac{1}{(2\pi\mathtt{i})^n}\left(J(\tau,-z),\ z^{n-\frac{\deg}{2}}z^{\rho}\Psi(\mathcal{E})\right)_{\mathrm{orb}},$$

where $J(\tau,z) = L(\tau,z)^{-1}\mathbf{1}$ is the untwisted J-function of $\mathcal{X}$ with $Q=1$.

4. Mirror Theorem for Toric Complete Intersections

In this section we state a Givental-style mirror theorem for complete intersections in toric orbifolds. By the mirror theorem we can calculate the J-function or the fundamental solution in terms of explicit hypergeometric series.

4.1. Notation on Toric Orbifolds

Toric orbifolds or *toric Deligne-Mumford stacks* were introduced by Borisov-Chen-Smith [7] in terms of a *stacky fan*. Here we fix notation for toric orbifolds and state basic facts. We only consider compact weak Fano toric orbifolds without generic stabilizers. See [21, 48, 7] for the basics of toric varieties and stacks. A similar but more detailed account was given in [37, Section 3.1] with a little different notation.

Let $\mathbf{N} \cong \mathbb{Z}^n$ be a free abelian group. Set $\mathbf{N}_{\mathbb{R}} = \mathbf{N}\otimes_{\mathbb{Z}}\mathbb{R}$. Let $\Delta \subset \mathbf{N}_{\mathbb{R}}$ be an integral convex polytope containing the origin 0 in its interior. We choose a stacky fan (Σ,β) on $\mathbf{N}$ *adapted to* Δ. It consists of the data

- a rational simplicial fan Σ in the vector space $\mathbf{N}_{\mathbb{R}}$;

- a homomorphism $\beta\colon \mathbb{Z}^m \to \mathbf{N}$ such that $\{\mathbb{R}_{\geqslant 0} b_1, \dots, \mathbb{R}_{\geqslant 0} b_m\}$ is the set $\Sigma^{(1)}$ of one-dimensional cones of Σ, where $b_i = \beta(e_i)$ is the image of the standard basis $e_i \in \mathbb{Z}^m$

which are adapted to Δ in the sense that Δ is the convex hull of $b_1, b_2, \dots, b_m$ and that $b_1, \dots, b_m$ are on the boundary of Δ. We call Δ the *fan polytope*. These data give rise to a weak Fano (*i.e.*, $c_1(\mathcal{X})$ is nef) toric orbifold $\mathcal{X}$. The coarse moduli space X of $\mathcal{X}$ is the toric variety associated with the fan Σ. We furthermore assume that

- the fan Σ admits a strictly convex piecewise linear function(7) $\varphi\colon N_{\mathbb{R}} \to \mathbb{R}$;
- the set $\Delta \cap \mathbf{N}$ generate $\mathbf{N}$ as a $\mathbb{Z}$-module.

The first condition means that the underlying toric variety X is projective. The second condition(8) ensures that the quantum D-module of $\mathcal{X}$ over the small parameter space $H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X})$ is generated by the I-function (see [37, Lemma 4.7]). Essentially the same assumption was made in [37] (see Remark 3.4 *ibid*). We usually identify a cone σ of Σ with the subset $\{i \mid b_i \subset \sigma\}$ of $\{1, \dots, m\}$.

Remark 4.1. — Borisov-Chen-Smith [7] allowed $\mathbf{N}$ to have torsion and the torsion part of $\mathbf{N}$ equals the group of generic stabilizers of $\mathcal{X}$. In this case the mirror of $\mathcal{X}$ becomes disconnected [37]. We will restrict to the free $\mathbf{N}$ to reduce technical complications.

Take a subset $\{b_{m+1}, \dots, b_{m+s}\}$ of $(\mathbf{N} \cap \Delta) \setminus \{b_1, \dots, b_m\}$ such that $b_1, \dots, b_{m+s}$ generate $\mathbf{N}$ as an abelian group. These are called *extended ray vectors*. They define an *extended stacky fan* in the sense of Jiang [39]. Let $\hat{\beta}\colon \mathbb{Z}^{m+s} \to \mathbf{N}$ be the homomorphism sending the standard basis vectors $e_1, \dots, e_{m+s}$ to $b_1, \dots, b_{m+s}$. Then $\hat{\beta}$ is surjective by the assumption. Define $\mathbb{L} := \operatorname{Ker} \hat{\beta}$. The *(extended) fan sequence* is the exact sequence:

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^{m+s} \xrightarrow{\hat{\beta}} \mathbf{N} \longrightarrow 0$$

and the *(extended) divisor sequence* is its dual:

$$0 \longrightarrow \mathbf{M} \xrightarrow{\hat{\beta}^*} (\mathbb{Z}^{m+s})^* \xrightarrow{D} \mathbb{L}^* \longrightarrow 0.$$

(7) A piecewise linear function is a continuous function on $\mathbf{N}_{\mathbb{R}}$ which is linear on each cone of Σ. See [48] for the (strict) convexity.

(8) This second assumption is satisfied if $\pi_1^{\mathrm{orb}}(\mathcal{X})$ is trivial; in particular if $\mathcal{X}$ is a weighted projective space. We will state the toric mirror theorem without this assumption in [15]. The results in this paper should be generalized also without this assumption, but we will stick to this case for simplicity.

Here $\mathbf{M} := \mathrm{Hom}(\mathbf{N}, \mathbb{Z})$. Let $D_i = D(e_i^*) \in \mathbb{L}^*$ be the image of the standard basis $e_i^* \in (\mathbb{Z}^{m+s})^*$. The Picard group $\mathrm{Pic}(\mathcal{X})$ on the *stack* $\mathcal{X}$ is given by

$$\mathrm{Pic}(\mathcal{X}) \cong H^2(\mathcal{X}; \mathbb{Z}) \cong \mathbb{L}^* \Big/ \sum_{i=m+1}^{m+s} \mathbb{Z} D_i.$$

The image $\overline{D}_i$ of D_i in $\mathrm{Pic}(\mathcal{X})$ is the class of a torus invariant divisor. We call D_i the *extended toric divisor class*. The anticanonical class is given by $\rho := c_1(\mathcal{X}) = -K_{\mathcal{X}} = \sum_{i=1}^m \overline{D}_i$. The *extended anticanonical class* is defined by $\hat{\rho} := \sum_{i=1}^{m+s} D_i$. Every element of $\mathrm{Pic}(\mathcal{X})$ is represented by an integral linear combination of toric divisors $\overline{D}_1, \dots, \overline{D}_m$. For an expression $\xi = \sum_{i=1}^m n_i \overline{D}_i$, define a piecewise linear function $\varphi_\xi \colon \mathbf{N}_\mathbb{R} \to \mathbb{R}$ by $\varphi_\xi(b_i) = n_i$ for $1 \leqslant i \leqslant m$. The function φ_ξ is ambiguous up to an integral linear function in $\mathbf{M} = \mathrm{Hom}(\mathbf{N}, \mathbb{Z})$. We have the following:

- ξ is nef (resp. ample) $\Longleftrightarrow$ φ_ξ is convex (resp. strictly convex);
- For $v \in \mathrm{Box}$, $\{\varphi_\xi(v)\}$ is the age $f_v(\xi)$ of the line bundle $\mathcal{L}_\xi$ along $\mathcal{X}_v$.

Define the set Box by

$$\mathrm{Box} := \Big\{ v \in \mathbf{N} \;\Big|\; \exists \sigma \in \Sigma,\ 0 \leqslant \exists c_i < 1,\ v = \sum_{i \in \sigma} c_i b_i \Big\}.$$

This parametrizes connected components of $\mathcal{IX}$ [7]. For $v \in \mathrm{Box}$, let $\mathcal{X}_v$ denote the corresponding component of $\mathcal{IX}$ and $\mathbf{1}_v \in H^0(\mathcal{X}_v) \subset H^{2\,\mathrm{age}(v)}_{\mathrm{orb}}(\mathcal{X})$ denote the unit class supported on $\mathcal{X}_v$. Here $\mathrm{age}(v)$ is given by $\mathrm{age}(v) = \sum_{i \in \sigma} c_i$ when v is written as $v = \sum_{i\in\sigma} c_i b_i$ for some cone $\sigma \in \Sigma$ and $c_i \geqslant 0$. The extended divisors $D_{m+1}, \dots, D_{m+s}$ correspond to the classes $\mathbf{1}_{b_{m+1}}, \dots, \mathbf{1}_{b_{m+s}}$ in $H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X})$.

Note that $H_2(\mathcal{X}; \mathbb{Q}) \cong (\bigoplus_{i=m+1}^{m+s} \mathbb{Q} D_i)^\perp \subset \mathbb{L}_\mathbb{Q} := \mathbb{L} \otimes \mathbb{Q}$. We see that $H_2(\mathcal{X}; \mathbb{Q})$ has a canonical complementary subspace in $\mathbb{L}_\mathbb{Q}$. For $m+1 \leqslant j \leqslant m+s$, b_j is contained in a cone σ of Σ and we can write $b_j = \sum_{i\in\sigma} c_{ji} b_i$ for some $c_{ji} \geqslant 0$. Then $\delta_j := e_j - \sum_{i \in \sigma} c_{ji} e_i \in \mathbb{Q}^{m+s}$ belongs to $\mathbb{L}_\mathbb{Q}$. We have

$$\mathbb{L}_\mathbb{Q} = H_2(\mathcal{X}; \mathbb{Q}) \oplus \bigoplus_{j=m+1}^{m+s} \mathbb{Q} \delta_j. \tag{4.1}$$

The elements $\delta_{m+1}, \dots, \delta_{m+s}$ are dual to $D_{m+1}, \dots, D_{m+s}$ and regarded as orbifold *homology* classes (of degree $\leqslant 2$). Set $\mathrm{NE}_{\mathcal{X},\sigma} := \{ d \in H_2(\mathcal{X}; \mathbb{R}) \mid \forall i \in \{1, \dots, m\} \setminus \sigma,\ \langle \overline{D}_i, d \rangle \geqslant 0 \}$ for a cone σ. The *Mori cone* $\mathrm{NE}_\mathcal{X} \subset H_2(\mathcal{X}; \mathbb{R})$ is given by

$$\mathrm{NE}_\mathcal{X} = \sum_{\sigma \in \Sigma} \mathrm{NE}_{\mathcal{X},\sigma}\,.$$

The *extended Mori cone* $\widehat{\mathrm{NE}}_{\mathcal{X}} \subset \mathbb{L}_{\mathbb{R}} := \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{R}$ is defined to be

$$\widehat{\mathrm{NE}}_{\mathcal{X}} := \mathrm{NE}_{\mathcal{X}} + \sum_{m+1 \leqslant j \leqslant m+s} \mathbb{R}_{\geqslant 0} \delta_j.$$

For $v \in \mathrm{Box}$, we define $\mathbb{K}_v$ to be the subset of $\mathbb{Q}^m \times \mathbb{Z}^s \subset \mathbb{Q}^{m+s}$ consisting of all $d \in \mathbb{Q}^m \times \mathbb{Z}^s$ such that $\sum_{i=1}^{m+s} d_i b_i + v = 0$ and that $\{1 \leqslant i \leqslant m \mid d_i \notin \mathbb{Z}\}$ is a cone of Σ. Let us write $v = \sum_{i \in \sigma} c_i b_i$ for some cone σ and $c_i \in [0,1)$ and set $c_i = 0$ for $i \notin \sigma$. Then we have a relation $\sum_{i=1}^{m+s}(d_i + c_i) b_i = 0$ for $d \in \mathbb{K}_v$. We denote by $d + v$ the element of $\mathbb{L}_{\mathbb{Q}}$ defined by this relation. The lattice $\mathbb{L}$ acts on $\mathbb{K}_v$ by addition and $\mathbb{K}_0 \subset \mathbb{L}_{\mathbb{Q}}$. We define the *reduction function* $\{-\cdot\} \colon \mathbb{K}_v \to \mathrm{Box}$ by

$$\{-d\} := \sum_{i=1}^{m} \{-d_i\} b_i$$

where $\{r\}$ denote the fractional part of r. Because $\sum_{i=1}^{m+s} d_i b_i + v = 0$, we have $\{-d\} = \sum_{i=1}^{m+s} \lceil d_i \rceil b_i + v$ and so $\{-d\} \in \mathbf{N}$. The reduction function in fact induces an isomorphism $\mathbb{K}_v / \mathbb{L} \cong \mathrm{Box}$.

4.2. Mirror Theorem I: Toric Orbifolds

Let $\mathcal{X}$ be a toric orbifold as in the previous section. Define $\mathcal{M} := \mathrm{Spec}\, \mathbb{C}[\mathbb{L}] = \mathrm{Hom}(\mathbb{L}, \mathbb{C}^\times)$. For $d \in \mathbb{L}$, let q^d denote the corresponding element in $\mathbb{C}[\mathbb{L}]$. This is a function $q^d \colon \mathcal{M} \to \mathbb{C}^\times$. The space $\mathcal{M}$ has a partial (possibly singular) compactification $\overline{\mathcal{M}} := \mathrm{Spec}\, \mathbb{C}[\mathbb{L} \cap \widehat{\mathrm{NE}}_{\mathcal{X}}]$. It has a special point (large radius limit point) $\mathbf{0}$ defined by $q^d = 0$ for all nonzero $d \in \mathbb{L} \cap \widehat{\mathrm{NE}}_{\mathcal{X}}$. We choose a $\mathbb{Z}$-basis $p_1, \ldots, p_{r+s}$ of $\mathbb{L}^*$ (here $r := m - n$) such that each p_a is *extended nef i.e.,* p_a is semi-positive on $\widehat{\mathrm{NE}}_{\mathcal{X}}$ and $p_{r+1}, \ldots, p_{r+s} \in \sum_{j=m+1}^{m+s} \mathbb{Q}_{\geqslant 0} D_j$. Then we have the corresponding coordinates $q_1, \ldots, q_{r+s}$ on $\mathcal{M}$ such that $q^d = q_1^{\langle p_1, d\rangle} \cdots q_{r+s}^{\langle p_{r+s}, d\rangle}$. These coordinates $(q_1, \ldots, q_{r+s})$ give a desingularization $\mathbb{C}^{r+s} \to \overline{\mathcal{M}}$ such that $\mathbf{0}$ corresponds to the origin of $\mathbb{C}^{r+s}$. For $d \in \mathbb{L}_{\mathbb{Q}}$, q^d defines a possibly multi-valued function on $\mathcal{M}$. Let $\overline{p}_a \in H^2(\mathcal{X}; \mathbb{Z})$ denote the image of $p_a \in \mathbb{L}^*$. We write $\overline{p} \log q := \sum_{a=1}^{r} \overline{p}_a \log q_a$. This is an $H^2(\mathcal{X}; \mathbb{C})$-valued (multi-valued) function on $\mathcal{M}$.

Definition 4.2 ([15]; See also [37, Section 4.1]). — *Take* $v \in \mathrm{Box}$. *Define an* $H^*_{\mathrm{orb}}(\mathcal{X})$*-valued (multi-valued) function* $I^v(q,z)$ *on an open subset* *of $\mathcal{M}\times\mathbb{C}^\times$ by*

$$I^v(q,z)=e^{\overline{p}\log q/z}\sum_{d\in\mathbb{K}_v} q^{d+v}\prod_{i=1}^{m+s}\frac{\prod_{k>d_i,\{k\}=\{d_i\}}(\overline{D}_i+kz)}{\prod_{k>0,\{k\}=\{d_i\}}(\overline{D}_i+kz)}\,\mathbf{1}_{\{-d\}}\,.$$

Here all but finite terms in the infinite product cancel and $\overline{D}_j=0$ for $m+1\leqslant j\leqslant m+s$. The terms with $d+v\notin\widehat{\mathrm{NE}}_{\mathcal{X}}$ automatically vanish and $I^v(q,z)$ is convergent in a neighborhood of $\mathbf{0}$. Apart from the prefactor $e^{\overline{p}\log q/z}$, it is homogeneous of degree $2\,\mathrm{age}(v)$ with respect to the grading of $H^*_{\mathrm{orb}}(\mathcal{X})$, $\deg(q^d):=2\langle\hat{\rho},d\rangle$ and $\deg z:=2$. The series $I(q,z):=I^0(q,z)$ is called the I-function. We have the asymptotics

$$I^v(q,z)=e^{\overline{p}\log q/z}(\mathbf{1}_v+O(q))$$
$$I(q,z)=\mathbf{1}+\frac{\tau(q)}{z}+O(z^{-2})$$

where $O(q)$ denotes a function vanishing at $\mathbf{0}$ and $\tau(q)$ is a multi-valued map with values in $H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X})$, called the mirror map. The map $\tau(q)$ induces a single-valued map

$$\tau(q)\colon\{(q_1,\dots,q_r)\mid 0<|q_a|<\epsilon\}\to H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X};\mathbb{C})/H^2(\mathcal{X};\mathbb{Z})\tag{4.2}$$

for some $\epsilon>0$. Here $H^2(\mathcal{X};\mathbb{Z})$ acts on $H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X})$ by the Galois action $\xi\mapsto G(\xi)$.

The following will be shown in joint work with Coates, Corti and Tseng [15] (see [18] for the case of weighted projective spaces):

Theorem 4.3 ([15]). — *Let $\mathcal{X}$ be a toric orbifold in Section 4.1 and $J(\tau,z)$ be the untwisted J-function of $\mathcal{X}$ with $Q=1$. Then we have $I(q,z)=J(\tau(q),z)$.*

The function $I^v(q,z)$ can be obtained from $I(q,z)=I^0(q,z)$ by differentiation. Writing $D_i=\sum_{a=1}^{r+s}\mathsf{m}_{ia}p_a$, we define the ($z$-decorated) logarithmic vector field $\boldsymbol{D}_i$ on $\mathcal{M}$ by $\boldsymbol{D}_i:=z\sum_{a=1}^{r+s}\mathsf{m}_{ia}q_a(\partial/\partial q_a)$. Taking $\delta\in\mathbb{K}_0$ such that $v=\{-\delta\}$ and $\lceil\delta_i\rceil\geqslant 0$ for all i, we can easily see that (see also [37, Lemma 4.7])

$$I^v(q,z)=q^{-\delta}\left(\prod_{i=1}^{m+s}\prod_{\nu=0}^{\lceil\delta_i\rceil-1}(\boldsymbol{D}_i-\nu z)\right)I(q,z).\tag{4.3}$$

In the terminology of Givental's Lagrangian cone [17, 16], $I^v(q,-z)$ is in the tangent space to the cone at $-zI(q,-z)$. Therefore, I^v appears as a column vector of the inverse fundamental solution.

Corollary 4.4 ([37, Eqn (65)]). — *Let $L(\tau,z)$ denote the fundamental solution* (2.8) *of the untwisted* ($\boldsymbol{s}=0$) *theory of $\mathcal{X}$. There exists an $H^*_{\mathrm{orb}}(\mathcal{X})$-valued function $\theta_v(q,z)\in H^*_{\mathrm{orb}}(\mathcal{X})\otimes\mathcal{O}_{\widetilde{\mathcal{M}}}[z]$ defined on a finite cover $\widetilde{\mathcal{M}}$ of $\mathcal{M}$ and in a neighborhood of $\mathbf{0}$ such that*

$$I^v(q,z)=L(\tau(q),z)^{-1}\theta_v(q,z),\quad \theta_v(q,z)=\mathbf{1}_v+O(q).$$

Also $\theta_v(q,z)$ is homogeneous of degree $2\,\mathrm{age}(v)$ and $\theta_0(q,z)=\mathbf{1}$.

