

QUANTUM COHOMOLOGY AND THE k -SCHUR BASIS

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ABSTRACT. We prove that structure constants related to Hecke algebras at roots of unity are special cases of k -Littlewood-Richardson coefficients associated to a product of k -Schur functions. As a consequence, both the 3-point Gromov-Witten invariants appearing in the quantum cohomology of the Grassmannian, and the fusion coefficients for the WZW conformal field theories associated to $\widehat{\mathfrak{su}}(\ell)$ are shown to be k -Littlewood-Richardson coefficients. From this, Mark Shimozono conjectured that the k -Schur functions form the Schubert basis for the homology of the loop Grassmannian, whereas k -Schur coproducts correspond to the integral cohomology of the loop Grassmannian. We introduce dual k -Schur functions defined on weights of k -tableaux that, given Shimozono's conjecture, form the Schubert basis for the cohomology of the loop Grassmannian. We derive several properties of these functions that extend those of skew Schur functions.

1. INTRODUCTION

The study of Macdonald polynomials led to the discovery of symmetric functions, $s_\lambda^{(k)}$, indexed by partitions whose first part is no larger than a fixed integer $k \geq 1$. Experimentation suggested that these functions play the fundamental combinatorial role of the Schur basis in the symmetric function subspace $\Lambda^k = \mathbb{Z}[h_1, \dots, h_k]$; that is, they satisfy properties generalizing classical properties of Schur functions such as Pieri and Littlewood-Richardson rules. The study of the $s_\lambda^{(k)}$ led to several different characterizations [19], [20], [23] (conjecturally equivalent) and to the proof of many of these combinatorial conjectures. We thus generically call the functions k -Schur functions, but in this article consider only the definition presented in [23].

Although prior work with k -Schur functions concentrated on proving that they act as the “Schur basis” for Λ^k , the analogy was so striking that it seemed likely to extend beyond combinatorics to fields such as algebraic geometry and representation theory. Our main finding in this direction is that the k -Schur functions are connected to representations of Hecke algebras $H_\infty(q)$, where q is a root of unity, and they provide the natural basis for work in the quantum cohomology of the Grassmannian just as the Schur functions do for the usual cohomology. In particular, the 3-point Gromov-Witten invariants are none other than relevant cases of

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“ k -Littlewood-Richardson coefficients”, the expansion coefficients in

$$(1.1) \quad s_\lambda^{(k)} s_\mu^{(k)} = \sum_{\nu: \nu_1 \leq k} c_{\lambda\mu}^{\nu, k} s_\nu^{(k)}.$$

To be precise, in Schubert calculus, the cohomology ring of the Grassmannian $Gr_{\ell n}$ (the manifold of ℓ -dimensional subspaces of \mathbb{C}^n) has a basis given by Schubert classes σ_λ that are indexed by partitions $\lambda \in \mathcal{P}^{\ell n}$ that fit inside an $\ell \times (n - \ell)$ rectangle. There is an isomorphism,

$$H^*(Gr_{\ell n}) \cong \Lambda^\ell / \langle e_{n-\ell+1}, \dots, e_n \rangle,$$

where the Schur function s_λ maps to the Schubert class σ_λ when $\lambda \in \mathcal{P}^{\ell n}$. Since s_λ is zero modulo the ideal when $\lambda \notin \mathcal{P}^{\ell n}$, the structure constants of $H^*(Gr_{\ell n})$ in the basis of Schubert classes:

$$\sigma_\lambda \sigma_\mu = \sum_{\nu \in \mathcal{P}^{\ell n}} c_{\lambda\mu}^\nu \sigma_\nu,$$

can be obtained from the Littlewood-Richardson coefficients for Schur functions,

$$s_\lambda s_\mu = \sum_{\nu \in \mathcal{P}^{\ell n}} c_{\lambda\mu}^\nu s_\nu + \sum_{\nu \notin \mathcal{P}^{\ell n}} c_{\lambda\mu}^\nu s_\nu,$$

which have well known combinatorial interpretations.