Proof. — We differentiate $I(q,z)=J(\tau(q),z)=L(\tau(q),z)^{-1}\,\mathbf{1}$ by the differential operator appearing in (4.3). Here notice that $z\partial_a\circ L(\tau(q),z)^{-1}=L(\tau(q),z)^{-1}\circ(z\partial_a+(\partial_a\tau)\circ_{\tau(q)})$ for $\partial_a=q_a(\partial/\partial q_a)$. $\square$

4.3. Mirror Theorem II: Toric Complete Intersection

As before, let $\mathcal{V}$ be the sum of line bundles $\mathcal{L}_1\oplus\mathcal{L}_2\oplus\cdots\oplus\mathcal{L}_c$ over a toric orbifold $\mathcal{X}$ and $\mathcal{Y}\subset\mathcal{X}$ be a quasi-smooth complete intersection with respect to a regular section of $\mathcal{V}$. Let $\iota\colon\mathcal{Y}\to\mathcal{X}$ be the inclusion. Let ξ_i be the class of $\mathcal{L}_i$ in $\mathrm{Pic}(\mathcal{X})\cong H^2(\mathcal{X};\mathbb{Z})$. We assume that

- The classes $\xi_1,\dots,\xi_c$ and $c_1(\mathcal{Y})=c_1(\mathcal{X})-\sum_{i=1}^c\xi_i$ are nef.
- The line bundles $\mathcal{L}_1,\dots,\mathcal{L}_n$ are pulled back from the coarse moduli space X, *i.e.*, $\xi_i\in H^2(X,\mathbb{Z})$.

Let $\varphi_i\colon\mathbf{N}_{\mathbb{R}}\to\mathbb{R}$ be the piecewise linear function corresponding to ξ_i (see Section 4.1). By the second assumption, we have $\{\varphi_i(v)\}=f_v(\xi_i)=0$ for all $1\leqslant i\leqslant c$ and $v\in\mathrm{Box}$. Define a lift $\tilde{\xi}_i\in\mathbb{L}^*$ of ξ_i by $\tilde{\xi}_i:=\sum_{j=1}^{m+s}\varphi_i(b_j)D_j$. The lift $\tilde{\xi}_i$ does not depend on the choice of φ_i. Then $\tilde{\xi}_i$ is extended nef (semi-positive on $\widehat{\mathrm{NE}}_{\mathcal{X}}$) since $\langle\tilde{\xi}_i,\delta_j\rangle=0$. Set $\hat{\rho}_{\mathcal{Y}}:=\hat{\rho}-\sum_{i=1}^c\tilde{\xi}_i$. This is also extended nef.

Definition 4.5. — *Let us write $I^v(q,z)=e^{\overline{p}\log q/z}\sum_{d\in\mathbb{K}_v}q^{d+v}\square_d\,\mathbf{1}_{\{-d\}}$. For $v\in\mathrm{Box}$, we define an $H^*_{\mathrm{orb}}(\mathcal{X})$-valued function $I^v_{\mathcal{V}}(q,z)$ by*

$$I^v_{\mathcal{V}}(q,z)=e^{\overline{p}\log q/z}\sum_{d\in\mathbb{K}_v}q^{d+v}\prod_{i=1}^{c}\prod_{k=1}^{\langle\tilde{\xi}_i,d+v\rangle}(\xi_i+kz)\cup\square_d\,\mathbf{1}_{\{-d\}}\,.$$

Note that $\square_d\,\mathbf{1}_{\{-d\}}=0$ for $d+v\notin\widehat{\mathrm{NE}}_{\mathcal{X}}$ and $\langle\tilde{\xi}_i,d+v\rangle\geqslant 0$ otherwise. Also it is easy to see that $\langle\tilde{\xi}_i,d+v\rangle$ is an integer. Under the above assumption, $I^v_{\mathcal{V}}(q,z)$ is convergent near $\mathbf{0}$. Apart from the prefactor $e^{\overline{p}\log q/z}$, it is homogeneous of degree $2\,\mathrm{age}(v)$ with respect to the grading of $H^*_{\mathrm{orb}}(\mathcal{X})$,

$\deg_{\mathcal{Y}} q^d := 2\langle \hat{\rho}_{\mathcal{Y}}, d\rangle$ *and* $\deg z := 2$. *We set* $I_{\mathcal{V}}(q,z) := I^0_{\mathcal{V}}(q,z)$. *We have the asymptotics:*

$$(4.4)\qquad \begin{aligned} I^v_{\mathcal{V}}(q,z) &= e^{\overline{p}\log q/z}(\mathbf{1}_v + O(q)) \\ I_{\mathcal{V}}(q,z) &= F(q)\,\mathbf{1} + \frac{G(q)}{z} + O(z^{-2}) \end{aligned}$$

where $F(q)$ *is a power series of the form* $1+\sum_{d\neq 0} c_d q^d$, $c_d \in \mathbb{Q}$ *with integral exponents* $d \in \mathbb{L} \cap \widehat{\mathrm{NE}}_{\mathcal{X}}$ *and* $G(q)$ *is an* $H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X})$*-valued map. The mirror map*

$$(4.5)\qquad \tilde{\varsigma}(q) := \frac{G(q)}{F(q)}$$

defines a single valued map from a neighborhood of $\mathbf{0}$ *to* $H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X};\mathbb{C})/H^2(\mathcal{X};\mathbb{Z})$.

Theorem 4.6. — *Let* $L^{\mathbf{e}}(\tau,z)$, $J^{\mathbf{e}}(\tau,z)$ *be the fundamental solution and the* J*-function of the* $(\mathbf{e},\mathcal{V})$*-twisted theory of* $\mathcal{X}$. *For* $v \in \mathrm{Box}$, *there exists an* $H^*_{\mathrm{orb}}(\mathcal{X})$*-valued function* $\widetilde{\Upsilon}_v(q,z) \in H^*_{\mathrm{orb}}(\mathcal{X}) \otimes \mathcal{O}_{\widetilde{\mathcal{M}}}[z]$ *defined on a finite cover* $\widetilde{\mathcal{M}}$ *of* $\mathcal{M}$ *and in a neighborhood of* $\mathbf{0}$ *such that*

$$(4.6)\qquad I^v_{\mathcal{V}}(q,z) = L^{\mathbf{e}}(\tilde{\varsigma}(q),z)^{-1}\widetilde{\Upsilon}_v(q,z), \quad \widetilde{\Upsilon}_v(q,z) = \mathbf{1}_v + O(q).$$

Also $\widetilde{\Upsilon}_v(q,z)$ *is homogeneous of degree* $2\,\mathrm{age}(v)$ *for the grading* $\deg_{\mathcal{Y}}(q^d) = 2\langle\hat{\rho}_{\mathcal{Y}}, d\rangle$. *We find that* $\widetilde{\Upsilon}_0 = F(q)\,\mathbf{1}$ *by comparing the asymptotics in* z. *Therefore,*

$$I_{\mathcal{V}}(q,z) = F(q) J^{\mathbf{e}}(\tilde{\varsigma}(q),z).$$

Proof. — When the mirror map $\tau(q)$ for $\mathcal{X}$ is "linear", the last statement follows from the quantum Lefschetz theorem [16, Corollary 5.1] applied to the previous theorem 4.3. First we see how to modify the proof of quantum Lefschetz in [16] to calculate a convenient slice (I-function) of the twisted Lagrangian cone. Let $\mathscr{L}_{\boldsymbol{s}}$ denote the $(\mathbf{c},\mathcal{V})$-twisted Lagrangian cone [16, Section 3] of $\mathcal{X}$. Define

$$\mathbf{I}_{\boldsymbol{s}}(q,z) = e^{\overline{p}\log q/z} \sum_{d\in\mathbb{K}_0} q^d Q^{\overline{d}} \prod_{i=1}^{c} \prod_{k=1}^{\langle\tilde{\xi}_i,d\rangle} \exp\left(\boldsymbol{s}(\xi_i + kz)\right) \Box_d \,\mathbf{1}_{\{-d\}}\,.$$

Here $\overline{d} \in H_2(\mathcal{X};\mathbb{Q})$ is the $H_2(\mathcal{X};\mathbb{Q})$-component of $d \in \mathbb{L}_{\mathbb{Q}}$ under the decomposition (4.1) and $\boldsymbol{s}(x) = \sum_{k\geqslant 0} s_k x^k/k!$. We claim that $-z\mathbf{I}_{\boldsymbol{s}}(q,-z)$ is on the cone $\mathscr{L}_{\boldsymbol{s}}$. Here we regard $\mathbf{I}_{\boldsymbol{s}}$ as a $\Lambda_{\boldsymbol{s}}[\![\log q_1,\dots,\log q_r, q_{r+1}^{1/e},\dots,q_{r+s}^{1/e}]\!]$-valued point on Givental's loop space $\mathcal{H}$ for $e\in\mathbb{N}$ such that $e\mathbb{K}_0\subset\mathbb{L}$. (See the definition of $\mathcal{H}$ and $\mathscr{L}_{\boldsymbol{s}}$ as formal schemes in [16, Appendix B].) At $\boldsymbol{s}=0$, $-z\mathbf{I}_0(q,-z)$ is on the untwisted cone $\mathscr{L}_0$ by Theorem 4.3. Write $\tilde{\xi}_i = \sum_{a=1}^{r+s} v_{ia}p_a$ and define the logarithmic vector field $\tilde{\boldsymbol{\xi}}_i := z\sum_{a=1}^{r+s} v_{ia} q_a(\partial/\partial q_a)$.

Then the same argument as the last paragraph of the proof of Theorem 4.8 in [16] shows that

$$\mathbf{f}_{\boldsymbol{s}}(q) = \exp\left(-\sum_{i=1}^{c} G_0(-\tilde{\boldsymbol{\xi}}_i, z)\right)(-z\mathbf{I}_0(q,-z))$$

is on the untwisted cone $\mathscr{L}_0$, where $G_y(x,z)$ is a formal power series depending on $\boldsymbol{s}$ defined in [16]. Applying Tseng's symplectic operator Δ^{tw} ([55]; we use the convention in [16, Theorem 4.1]), we get an element $\Delta^{\mathrm{tw}}(\mathbf{f}_{\boldsymbol{s}}(q))$ on $\mathscr{L}_{\boldsymbol{s}}$. Using the property of the function $G_y(x,z)$ and $\tilde{\boldsymbol{\xi}}_i(e^{\overline{p}\log q/z}q^d) = (\xi_i + z\langle\tilde{\xi}_i, d\rangle)e^{\overline{p}\log q/z}q^d$ (see Eqns (12), (13) in [16] and the discussion following them), we find that this equals $-z\mathbf{I}_{\boldsymbol{s}}(q,-z)$. This proves the claim. Taking $\mathbf{c} = \mathbf{e}_\lambda$, we obtain a vector $\mathbf{I}_\lambda$ on the $(\mathbf{e}_\lambda, \mathcal{V})$-twisted Lagrangian cone:

$$\mathbf{I}_\lambda(q,z) := \mathbf{I}_{\boldsymbol{s}}(q,z)|_{\mathbf{c}=\mathbf{e}_\lambda} = e^{\overline{p}\log q/z}\sum_{d\in\mathbb{K}_0} q^d Q^{\overline{d}} \prod_{i=1}^{c}\prod_{k=1}^{\langle\tilde{\xi}_i,d\rangle}(\lambda+\xi_i+kz)\Box_d\,\mathbf{1}_{\{-d\}}\,.$$

By the discussion as in [16, Section 5.2], we know that $\mathbf{I}_\lambda$ and $\mathbf{J}^{\mathbf{e}_\lambda}$ are related as $\mathbf{I}_\lambda(q,z) = \boldsymbol{F}(q)\mathbf{J}^{\mathbf{e}_\lambda}(\tilde{\boldsymbol{\varsigma}}(q;\lambda),z)$ where $\boldsymbol{F}(q)$, $\boldsymbol{\varsigma}(q;\lambda)$ are determined by the z-asymptotics of $\mathbf{I}_\lambda$ in the same way as (4.4) and (4.5). (Here $F(q) = \boldsymbol{F}(q)|_{Q=1}$, $\tilde{\varsigma}(q) = \tilde{\boldsymbol{\varsigma}}(q;0)|_{Q=1}$.) Now we differentiate $\mathbf{I}_\lambda$ by the differential operator appearing in (4.3). We find

$$\begin{aligned} &q^{-\delta}\left(\prod_{i=1}^{m+s}\prod_{\nu=0}^{\lceil\delta_i\rceil-1}(\boldsymbol{D}_i-\nu z)\right)\mathbf{I}_\lambda \\ &= e^{\overline{p}\log q/z}\sum_{d\in\mathbb{K}_v} q^{d+v}Q^{\overline{d+v}}\prod_{i=1}^{c}\prod_{k=1}^{\langle\tilde{\xi}_i,d+\lceil\delta\rceil\rangle}(\lambda+\xi_i+kz)\Box_d\,\mathbf{1}_{\{-d\}} \end{aligned} \tag{4.7}$$

where $d+\lceil\delta\rceil = (d_i+\lceil\delta_i\rceil)_{i=1}^{m+s}$ is an element of $\mathbb{K}_0$. Applying the infinite-rank differential operator $\prod_{i=1}^{c}\prod_{k=1}^{\langle\tilde{\xi}_i,\delta\rangle}(\lambda+\tilde{\boldsymbol{\xi}}_i+kz)^{-1}$ to the above element (here note that $\langle\tilde{\xi}_i,\delta\rangle\in\mathbb{Z}_{\geqslant 0}$ since $\delta\in\widehat{\mathrm{NE}}_{\mathcal{X}}\cap\mathbb{K}_0$), we obtain

$$\mathbf{I}_\lambda^v(q,z) := e^{\overline{p}\log q/z}\sum_{d\in\mathbb{K}_v} q^{d+v}Q^{\overline{d+v}}\prod_{i=1}^{c}\prod_{k=1}^{\langle\tilde{\xi}_i,d+v\rangle}(\lambda+\xi_i+kz)\Box_d\,\mathbf{1}_{\{-d\}}\,.$$

Here we expand $(\lambda+\tilde{\boldsymbol{\xi}}_i+kz)^{-1}$ as $\sum_{k=0}^{\infty}\lambda^{-k-1}(-\tilde{\boldsymbol{\xi}}_i-kz)^k$. Because

$$\mathbf{I}_\lambda = \boldsymbol{F}(q)\mathbf{J}^{\mathbf{e}_\lambda}(\tilde{\boldsymbol{\varsigma}}(q;\lambda),z) = \mathbf{L}^{\mathbf{e}_\lambda}(\tilde{\boldsymbol{\varsigma}}(q;\lambda),z)^{-1}\boldsymbol{F}(q)\,\mathbf{1}$$

and $\mathbf{I}_\lambda^v$ is obtained from $\mathbf{I}_\lambda$ by differentiation, $\mathbf{I}_\lambda^v = \mathbf{L}^{\mathbf{e}_\lambda}(\tilde{\boldsymbol{\varsigma}}(q;\lambda),z)^{-1}\widetilde{\boldsymbol{\Upsilon}}_v$ for an $H^*_{\mathrm{orb}}(\mathcal{X})$-valued function $\widetilde{\boldsymbol{\Upsilon}}_v(q,z;\lambda)$ which is regular at $z=0$ (see the proof of Corollary 4.4). Here $\widetilde{\boldsymbol{\Upsilon}}_v$ is defined over the ring $\mathbb{C}[z]((\lambda^{-1}))[\![\mathrm{Eff}_{\mathcal{X}}]\!]$ $[\![\log q_1, \dots, \log q_r, q_{r+1}^{1/e}, \dots, q_{r+s}^{1/e}]\!]$. But $\mathbf{I}^v_\lambda$, $\mathbf{L}^{\mathbf{e}_\lambda}$, $\tilde{\boldsymbol{\varsigma}}(q;\lambda)$ do not contain negative powers of λ, so it follows that $\widetilde{\boldsymbol{\Upsilon}}_v$ is also regular at $\lambda = 0$. Now the conclusion follows by setting $\lambda = 0$, $Q = 1$. □

Remark 4.7. — Recall that I^v was obtained from I by differentiation (see (4.3)). In the twisted case, in general, $I^v_{\mathcal{V}}$ cannot be written in the form $I^v_{\mathcal{V}} = P_v I_{\mathcal{V}}$ for some differential operator $P_v \in \mathcal{O}_{\widetilde{\mathcal{M}}}[z]\langle z\partial_1, \dots, z\partial_{r+s}\rangle$. This means that the twisted quantum D-module over $H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X})$ may not be generated by the unit section $\mathbf{1}$ as an $\mathcal{O}[z]\langle z\partial\rangle$-module, where ∂ denotes the derivative in the $H^{\leqslant 2}_{\mathrm{orb}}(\mathcal{X})$ direction. Differentiating $I_{\mathcal{V}}$ by the same differential operator as in (4.3), we obtain (*cf.* (4.7))

(4.8)
$$q^{-\delta}\left(\prod_{i=1}^{m+s}\prod_{\nu=0}^{\lceil\delta_i\rceil-1}(\boldsymbol{D}_i - \nu z)\right) I_{\mathcal{V}} = e^{\overline{p}\log q/z}\left(\prod_{i=1}^{c}\prod_{k=1}^{\langle\tilde{\xi}_i,\delta\rangle}(\xi_i + kz)\,\mathbf{1}_v + O(q)\right).$$

This equals $I^v_{\mathcal{V}}$ if $\langle\tilde{\xi}_i, \delta\rangle = 0$ for all i. The equalities $\langle\tilde{\xi}_i,\delta\rangle = 0$ $(\forall i)$ can be achieved (for some δ) if there exists a cone σ in Σ such that $v \in \sigma$ and v is in the monoid generated by $\{b_1, \dots, b_{m+s}\} \cap \sigma$. On the other hand, if we invert the variable z, *i.e.*, restrict the D-module to the complement of $z = 0$, we can see from (4.8) that the twisted quantum D-module is still generated by $\mathbf{1}$. Such a non-generation phenomenon first appeared in the work of Guest-Sakai [29].

We remark that one can calculate $L^{\mathbf{e}}$ and $\widetilde{\Upsilon}_v$ from the functions $I^v_{\mathcal{V}}$ using the *Birkhoff factorization* in the theory of loop groups, as observed by Coates-Givental [17] and Guest [28]. Using the fact that $H^*(\mathcal{X}_v)$ is generated by $\mathbf{1}_v$ as a $\mathbb{C}[\overline{p}_1, \dots, \overline{p}_r]$-module, we can find differential operators $P_{v,i}(z\partial) \in \mathbb{C}[z\partial_1, \dots, z\partial_r]$, $i = 1, \dots, l_v$ (where $\partial_a = q_a(\partial/\partial q_a)$) such that $\phi_{v,i} = P_{v,i}(\overline{p}_1, \dots, \overline{p}_r)\,\mathbf{1}_v$, $v \in \mathrm{Box}$, $i = 1, \dots, l_v$ form a basis of $H^*_{\mathrm{orb}}(\mathcal{X})$. Then by the asymptotics (4.4) and the previous theorem, we have

$$P_{v,i}(z\partial) I^v_{\mathcal{V}}(q,z) = e^{\overline{p}\log q/z}(\phi_{v,i} + O(q)) = L^{\mathbf{e}}(\tilde{\varsigma}(q), z)^{-1}\widetilde{\Upsilon}_{v,i}(q,z).$$

Here $\widetilde{\Upsilon}_{v,i}(q,z) = P_{v,i}(z\tilde{\varsigma}^*\nabla^{\mathbf{e}})\widetilde{\Upsilon}_v(q,z) = \phi_{v,i} + O(q)$ is an $H^*_{\mathrm{orb}}(\mathcal{X})$-valued function regular at $z = 0$. We consider the matrix formed by the column vectors $P^i_v(z\partial) I^v_{\mathcal{V}}$ and regard it as an element of the loop group with loop parameter z. Then the above equation shows that $(L^{\mathbf{e}})^{-1}$ and $(\widetilde{\Upsilon}_{v,i})_{v,i}$ are obtained from it by the Birkhoff factorization [51]. Here we use the fact that $L^{\mathbf{e}} = \mathrm{id} + O(z^{-1})$ and $\widetilde{\Upsilon}_{v,i}(q,z)$ is regular at $z = 0$. This also gives a proof that $\widetilde{\Upsilon}_v(q,z)$ and $L^{\mathbf{e}}(\varsigma(q), z)$ are analytic near $q = \mathbf{0}$.

Using Proposition 2.4, we get the following corollary of Theorem 4.6.

Corollary 4.8. — *Let $L_{\mathcal{Y}}(\tau,z)$, $J_{\mathcal{Y}}(\tau,z)$ denote the fundamental solution and the J-function of the untwisted theory of $\mathcal{Y}$. Set $\Upsilon_v(q,z) = \iota^*\widetilde{\Upsilon}_v(q,z)$ and $\varsigma(q) = \iota^*\tilde{\varsigma}(q)$. Then we have $\iota^* I^v_{\mathcal{V}}(q,z) = L_{\mathcal{Y}}(\varsigma(q),z)^{-1}\Upsilon_v(q,z)$ and $\iota^* I_{\mathcal{V}}(q,z) = F(q) J_{\mathcal{Y}}(\varsigma(q),z)$.*

5. Equality of Periods: A-periods = B-periods

In this section we show that the A-periods of $\mathcal{X}$ equal ordinary periods (or oscillatory integral) of the mirror. The key point — the hypersurface J-function is a Laplace transform of the ambient one and the same for mirror oscillatory integrals — had been observed in Givental's paper [25] on toric mirror theorem and in the Coates-Givental proof [17] of quantum Lefschetz.

5.1. Laplace Transform of A-Periods

Let $\mathcal{X}$ be a toric orbifold in Section 4.1 and $\mathcal{Y}$ be a complete intersection in $\mathcal{X}$ with respect to $\mathcal{V} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_c$ in Section 4.3. Here we show that the Laplace transforms of the A-periods of $\mathcal{X}$ give precisely those of $\mathcal{Y}$. We choose a lift $\tilde{\xi}_j \in \mathbb{L}^*$ of ξ_j for $1 \leqslant j \leqslant c$ as in Section 4.3. Then $\tilde{\xi}_j$ defines a one-parameter subgroup $\mathbb{C}^\times \ni r \mapsto r^{\tilde{\xi}_i}$ of $\mathcal{M} = \mathbb{L}^* \otimes \mathbb{C}^\times$. In co-ordinates, $r^{\tilde{\xi}_j} = (r^{v_{j1}}, \ldots, r^{v_{j,r+s}})$ when we set $\tilde{\xi}_j = \sum_{a=1}^{r+s} v_{ja} p_a$. By the formula (3.6) and Corollary 4.4, the A-period $\Pi(\theta_v, \mathcal{E})$ of $\mathcal{X}$ is given by

$$\Pi(\theta_v,\mathcal{E})(\tau(q),z) = \Big(I^v(q,-z),\ z^{n-\frac{\deg}{2}} z^\rho \Psi(\mathcal{E})\Big)_{\mathrm{orb}}. \tag{5.1}$$

Here θ_v is the section[9] of the quantum D-module $QDM(\mathcal{X})$ of $\mathcal{X}$ in Corollary 4.4. Define the (partial) Laplace transform $\widehat{\Pi}(\phi,\mathcal{E})$ by

$$\widehat{\Pi}(\phi,\mathcal{E})(q,s,z) :=$$
$$\Big(\prod_{j=1}^c s_j\Big) \int_0^\infty \cdots \int_0^\infty \Pi(\phi,\mathcal{E}) \Big(\tau(\textstyle\prod_{j=1}^c (zr_j)^{\tilde{\xi}_j} \cdot q), z\Big) e^{-\sum_{j=1}^c r_j s_j} dr_1 \cdots dr_c.$$

where $s = (s_1, \ldots, s_c) \in (\mathbb{R}_{>0})^c$. Note that $\Pi(\phi,\mathcal{E})(\tau(q),z)$, $\widehat{\Pi}(\phi,\mathcal{E})(q,s,z)$ are multi-valued. We can regard them as a single-valued function in $(\log q_1, \ldots, \log q_{r+s}, \log z)$.

[9] More precisely, θ_v is a section of $\tau^* QDM(\mathcal{X})$. The A-period $\Pi(\theta_v,\mathcal{E})$ should be understood as the pairing of $\theta_v(q,-z)$ and $(\tau^*\mathfrak{s}(\mathcal{E}))(q,z)$ in the pulled-back quantum D-module.

Proposition 5.1. — *For $\mathcal{E} \in K(\mathcal{X})$ and $v \in \mathrm{Box}$, we have*

$$\widehat{\Pi}(\theta_v, \mathcal{E})(q,s,z) = \left(I^v_{\mathcal{V}}\left(q'_s, -z\right),\ z^{n-\frac{\deg}{2}} z^{\rho_{\mathcal{Y}}} \Psi_{\mathcal{V}}(\mathcal{E})\right)_{\mathrm{orb}}$$

where we set $\Psi_{\mathcal{V}}(\mathcal{E}) := e^{\pi \mathbf{i} c_1(\mathcal{V})}\widehat{\Gamma}(\mathcal{V}^\vee) \cup \Psi(\mathcal{E})$, $\rho_{\mathcal{Y}} := c_1(\mathcal{Y}) = \rho - \sum_{j=1}^c \xi_j$ and

$$q'_s = \prod_{j=1}^c (e^{\pi\mathbf{i}} s_j^{-1})^{\tilde{\xi}_j} \cdot q. \quad \textit{i.e.,}\ \log q'_{s,a} = \log q_a + \sum_{j=1}^c (\pi\mathbf{i} - \log s_j) v_{ja},\ \log s_j > 0.$$

Note that the right-hand side also gives the analytic continuation of $\widehat{\Pi}(\theta_v,\mathcal{E})$ in s.

Proof. — First we calculate the Laplace transform of $I^v(q,-z)$. Writing $I^v(q,-z) = e^{-\overline{p}\log q/z} \sum_{d\in\mathbb{K}_v} q^{d+v} \overline{\Box}_d \mathbf{1}_{\{-d\}}$, we have

$$\begin{aligned}
\int_0^\infty \cdots \int_0^\infty I^v\left(\prod (r_j z)^{\tilde{\xi}_j} \cdot q, -z\right) e^{-\sum_{j=1}^c r_j s_j} dr_1 \cdots dr_c \\
= \sum_{d \in \mathbb{K}_v, d+v \in \widehat{\mathrm{NE}}_{\mathcal{X}}} e^{-\overline{p}\log q/z} q^{d+v} \overline{\Box}_d \\
\times \prod_{j=1}^c \int_0^\infty e^{-\xi_j \log(r_j z)/z} (r_j z)^{\langle \tilde{\xi}_j, d+v\rangle} e^{-r_j s_j} dr_j\, \mathbf{1}_{\{-d\}} \\
= \sum_{d \in \mathbb{K}_v, d+v \in \widehat{\mathrm{NE}}_{\mathcal{X}}} e^{-\overline{p}\log q/z} q^{d+v} \overline{\Box}_d \\
\times \prod_{j=1}^c (z/s_j)^{\langle \tilde{\xi}_j, d+v\rangle - \frac{\xi_j}{z}} s_j^{-1} \Gamma\left(1 + \langle \tilde{\xi}_j, d+v\rangle - \tfrac{\xi_j}{z}\right) \mathbf{1}_{\{-d\}}\,.
\end{aligned}$$

Using $\Gamma(1+x) = x\Gamma(x)$ and $\langle \tilde{\xi}_j, d+v\rangle \in \mathbb{Z}_{\geqslant 0}$, we find that this is $(s_1\cdots s_c)^{-1}$ times

$$\prod_{j=1}^c e^{(\pi\mathbf{i} - \log z)\xi_j/z} \Gamma\left(1 - \xi_j/z\right) \cdot I^v_{\mathcal{V}}\left(\prod_{j=1}^c (e^{\pi\mathbf{i}}/s_j)^{\tilde{\xi}_j} \cdot q, -z\right).$$

The conclusion now easily follows from this and (5.1). □

Remark 5.2. — The integral in r_j yielding the factor $\Gamma_j = \Gamma(1 + \langle \tilde{\xi}_j, d+v\rangle - \xi_j/z)$ in the above calculation should be understood as a vector-valued integration. The exchange of sum and integral is justified by the estimates $\|\Box_d\| \leqslant C_1 C_2^{|d|}/\langle \hat{\rho}, d\rangle!$ and $\|\Box_d \prod_{j=1}^c \Gamma_j\| \leqslant C_1 C_2^{|d|}/\langle \hat{\rho}_{\mathcal{Y}}, d\rangle!$ for some $C_1, C_2 > 0$. Here $|d| = \sum_{a=1}^{r+s} \langle p_a, d\rangle$ and $\|\cdot\|$ is the operator norm with respect to some norm on $H^*_{\mathrm{orb}}(\mathcal{X})$. Note that we need the fact that $\hat{\rho}, \hat{\rho}_{\mathcal{Y}}$ are extended nef.

Remark 5.3. — We can view the map $\Psi_{\mathcal{V}}(\mathcal{E})$ as defining a dual[10] integral structure of the Euler twisted theory.

Using the mirror theorem (Corollary 4.8) we obtain the following corollary.

Corollary 5.4. — *For an algebraic vector bundle $\mathcal{E}$ on $\mathcal{Y}$, we have*

$$(2\pi\mathbf{i})^c \Pi_{\mathcal{Y}}(\Upsilon_v, \mathcal{E})\left(\varsigma\left(q'_s\right), z\right) = \widehat{\Pi}(\theta_v, \iota_*\mathcal{E})(q, s, z),$$

where $\Pi_{\mathcal{Y}}(\Upsilon_v, \mathcal{E})$ *denotes the A-period* (3.5) *for* $\mathcal{Y}$ *and* Υ_v *is the section of the quantum D-module of* $\mathcal{Y}$ *appearing in Corollary 4.8 and* (4.6).

Proof. — By Toen's Grothendieck-Riemann-Roch [54] we have

$$\widetilde{\mathrm{ch}}(\iota_*\mathcal{E}) = \iota_*\left(\prod_{j=1}^{c} \frac{1-e^{-\xi_j}}{\xi_j} \cdot \widetilde{\mathrm{ch}}(\mathcal{E})\right). \tag{5.2}$$

Using this, $\iota^*\widehat{\Gamma}_{\mathcal{X}} = \widehat{\Gamma}_{\mathcal{Y}} \cup \iota^*\widehat{\Gamma}(\mathcal{V})$ and $\Gamma(1-x)\Gamma(1+x) = \pi x/\sin(\pi x)$, we find

$$\Psi_{\mathcal{V}}(\iota_*\mathcal{E}) = (2\pi\mathbf{i})^c \iota_* \Psi_{\mathcal{Y}}(\mathcal{E}).$$

Here $\Psi_{\mathcal{Y}}$ denotes the map Ψ (3.3) for $\mathcal{Y}$. The conclusion follows from the previous proposition and Corollary 4.8. $\square$

5.2. Mirror Construction

In toric geometry, various mirror constructions have been found by Batyrev [4], Batyrev-Borisov [5], Givental [26] and Hori-Vafa [32]. Following Givental and Hori-Vafa, we shall construct mirrors for nef complete intersections in toric orbifolds as Landau-Ginzburg models.

Let $\mathcal{X}$ be a toric orbifold in Section 4.1. A *nef partition* is a partition $\{1,\dots,m\} = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_c$ such that $\xi_j := \sum_{i\in I_j} \overline{D}_i$ is nef for all $0 \leqslant j \leqslant c$ and that $\xi_1,\dots,\xi_c$ are pulled back from the coarse moduli space X, *i.e.,* they are Cartier divisors on X. This is a special case of the situation in Section 4.3. In the case of the original nef partition due to Batyrev-Borisov [5], I_0 is assumed to be empty. We need not to assume that ξ_0 is Cartier on X. As before, we assume the existence of a quasi-smooth complete intersection $\mathcal{Y} \subset \mathcal{X}$ with respect to $\mathcal{V} = \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_c$

[10] Because the Euler twisted theory has a degenerate pairing, we have to distinguish the $\widehat{\Gamma}$-integral structure from its dual: the integral structure itself should be given by $\Psi^{\mathcal{V}}(\mathcal{E}) = \widehat{\Gamma}(\mathcal{V})^{-1}\Psi(\mathcal{E})$.

where $\mathcal{L}_i$ is the line bundle in the class $\xi_i \in \mathrm{Pic}(\mathcal{X})$. Let $\varphi_j \colon \mathbf{N}_\mathbb{R} \to \mathbb{R}$ be the piecewise linear function defined by $\varphi_j(b_i) = 1$ for $i \in I_j$ and $\varphi_j(b_i) = 0$ for $i \in \{1, \dots, m\} \setminus I_j$. By the assumption $\varphi_j(b_i)$ is 0 or 1 for all $1 \leqslant i \leqslant m+s$ and $j \geqslant 1$. We set $\widehat{I}_j := \{1 \leqslant i \leqslant m+s \mid \varphi_j(b_i) = 1\}$ for $j \geqslant 1$. The sets $\widehat{I}_1, \dots, \widehat{I}_c$ are mutually disjoint. Set $\widehat{I}_0 := \{1, \dots, m+s\} \setminus \bigcup_{j=1}^c \widehat{I}_j$.

Consider the torus $\check{\mathbb{T}} := \mathrm{Hom}(\mathbf{N}, \mathbb{C}^\times) = \mathbf{M} \otimes_\mathbb{Z} \mathbb{C}^\times$. Let t denote a point on $\check{\mathbb{T}}$. Each element $b \in \mathbf{N}$ defines a function $t^b \colon \check{\mathbb{T}} \to \mathbb{C}^\times$. Take $\alpha = (\alpha_1, \dots, \alpha_{m+s})$ in $(\mathbb{C}^\times)^{m+s}$ and define the Laurent polynomials $W_\alpha^{(0)}(t), \dots, W_\alpha^{(c)}(t)$ on $\check{\mathbb{T}}$ as $W_\alpha^{(j)}(t) = \sum_{i \in \widehat{I}_j} \alpha_i t^{b_i}$. A mirror of $\mathcal{Y}$ is given by the complete intersection in $\check{\mathbb{T}}$

$$\check{Y}_\alpha = \{t \in \check{\mathbb{T}} \mid W_\alpha^{(1)}(t) = \cdots = W_\alpha^{(c)}(t) = 1\}$$

endowed with a holomorphic function $W_\alpha^{(0)} \colon \check{Y}_\alpha \to \mathbb{C}$. The pair $(\check{Y}_\alpha, W_\alpha^{(0)})$ is called the *Landau-Ginzburg model*. We assume that $\check{Y}_\alpha$ is a non-empty smooth complete intersection for generic α. The translation of the torus $\check{\mathbb{T}}$ induces the $\check{\mathbb{T}}$-action on the parameter space: $\alpha \mapsto t \cdot \alpha := (t^{b_1}\alpha_1, \dots, t^{b_{m+s}} \alpha_{m+s})$. Then $(\check{Y}_\alpha, W_\alpha^{(0)}) \cong (\check{Y}_{t\cdot\alpha}, W_{t\cdot\alpha}^{(0)})$. Therefore the parameter space of the mirror family descends to $\mathcal{M}$ (in Section 4.2) via the exact sequence (the divisor sequence tensored with $\mathbb{C}^\times$):

$$1 \longrightarrow \check{\mathbb{T}} = \mathbf{M} \otimes_\mathbb{Z} \mathbb{C}^\times \longrightarrow (\mathbb{C}^\times)^{m+s} \longrightarrow \mathcal{M} = \mathbb{L}^* \otimes_\mathbb{Z} \mathbb{C}^\times \longrightarrow 1.$$

In [37] we considered the mirror of a toric orbifold $\mathcal{X}$ itself. In this case $I_0 = \{1, \dots, m\}$ and the mirror is the family of functions $\sum_{j=1}^{m+s} \alpha_j t^{b_j} : \check{\mathbb{T}} \to \mathbb{C}$.

Remark 5.5. — Batyrev and Borisov [5] dealt with the case where I_0 is empty. In this case $\mathcal{Y}$ is Calabi-Yau. They considered a Calabi-Yau compactification of $\check{Y}_\alpha$ in a toric variety $\widehat{\mathbb{P}}_\nabla$. Here $\widehat{\mathbb{P}}_\nabla$ is a crepant partial resolution of the toric variety $\mathbb{P}_\nabla$ associated with the polytope $\nabla = \nabla_1 + \cdots + \nabla_c \subset \mathbf{N}_\mathbb{R}$, where ∇_i is the convex hull of $\{b_j \mid j \in I_i\} \cup \{0\}$. It would be interesting to find a partial compactification of $(\check{Y}_\alpha, W_\alpha^{(0)})$ with good topological properties.

Remark 5.6. — We hope that the existence of a quasi-smooth complete intersection $\mathcal{Y}$ and that of a smooth complete intersection $\check{Y}_\alpha$ are related. In the Batyrev-Borisov construction [5], it was shown that a general complete intersection $\mathcal{Y}$ is quasi-smooth if and only if the compactification of a general $\check{Y}_\alpha$ is quasi-smooth.

5.3. A-Periods and B-Periods

Take $\mathbb{C}^\times$-co-ordinates $(t_1, \dots, t_n)$ on $\check{\mathbb{T}}$ associated with a basis of $\mathbf{N}$. Define a holomorphic volume form Ω_α on $\check{Y}_\alpha$ by

$$\Omega_\alpha = \frac{\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}}{dW_\alpha^{(1)} \wedge \cdots \wedge dW_\alpha^{(c)}}.$$

We shall consider the following oscillatory integral (B-periods):

$$\int_{\Gamma(\alpha)} \phi(t) e^{-W_\alpha^{(0)}(t)/z} \Omega_\alpha \tag{5.3}$$

for a Laurent polynomial $\phi\colon \check{\mathbb{T}} \to \mathbb{C}$ and a possibly noncompact cycle $\Gamma(\alpha) \subset \check{Y}_\alpha$ such that $\Re(W_\alpha^{(0)}(t)/z) \to \infty$ in the end of $\Gamma(\alpha)$. More generally, for $\vec{k} = (k_1, \dots, k_c) \in (\mathbb{Z}_{\geqslant 0})^c$, we introduce the residue symbol

$$\operatorname{Osc}(\phi, \vec{k}; \alpha) = \left(\prod_{j=1}^c z^{k_j} k_j!\right) \operatorname{Res}_{\check{Y}_\alpha} \left(\frac{\phi(t) e^{-W_\alpha^{(0)}(t)/z} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}}{\prod_{j=1}^c (W_\alpha^{(j)}(t) - 1)^{k_j+1}} \right)$$

and define the "oscillatory" residue integral

$$\int_{\Gamma(\alpha)} \operatorname{Osc}(\phi, \vec{k}; \alpha) = \frac{\prod_{j=1}^c z^{k_j} k_j!}{(2\pi \mathtt{i})^c} \int_{T(\Gamma(\alpha))} \frac{\phi(t) e^{-W_\alpha^{(0)}(t)/z} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}}{\prod_{j=1}^c (W_\alpha^{(j)}(t) - 1)^{k_j+1}}. \tag{5.4}$$

Here $T(\Gamma(\alpha)) \subset \check{\mathbb{T}}$ is a cycle given as follows: Take a small tubular neighbourhood N of $\check{Y}_\alpha$ in $\check{\mathbb{T}}$. Then $N \setminus \bigcup_{j=1}^c (W_\alpha^{(j)})^{-1}(1)$ has a deformation retraction to an $(S^1)^c$-bundle $T \to \check{Y}_\alpha$. We take $T(\Gamma(\alpha))$ to be the total space of $T|_{\Gamma(\alpha)}$. Note that (5.4) equals (5.3) when $\vec{k} = 0$.