The small quantum cohomology ring of the Grassmannian $QH^*(Gr_{\ell n})$ is a deformation of the usual cohomology that has become the object of much recent attention (e.g. [1], [33]). As a linear space, this is the tensor product $H^*(Gr_{\ell n}) \otimes \mathbb{Z}[q]$ and the σ_λ with $\lambda \in \mathcal{P}^{\ell n}$ form a $\mathbb{Z}[q]$ -linear basis of $QH^*(Gr_{\ell n})$. Multiplication is a q -deformation of the product in $H^*(Gr_{\ell n})$, defined by

$$\sigma_\lambda * \sigma_\mu = \sum_{\substack{\nu \in \mathcal{P}^{\ell n} \\ |\nu| = |\lambda| + |\mu| - dn}} q^d C_{\lambda\mu}^{\nu, d} \sigma_\nu.$$

The $C_{\lambda\mu}^{\nu, d}$ are the 3-point Gromov-Witten invariants, which count the number of rational curves of degree d in $Gr_{\ell n}$ that meet generic translates of the Schubert varieties associated to λ, μ , and ν . Finding a combinatorial interpretation for these constants is an interesting open problem that would have applications to many areas, including the study of the Verlinde fusion algebra [29] as well as the computation of certain knot invariants [31].

As with the usual cohomology, quantum cohomology can be connected to symmetric functions by:

$$QH^*(Gr_{\ell n}) \cong (\Lambda^\ell \otimes \mathbb{Z}[q]) / J_q^{\ell n},$$

where $J_q^{\ell n} = \langle e_{n-\ell+1}, \dots, e_{n-1}, e_n + (-1)^\ell q \rangle$. When $\lambda \in \mathcal{P}^{\ell n}$, the Schubert class σ_λ still maps to the Schur function s_λ , but unfortunately when $\lambda \notin \mathcal{P}^{\ell n}$, some s_λ are not zero modulo the ideal. Thus, the Schur functions cannot be used to directly obtain the quantum structure constants. Instead, these Gromov-Witten invariants arise as the expansion coefficients in

$$s_\lambda s_\mu = \sum_{\substack{\nu \in \mathcal{P}^{\ell n} \\ |\nu| = |\lambda| + |\mu| - dn}} q^d C_{\lambda\mu}^{\nu, d} s_\nu \pmod{J_q^{\ell n}},$$

and to compute the coefficients, an algorithm involving negatives [9], [12], [32] must be used to reduce a Schur function modulo the ideal $J_q^{\ell n}$.

Remarkably, by first working with an ideal that arises in the context of Hecke algebras at roots of unity, we find that the k -Schur functions circumvent this problem: a k -Schur function maps to a single Schur function times a q power (with no negatives) or to zero, modulo the ideal. To be more precise, let $I^{\ell n}$ denote the ideal

$$I^{\ell n} = \left\langle s_\lambda \mid \#\{j \mid \lambda_j < \ell\} = n - \ell + 1 \right\rangle.$$

A basis for $\Lambda^\ell/I^{\ell n}$ is given by the Schur functions indexed by partitions in $\Pi^{\ell n}$, the set of partitions with no part larger than ℓ and no more than $n - \ell$ rows of length smaller than ℓ . In [10], certain structure constants associated to representations of Hecke algebras at roots of unity are shown to be the expansion coefficients in

$$s_\lambda s_\mu = \sum_{\nu \in \Pi^{\ell n}} a_{\lambda\mu}^\nu s_\nu \pmod{I^{\ell n}}.$$

We prove that the $a_{\lambda\mu}^\nu$ are just special cases of k -Littlewood-Richardson coefficients by showing that when $\nu \in \Pi^{\ell n}$, the k -Schur function $s_\nu^{(k=n-1)}$ modulo the ideal $I^{\ell n}$ is simply s_ν , and is zero otherwise. Thus it is revealed that the $a_{\lambda\mu}^\nu$ are coefficients in the expansion:

$$s_\lambda^{(k)} s_\mu^{(k)} = \sum_{\nu \in \Pi^{\ell n}} a_{\lambda\mu}^\nu s_\nu^{(k)} + \sum_{\nu \notin \Pi^{\ell n}} c_{\lambda\mu}^{\nu,k} s_\nu^{(k)}.$$

We can then obtain the 3-point Gromov-Witten invariants from this result by simply computing s_ν modulo $J_{\ell n}^q$ for $\nu \in \Pi^{\ell n}$, since $I^{\ell n}$ is a subideal of $J_{\ell n}^q$. In this case, s_ν beautifully reduces to positive $s_{\tau(\nu)}$ times a q power, where $\tau(\nu)$ is the n -core of ν . Consequently, we prove that the 3-point Gromov-Witten invariants are none other than certain k -Schur function Littlewood-Richardson coefficients. To be more specific,

$$C_{\lambda\mu}^{\nu,d} = c_{\lambda\mu}^{\hat{\nu},n-1},$$

where the value of d associates a unique element $\hat{\nu} \in \Pi^{\ell n}$ (given explicitly in Theorem 5.6) to each $\nu \in \mathcal{P}^{\ell n}$.

It also follows from our results that the k -Littlewood-Richardson coefficients, when $k = n - 1$, include the fusion rules for the Wess-Zumino-Witten conformal field theories associated to $\widehat{su}(\ell)$ at level $n - \ell$, since the algorithm given by Kac [12] and Walton [32] for computing in the fusion algebra reduces to the one given by Goodman and Wenzl [10] for computing the Hecke algebra structure constants.

It is important to note that since the Gromov-Witten invariants under consideration are indexed by partitions fitting inside a rectangle, they are given by only a subset of the k -Littlewood-Richardson coefficients. We thus naturally sought the larger picture that would be explained by the complete set of k -Littlewood-Richardson coefficients. In discussion with Mark Shimozono about this problem, he conjectured that the k -Schur functions form the Schubert basis for the homology of the affine (loop) Grassmannian of GL_{k+1} , and that the k -Schur expansion coefficients of the k -Schur coproduct give the integral cohomology of the loop Grassmannian.¹ We introduce in the final section of this article, a family of symmetric functions dual to the k -Schur functions, defined by the weight of certain “ k -tableaux” related to the affine symmetric group [22]. Following the theory of skew Schur functions, we

¹ Since the submission of this article, Thomas Lam proved these two conjectures [18].

prove a number of results about these dual k -Schur functions. In particular, we show that the coefficients in a product of dual k -Schur functions are the structure constants in the k -Schur coproduct, implying from Shimozono’s conjecture that the dual k -Schur functions form the Schubert basis for the cohomology of the loop Grassmannian.

In addition to using the k -Schur functions to study the Gromov-Witten invariants and the loop Grassmannian, they are a natural tool to seek “affine Schubert polynomials”. Our results strongly support the idea of Michelle Wachs that the (dual) k -Schur functions provide the symmetric Grassmannian component of a larger family of polynomials that are analogous to Schubert polynomials, but indexed instead by affine permutations. After discussion with Thomas Lam of the work presented here, he made a beautiful step in this direction by introducing a family of “affine Stanley symmetric functions” that reduce in special cases to the dual k -Schur functions (called “affine Schur functions” in [17]). Details of a connection between the dual k -Schur functions and the cylindric Schur functions of [25] is also carried out in [17].