In this section we consider the integral over the *real locus*. Set[11] $\check{\mathbb{T}}_\mathbb{R} := \operatorname{Hom}(\mathbf{N}, \mathbb{R}_{>0}) = \{(t_1, \dots, t_n) \,|\, t_i > 0 \,(\forall i)\}$. When $\alpha \in (\mathbb{R}_{>0})^{m+s}$, we can define the real cycle $\Gamma_\mathbb{R}(\alpha)$ in $\check{Y}_\alpha$ by $\Gamma_\mathbb{R}(\alpha) := \check{Y}_\alpha \cap \check{\mathbb{T}}_\mathbb{R}$. Similarly we define $\mathcal{M}_\mathbb{R} := \operatorname{Hom}(\mathbb{L}, \mathbb{R}_{>0}) \subset \mathcal{M}$. For $\alpha \in (\mathbb{R}_{>0})^{m+s}$, we have the estimate[12]:

$$\sum_{j=0}^c W_\alpha^{(j)}(t) = \sum_{i=1}^{m+s} \alpha_i t^{b_i} \geqslant \epsilon(\alpha) \max_{1 \leqslant i \leqslant n} \left\{ t_i^{1/N}, t_i^{-1/N} \right\} \qquad \forall t \in \check{\mathbb{T}}_\mathbb{R}$$

for some $\epsilon(\alpha) > 0$ and $N \in \mathbb{N}$. Restricting this to $\Gamma_\mathbb{R}(\alpha) = \check{\mathbb{T}}_\mathbb{R} \cap \check{Y}_\alpha$, we get $W_\alpha^{(0)}(t) + c \geqslant \epsilon(\alpha) \max_{1 \leqslant i \leqslant n} \{t_i^{1/N}, t_i^{-1/N}\}$. Consider the integrals (5.3), (5.4) with $\Re(z) > 0$, $\Gamma(\alpha) = \Gamma_\mathbb{R}(\alpha)$ and $\alpha \in (\mathbb{R}_{>0})^{m+c}$. Take $\mathbb{P}^n$

[11] Note that this does not depend on the choice of co-ordinates t_i on $\check{\mathbb{T}}$.

[12] Use the fact that 0 is in the interior of Δ and the inequality $\beta_1 x_1 + \cdots + \beta_k x_k \geqslant x_1^{\beta_1} \cdots x_k^{\beta_k}$ for $x_i > 0$, $\beta_i \geqslant 0$, $\sum_{i=1}^k \beta_i = 1$.

as a compactification of $\check{\mathbb{T}}$. Then the convergence of the integral (5.3) is ensured by the exponential factor $e^{-W_\alpha^{(0)}(t)/z}$ because $\phi(t)\Omega_\alpha$ grows at most polynomially near the infinity $\mathbb{P}^n \setminus \check{\mathbb{T}}$. By taking $T(\Gamma_\mathbb{R}(\alpha))$ to be a semi-algebraic cycle (as in [50, Appendix]) which is sufficiently close to $\Gamma_\mathbb{R}(\alpha)$, we have the convergence of (5.4) similarly.

For $v \in \mathrm{Box}$, we set $\alpha^v := \prod_{j\in\sigma} \alpha_j^{c_j}$ when $v = \sum_{j\in\sigma} c_j b_j$ for some cone σ and $c_j \in [0,1)$. The following is the first main theorem of this paper.

Theorem 5.7. — *Let $\mathcal{Y}$ be a toric complete intersection in Section 5.2. The A-periods* (3.5) *of the structure sheaf $\mathcal{O}_\mathcal{Y}$ equal the oscillatory residue integrals over $\Gamma_\mathbb{R}(\alpha)$.*

$$\Pi_\mathcal{Y}(\Upsilon_v, \mathcal{O}_\mathcal{Y})(\varsigma(q), z) = \int_{\Gamma_\mathbb{R}(\alpha)} \mathrm{Osc}(\alpha^v t^v, \vec{\varphi}(v); \alpha).$$

Here $\vec{\varphi}(v) = (\varphi_1(v), \dots, \varphi_c(v))$, $v \in \mathrm{Box}$, $\alpha \in (\mathbb{R}_{>0})^{m+s}$, $q = [\alpha] \in \mathcal{M}_\mathbb{R}$, $z > 0$ and the functions $\varsigma(q)$, $\Upsilon_v(q)$ are as in Corollary 4.8 (see also (4.5), (4.6)*). In particular, the quantum cohomology central charge $Z_\mathcal{Y}(\mathcal{O}_\mathcal{Y}) = (2\pi\mathtt{i})^{-\dim\mathcal{Y}}\Pi_\mathcal{Y}(\mathbf{1}, \mathcal{O}_\mathcal{Y})$ is given by*

$$Z_\mathcal{Y}(\mathcal{O}_\mathcal{Y})(\varsigma(q), z) = \frac{1}{(2\pi\mathtt{i})^{\dim\mathcal{Y}} F(q)} \int_{\Gamma_\mathbb{R}(\alpha)} e^{-W_\alpha^{(0)}(t)/z}\Omega_\alpha.$$

Moreover, the A-period $\Pi_\mathcal{Y}(\Upsilon_v, \iota^*\mathcal{E})$ for $\mathcal{E} \in K(\mathcal{X})$ is in the $\mathbb{Z}$-span of the monodromy transforms of $\Pi_\mathcal{Y}(\Upsilon_v, \mathcal{O}_\mathcal{Y})$ with respect to the monodromy around $q = \mathbf{0}$.

Remark 5.8. — The A-periods $\Pi_\mathcal{Y}(\Upsilon_\gamma, \iota^*\mathcal{E})$ should be written as an oscillatory integral over an integral cycle $\Gamma_\mathcal{E}$ which is monodromy-generated by $\Gamma_\mathbb{R}$, but we do not know its explicit representative.

Remark 5.9. — By Corollary 4.8, the A-period $\Pi_\mathcal{Y}(\Upsilon_v, \mathcal{E})$ for $\mathcal{E} \in K(\mathcal{Y})$ can be expressed in terms of the explicit hypergeometric function $I_\mathcal{Y}^v$:

$$\Pi_\mathcal{Y}(\Upsilon_v, \mathcal{E})(\varsigma(q), z) = \left(\iota^* I_\mathcal{Y}^v(q, -z), z^{\dim\mathcal{Y} - \frac{\deg}{2}} z^{\rho_\mathcal{Y}} \Psi_\mathcal{Y}(\mathcal{E})\right)_{\mathrm{orb}}.$$

Hence theorem 5.7 gives equalities of oscillatory integrals and hypergeometric series.

Proof of Theorem 5.7. — The case $\mathcal{Y} = \mathcal{X}$ was proved in [37, Theorem 4.14]. In this case we have the following:

$$\Pi(\mathbf{1}, \mathcal{O}_\mathcal{X})(\tau(q), z) = \int_{\check{\mathbb{T}}_\mathbb{R}} e^{-\sum_{j=0}^c W_\alpha^{(j)}(t)/z} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n},$$

where $q = [\alpha] \in \mathcal{M}_\mathbb{R}$, $\alpha \in (\mathbb{R}_{>0})^{m+s}$. By (4.3) and (5.1), we know that $\Pi(\theta_v, \mathcal{O}_\mathcal{X})$ can be obtained from $\Pi(\mathbf{1}, \mathcal{O}_\mathcal{X})$ by differentiation. Using the fact that the vector field $\boldsymbol{D}_i$ there is lifted to $z\alpha_i(\partial/\partial\alpha_i)$ on the α-space, we calculate

$$\Pi(\theta_v,\mathcal{O}_{\mathcal{X}})(\tau(q),z)=\int_{\check{\mathbb{T}}_{\mathbb{R}}}\alpha^v t^v e^{-\sum_{j=0}^{c}W_\alpha^{(j)}(t)/z}\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_n}{t_n}.$$

By the assumption that $\check{Y}_\alpha$ is smooth for generic α, we can see that the map

$$\vec{W}_\alpha:=(W_\alpha^{(1)},\ldots,W_\alpha^{(c)})\colon\check{\mathbb{T}}_{\mathbb{R}}\longrightarrow(\mathbb{R}_{>0})^c$$

is generically submersive for generic $\alpha\in(\mathbb{R}_{>0})^{m+s}$. Hence the above oscillatory integral can be rewritten as

$$\Pi(\theta_v,\mathcal{O}_{\mathcal{X}})(\tau(q),z)=\int_{u\in(\mathbb{R}_{>0})^c}\prod_{j=1}^{c}du_j e^{-\sum_{j=1}^{c}u_j/z}P_v(\alpha,u),$$

where $P_v(\alpha,u):=\int_{\check{\mathbb{T}}_{\mathbb{R}}\cap\{\vec{W}_\alpha(t)=u\}}\alpha^v t^v e^{-W_\alpha^{(0)}(t)/z}\frac{\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_n}{t_n}}{dW_\alpha^{(1)}\wedge\cdots\wedge dW_\alpha^{(r)}}$. Here we set $P_v(\alpha,u)=0$ if u is not in the image of $\vec{W}_\alpha$. We take the partial Laplace transform of the both-hand sides. Under the divisor map $D\colon\mathbb{Z}^{m+s}\to\mathbb{L}^*$, $\tilde{\xi}_j$ can be lifted to the sum $\sum_{i\in\widehat{I}_j}e_i\in\mathbb{Z}^{m+s}$. Therefore the c-dimensional flow $q\mapsto\prod_{j=1}^{c}(zr_j)^{\tilde{\xi}_j}\cdot q$ on $\mathcal{M}$ can be lifted to the flow on the α-space scaling $W_\alpha^{(j)}$ by zr_j for $1\leqslant j\leqslant c$ and leaving $W_\alpha^{(0)}$ invariant. Hence $\widehat{\Pi}(\theta_v,\mathcal{O}_{\mathcal{X}})(q,s,z)$ with $s\in(\mathbb{R}_{>0})^c$ equals

$$\begin{aligned}\left(\prod_{k=1}^{c}\int_0^\infty s_k dr_k\right)\int_{u\in(\mathbb{R}_{>0})^c}&\left(\prod_{j=1}^{c}du_j e^{-(u_j+s_j)r_j}(zr_j)^{\varphi_j(v)}\right)P_v(\alpha,u)\\&=\int_{u\in(\mathbb{R}_{>0})^c}\left(\prod_{j=1}^{c}\frac{du_j s_j z^{\varphi_j(v)}\varphi_j(v)!}{(u_j+s_j)^{1+\varphi_j(v)}}\right)P_v(\alpha,u).\end{aligned}$$

We can change the order of the integration by Fubini since the integrand can be viewed as a non-negative measure. Because $\widehat{\Pi}(\theta_v,\mathcal{O}_{\mathcal{X}})$ is well-defined by Proposition 5.1, the integral in the right-hand side also converges for $s_j>0$. If $s_j\in\mathbb{C}\setminus\mathbb{R}_{\leqslant 0}$, there exists a constant $C(s_j)>0$ such that

$$\left|\frac{1}{(u_j+s_j)^{1+\varphi_j(v)}}\right|\leqslant\frac{C(s_j)}{(u_j+1)^{1+\varphi_j(v)}}\quad\text{for all } u_j>0.$$

Therefore the above integral can be analytically continued in s and makes sense for $s_j\in\mathbb{C}\setminus\mathbb{R}_{<0}$. The jump of the values across the branch cut $\mathbb{R}_{<0}$ can be calculated by the Cauchy integral formula. For a function $f(s_j)$ on

$\mathbb{C} \setminus \mathbb{R}_{<0}$ we set $(\boldsymbol{\Delta}_j f)(-s_j) := \lim_{\epsilon \to +0}(f(-s_j - \mathbf{i}\epsilon) - f(-s_j + \mathbf{i}\epsilon))$ for $s_j > 0$. Then we have

$$\left(\boldsymbol{\Delta}_1 \cdots \boldsymbol{\Delta}_c \widehat{\Pi}(\theta_v, \mathcal{O}_{\mathcal{X}})\right)(q, -s_1, \ldots, -s_c, z) = \prod_{j=1}^{c}(-2\pi\mathbf{i}s_j) \prod_{j=1}^{c} \left(z \frac{\partial}{\partial s_j}\right)^{\varphi_j(v)} P_v(\alpha, s).$$

On the other hand, we can calculate the left-hand side using Proposition 5.1 when q is sufficiently close to $\mathbf{0}$ and $s_j \geqslant 1$. It is

$$\begin{aligned}
&\left(\textstyle\prod_{j=1}^{c}(e^{-2\pi\mathbf{i}\xi_j/z} - 1) \cup I_{\mathcal{V}}^{v}\left(\prod_{j=1}^{c} s_j^{-\tilde{\xi}_j} \cdot q, -z\right),\ z^{n-\frac{\deg}{2}} z^{\rho_{\mathcal{Y}}} \Psi_{\mathcal{V}}(\mathcal{O}_{\mathcal{X}})\right)_{\mathrm{orb}} \\
&\quad = (-1)^c \widehat{\Pi}(\theta_v, \iota_* \mathcal{O}_{\mathcal{Y}})(q, e^{\pi\mathbf{i}}s, z) \qquad \text{by (5.2) and Proposition 5.1} \\
&\quad = (-2\pi\mathbf{i})^c \Pi_{\mathcal{Y}}(\Upsilon_v, \mathcal{O}_{\mathcal{Y}}) \left(\varsigma\left(\textstyle\prod_{j=1}^{c} s_j^{-\tilde{\xi}_j} \cdot q\right), z\right) \qquad \text{by Corollary 5.4.}
\end{aligned}$$

We arrive at the formula in the theorem by differentiating

$$P_v(\alpha, s) = \int_{T(\check{\mathbb{T}}_{\mathbb{R}} \cap \{\vec{W}_\alpha(t) = s\})} \frac{\alpha^v t^v e^{-W_\alpha^{(0)}(t)/z} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}}{(W_\alpha^{(1)}(t) - s_1) \cdots (W_\alpha^{(c)}(t) - s_c)}$$

in $s_1, \ldots, s_c$ and setting $s_1 = \cdots = s_c = 1$. The last statement follows from the fact that $K(\mathcal{X})$ is generated by line bundles [10] and the monodromy formula

$$\Pi_{\mathcal{Y}}(\phi, \mathcal{E})(\varsigma(e^{2\pi\mathbf{i}\tilde{\xi}} \cdot q), z) = \Pi_{\mathcal{Y}}(\phi, \iota^*(\mathcal{L}_\xi^\vee) \otimes \mathcal{E})(\varsigma(q), z)$$

where $\mathcal{E} \in K(\mathcal{Y})$, $\tilde{\xi} \in \mathbb{L}^*$ and $\xi \in \mathrm{Pic}(\mathcal{X})$ is its image. (It follows from Definition 3.6, (ii) and $\varsigma(e^{2\pi\mathbf{i}\tilde{\xi}} \cdot q) = G(\iota^*\xi)^{-1}\varsigma(q)$.) □

6. Toric Calabi-Yau Hypersurfaces

In this section we restrict our attention to a Calabi-Yau hypersurface $\mathcal{Y}$ in a Gorenstein weak Fano toric orbifold $\mathcal{X}$. Based on the period calculation in Theorem 5.7, we study mirror symmetry of $\mathcal{Y}$ as an isomorphism of variations of Hodge structure and compare the integral structures on the both sides. We set $n := \dim_{\mathbb{C}} \mathcal{X}$ as before.

6.1. Batyrev Mirror

Batyrev [4] constructed mirror pairs of Calabi-Yau hypersurfaces based on the duality of reflexive polytopes. This is the case where the ambient toric orbifold $\mathcal{X}$ is Gorenstein (*i.e.,* $K_{\mathcal{X}}$ is pulled back from the coarse moduli space) and we take the nef partition $I_0 = \emptyset$, $I_1 = \{1, \dots, m\}$ in Section 5.2. We refer the reader to [4], [20, Section 4] for details. The fan polytope Δ is said to be *reflexive* if the integral distance between 0 and all hyperplanes generated by codimension one faces equal 1. If Δ is reflexive, the dual polytope $\Delta^* = \{y \in \mathbf{M}_{\mathbb{R}} \mid \langle x, y\rangle \geqslant -1\,(\forall x \in \Delta)\}$ have integral points as its vertices and $(\Delta^*)^* = \Delta$. A weak Fano toric orbifold is Gorenstein if and only if its fan polytope is reflexive. A general anticanonical section $\mathcal{Y}$ in a weak Fano Gorenstein toric orbifold $\mathcal{X}$ is a quasi-smooth Calabi-Yau orbifold. As in Section 5.2, the (uncompactified) mirror $\check{Y}_\alpha$ of $\mathcal{Y}$ is given as a hypersurface in the torus $\check{\mathbb{T}}$:

$$\check{Y}_\alpha = W_\alpha^{-1}(1) \subset \check{\mathbb{T}}, \quad W_\alpha(t) := W_\alpha^{(1)}(t) = \sum_{i=1}^{m+s} \alpha_i t^{b_i}.$$

Take $\mathbb{C}^\times$ co-ordinates $(t_1, \dots, t_n)$ on $\check{\mathbb{T}}$ as before. Introduce another variable t_0 and define the subring S_Δ of $\mathbb{C}[t_0, t_1^\pm, \dots, t_n^\pm]$ by

$$S_\Delta = \bigoplus_{k \geqslant 0} S_\Delta^k, \quad S_\Delta^k = \bigoplus_{b \in k\Delta \cap \mathbf{N}} \mathbb{C} t^b t_0^k.$$

It is graded by the degree of t_0. The toric variety $\mathbb{P}_\Delta := \operatorname{Proj} S_\Delta$ is a compactification of $\check{\mathbb{T}}$. The variety $\mathbb{P}_\Delta$ is associated with the normal fan of Δ and its fan polytope is Δ^*. The closure $\overline{\check{Y}}_\alpha$ of $\check{Y}_\alpha$ in $\mathbb{P}_\Delta$ is an anticanonical section of $\mathbb{P}_\Delta$. We say that $\check{Y}_\alpha$ is Δ-*regular* if the intersection of $\overline{\check{Y}}_\alpha$ and every torus orbit O in $\mathbb{P}_\Delta$ is a smooth subvariety of codimension 1 in O. The set of parameters α for which $\check{Y}_\alpha$ is Δ-regular is a non-empty Zariski open subset $(\mathbb{C}^\times)^{m+s}_{\mathrm{reg}}$ of $(\mathbb{C}^\times)^{m+s}$. This is invariant under the action of $\check{\mathbb{T}}$ and descends to a Zariski open subset $\mathcal{M}_{\mathrm{reg}}$ in $\mathcal{M}$. Let $\check{Y}_\alpha$ be Δ-regular. A resolution of $\overline{\check{Y}}_\alpha$ by a Calabi-Yau orbifold is constructed as follows. Choose a projective crepant resolution $\check{\mathcal{X}} \to \mathbb{P}_\Delta$ by a toric orbifold $\check{\mathcal{X}}$. This amounts to choosing a triangulation of the fan polytope Δ^*. The fan polytope of $\check{\mathcal{X}}$ is still Δ^*. Then the pull-back $\check{\mathcal{Y}}_\alpha \subset \check{\mathcal{X}}$ of $\overline{\check{Y}}_\alpha$ is a quasi-smooth Calabi-Yau hypersurface which gives a crepant resolution of $\overline{\check{Y}}_\alpha$.

Remark 6.1. — Batyrev [4] showed that one can choose $\check{\mathcal{X}}$ with only terminal singularities (MPCP desingularization). In this case $\check{\mathcal{Y}}_\alpha$ becomes also terminal. In particular, we can take $\check{\mathcal{Y}}_\alpha$ to be smooth in dimension 3 because terminal Gorenstein orbifolds in dimension 3 are all smooth. From a viewpoint of orbifold mirror symmetry, we do not need to restrict our attention to terminal partial resolutions.

6.2. A-Model VHS of a Calabi-Yau Hypersurface

By the Gorenstein condition, the orbifold cohomology of $\mathcal{Y}$ is graded by even integers[(13)]. Since $\mathcal{Y}$ is Calabi-Yau, the quantum D-module $QDM(\mathcal{Y})$ of $\mathcal{Y}$ in Definition 3.1 restricted to $H^2_{\mathrm{orb}}(\mathcal{Y})$ and $z=1$ reduces to a *variation of Hodge structure* (VHS) in the classical sense (*cf.* Remark 3.3). See [20, Section 8.5]. We furthermore restrict our attention to the "ambient part". Set

$$H^*_{\mathrm{amb}}(\mathcal{Y}) := \mathrm{Im}(\iota^*\colon H^*_{\mathrm{orb}}(\mathcal{X}) \to H^*_{\mathrm{orb}}(\mathcal{Y})).$$

By Corollary 2.5, $H^*_{\mathrm{amb}}(\mathcal{Y})$ is closed under quantum product. For the convergence domain $U \subset H^*_{\mathrm{orb}}(\mathcal{Y})$ in Section 3.1, we set $U' = H^2_{\mathrm{amb}}(\mathcal{Y}) \cap U$. Take a $\mathbb{C}$-basis $\eta_1, \dots, \eta_\ell$ in $H^2_{\mathrm{amb}}(\mathcal{Y})$. Let $\tau^1, \dots, \tau^\ell$ be the corresponding co-ordinates on $H^2_{\mathrm{amb}}(\mathcal{Y})$ and $\tau = \sum_{i=1}^\ell \tau^i \eta_i$ be a general point on $H^2_{\mathrm{amb}}(\mathcal{Y})$.

Definition 6.2. — *The ambient A-model VHS of $\mathcal{Y}$ is the tuple $(\mathscr{H}_{\mathrm{A}}, \nabla^{\mathrm{A}}, \mathscr{F}^\bullet_{\mathrm{A}}, Q_{\mathrm{A}})$ consisting of the locally free sheaf $\mathscr{H}_{\mathrm{A}} = H^*_{\mathrm{amb}}(\mathcal{Y}) \otimes \mathcal{O}_{U'}$ over U', the flat Dubrovin connection $\nabla^{\mathrm{A}}\colon \mathscr{H}_{\mathrm{A}} \to \mathscr{H}_{\mathrm{A}} \otimes \Omega^1_{U'}$*

$$\nabla^{\mathrm{A}} = d + \sum_{i=1}^{\ell} (\eta_i \circ_\tau) d\tau^i,$$

the decreasing Hodge filtration $\mathscr{F}^p_{\mathrm{A}} = H^{\leqslant 2(n-1-p)}_{\mathrm{amb}}(\mathcal{Y}) \otimes \mathcal{O}_{U'}$ on $\mathscr{H}_{\mathrm{A}}$ and the ∇^{A}-flat $(-1)^{n-1}$-symmetric pairing $Q_{\mathrm{A}}\colon \mathscr{H}_{\mathrm{A}} \otimes \mathscr{H}_{\mathrm{A}} \to \mathcal{O}_{U'}$

$$Q_{\mathrm{A}}(\alpha, \beta) = (2\pi \mathtt{i})^{n-1} ((-1)^{\frac{\deg}{2}} \alpha, \beta)_{\mathrm{orb}}.$$

The Galois action of $\iota^* H^2(\mathcal{X}; \mathbb{Z})$ on the A-model VHS is defined similarly to (3.1), (3.2).

The A-model VHS satisfies Griffiths transversality and the Riemann bilinear relation:

$$\nabla^{\mathrm{A}} \mathscr{F}^p_{\mathrm{A}} \subset \mathscr{F}^{p-1}_{\mathrm{A}} \otimes \Omega^1_{U'}, \quad Q_{\mathrm{A}}(\mathscr{F}^p_{\mathrm{A}}, \mathscr{F}^{n-p}_{\mathrm{A}}) = 0. \tag{6.1}$$

By a result of Mavlyutov [47, Theorem 5.1], we have the decomposition $H^*_{\mathrm{orb}}(\mathcal{Y}) = H^*_{\mathrm{amb}}(\mathcal{Y}) \oplus \mathrm{Ker}(\iota_*)$. Since the two summands are orthogonal to each other with respect to the orbifold Poincaré pairing, we know that the polarization form Q_{A} is non-degenerate on the ambient part $\mathscr{H}_{\mathrm{A}}$.

The $\widehat{\Gamma}$-integral structure in Definition 3.6 induces an integral structure on the above A-model VHS. Let $L_{\mathcal{Y}}(\tau) := L_{\mathcal{Y}}(\tau, z=1)$ denote the fundamental solution of the quantum differential equation (with $Q=1$) of $\mathcal{Y}$. By

(13) Recall that we ignore odd cohomology classes on the A-side.

Proposition 2.4, we know that $L_{\mathcal{Y}}(\tau)$ with $\tau \in U'$ maps a class in $H^*_{\mathrm{amb}}(\mathcal{Y})$ to a flat section of $\mathscr{H}_{\mathrm{A}}$. Therefore, if $\mathcal{E} \in K(\mathcal{Y})$ satisfies $\widetilde{\mathrm{ch}}(\mathcal{E}) \in H^*_{\mathrm{amb}}(\mathcal{Y})$, we have a flat section $\mathfrak{s}(\mathcal{E})(\tau) = \mathfrak{s}(\mathcal{E})(\tau, z = 1)$ of $\mathscr{H}_{\mathrm{A}}$ in the same way as (3.3):

$$\mathfrak{s}(\mathcal{E})(\tau) = (2\pi\mathtt{i})^{-(n-1)} L_{\mathcal{Y}}(\tau)\Psi_{\mathcal{Y}}(\mathcal{E}),$$
$$\Psi_{\mathcal{Y}}(\mathcal{E}) = \widehat{\Gamma}_{\mathcal{Y}} \cup (2\pi\mathtt{i})^{\frac{\deg_0}{2}} \operatorname{inv}^* \widetilde{\mathrm{ch}}(\mathcal{E}).$$

Definition 6.3. — *Set $H_{\mathrm{A}} := \operatorname{Ker} \nabla^{\mathrm{A}} \subset \mathscr{H}_{\mathrm{A}}$. Define the local subsystem $H^{\mathrm{amb}}_{\mathrm{A},\mathbb{Z}}$ of H_{A} as*

$$H^{\mathrm{amb}}_{\mathrm{A},\mathbb{Z}} := \{\mathfrak{s}(\iota^*\mathcal{E}) \,|\, \mathcal{E} \in K(\mathcal{X})\}.$$

It is a $\mathbb{Z}$-lattice of H_{A} and preserved under the Galois action by $\iota^* H^2(\mathcal{X};\mathbb{Z})$. For $\mathfrak{s}(\mathcal{E}_1), \mathfrak{s}(\mathcal{E}_2) \in H^{\mathrm{amb}}_{\mathrm{A},\mathbb{Z}}$, we have

$$Q_{\mathrm{A}}(\mathfrak{s}(\mathcal{E}_1), \mathfrak{s}(\mathcal{E}_2)) = \chi_{\mathcal{Y}}(\mathcal{E}_1, \mathcal{E}_2).$$

We call $H^{\mathrm{amb}}_{\mathrm{A},\mathbb{Z}}$ the ambient $\widehat{\Gamma}$-integral structure on the ambient A-model VHS of $\mathcal{Y}$.

Remark 6.4. — The above integral structure also induces rational and real structures $H_{\mathrm{A},\mathbb{Q}}$, $H_{\mathrm{A},\mathbb{R}}$ on the A-model VHS. With respect to the real involution κ defined by $H_{\mathrm{A},\mathbb{R}}$, we hope to have the Hodge decomposition and the Riemann bilinear inequality:

$$\mathscr{H}_{\mathrm{A}} = \mathscr{F}^p_{\mathrm{A}} \oplus \kappa(\mathscr{F}^{n-p}_{\mathrm{A}}), \quad \mathtt{i}^{p-q} Q_{\mathrm{A}}(\phi, \kappa(\phi)) > 0 \tag{6.2}$$

for $\phi \in \mathscr{H}^{p,q}_{\mathrm{A}} = \mathscr{F}^p_{\mathrm{A}} \cap \kappa(\mathscr{F}^q_{\mathrm{A}}) \setminus \{0\}$, $q = n - 1 - p$. From a result in [35], it follows that these properties hold in a neighbourhood of the large radius limit(14). In fact, by mirror symmetry, we will see that these properties hold globally.

6.3. B-Model VHS

As we saw in Section 5.2, the parameter space of the mirror $\check{Y}_\alpha$ (or its compactification $\check{\mathcal{Y}}_\alpha$) descends to $\mathcal{M} = \mathbb{L}^* \otimes \mathbb{C}^\times$. We are interested in the VHS associated with the family of Δ-regular Calabi-Yau hypersurfaces:

$$\mathrm{pr}_2 \colon \check{\mathfrak{Y}} \overset{\mathrm{def}}{=} \left\{(t, \alpha) \in \check{\mathcal{X}} \times (\mathbb{C}^\times)^{m+s}_{\mathrm{reg}} \;\middle|\; t \in \check{\mathcal{Y}}_\alpha\right\} / \check{\mathbb{T}} \longrightarrow \mathcal{M}_{\mathrm{reg}}.$$

(14) We showed the Riemann bilinear inequality for the (p,p) part (or algebraic part) of quantum cohomology VHS in [35]. Note that the ambient part $H^*_{\mathrm{amb}}(\mathcal{Y})$ is contained in the (p,p) part.

In this paper, we restrict our attention to the VHS on the untwisted middle cohomology $H^{n-1}(\check{\mathcal{Y}}_\alpha)$. Furthermore, we only consider classes obtained as residues. Let $\mathrm{Res}\colon H^n(\check{\mathcal{X}} \setminus \check{\mathcal{Y}}_\alpha) \to H^{n-1}(\check{\mathcal{Y}}_\alpha)$ be the Poincaré residue map:

$$\int_\gamma \mathrm{Res}(\omega) = \frac{1}{2\pi \mathtt{i}} \int_{T(\gamma)} \omega, \quad \omega \in H^n(\check{\mathcal{X}} \setminus \check{\mathcal{Y}}_\alpha),$$

where $T(\gamma) \subset \check{\mathcal{X}} \setminus \check{\mathcal{Y}}_\alpha$ is a tube of an $(n-1)$-cycle $\gamma \subset \check{\mathcal{Y}}_\alpha$ (as appeared in (5.4)). Define the residue part of $H^{n-1}(\check{\mathcal{Y}}_\alpha)$ by

$$H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha) := \mathrm{Im}(\mathrm{Res}\colon H^0(\check{\mathcal{X}}, \Omega^n_{\check{\mathcal{X}}}(*\check{\mathcal{Y}}_\alpha)) \to H^{n-1}(\check{\mathcal{Y}}_\alpha)).$$

Here $H^0(\check{\mathcal{X}}, \Omega^n_{\check{\mathcal{X}}}(*\check{\mathcal{Y}}_\alpha))$ is the space of algebraic n-forms with arbitrary poles along $\check{\mathcal{Y}}_\alpha$. Let $\check{D}_1, \dots, \check{D}_N$ be the toric divisors of $\check{\mathcal{X}}$. We claim that we have the orthogonal decomposition with respect to the intersection pairing:

$$H^{n-1}(\check{\mathcal{Y}}_\alpha) = H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha) \oplus \left(\sum_{i=1}^N f_{i*} H^{n-3}(\check{\mathcal{Y}}_\alpha \cap \check{D}_i) \right) \tag{6.3}$$

where $f_i\colon \check{D}_i \cap \check{\mathcal{Y}}_\alpha \hookrightarrow \check{\mathcal{Y}}_\alpha$ is the inclusion. To see this, we use the Gysin exact sequence (see [46, Eqn (7)]):

$$\bigoplus_{i=1}^N H^{n-3}(\check{\mathcal{Y}}_\alpha \cap \check{D}_i) \xrightarrow{\bigoplus f_{i*}} H^{n-1}(\check{\mathcal{Y}}_\alpha) \to W_{n-1}(H^{n-1}(\check{Y}_\alpha)) \to 0. \tag{6.4}$$

Here $W_\bullet(H^{n-1}(\check{Y}_\alpha))$ denotes the weight filtration of Deligne's mixed Hodge structure. In the proof of [46, Theorem 4.4], it is shown that the composition $H^0(\check{\mathcal{X}}, \Omega^n_{\check{\mathcal{X}}}(*\check{\mathcal{Y}}_\alpha)) \to H^{n-1}(\check{\mathcal{Y}}_\alpha) \to W_{n-1}(H^{n-1}(\check{Y}_\alpha))$ is surjective. Hence $H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha)$ and $\sum_{i=1}^N f_{i*} H^{n-3}(\check{\mathcal{Y}}_\alpha \cap \check{D}_i)$ generate $H^{n-1}(\check{\mathcal{Y}}_\alpha)$. On the other hand, $H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha)$ and $f_{i*} H^{n-3}(\check{\mathcal{Y}}_\alpha \cap \check{D}_i)$ are orthogonal to each other because $\mathrm{Res}(\omega)$ for a holomorphic n-form ω on $\check{\mathcal{X}} \setminus \check{\mathcal{Y}}_\alpha$ vanishes on $\check{\mathcal{Y}}_\alpha \cap \check{D}_i$. This proves the claim. The decomposition (6.3) gives a topological characterization of $H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha)$: It consists of degree $n-1$ classes on $\check{\mathcal{Y}}_\alpha$ which vanish on the toric boundaries $\check{\mathcal{Y}}_\alpha \cap \check{D}_i$. Therefore the subspace $H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha)$ is defined over $\mathbb{Q}$ and is preserved by the Gauss-Manin connection. We can also see that each class $\alpha \in H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}})$ is primitive, *i.e.*, $\alpha \cup H = 0$ for an ample hyperplane class H. By (6.3) and (6.4), we have the identification

$$H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha) \cong W_{n-1}(H^{n-1}(\check{Y}_\alpha)).$$

Since $W_{n-1}(H^{n-1}(\check{Y}_\alpha))$ is the lowest weight component, this identification induces a $\mathbb{Q}$-Hodge structure of weight $n-1$ on $H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha)$.

Definition 6.5. — *The residual B-model VHS of the family* $\mathrm{pr}_2\colon \check{\mathfrak{Y}} \to \mathcal{M}_{\mathrm{reg}}$ *is a tuple* $(\mathscr{H}_{\mathrm{B}}, \nabla^{\mathrm{B}}, H_{\mathrm{B},\mathbb{Q}}, \mathscr{F}^{\bullet}_{\mathrm{B}}, Q_{\mathrm{B}})$ *where* $\mathscr{H}_{\mathrm{B}}$ *is the locally free subsheaf of* $(R^{n-1}\mathrm{pr}_{2*}\mathbb{C}_{\check{\mathfrak{Y}}}) \otimes \mathcal{O}_{\mathcal{M}_{\mathrm{reg}}}$ *over* $\mathcal{M}_{\mathrm{reg}}$ *whose fiber at* $[\alpha]$ *is the residue part* $H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha)$, ∇^{B} *is the Gauss-Manin connection,* $H_{\mathrm{B},\mathbb{Q}} \subset \mathrm{Ker}\,\nabla^{\mathrm{B}}$ *is the rational structure explained above,* $\mathscr{F}^p_{\mathrm{B},[\alpha]} = \bigoplus_{j\geqslant p} H^{j,n-1-j}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha)$ *is the standard Hodge filtration and* Q_{B} *is the intersection form:*

$$Q_{\mathrm{B}}(\omega_1,\omega_2) = (-1)^{(n-1)(n-2)/2}\int_{\check{\mathcal{Y}}_\alpha} \omega_1 \cup \omega_2.$$

The residual B-model VHS satisfies the usual properties of a variation of *polarized* Hodge structure as given in (6.1) and (6.2).

Consider the relative homology group $H_*(\check{\mathbb{T}}, \check{Y}_\alpha)$. The Morse-theoretic argument in [37, Section 3.3.1] (see also [22]) shows that

$$H_k(\check{\mathbb{T}}, \check{Y}_\alpha;\mathbb{Z}) \cong H_k(\check{\mathbb{T}}, \{\Re(W_\alpha(t)) \gg 0\};\mathbb{Z}) \cong \begin{cases} 0 & k \neq n \\ \mathbb{Z}^{\mathrm{Vol}(\Delta)} & k = n \end{cases} \tag{6.5}$$

where $\mathrm{Vol}(\Delta)$ is the normalized volume of Δ such that the volume of the standard n-simplex is 1. The group $H_n(\check{\mathbb{T}}, \{\Re(W_\alpha(t)) \gg 0\})$ is generated by *Lefschetz thimbles* emanating from critical points of $W_\alpha(t)$; $\mathrm{Vol}(\Delta)$ is the number of critical points of $W_\alpha(t)$ (with multiplicities). By the relative homology exact sequence, we have

$$0 \longrightarrow H_n(\check{\mathbb{T}}) \longrightarrow H_n(\check{\mathbb{T}}, \check{Y}_\alpha) \xrightarrow{\partial} H_{n-1}(\check{Y}_\alpha) \longrightarrow H_{n-1}(\check{\mathbb{T}}) \longrightarrow 0.$$

The image of $H_n(\check{\mathbb{T}}, \check{Y}_\alpha;\mathbb{Z})$ under ∂ consists of the *vanishing cycles* of $W_\alpha(t)$.

Lemma 6.6. — *The image of the composition*

$$H_n(\check{\mathbb{T}}, \check{Y}_\alpha) \xrightarrow{\ \partial\ } H_{n-1}(\check{Y}_\alpha) \longrightarrow H_{n-1}(\check{\mathcal{Y}}_\alpha) \xrightarrow{\ \mathrm{PD}\ } H^{n-1}(\check{\mathcal{Y}}_\alpha)$$

is $H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha)$. *Here* PD *is the Poincaré duality isomorphism (defined over* $\mathbb{Q}$*). We denote by* $\mathrm{VC}\colon H_n(\check{\mathbb{T}}, \check{Y}_\alpha) \to H^{n-1}_{\mathrm{res}}(\check{Y}_\alpha)$ *the resulting surjection.*

Proof. — It is clear that an image of the above map vanishes on the toric boundaries $\check{\mathcal{Y}}_\alpha \cap \check{D}_i$. Thus the image is contained in $H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha)$. The dual of this map is given by

$$H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha)^\vee \cong H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha) \cong W_{n-1}H^{n-1}(\check{Y}_\alpha) \hookrightarrow H^{n-1}(\check{Y}_\alpha) \xrightarrow{\delta} H^n(\check{\mathbb{T}}, \check{Y}_\alpha) \tag{6.6}$$

where the first isomorphism is via the intersection pairing. The map δ is dual to ∂ and its kernel is $H^{n-1}(\check{\mathbb{T}})$. Because the intersection of $H^{n-1}(\check{\mathbb{T}})$ and $W^{n-1}H^{n-1}(\check{Y}_\alpha)$ is zero for the weight reason, the above dual map is injective. $\square$

Definition 6.7. — *The vanishing cycle integral structure* $H^{\mathrm{vc}}_{\mathrm{B},\mathbb{Z}} \subset H_{\mathrm{B},\mathbb{Q}}$ *on the residual B-model VHS is defined to be the image of* $H_n(\check{\mathbb{T}}, \check{Y}_\alpha; \mathbb{Z})$ *under the vanishing cycle map* $\mathrm{VC}\colon H_n(\check{\mathbb{T}}, \check{Y}_\alpha) \to H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha)$.

Remark 6.8. — The mixed Hodge structure of affine hypersurfaces in the algebraic tori was studied by Danilov-Khovanskii [22] and Batyrev [3]. The Hodge structure of toric hypersurfaces has been studied by Batyrev-Cox [6] and Mavlyutov [46, 47].