2. DEFINITIONS

Let Λ denote the ring of symmetric functions over \mathbb{Z} , generated by the elementary symmetric functions $e_r = \sum_{i_1 < \dots < i_r} x_{i_1} \cdots x_{i_r}$, or equivalently by the complete symmetric functions $h_r = \sum_{i_1 \leq \dots \leq i_r} x_{i_1} \cdots x_{i_r}$, and let $\Lambda^k = \mathbb{Z}[h_1, \dots, h_k]$. Bases for Λ are indexed by partitions $\lambda = (\lambda_1 \geq \dots \geq \lambda_m > 0)$ whose degree $|\lambda|$ is $|\lambda| = \lambda_1 + \dots + \lambda_m$ and whose length $\ell(\lambda)$ is the number of parts m . Each partition λ has an associated Ferrers diagram with λ_i lattice squares in the i th row, from the bottom to top. Any lattice square in the Ferrers diagram is called a cell, where the cell (i, j) is in the i th row and j th column of the diagram. Given a partition λ , its conjugate λ' is the diagram obtained by reflecting λ about the main diagonal. A partition λ is “ k -bounded” if $\lambda_1 \leq k$, and the set of all such partitions is denoted \mathcal{P}^k . The set $\mathcal{P}^{\ell n}$ is the partition fitting inside an $\ell \times (n - \ell)$ rectangle (with $n - \ell$ rows of size ℓ). We say that $\lambda \subseteq \mu$ when $\lambda_i \leq \mu_i$ for all i . Dominance order \supseteq on partitions is defined by $\lambda \supseteq \mu$ when $\lambda_1 + \dots + \lambda_i \geq \mu_1 + \dots + \mu_i$ for all i , and $|\lambda| = |\mu|$.

More generally, for $\rho \subseteq \gamma$, the skew shape γ/ρ is identified with its diagram $\{(i, j) : \rho_i < j \leq \gamma_i\}$. We say that any $c \in \rho$ lies “below” γ/ρ . The “hook” of any lattice square $s \in \gamma$ is defined as the collection of cells of γ/ρ that lie inside the L with s as its corner. This is intended to apply to all $s \in \gamma$ including those below γ/ρ . For example, the hook of $s = (1, 3)$ is depicted by the framed cells:

$$(2.1) \quad \gamma/\rho = (5, 5, 4, 1)/(4, 2) = \begin{array}{cccc} & & & \square \\ & & & \square \\ & & \square & \square \\ & \square & \square & \square \\ \square & \square & \square & \square \end{array}.$$

The “hook-length” of s , denoted $h_s(\gamma/\rho)$, is the number of cells in the hook of s . In the preceding example, $h_{(1,3)}((5, 5, 4, 1)/(4, 2)) = 3$ and $h_{(3,1)}((5, 5, 4, 1)/(4, 2)) = 5$. A cell or lattice square has a k -bounded hook if its hook-length is no larger than k .

A “ p -core” is a partition that does not contain any hooks of length p , and \mathcal{C}^p will denote the set of all p -cores. The “ p -residue” of lattice square (i, j) is $j - i \pmod p$; that is, the label of this square when squares are periodically labeled with

$0, 1, \dots, p - 1$, where zeros lie on the main diagonal (see [16] for more on cores and residues). The 5-residues associated to the 5-core $(6, 4, 3, 1, 1, 1)$ are

$$\begin{array}{cccccccc}
 & & & & & & & 4 \\
 & & & & & & & \boxed{0} \\
 & & & & & & & \boxed{1} \\
 & & & & & & & \boxed{2} \quad 3 \\
 & & & & & & & \boxed{3} \quad \boxed{4} \quad \boxed{0} \quad 1 \\
 & & & & & & & \boxed{4} \quad \boxed{0} \quad \boxed{1} \quad \boxed{2} \quad 3 \\
 & & & & & & & \boxed{0} \quad \boxed{1} \quad \boxed{2} \quad \boxed{3} \quad \boxed{4} \quad \boxed{0} \quad 1
 \end{array}$$

A “tableau” is a filling of a Ferrers diagram with integers that strictly increase in columns and weakly increase in rows. The “weight” of a given tableau is the composition α where α_i is the multiplicity of i in the tableau. A “Schur function” can be defined by

$$(2.2) \quad s_\lambda = \sum_T x^T,$$

where the sum is over all tableaux of shape λ , and where $x^T = x^{\text{weight}(T)}$.

3. k -SCHUR FUNCTIONS

There are several conjecturally equivalent characterizations for the k -Schur functions. Here we use the definition explored in [23] that relies on a family of tableaux related to the affine symmetric group.

Definition 3.1 ([22]). Let γ be a $k + 1$ -core, m be the number of k -bounded hooks of γ , and $\alpha = (\alpha_1, \dots, \alpha_r)$ be a composition of m . A “ k -tableau” of shape γ and “ k -weight” α is a filling of γ with integers $1, 2, \dots, r$ such that

- (i) rows are weakly increasing and columns are strictly increasing,
- (ii) the collection of cells filled with the letter i are labeled with exactly α_i distinct $k + 1$ -residues.

Example 3.2. The 3-tableaux of 3-weight $(1, 3, 1, 2, 1, 1)$ and shape $(8, 5, 2, 1)$ are:

$$(3.1) \quad
 \begin{array}{|c|c|c|c|c|c|c|c|}
 \hline
 5 & & & & & & & \\
 \hline
 4 & 6 & & & & & & \\
 \hline
 2 & 3 & 4 & 4 & 6 & & & \\
 \hline
 1 & 2 & 2 & 2 & 3 & 4 & 4 & 6
 \end{array}
 \quad
 \begin{array}{|c|c|c|c|c|c|c|c|}
 \hline
 6 & & & & & & & \\
 \hline
 4 & 5 & & & & & & \\
 \hline
 2 & 3 & 4 & 4 & 5 & & & \\
 \hline
 1 & 2 & 2 & 2 & 3 & 4 & 4 & 5
 \end{array}
 \quad
 \begin{array}{|c|c|c|c|c|c|c|c|}
 \hline
 4 & & & & & & & \\
 \hline
 3 & 6 & & & & & & \\
 \hline
 2 & 4 & 4 & 5 & 6 & & & \\
 \hline
 1 & 2 & 2 & 2 & 4 & 4 & 5 & 6
 \end{array}$$

The definition of k -tableaux easily extends.

Definition 3.3. Let $\delta \subseteq \gamma$ be $k + 1$ -cores with m_1 and m_2 k -bounded hooks respectively, and let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a composition of $m_1 - m_2$. A “skew k -tableau” of shape γ/δ and “ k -weight” α is a filling of γ/δ with integers $1, 2, \dots, r$ such that

- (i) rows are weakly increasing and columns are strictly increasing,
- (ii) the collection of cells filled with letter i are labeled by exactly α_i distinct $k + 1$ -residues.

Although a k -tableau is associated to a shape γ and weight α , in contrast to usual tableaux, $|\alpha|$ does not equal $|\gamma|$. Instead, $|\alpha|$ is the number of k -bounded hooks in γ . This distinction becomes natural through a correspondence between $k + 1$ -cores and k -bounded diagrams. This bijection \mathfrak{c} between \mathcal{C}^{k+1} and \mathcal{P}^k was defined in [22] by its inverse map

$$\mathfrak{c}^{-1}(\gamma) = (\lambda_1, \dots, \lambda_\ell)$$

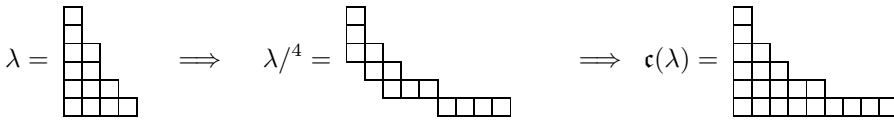
where λ_i is the number of cells with a k -bounded hook in row i of γ . Note that the number of k -bounded hooks in γ is $|\lambda|$. The inverse map relies on constructing a certain “ k -skew diagram” $\lambda/k = \gamma/\rho$ from λ , and setting $\mathbf{c}(\lambda) = \gamma$. These special skew diagrams are defined:

Definition 3.4. For $\lambda \in \mathcal{P}^k$, the “ k -skew diagram of λ ” is the diagram λ/k where

- (i) row i has length λ_i for $i = 1, \dots, \ell(\lambda)$
- (ii) no cell of λ/k has hook-length exceeding k
- (iii) all lattice squares below λ/k have hook-length exceeding k .