6.4. Mirror Isomorphism with Integral Structures

In this section, as we did in Section 4.1, we assume that $\mathbf{N}$ is generated by $\Delta \cap \mathbf{N}$ as a $\mathbb{Z}$-module. Also we choose *non-zero* vectors $\{b_{m+1}, \dots, b_{m+s}\} \subset \Delta \cap \mathbf{N} \setminus \{b_1, \dots, b_m\}$ such that $b_1, \dots, b_{m+s}$ generate $\mathbf{N}$ over $\mathbb{Z}$. Because Δ is reflexive, every b_i has to be on the boundary of Δ. Then the lift $\tilde{\xi}_1$ of $\xi_1 = \overline{D}_1 + \dots + \overline{D}_m$ defined in Section 4.3 equals $\sum_{i=1}^{m+s} D_i$ and so $\hat{\rho}_{\mathcal{Y}} = \hat{\rho} - \tilde{\xi}_1 = 0$. Thus the degrees of the variables $q_1, \dots, q_{r+s}$ are zero. By the homogeneity of the I-function $I_{\mathcal{Y}}$, the mirror map $\varsigma(q) = \iota^* \tilde{\varsigma}(q)$ for $\mathcal{Y}$ (see (4.5)) takes values in $H^2_{\mathrm{amb}}(\mathcal{Y})$.

We briefly review the mirror isomorphism of D-modules for a toric orbifold $\mathcal{X}$ in [37]. We can associate the *B-model D-module* $(\mathcal{R}^{(0)}, \nabla, (\cdot,\cdot)_{\mathcal{R}^{(0)}})$ to the Landau-Ginzburg mirror $(\check{\mathbb{T}}, W_\alpha(t))$ of $\mathcal{X}$. It is a meromorphic flat connection over $\mathcal{M}^{\mathrm{o}} \times \mathbb{C}$ with poles along $z = 0$ such that the underlying $\mathbb{Z}$-local system at $([\alpha], z) \in \mathcal{M}^{\mathrm{o}} \times \mathbb{C}^\times$ is given by the lattice $H^n(\check{\mathbb{T}}, \{\Re(W_\alpha(t)/z) \ll 0\}; \mathbb{Z})$. Here $\mathcal{M}^{\mathrm{o}}$ is a Zariski open subset of $\mathcal{M}$ containing $\mathcal{M}_{\mathrm{reg}}$. The oscillatory form $\phi(t) e^{W_\alpha(t)/z} \frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_n}{t_n}$ appearing in Section 5.3 gives a section of the B-model D-module $\mathcal{R}^{(0)}$. We have the mirror isomorphism(15) [37, Proposition 4.8] in a neighbourhood of $q = \mathbf{0}$

$$\mathrm{Mir}_{\mathcal{X}}\colon (\tau \times \mathrm{id})^* (QDM(\mathcal{X})/H^2(\mathcal{X};\mathbb{Z})) \cong (\mathcal{R}^{(0)}, \nabla, (\cdot,\cdot)_{\mathcal{R}^{(0)}})$$

sending the unit section $\mathbf{1}$ to $[e^{W_\alpha(t)/z} \frac{dt_1}{t_1} \wedge \dots \wedge \frac{dt_n}{t_n}]$. Here τ is the mirror map in (4.2) and $H^2(\mathcal{X};\mathbb{Z})$ acts by Galois action. This induces an inclusion

(15) Because we changed our convention of $QDM(\mathcal{X})$ from [37] (see Remark 3.2), we also need to modify the definition of the B-model D-module accordingly. The necessary modification makes the B-model D-module more natural. We defined the integration pairing of a section $[\phi(t) e^{W(t)/z} \frac{dt_1}{t_1} \cdots \frac{dt_n}{t_n}]$ of $\mathcal{R}^{(0)}$ and a Lefschetz thimble with the additional factor of $(-2\pi z)^{-n/2}$ in [37, Eqn. (53)]. Then the integral structure, the flat connection and the pairing on $\mathcal{R}^{(0)}$ were introduced *through* this integration pairing. We just need to remove the factor $(-2\pi z)^{-n/2}$ there to redefine these ingredients.

of $\mathbb{Z}$-lattices:

$$\mathrm{Mir}_{\mathcal{X}}^{\mathbb{Z}} \colon K(\mathcal{X}) \hookrightarrow H_n(\check{\mathbb{T}}, \{\Re(W_\alpha(t)/z) \gg 0\}; \mathbb{Z})$$

such that for $\Gamma_{\mathcal{E}} = \mathrm{Mir}_{\mathcal{X}}^{\mathbb{Z}}(\mathcal{E})$,

$$\big(\phi(q,-z), \mathfrak{s}(\mathcal{E})(\tau(q), z)\big)_F = \int_{\Gamma_{\mathcal{E}}} \mathrm{Mir}_{\mathcal{X}}(\phi(q,-z))$$

for any section $\phi(q,z)$ of $(\tau \times \mathrm{id})^* QDM(\mathcal{X})$. (It is known by Hua [34] that $K(\mathcal{X})$ is a free $\mathbb{Z}$-module and thus $K(\mathcal{X}) \cong \mathcal{S}(\mathcal{X})$.) For $z > 0$ and $\alpha \in (\mathbb{R}_{>0})^{m+s}$, $\mathrm{Mir}_{\mathcal{X}}^{\mathbb{Z}}$ sends the structure sheaf $\mathcal{O}_{\mathcal{X}}$ to the real Lefschetz thimble $\check{\mathbb{T}}_{\mathbb{R}}$. By [37, Theorem 4.11], the map $\mathrm{Mir}_{\mathcal{X}}^{\mathbb{Z}}$ gives an isomorphism of lattices under the assumption

$$K(\mathcal{X}) \to \mathrm{Hom}(K(\mathcal{X}), \mathbb{Z}), \ \mathcal{E} \mapsto \chi(\cdot, \mathcal{E}), \text{ is surjective.}$$

This holds true in our case because $\mathcal{X}$ does not have generic stabilizers and we can apply the result of Kawamata [41] that the derived category $D^b(\mathcal{X})$ of coherent sheaves has a full exceptional collection $\{\mathcal{E}\}_{i=1}^N$. (In this case, $\{\mathcal{E}_i\}_{i=1}^N$ forms a $\mathbb{Z}$-basis of $K(\mathcal{X})$ and the Gram matrix $\chi(\mathcal{E}_i, \mathcal{E}_j)$ is an upper-triangular matrix with diagonal entries all equal to one.) Therefore $\mathrm{Mir}_{\mathcal{X}}^{\mathbb{Z}}$ is an isomorphism. If $q = [\alpha]$ is sufficiently close to $\mathbf{0}$, all the critical values of $W_\alpha(t)$ are contained in the ball $\{u \in \mathbb{C} \mid |u| \leqslant 1/2\}$. Then we have the canonical identification for $z > 0$:

(6.7)
$$H_n(\check{\mathbb{T}}, \{\Re(W_\alpha(t)/z) \gg 0\}; \mathbb{Z}) \cong H_n(\check{\mathbb{T}}, \{\Re(W_\alpha(t)) \geqslant 1\}; \mathbb{Z}) \cong H_n(\check{\mathbb{T}}, \check{Y}_\alpha; \mathbb{Z}).$$

The following is the second main theorem of the paper.

THEOREM 6.9. — *Let $(\mathcal{Y}, \check{\mathcal{Y}}_\alpha)$ be a Batyrev's mirror pair of Calabi-Yau hypersurfaces. The ambient A-model VHS of $\mathcal{Y}$ and the residual B-model VHS of $\check{\mathcal{Y}}_\alpha$ are isomorphic:*

$$\mathrm{Mir}_{\mathcal{Y}} \colon \varsigma^* \left((\mathscr{H}_{\mathrm{A}}, \nabla^{\mathrm{A}}, \mathscr{F}_{\mathrm{A}}^{\bullet}, Q_{\mathrm{A}}) / \iota^* H^2(\mathcal{X}; \mathbb{Z}) \right) \cong (\mathscr{H}_{\mathrm{B}}, \nabla^{\mathrm{B}}, \mathscr{F}_{\mathrm{B}}^{\bullet}, Q_{\mathrm{B}})$$

in a neighbourhood of $q = \mathbf{0}$. We have the following correspondence under $\mathrm{Mir}_{\mathcal{Y}}$:

$$\mathrm{Mir}_{\mathcal{Y}} \colon \Upsilon_v(q, z=1) \longmapsto (-1)^{\mathrm{age}(v)} \, \mathrm{age}(v)! \, \mathrm{Res}\left(\frac{\alpha^v t^v \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}}{(W_\alpha(t) - 1)^{\mathrm{age}(v)+1}} \right),$$

where $v \in \mathrm{Box}$, $q = [\alpha]$ and $\Upsilon_v(q,1)$ is the section of $\varsigma^* \mathscr{H}_{\mathrm{A}}$ in Corollary 4.8 (see (4.6)). In particular, $F(q)\,\mathbf{1}$ corresponds to the holomorphic volume form $\Omega_\alpha = \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n} / dW_\alpha$ on $\check{\mathcal{Y}}_\alpha$. The mirror isomorphism $\mathrm{Mir}_{\mathcal{Y}}$ induces an isomorphism

$$\mathrm{Mir}_{\mathcal{Y}}^{\mathbb{Z}} \colon H_{\mathrm{A},\mathbb{Z}}^{\mathrm{amb}} \cong H_{\mathrm{B},\mathbb{Z}}^{\mathrm{vc}}$$

of $\mathbb{Z}$-local systems. Moreover, when $z>0$ and $q=[\alpha]$ is sufficiently close to $\mathbf{0}$, we have the commutative diagram:

$$\begin{tikzcd} K(\mathcal{X}) \arrow[r, "\mathrm{Mir}^{\mathbb{Z}}_{\mathcal{X}}", "\cong"'] \arrow[d, "\iota^*"'] & H_n(\check{\mathbb{T}}, \{\Re(W_\alpha(t)/z) \gg 0\}; \mathbb{Z}) \arrow[d, "\epsilon_{n-1}\,\mathrm{VC}"] \\ H^{\mathrm{amb}}_{\mathrm{A},\mathbb{Z}} \arrow[r, "\mathrm{Mir}^{\mathbb{Z}}_{\mathcal{Y}}", "\cong"'] & H^{\mathrm{vc}}_{\mathrm{B},\mathbb{Z}} \end{tikzcd} \tag{6.8}$$

where $\epsilon_n=(-1)^{n(n-1)/2}$. Here we used the identification (6.7) *to define the right vertical arrow* VC *(see Lemma 6.6). The left vertical arrow ι^* sends $\mathcal{E}$ to $\mathfrak{s}(\iota^*\mathcal{E})(\varsigma(q))$ for $\mathcal{E}\in K(\mathcal{X})$.*

Proof. — Consider the map $\mathscr{R}\colon S^+_\Delta \to H^0(\check{\mathbb{T}}, \Omega^n_{\check{\mathbb{T}}}(*\check{Y}_\alpha))$ [3] defined by

$$\mathscr{R}(t^b t_0^k) = (-1)^{k-1}(k-1)!\frac{t^b}{(W_\alpha(t)-1)^k}\frac{dt_1}{t_1}\wedge\cdots\wedge\frac{dt_n}{t_n}, \quad k>0.$$

Let $I^{(1)}_\Delta$ be the ideal of S_Δ spanned by the monomials $t^b t_0^k$ such that b is in the interior of $k\Delta$. Batyrev showed [3, Theorem 8.1, 8.2] that $\mathscr{R}(t^b t_0^k)$ with $t^b t_0^k \in I^{(1)}_\Delta$, $k\leqslant p$ generate the Hodge filter $F^{n+1-p}W_{n+1}(H^n(\check{\mathbb{T}}\setminus \check{Y}_\alpha))$. Because the Poincaré residue map $H^n(\check{\mathbb{T}}\setminus\check{Y}_\alpha)\to H^{n-1}(\check{Y}_\alpha)$ is a surjective morphism of mixed Hodge structures of the Hodge type $(-1,-1)$ [3, Section 5], we know that $\mathrm{Res}(\mathscr{R}(t^b t_0^k))$, $t^b t_0^k\in I^{(1)}_\Delta$, $k\leqslant p$ generate $F^{n-p}W_{n-1}(H^n(\check{Y}_\alpha))$. (One can also see that $\mathscr{R}(t^b t_0^k)$ for $t^b t_0^k\in I^{(1)}_\Delta$ extends to a holomorphic n-form on $\check{\mathcal{X}}\setminus\check{\mathcal{Y}}_\alpha$. See the proof of [46, Theorem 4.4].)

By the homogeneity of $\Upsilon_v(q,z)$, we have $(-1)^{\deg/2}\Upsilon_v(q,1)=(-1)^{\mathrm{age}(v)}$ $\Upsilon(q,-1)$. Using this and Theorem 5.7, we have

$$\begin{aligned}\varsigma^* Q_{\mathrm{A}}(\Upsilon_v|_{z=1}, \mathfrak{s}(\iota^*\mathcal{O}_\mathcal{X})) &= (-1)^{\mathrm{age}(v)}\varsigma^*\Pi_\mathcal{Y}(\Upsilon_v,\mathcal{O}_\mathcal{Y})\big|_{z=1} \\ &= Q_{\mathrm{B}}(\mathrm{Res}(\mathscr{R}(\alpha^v t^v t_0^{\mathrm{age}(v)+1})), \epsilon_{n-1}\,\mathrm{VC}(\check{\mathbb{T}}_\mathbb{R}))\end{aligned}$$

for $v\in\mathrm{Box}$ and q sufficiently close to $\mathbf{0}$. Here $\mathrm{VC}(\check{\mathbb{T}}_\mathbb{R})=\check{Y}_\alpha\cap\check{\mathbb{T}}_\mathbb{R}$ is what we denote by $\Gamma_\mathbb{R}(\alpha)$ before. We consider the monodromy transforms of the both-hand sides around $q=\mathbf{0}$. By the last statement in Theorem 5.7, $\varsigma^*Q_A(\Upsilon_v,\mathfrak{s}(\iota^*\mathcal{E}))$ is monodromy generated by $\varsigma^*Q_A(\Upsilon_v,\mathfrak{s}(\iota^*\mathcal{O}_\mathcal{X}))$. Moreover, $\mathrm{Mir}_\mathcal{X}$ and the vertical arrows in the diagram (6.8) is equivariant with respect to the monodromy transformation around $q=\mathbf{0}$ (which is the tensor product of line bundles on $K(\mathcal{X})$; see Definition 3.6, (ii)). Therefore, we have for any $\mathcal{E}\in K(\mathcal{X})$ and q sufficiently close to $\mathbf{0}$,

$$\varsigma^* Q_{\mathrm{A}}(\Upsilon_v|_{z=1}, \mathfrak{s}(\iota^*\mathcal{E})) = Q_{\mathrm{B}}(\mathrm{Res}(\mathscr{R}(\alpha^v t^v t_0^{\mathrm{age}(v)+1})), \epsilon_{n-1}\,\mathrm{VC}(\Gamma_\mathcal{E})) \tag{6.9}$$

where $\Gamma_{\mathcal{E}} := \mathrm{Mir}^{\mathbb{Z}}_{\mathcal{X}}(\mathcal{E})$. The A-model VHS $\varsigma^*\mathscr{H}_{\mathrm{A}}$ is generated by $\Upsilon_v|_{z=1}$, $v \in \mathrm{Box}$ and their covariant derivatives near $q = \mathbf{0}$ as an $\mathcal{O}_{\mathcal{M}_{\mathrm{reg}}}$-module (see the discussion before Corollary 4.8). Likewise we claim that the B-model VHS $\mathscr{H}_{\mathrm{B}}$ is generated by $\mathrm{Res}(\mathscr{R}(\alpha^v t^v t_0^{\mathrm{age}(v)+1}))$, $v \in \mathrm{Box}$ and their derivatives. Take any element $t^b t_0^k \in I_\Delta^{(1)}$. Take a cone $\sigma \in \Sigma$ such that $b \in \sigma$. Then we can write $b = \sum_{i\in\sigma} l_i b_i + v$ for some $v \in \mathrm{Box} \cap \sigma$ and $l_i \in \mathbb{Z}_{\geqslant 0}$. The piecewise linear function φ_1 defined in Section 5.2 satisfies $\varphi_1(b_j) = 1$ for all $1 \leqslant j \leqslant m+s$. So we have $\varphi_1(b) = \sum_{i\in\sigma} l_i + \mathrm{age}(v)$. Since $t^b t_0^k \in I_\Delta^{(1)}$, we have $\varphi_1(b) + 1 \leqslant k$. By a direct calculation, we find

$$\prod_{i\in\sigma} (\nabla^{\mathrm{B}}_{\partial/\partial\alpha_i})^{l_i} \,\mathrm{Res}\left(\mathscr{R}(t^v t_0^{\mathrm{age}(v)+1})\right) = \mathrm{Res}\left(\mathscr{R}(t^b t_0^{\varphi_1(b)+1})\right).$$

Using the Euler vector field $\widetilde{E} = \sum_{i=1}^{m+s} \alpha_i(\partial/\partial\alpha_i)$, we find

$$(\nabla^{\mathrm{B}}_{\widetilde{E}} + k - 1)\cdots(\nabla^{\mathrm{B}}_{\widetilde{E}} + \varphi_1(b) + 1)\,\mathrm{Res}\left(\mathscr{R}(t^b t_0^{\varphi_1(b)+1})\right) = \mathrm{Res}\left(\mathscr{R}(t^b t_0^k)\right).$$

Now the claim follows. By taking the derivatives of (6.9), we know that the full *period matrices* of $\varsigma^*(\mathscr{H}_{\mathrm{A}}, \nabla^{\mathrm{A}})$ and $(\mathscr{H}_{\mathrm{B}}, \nabla^{\mathrm{B}})$ are the same. This shows that we have an isomorphism $\mathrm{Mir}_{\mathcal{Y}} \colon \varsigma^*(\mathscr{H}_{\mathrm{A}}, \nabla^{\mathrm{A}}) \cong (\mathscr{H}_{\mathrm{B}}, \nabla^{\mathrm{B}})$ sending $\Upsilon_v(q,1)$ to $\mathrm{Res}(\mathscr{R}(t^v t_0^{\mathrm{age}(v)+1}))$. Moreover, from the above calculation, it turns out that the generators $\mathrm{Res}(\mathscr{R}(t^b t_0^k))$, $t^b t_0^k \in I_\Delta^{(1)}$, $k \leqslant p$ of $F^{n-p} W_{n-1}(H^n(\check{Y}_\alpha)) \cong \mathscr{F}^{n-p}_{\mathrm{B},[\alpha]}$ correspond via $\mathrm{Mir}_{\mathcal{Y}}$ to elements in $(\varsigma^*\mathscr{F}^{n-p}_{\mathrm{A}})_{[\alpha]} \cong H^{\leqslant 2(p-1)}_{\mathrm{amb}}(\mathcal{Y})$. Hence $\mathrm{Mir}_{\mathcal{Y}}(\varsigma^*\mathscr{F}^{n-p}_{\mathrm{A}}) \supset \mathscr{F}^{n-p}_{\mathrm{B}}$. The other inclusion $\mathrm{Mir}_{\mathcal{Y}}(\varsigma^*\mathscr{F}^{n-p}_{\mathrm{A}}) \subset \mathscr{F}^{n-p}_{\mathrm{B}}$ can be easily seen by taking a basis of $\mathscr{F}^{n-p}_{\mathrm{A}}$ given by the covariant derivatives of $\Upsilon_v|_{z=1}$. Therefore $\varsigma^*(\mathscr{H}_{\mathrm{A}}, \nabla^{\mathrm{A}}, \mathscr{F}^{\bullet}_{\mathrm{A}}) \cong (\mathscr{H}_{\mathrm{B}}, \nabla^{\mathrm{B}}, \mathscr{F}^{\bullet}_{\mathrm{B}})$.