A convenient algorithm for constructing the diagram of λ/k is given by successively attaching a row of length λ_i to the bottom of $(\lambda_1, \dots, \lambda_{i-1})/k$ in the leftmost position so that no hook-lengths exceeding k are created.

Example 3.5. Given $\lambda = (4, 3, 2, 2, 1, 1)$ and $k = 4$,



The analogy with usual tableaux is now more apparent, and we let $\mathcal{T}_\alpha^k(\mu)$ denote the set of all k -tableaux of shape $\mathbf{c}(\mu)$ and k -weight α . When the k -weight is (1^n) , a k -tableau is called “standard”. The “ k -Kostka numbers” $K_{\mu\alpha}^{(k)} = |\mathcal{T}_\alpha^k(\mu)|$ satisfy a triangularity property [22] similar to that of the Kostka numbers: for k -bounded partitions λ and μ ,

$$(3.2) \quad K_{\mu\lambda}^{(k)} = 0 \quad \text{when} \quad \mu \not\triangleright \lambda \quad \text{and} \quad K_{\mu\mu}^{(k)} = 1.$$

Given this triangularity, the inverse of $\|K_{\mu\lambda}^{(k)}\|_{\lambda, \mu \in \mathcal{P}^k}$ exists. Our main object of study can now be defined by $\|K^{(k)}\|^{-1}$, denoted $\|\bar{K}^{(k)}\|$.

Definition 3.6. For any $\lambda \in \mathcal{P}^k$, the “ k -Schur function” is defined

$$(3.3) \quad s_\lambda^{(k)} = \sum_{\mu \triangleright \lambda} \bar{K}_{\mu\lambda}^{(k)} h_\mu.$$

A number of properties held by k -Schur functions suggest that these elements play the role of the Schur functions in the subspace Λ^k . First, the definition implies that the set $\{s_\lambda^{(k)}\}_{\lambda_1 \leq k}$ forms a basis of Λ^k , and that for any $\lambda \in \mathcal{P}^k$,

$$(3.4) \quad h_\lambda = \sum_{\mu \triangleright \lambda} K_{\mu\lambda}^{(k)} s_\mu^{(k)}.$$

More generally, it was shown in [23] that if $K_{\nu/\mu, \lambda}^{(k)}$ is the number of skew tableaux of shape $\mathbf{c}(\mu)/\mathbf{c}(\nu)$ and k -weight λ , then

$$(3.5) \quad h_\lambda s_\nu^{(k)} = \sum_{\mu} K_{\mu/\nu, \lambda}^{(k)} s_\mu^{(k)}.$$

Another example of a Schur property held by k -Schur functions is drawn from the ω -involution, defined as the homomorphism $\omega(h_i) = e_i$. In particular, ω maps a Schur function s_λ to its conjugate $s_{\lambda'}$. Using a refinement of partition conjugation that arose in [19], [20], it was shown in [23] that

$$(3.6) \quad \omega s_\lambda^{(k)} = s_{\lambda^{\omega_k}}^{(k)},$$

where $\lambda^{\omega_k} = \mathfrak{c}^{-1}(\mathfrak{c}(\lambda)')$ is the “ k -conjugate” of λ . This result led to the property:

$$(3.7) \quad s_\lambda^{(k)} = s_\lambda \text{ when } h_{(1,1)}(\lambda) \leq k,$$

that is, when the hook-length of the cell $(1,1)$ is not larger than k .