Next we show that $\varsigma^* Q_{\mathrm{A}} = Q_{\mathrm{B}}$ under $\mathrm{Mir}_{\mathcal{Y}}$. We know by (6.9) that the "dual" flat sections $Q_{\mathrm{A}}(\cdot, \mathfrak{s}(\iota^*\mathcal{E}))$ and $Q_{\mathrm{B}}(\cdot, \epsilon_{n-1}\,\mathrm{VC}(\Gamma_{\mathcal{E}}))$ correspond to each other under $\mathrm{Mir}_{\mathcal{Y}}$. Therefore it suffices to show that $Q_{\mathrm{A}}(\mathfrak{s}(\iota^*\mathcal{E}_1), \mathfrak{s}(\iota^*\mathcal{E}_2))$ and $Q_{\mathrm{B}}(\mathrm{VC}(\Gamma_{\mathcal{E}_1}), \mathrm{VC}(\Gamma_{\mathcal{E}_2}))$ are equal for $\mathcal{E}_1, \mathcal{E}_2 \in K(\mathcal{X})$. We have

$$Q_{\mathrm{A}}(\mathfrak{s}(\iota^*\mathcal{E}_1), \mathfrak{s}(\iota^*\mathcal{E}_2)) = \chi_{\mathcal{Y}}(\iota^*\mathcal{E}_1, \iota^*\mathcal{E}_2) = \chi(\mathcal{E}_1, \mathcal{E}_2) - (-1)^n \chi(\mathcal{E}_2, \mathcal{E}_1).$$

Let $e^{\pi\mathbf{i}}\Gamma_{\mathcal{E}_i}$ denote the parallel translate of $\Gamma_{\mathcal{E}_i} \in H_n(\check{\mathbb{T}}, \{\Re(W_\alpha(t)/z) \gg 0\})$ along the path $[0,1] \ni \theta \mapsto e^{\pi\mathbf{i}\theta} z$. Then we have

$$\begin{aligned} Q_{\mathrm{B}}(\mathrm{VC}(\Gamma_{\mathcal{E}_1}), \mathrm{VC}(\Gamma_{\mathcal{E}_2})) &= \epsilon_{n-1}\,\mathrm{VC}(\Gamma_{\mathcal{E}_1}) \cdot \mathrm{VC}(\Gamma_{\mathcal{E}_2}) \\ &= \epsilon_n(\Gamma_{\mathcal{E}_1} \cdot e^{\pi\mathbf{i}}\Gamma_{\mathcal{E}_2} + (-1)^{n-1}\Gamma_{\mathcal{E}_2} \cdot e^{\pi\mathbf{i}}\Gamma_{\mathcal{E}_1}). \end{aligned}$$

Since $\mathrm{Mir}_{\mathcal{X}}$ preserves the pairing[16], we have

$$\chi(\mathcal{E}_1,\mathcal{E}_2) = \epsilon_n \Gamma_{\mathcal{E}_1} \cdot e^{\pi \mathtt{i}} \Gamma_{\mathcal{E}_2}$$

and $\varsigma^* Q_{\mathrm{A}} = Q_{\mathrm{B}}$ follows. Therefore, $\mathfrak{s}(\iota^*\mathcal{E})$ corresponds to $\mathrm{VC}(\Gamma_{\mathcal{E}})$ under $\mathrm{Mir}_{\mathcal{Y}}$ when q is sufficiently close to $\mathbf{0}$. This shows the commutative diagram (6.8). □

The class $[\mathcal{O}_{\mathrm{pt}}] \in K(\mathcal{Y})$ of a skyscraper sheaf at a non-stacky point on $\mathcal{Y}$ gives a flat section $\mathfrak{s}(\mathcal{O}_{\mathrm{pt}})$ of $\mathscr{H}_{\mathrm{A}}$. Usually $[\mathcal{O}_{\mathrm{pt}}]$ is not contained in $\iota^* K(\mathcal{X})$, but we can still find an integral cycle on $\check{\mathcal{Y}}_\alpha$ corresponding to it.

Theorem 6.10. — *Under the mirror isomorphism* $\mathrm{Mir}_{\mathcal{Y}}$ *in Theorem 6.9, the flat section* $\mathfrak{s}(\mathcal{O}_{\mathrm{pt}})$ *corresponds to an integral compact* $(n-1)$*-cycle* $C(\alpha) \subset \check{Y}_\alpha$ *i.e.,*

$$Q_{\mathrm{A}}(\phi, \mathfrak{s}(\mathcal{O}_{\mathrm{pt}})) = \int_{C(\alpha)} \mathrm{Mir}_{\mathcal{Y}}(\phi) \tag{6.10}$$

for any section ϕ *of* $\varsigma^*\mathscr{H}_{\mathrm{A}}$*. In particular, we have* $(2\pi\mathtt{i})^{n-1} F(q) = \int_{C(\alpha)} \Omega_\alpha$.

Proof. — It suffices to prove (6.10) for $\phi = \Upsilon_v(q,1)$, $v \in \mathrm{Box}$. We first show that for sufficiently small $\alpha \in (\mathbb{C}^\times)^{m+s}$ and $q = [\alpha]$,

$$Q_{\mathrm{A}}\left(\Upsilon_v(q,1), \mathfrak{s}(\mathcal{O}_{\mathrm{pt}})(\varsigma(q))\right) = -\frac{1}{2\pi\mathtt{i}} \int_{(S^1)^n} \frac{(-1)^{\mathrm{age}(v)}\, \mathrm{age}(v)! \alpha^v t^v}{(W_\alpha(t)-1)^{1+\mathrm{age}(v)}} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}$$

where $(S^1)^n = \{|t_1| = \cdots = |t_n| = 1\}$. By (4.6) and the formula of $I^v_{\mathcal{Y}}$, we find that the left-hand side equals

$$(2\pi\mathtt{i})^{n-1} \sum_{\substack{\sum_{j=1}^{m+s} d_j b_j + v = 0 \\ d_j \in \mathbb{Z}_{\geqslant 0}}} \frac{\left(\sum_{j=1}^{m+s} d_j + \mathrm{age}(v)\right)!}{\prod_{j=1}^{m+s} d_j!} q^{d+v}.$$

On the other hand, by expanding $1/(1-W_\alpha(t))$ in geometric series, the right-hand side can be calculated as the residue at $t=0$. This is possible under the assumption that $|W_\alpha(t)| < 1$ for all $t \in (S^1)^n$ (this holds when all α_j are sufficiently small). It is easy to see that the residue calculation

(16) We made a sign error in [37] in matching the pairings under mirror symmetry. In [37, Appendix A.3] we showed that the A-model and B-model pairings differ only by a constant. The constant is fixed by comparing $\Gamma_{\mathbb{R}} \cdot \Gamma_{\mathrm{c}}$ with $\chi(\mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathrm{pt}}) = 1$, where $\Gamma_{\mathbb{R}} = \check{\mathbb{T}}_{\mathbb{R}}$ and $\Gamma_{\mathrm{c}} \cong (S^1)^n$. Taking the orientation into account, we find that $\Gamma_{\mathbb{R}} \cdot \Gamma_{\mathrm{c}} = (-1)^{n(n-1)/2}$ instead of 1. So the B-model pairing should be multiplied by $\epsilon_n = (-1)^{n(n-1)/2}$ to have the complete match with the A-side.

gives the same answer as above. We now use the following argument by Przyjalkowski [52, Section 2.5]. For a fixed such α, we consider the family of compact tori $(S^1_\epsilon)^n = \{|t_1| = \cdots = |t_n| = \epsilon\}$ for $0 < \epsilon \leqslant 1$. At $\epsilon = 1$, we have $(S^1_1)^n \cap \check{Y}_\alpha = \emptyset$. While ϵ decreases from 1, this family of tori slices the hypersurface $\check{Y}_\alpha$. For sufficiently small ϵ, $(S^1_\epsilon)^n$ does not intersect $\check{Y}_\alpha$ again. Let $0 < \delta < 1$ be such that $(S^1_\epsilon)^n \cap \check{Y}_\alpha = \emptyset$ for $\epsilon \leqslant \delta$. Then one can use $(S^1_\delta)^n - (S^1_1)^n$ as a tube cycle of the slice $C(\alpha) := \bigcup_{\delta<\epsilon<1} \check{Y}_\alpha \cap (S^1_\epsilon)^n$. We can see that the integral over $(S^1_\delta)^n$ of the same integrand tends to zero as $\delta \to 0$ since the denominator grows faster than the numerator. From this it follows that the integral over $(S^1_\delta)^n$ is in fact zero and the right-hand side equals $\int_{C(\alpha)} \mathrm{Mir}_{\mathcal{Y}}(\Upsilon_v(q,1))$. The last statement follows from the case $v = 0$. $\square$

6.5. Multi-GKZ System

Batyrev [3] showed that a rational period of affine hypersurfaces in $\check{\mathbb{T}}$ satisfies the Gelfand-Kapranov-Zelevinsky (GKZ) hypergeometric differential system [24]. Borisov-Horja [8] showed that a certain collection of periods satisfies a multi-generator version of GKZ system, which they called *better behaved GKZ system*. Here we see that the residual B-model VHS can be realized as a sub D-module of the D-module defined by the multi-GKZ system. This is related to the multi-generation phenomenon explained in the Introduction and Remark 4.7. In joint work [15] with Coates, Corti and Tseng, we also found that the multi-GKZ system arises for the quantum D-module of a toric orbifold $\mathcal{X}$ *itself* [(17)].

Set $\widehat{\mathbf{N}} := \mathbf{N} \oplus \mathbb{Z}$ and $\hat{b}_i = (b_i, 1) \in \widehat{\mathbf{N}}$. We still assume $b_i \neq 0$ for all i and set $\hat{b}_0 = (0,1)$. For simplicity we set $N := m + s$. Let $\widehat{\Delta}$ be the cone in the vector space $\widehat{\mathbf{N}}_\mathbb{R} = \widehat{\mathbf{N}} \otimes \mathbb{R}$ generated by $\Delta \times \{1\}$. Let $\mathbf{m}_1, \dots, \mathbf{m}_n$ be a basis of $\mathbf{M} = \mathrm{Hom}(\mathbf{N}, \mathbb{Z})$. Let $\mathbf{m}_0 \in \mathrm{Hom}(\widehat{\mathbf{N}}, \mathbb{Z})$ be the projection to the second factor. We can then regard $\mathbf{m}_0, \mathbf{m}_1, \dots, \mathbf{m}_n$ as a basis of $\widehat{\mathbf{M}} := \mathrm{Hom}(\widehat{\mathbf{N}}, \mathbb{Z})$.

Definition 6.11 ([8, 15]). — *The multi-GKZ system associated to* $\{\hat{b}_0, \hat{b}_1, \dots, \hat{b}_N = \hat{b}_{m+s}\}$ *is the system of differential equations for a family* $\{\varpi_\mathbf{e}(\alpha_0, \dots, \alpha_N) \,|\, \mathbf{e} \in \widehat{\mathbf{N}} \cap \widehat{\Delta}\}$ *of functions on* $(\mathbb{C}^\times)^{N+1}$ *given by* $D_{\nu;\mathbf{e},\mathbf{e}'} =$

(17) The multi-generation occurs typically for non-compact $\mathcal{X}$. It can also occur for compact $\mathcal{X}$ which does not satisfy the assumption in Section 4.1.

$Z_{i,\mathbf{e}} = 0$, *where*

$$D_{\nu;\mathbf{e},\mathbf{e}'} := \prod_{i=0}^{N} \left(\frac{\partial}{\partial \alpha_i}\right)^{\nu_{+,i}} \varpi_{\mathbf{e}} - \prod_{i=0}^{N} \left(\frac{\partial}{\partial \alpha_i}\right)^{\nu_{-,i}} \varpi_{\mathbf{e}'}$$

$$Z_{i,\mathbf{e}} := \sum_{j=0}^{N} \langle \mathbf{m}_i, \hat{b}_j\rangle \alpha_j \frac{\partial \varpi_{\mathbf{e}}}{\partial \alpha_j} + (\langle \mathbf{m}_i, \mathbf{e}\rangle - \beta_i)\varpi_{\mathbf{e}}, \quad 0 \leqslant i \leqslant n,$$

ν runs through all elements in $\mathbb{Z}^{N+1}$ satisfying $\mathbf{e}' = \mathbf{e} + \sum_{i=0}^{N} \nu_i \hat{b}_i$ and $\nu_+, \nu_- \in (\mathbb{Z}_{\geqslant 0})^{N+1}$ are given by $\nu_{\pm,i} = \max(\pm\nu_i, 0)$ (then $\nu = \nu_+ - \nu_-$). The constants $\beta_0, \ldots, \beta_n$ are called exponents. Alternatively, we can regard the multi-GKZ system as a D-module on $(\mathbb{C}^\times)^{N+1}$ defined by

$$\bigoplus_{\mathbf{e}\in\widehat{\mathbf{N}}\cap\widehat{\Delta}} \mathcal{D}\varpi_{\mathbf{e}} \Big/ \sum_{\mathbf{e},\mathbf{e}',\nu:\text{ as above}} \mathcal{D}Z_{\nu;\mathbf{e},\mathbf{e}'} + \sum_{1\leqslant i\leqslant N, \mathbf{e}\in\widehat{\mathbf{N}}\cap\widehat{\Delta}} \mathcal{D}Z_{i,\mathbf{e}} \tag{6.11}$$

where $\varpi_{\mathbf{e}}$ here is a formal symbol and $\mathcal{D} = \mathbb{C}\left\langle \alpha_0^{\pm}, \ldots, \alpha_N^{\pm}, \partial_{\alpha_0}, \ldots, \partial_{\alpha_N}\right\rangle$.

It is easy to see that the multi-GKZ system is generated by finitely many $\varpi_{\mathbf{e}}$. In this section, for $\alpha = (\alpha_0, \ldots, \alpha_N) \in (\mathbb{C}^\times)^{N+1}$, we set $W_\alpha(t) = \alpha_0 + \sum_{i=1}^{N} \alpha_i t^{b_i}$ and $\check{Y}_\alpha := \{t \in \check{\mathbb{T}} \mid W_\alpha(t) = 0\}$. Then $W_\alpha(t)$ and $\check{Y}_\alpha$ in the previous section are recovered by setting $\alpha_0 = -1$. Note that $\check{Y}_{(\alpha_0,\alpha')} = \check{Y}_{-\alpha_0^{-1}\alpha'}$ for $\alpha_0 \in \mathbb{C}^\times$, $\alpha' \in (\mathbb{C}^\times)^N$. Take a locally constant section $\Gamma(\alpha) \in H_n(\check{\mathbb{T}}, \check{Y}_\alpha; \mathbb{Z})$ of the relative homology bundle over the α-space. Let $C(\alpha) := \partial\Gamma(\alpha) \in H_{n-1}(\check{Y}_\alpha; \mathbb{Z})$ be its boundary. For $\mathbf{e} = (b, k) \in \widehat{\Delta} \cap \widehat{\mathbf{N}}$ we set

$$\Pi_{\mathbf{e}}(\alpha) := (-1)^{k-1}(k-1)! \int_{C(\alpha)} \operatorname{Res}\left(\frac{t^b}{W_\alpha(t)^k} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}\right)$$

for $k > 0$ and for $k = 0$ (in this case $\mathbf{e} = 0$),

$$\Pi_0(\alpha) := -\int_{\Gamma(\alpha)} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}.$$

The integrands of these period integrals $\Pi_{\mathbf{e}}(\alpha)$ generate the relative cohomology bundle $\bigcup_\alpha H^n(\check{\mathbb{T}}, \check{Y}_\alpha)$ by the results of Batyrev [3] and Stienstra [53] (see also [44]).

Proposition 6.12. — *The set of functions $\Pi_{\mathbf{e}}(\alpha)$, $\mathbf{e} \in \widehat{\mathbf{N}} \cap \widehat{\Delta}$ satisfies the multi-GKZ system with exponent $\beta = (0, 0, \ldots, 0)$.*

Proof. — Borisov-Horja [8, Proposition 5.2] proved a similar result. Stienstra [53] proved that Π_0 satisfies the ordinary GKZ system. The proposition here can be proved by an easy direct calculation, using the formula for $\partial_{\alpha_j}\Pi_0(\alpha)$ of Konishi-Minabe [44, Section 4.3] and the method of Batyrev [3, Theorem 14.2]. □

The relative cohomology bundle $\bigcup_\alpha H^n(\check{\mathbb{T}}, \check{Y}_\alpha)$ has the rank $\mathrm{Vol}(\Delta)$ (6.5) and the multi-GKZ system has the same rank by Borisov-Horja [8, Section 3]. Thus they are isomorphic as a local system. Because $W_{n-1}(H^{n-1}(\check{Y}_\alpha)) \to H^n(\check{\mathbb{T}}, \check{Y}_\alpha)$ is injective (see (6.6)), the residual B-model VHS is embedded in the multi-GKZ system. Note that $\Pi_{\mathbf{e}}(\alpha)$ is a period of the residual B-model VHS if $\mathbf{e}$ is in the interior of $\widehat{\Delta}$ and also that the corresponding residue classes on $\check{\mathcal{Y}}_\alpha$ generate the B-model VHS. Therefore we have the following.

Theorem 6.13. — *Let $\pi\colon (\mathbb{C}^\times)^{N+1} \to \mathcal{M}$ be the map sending (α_0, α') to $[-\alpha_0^{-1}\alpha']$ and set $(\mathbb{C}^\times)^{N+1}_{\mathrm{reg}} = \pi^{-1}(\mathcal{M}_{\mathrm{reg}})$. Under the pull-back by $\pi\colon (\mathbb{C}^\times)^{N+1}_{\mathrm{reg}} \to \mathcal{M}_{\mathrm{reg}}$, the residual B-model VHS is isomorphic to the sub D-module of the multi-GKZ system* (6.11) *generated by $\varpi_{\mathbf{e}}$ such that* $\mathbf{e}$ *is in the interior of the cone $\widehat{\Delta}$. In particular, the ambient A-model VHS is embedded in the multi-GKZ system under mirror isomorphism of Theorem 6.9.*

Remark 6.14. — We will see that the multi-GKZ system here describes the quantum D-module of the total space of $K_{\mathcal{X}}$ in [15]. So $QDM_{\mathrm{amb}}(\mathcal{Y})$ is contained in $QDM(K_{\mathcal{X}})$.

Remark 6.15. — The above identification between the multi-GKZ and the relative cohomology introduces a mixed Hodge structure on the multi-GKZ system. In the context of orbifold mirror symmetry, Corti-Golyshev [19] studied the hypergeometric system associated to weighted projective Calabi-Yau hypersurfaces and its Hodge structure.

Remark 6.16. — Mann-Mignon [45] obtained another (but closely related) description of the quantum D-module of a nef complete intersection in a smooth toric variety.

6.6. Questions and Example

In Theorem 6.9, we showed the correspondence between *vanishing cycles* on $\check{\mathcal{Y}}_\alpha$ and *ambient K-classes* on $\mathcal{Y}$. This match of the integral structures is not completely satisfactory. For example, the class $[\mathcal{O}_{\mathrm{pt}}]$ on $\mathcal{Y}$ would not be contained in $\iota^* K(\mathcal{X})$, but the corresponding mirror cycle exists (see Theorem 6.10). We can also consider different integral structures which might be more natural. For example, on the A-side, we can take the integral structure

$$H_{\mathrm{A},\mathbb{Z}} = \{\mathfrak{s}(\mathcal{E}) \mid \widetilde{\mathrm{ch}}(\mathcal{E}) \in H^*_{\mathrm{amb}}(\mathcal{Y}), \mathcal{E} \in K(\mathcal{Y})\}.$$

This is bigger than $H^{\mathrm{amb}}_{\mathrm{A},\mathbb{Z}}$ in general. On the B-side we could consider the integral structure $W_{n-1}(H^{n-1}(\check{Y}_\alpha)) \cap H^{n-1}(\check{Y}_\alpha;\mathbb{Z})$. When we can choose a smooth compactification $\check{\mathcal{Y}}_\alpha$, we could also take the integral structure $H^{n-1}_{\mathrm{res}}(\check{\mathcal{Y}}_\alpha) \cap H^{n-1}(\check{\mathcal{Y}}_\alpha;\mathbb{Z})$.

Question 1. — *What is the integral structure in the B-model corresponding to $H_{\mathrm{A},\mathbb{Z}}$?*

Yongbin Ruan asked the following question to the author.

Question 2. — *What is the "correct" definition of the integral middle homology group of the orbifold $\check{\mathcal{Y}}_\alpha$ in this context?*

Mirror symmetry for the orbifold Hodge numbers of toric Calabi-Yau hypersurfaces was studied by Borisov-Mavlyutov [11].

Question 3. — *Can one extend the mirror isomorphism of VHS beyond the ambient part/residue part? Then what is the integral structure in the B-model VHS on the full orbifold cohomology of $\check{\mathcal{Y}}_\alpha$?*

Example 6.17. — We first consider the simplest example of an elliptic curve $\mathcal{Y}$ in $\mathbb{P}^2$. The mirror is defined by $W_\alpha(t_1,t_2) = \alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 (t_1 t_2)^{-1}$. The $\mathbb{C}^\times$-co-ordinate q on $\mathcal{M} \cong \mathbb{C}^\times$ is given by $q = \alpha_1\alpha_2\alpha_3$. We choose a section $(\alpha_1,\alpha_2,\alpha_3) = (1,1,q)$ and work with $W_q(t) = t_1 + t_2 + q(t_1t_2)^{-1}$. The mirror hypersurface $\check{Y}_q = \{W_q(t) = 1\}$ is an elliptic curve minus 3 points. The function $W_q(t)$ has the three critical values $3q^{1/3}$, $3\omega q^{1/3}$, $3\omega^2 q^{1/3}$, where $\omega = e^{2\pi \mathbf{i}/3}$. Let $q > 0$ be small and take vanishing paths from the three critical values to 1 as shown in Figure 6.1.

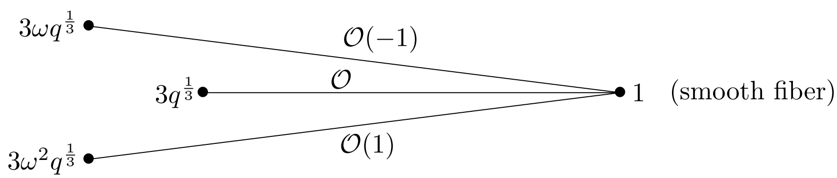

Figure 6.1. Vanishing Paths

The Lefschetz thimble Γ_i from $3\omega^{-i}q^{1/3}$ $(i = -1,0,1)$ along the given vanishing path corresponds to the line bundle $\mathcal{O}(i)$ on $\mathbb{P}^2$. The vanishing cycle $C_i = \partial\Gamma_i \subset \check{Y}_q$ corresponds to $\iota^*\mathcal{O}(i)$ under $\mathrm{Mir}^{\mathbb{Z}}_{\mathcal{Y}}$. For a suitable symplectic basis $\{A,B\}$ of $H_1(\overline{\check{Y}}_\alpha,\mathbb{Z})$, we have

$$C_{\pm 1} = A \pm 3B, \quad C_0 = A.$$

Similarly for a basis $\{\mathcal{O}_{\mathrm{pt}}, \mathcal{O}_{\mathcal{Y}}\}$ of the topological K-group $K(\mathcal{Y})$ of $\mathcal{Y}$, we have

$$\iota^*\mathcal{O}(\pm 1) = \mathcal{O}_{\mathcal{Y}} \pm 3\mathcal{O}_{\mathrm{pt}}, \quad \iota^*\mathcal{O} = \mathcal{O}_{\mathcal{Y}}$$

in $K(\mathcal{Y})$. Therefore $\mathrm{Mir}_{\mathcal{Y}}^{\mathbb{Z}}$ extends to an isomorphism of the overlattices in this case.

$$H_{\mathrm{A},\mathbb{Z}} \cong K(\mathcal{Y}) \cong H_1(\overline{\check{Y}}_\alpha; \mathbb{Z}).$$

Example 6.18. — Next we consider a quintic threefold $\mathcal{Y}$ in $\mathbb{P}^4$, famous example studied in [12]. The mirror of $\mathcal{Y}$ is defined by the function $W_q(x_1, \ldots, x_4) = x_1 + x_2 + x_3 + x_4 + x_5 + q/(x_1x_2x_3x_4x_5)$ with one complex parameter $q \in \mathbb{C}^\times$. The affine hypersurface $\check{Y}_q = \{z \in \check{\mathbb{T}} \mid W_q(x) = 1\}$ can be compactified to a smooth Calabi-Yau manifold $\check{\mathcal{Y}}_q$. In this case, the ambient A-model VHS of $\mathcal{Y}$ is a flat bundle of rank 4 with fiber $\bigoplus_{p=0}^3 H^{2p}(\mathcal{Y}) = \bigoplus_{p=0}^3 H^{p,p}(\mathcal{Y})$ and the residual B-model VHS of $\check{\mathcal{Y}}_q$ is a flat bundle with fiber $H^3(\check{\mathcal{Y}}_q)$. We shall show that the isomorphism of the $\mathbb{Z}$-structures in Theorem 6.9

$$\mathrm{Mir}_{\mathcal{Y}}^{\mathbb{Z}} \colon H_{\mathrm{A},\mathbb{Z}}^{\mathrm{amb}} = \iota^* K(\mathcal{X}) \xrightarrow{\cong} H_{\mathrm{B},\mathbb{Z}}^{\mathrm{vc}} \subset H_3(\check{\mathcal{Y}}_q; \mathbb{Z})$$

extends to an isomorphism $K(\mathcal{Y}) \cong H_3(\check{\mathcal{Y}}_q; \mathbb{Z})$ of the overlattices. (Here as usual $K(\mathcal{Y})$ denotes the topological K-group.) By the Atiyah-Hirzebruch spectral sequence, we know that $K(\mathcal{Y})$ is a free $\mathbb{Z}$-module generated by $\mathcal{O}_{\mathrm{pt}}$, $\mathcal{O}_C$, $\mathcal{O}_D$, $\mathcal{O}_{\mathcal{Y}}$ where $C \subset \mathcal{Y}$ is a line and $D = \mathcal{Y} \cap \mathbb{P}^3$ is a hyperplane section. Under the isomorphism $K(\mathcal{Y}) \otimes \mathbb{Q} \cong H_3(\check{\mathcal{Y}}_q; \mathbb{Q})$ of rational vector spaces, the dimension filtration $K_{\leqslant 0} \subset K_{\leqslant 1} \subset K_{\leqslant 2} \subset K_{\leqslant 3} = K(\mathcal{Y})$ induces the filtration $W_0 \subset W_1 \subset W_2 \subset W_3 = H_3(\check{\mathcal{Y}}_q; \mathbb{Q})$. For $\mathcal{E} \in K(\mathcal{Y})$, we denote the corresponding element in $H_3(\check{\mathcal{Y}}_q; \mathbb{Q})$ by the same symbol. We set $W_i^{\mathbb{Z}} = W_i \cap H_3(\check{\mathcal{Y}}_q; \mathbb{Z})$. We have $W_0 = \mathbb{Q}[\mathcal{O}_{\mathrm{pt}}]$. For $\alpha \in W_3^{\mathbb{Z}}$, we have $\alpha - (\alpha \cdot [\mathcal{O}_{\mathrm{pt}}])[\mathcal{O}_{\mathcal{Y}}] \in W_0^{\perp} = W_2$. Here the intersection number $\alpha \cdot [\mathcal{O}_{\mathrm{pt}}]$ is an integer since $[\mathcal{O}_{\mathrm{pt}}] \in H_3(\check{\mathcal{Y}}_q; \mathbb{Z})$ by Theorem 6.10. This shows that

$$W_3^{\mathbb{Z}} = W_2^{\mathbb{Z}} + \mathbb{Z}[\mathcal{O}_{\mathcal{Y}}].$$

It is easy to see that the perfect intersection pairing on $W_3^{\mathbb{Z}}$ induces a perfect pairing on $W_2^{\mathbb{Z}}/W_0^{\mathbb{Z}}$. We know that $[\mathcal{O}_D] = [\mathcal{O}_{\mathcal{Y}} - \mathcal{O}_{\mathcal{Y}}(-1)]$ and $[\mathcal{O}_D]^2 = 5([\mathcal{O}_C] - 2[\mathcal{O}_{\mathrm{pt}}])$ belong to $\iota^*K(\mathcal{X})$ and thus to $H_3(\check{\mathcal{Y}}_q, \mathbb{Z})$. They also form a rational basis of W_2/W_0. Take an element $a[\mathcal{O}_D] + b[\mathcal{O}_D]^2 \in W_2^{\mathbb{Z}}/W_0^{\mathbb{Z}}$. By taking the tensor product with $\mathcal{O}_{\mathcal{Y}}(-1)$ which corresponds to the monodromy transformation around $q = 0$, we have $a[\mathcal{O}_D]^2 \in W_2^{\mathbb{Z}}/W_0^{\mathbb{Z}}$. Hence $(a[\mathcal{O}_D] + b[\mathcal{O}_D]^2) \cdot (a[\mathcal{O}_D]^2) = 5a^2 \in \mathbb{Z}$. Thus $a \in \mathbb{Z}$. Therefore

$$W_2^{\mathbb{Z}} = W_1^{\mathbb{Z}} + \mathbb{Z}[\mathcal{O}_D].$$

Moreover the perfectness of the pairing on $W_2^{\mathbb{Z}}/W_0^{\mathbb{Z}}$ implies that $[\mathcal{O}_C] = [\mathcal{O}_D]^2/5 \in W_2^{\mathbb{Z}}/W_0^{\mathbb{Z}}$ and that $W_2^{\mathbb{Z}}/W_0^{\mathbb{Z}} = \mathbb{Z}[\mathcal{O}_C] \oplus \mathbb{Z}[\mathcal{O}_D]$. By pairing with $[\mathcal{O}_{\mathcal{Y}}]$ again we know that $W_0^{\mathbb{Z}} = \mathbb{Z}[\mathcal{O}_{\mathrm{pt}}]$. These show that $K(\mathcal{Y}) \cong W_3^{\mathbb{Z}} = H_3(\check{\mathcal{Y}}_q;\mathbb{Z})$.

Remark 6.19. — Hartmann [30] studied Hodge-theoretic mirror symmetry for a quartic K3 surface and identified the mirror periods with certain hypergeometric functions.

BIBLIOGRAPHY

[1] D. Abramovich, T. Graber & A. Vistoli, "Gromov-Witten theory of Deligne-Mumford stacks", *Amer. J. Math.* **130** (2008), no. 5, p. 1337-1398.

[2] S. Barannikov, "Quantum periods. I. Semi-infinite variations of Hodge structures", *Internat. Math. Res. Notices* (2001), no. 23, p. 1243-1264.

[3] V. V. Batyrev, "Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori", *Duke Math. J.* **69** (1993), no. 2, p. 349-409.

[4] ———, "Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties", *J. Algebraic Geom.* **3** (1994), no. 3, p. 493-535.

[5] V. V. Batyrev & L. A. Borisov, "On Calabi-Yau complete intersections in toric varieties", in *Higher-dimensional complex varieties (Trento, 1994)*, de Gruyter, Berlin, 1996, p. 39-65.

[6] V. V. Batyrev & D. A. Cox, "On the Hodge structure of projective hypersurfaces in toric varieties", *Duke Math. J.* **75** (1994), no. 2, p. 293-338.

[7] L. A. Borisov, L. Chen & G. G. Smith, "The orbifold Chow ring of toric Deligne-Mumford stacks", *J. Amer. Math. Soc.* **18** (2005), no. 1, p. 193-215 (electronic).

[8] L. A. Borisov & R. P. Horja, "On the better behaved version of the GKZ hypergeometric system", arXiv:1011.5720.

[9] ———, "Mellin-Barnes integrals as Fourier-Mukai transforms", *Adv. Math.* **207** (2006), no. 2, p. 876-927.

[10] ———, "On the K-theory of smooth toric DM stacks", in *Snowbird lectures on string geometry*, Contemp. Math., vol. 401, Amer. Math. Soc., Providence, RI, 2006, p. 21-42.

[11] L. A. Borisov & A. R. Mavlyutov, "String cohomology of Calabi-Yau hypersurfaces via mirror symmetry", *Adv. Math.* **180** (2003), no. 1, p. 355-390.

[12] P. Candelas, X. C. de la Ossa, P. S. Green & L. Parkes, "A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory", *Nuclear Phys. B* **359** (1991), no. 1, p. 21-74.

[13] W. Chen & Y. Ruan, "Orbifold Gromov-Witten theory", in *Orbifolds in mathematics and physics (Madison, WI, 2001)*, Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, p. 25-85.

[14] ———, "A new cohomology theory of orbifold", *Comm. Math. Phys.* **248** (2004), no. 1, p. 1-31.

[15] T. Coates, A. Corti, H. Iritani & H.-H. Tseng, in preparation.

[16] ———, "Computing genus-zero twisted Gromov-Witten invariants", *Duke Math. J.* **147** (2009), no. 3, p. 377-438.

[17] T. Coates & A. Givental, "Quantum Riemann-Roch, Lefschetz and Serre", *Ann. of Math. (2)* **165** (2007), no. 1, p. 15-53.

[18] T. Coates, Y.-P. Lee, A. Corti & H.-H. Tseng, "The quantum orbifold cohomology of weighted projective spaces", *Acta Math.* **202** (2009), no. 2, p. 139-193.

[19] A. Corti & V. Golyshev, "Hypergeometric Equations and Weighted Projective Spaces", arXiv:math.AG/0607016.

[20] D. A. Cox & S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs, vol. 68, American Mathematical Society, Providence, RI, 1999, xxii+469 pages.

[21] V. I. Danilov, "The geometry of toric varieties", *Uspekhi Mat. Nauk* **33** (1978), no. 2(200), p. 85-134, 247.

[22] V. I. Danilov & A. G. Khovanskiĭ, "Newton polyhedra and an algorithm for calculating Hodge-Deligne numbers", *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), no. 5, p. 925-945.

[23] W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998, xiv+470 pages.

[24] I. M. Gel'fand, A. V. Zelevinskiĭ & M. M. Kapranov, "Hypergeometric functions and toric varieties", *Funktsional. Anal. i Prilozhen.* **23** (1989), no. 2, p. 12-26.

[25] A. Givental, "A mirror theorem for toric complete intersections", in *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, p. 141-175.

[26] A. B. Givental, "Homological geometry and mirror symmetry", in *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)* (Basel), Birkhäuser, 1995, p. 472-480.

[27] ———, "Symplectic geometry of Frobenius structures", in *Frobenius manifolds*, Aspects Math., E36, Vieweg, Wiesbaden, 2004, p. 91-112.

[28] M. A. Guest, "Quantum cohomology via D-modules", *Topology* **44** (2005), no. 2, p. 263-281.

[29] M. A. Guest & H. Sakai, "Orbifold quantum D-modules associated to weighted projective spaces", arXiv:0810.4236.

[30] H. Hartmann, "Period- and mirror-map for the quartic K3", arXiv:1101.4601.

[31] C. Hertling, "tt^* geometry, Frobenius manifolds, their connections, and the construction for singularities", *J. Reine Angew. Math.* **555** (2003), p. 77-161.

[32] K. Hori & C. Vafa, "Mirror symmetry", arXiv:hep-th/0002222.

[33] S. Hosono, "Central charges, symplectic forms, and hypergeometric series in local mirror symmetry", in *Mirror symmetry. V*, AMS/IP Stud. Adv. Math., vol. 38, Amer. Math. Soc., Providence, RI, 2006, p. 405-439.

[34] Z. Hua, "On the Grothendieck groups of toric stacks", arXiv:0904.2824.

[35] H. Iritani, "tt^*-geometry in quantum cohomology", arXiv:0906.1307.

[36] ———, "Quantum D-modules and generalized mirror transformations", *Topology* **47** (2008), no. 4, p. 225-276.

[37] ———, "An integral structure in quantum cohomology and mirror symmetry for toric orbifolds", *Adv. Math.* **222** (2009), no. 3, p. 1016-1079.

[38] ———, "Ruan's conjecture and integral structures in quantum cohomology", in *New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008)*, Adv. Stud. Pure Math., vol. 59, Math. Soc. Japan, Tokyo, 2010, p. 111-166.

[39] Y. Jiang, "The orbifold cohomology ring of simplicial toric stack bundles", *Illinois J. Math.* **52** (2008), no. 2, p. 493-514.

[40] L. Katzarkov, M. Kontsevich & T. Pantev, "Hodge theoretic aspects of mirror symmetry", in *From Hodge theory to integrability and TQFT tt*-geometry*, Proc. Sympos. Pure Math., vol. 78, Amer. Math. Soc., Providence, RI, 2008, p. 87-174.

[41] Y. Kawamata, "Derived categories of toric varieties", *Michigan Math. J.* **54** (2006), no. 3, p. 517-535.

[42] T. Kawasaki, "The Riemann-Roch theorem for complex V-manifolds", *Osaka J. Math.* **16** (1979), no. 1, p. 151-159.

[43] B. Kim, A. Kresch & T. Pantev, "Functoriality in intersection theory and a conjecture of Cox, Katz, and Lee", *J. Pure Appl. Algebra* **179** (2003), no. 1-2, p. 127-136.

[44] Y. Konishi & S. Minabe, "Local B-model and mixed Hodge structure", *Adv. Theor. Math. Phys.* **14** (2010), no. 4, p. 1089-1145.

[45] E. Mann & T. Mignon, "Quantum D-modules for toric nef complete intersections", arXiv:1112.1552.

[46] A. R. Mavlyutov, "Semiample hypersurfaces in toric varieties", *Duke Math. J.* **101** (2000), no. 1, p. 85-116.

[47] ———, "On the chiral ring of Calabi-Yau hypersurfaces in toric varieties", *Compositio Math.* **138** (2003), no. 3, p. 289-336.

[48] T. Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 15, Springer-Verlag, Berlin, 1988, An introduction to the theory of toric varieties, Translated from the Japanese, viii+212 pages.

[49] R. Pandharipande, "Rational curves on hypersurfaces (after A. Givental)", *Astérisque* (1998), no. 252, p. Exp. No. 848, 5, 307-340, Séminaire Bourbaki. Vol. 1997/98.

[50] F. Pham, "La descente des cols par les onglets de Lefschetz, avec vues sur Gauss-Manin", *Astérisque* (1985), no. 130, p. 11-47, Differential systems and singularities (Luminy, 1983).

[51] A. Pressley & G. Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1986, Oxford Science Publications, viii+318 pages.

[52] V. Przyjalkowski, "On Landau-Ginzburg models for Fano varieties", *Commun. Number Theory Phys.* **1** (2007), no. 4, p. 713-728.

[53] J. Stienstra, "Resonant hypergeometric systems and mirror symmetry", in *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, World Sci. Publ., River Edge, NJ, 1998, p. 412-452.

[54] B. Toen, "Théorèmes de Riemann-Roch pour les champs de Deligne-Mumford", *K-Theory* **18** (1999), no. 1, p. 33-76.

[55] H.-H. Tseng, "Orbifold quantum Riemann-Roch, Lefschetz and Serre", *Geom. Topol.* **14** (2010), no. 1, p. 1-81.

[56] A. Vistoli, "Intersection theory on algebraic stacks and on their moduli spaces", *Invent. Math.* **97** (1989), no. 3, p. 613-670.

Manuscrit reçu le 3 janvier 2011,
accepté le 1er décembre 2011.

Hiroshi IRITANI
Kyoto University
Department of Mathematics
Kitashirakawa-Oiwake-cho
Sakyo-ku, Kyoto, 606-8502 (Japan)
iritani@math.kyoto-u.ac.jp