In (3.5), the case where $\lambda = (\ell)$ is a one-row partition of size not larger than k is called the “ k -Pieri formula”. Results of [22] laid the groundwork for another characterization of the k -Pieri formula that mimics that of the Pieri formula. In particular,

Theorem 3.7 ([23]). *For $\nu \in \mathcal{P}^k$ and $\ell \leq k$,*

$$(3.8) \quad h_\ell s_\nu^{(k)} = \sum_{\mu \in H_{\nu,\ell}^{(k)}} s_\mu^{(k)}$$

where the sum is over partitions of the form:

$$H_{\nu,\ell}^{(k)} = \left\{ \mu \in \mathcal{P}^k \mid \mu/\nu = \text{horizontal } \ell\text{-strip} \text{ and } \mu^{\omega_k}/\nu^{\omega_k} = \text{vertical } \ell\text{-strip} \right\}.$$

When $\mu \in H_{\nu,\ell}^{(k)}$, we say that “ μ/ν is a k -horizontal strip of size ℓ ”.

In the spirit of Schur function theory, it is conjectured that the “ k -Littlewood-Richardson coefficients” in

$$(3.9) \quad s_\lambda^{(k)} s_\mu^{(k)} = \sum_{\nu: \nu_1 \leq k} c_{\lambda\mu}^{\nu,k} s_\nu^{(k)}$$

are positive numbers. Our development here will prove that in certain cases these coefficients are Gromov-Witten invariants thus proving positivity in these cases. Note that given the action of the ω involution on k -Schur functions, the k -Littlewood-Richardson coefficients satisfy

$$(3.10) \quad c_{\lambda\mu}^{\nu,k} = c_{\lambda^{\omega_k}\mu^{\omega_k}}^{\nu^{\omega_k},k}.$$

4. HECKE ALGEBRAS, FUSION RULES, AND THE k -SCHUR FUNCTIONS

Presented in [10] are generalized Littlewood-Richardson coefficients for (ℓ, n) -representations of the Hecke algebras $H_\infty(q)$, when q is an n th root of unity. These coefficients are equal to the structure constants for the Verlinde (fusion) algebra associated to the $\widehat{su}(\ell)$ -Wess-Zumino-Witten conformal field theories at level $n - \ell$. In this section, we will use the k -Pieri rule to establish that for $k = n - 1$, the k -Littlewood-Richardson coefficients contain these constants as special cases.

4.1. The connection. From [10], we recall a simple interpretation for these “ (ℓ, n) -Littlewood-Richardson coefficients” given in the language of symmetric functions. For $n > \ell \geq 1$, consider the quotient $R^{\ell n} = \Lambda^\ell / I^{\ell n}$ where $I^{\ell n}$ is the ideal generated by Schur functions that have exactly $n - \ell + 1$ rows of length smaller than ℓ :

$$I^{\ell n} = \left\langle s_\lambda \mid \#\{j \mid \lambda_j < \ell\} = n - \ell + 1 \right\rangle.$$

A basis for $R^{\ell n}$ is given by the set $\{s_\lambda\}_{\lambda \in \Pi^{\ell n}}$ where the indices are partitions in:

$$\Pi^{\ell n} = \{\lambda \in \mathcal{P} : \lambda_1 \leq \ell \text{ and } \#\{j \mid \lambda_j < \ell\} \leq n - \ell\}.$$

Theorem 4.1 ([10]). *The (ℓ, n) -Littlewood-Richardson coefficients are the coefficients $a_{\lambda\mu}^\nu$ in the expansion*

$$(4.1) \quad s_\lambda s_\mu = \sum_{\nu} a_{\lambda\mu}^\nu s_\nu \pmod{I^{\ell n}}, \quad \text{where } \lambda, \mu, \nu \in \Pi^{\ell n}.$$

It is in this context that we prove the coefficients $a_{\lambda\mu}^\nu$ are none other than k -Littlewood-Richardson coefficients when $k = n - 1$.

Remark 4.2. The results of [10] are presented in a transposed form, where they instead work with the ideal $\langle s_\lambda \mid \lambda_1 - \lambda_\ell = n - \ell + 1 \rangle$ in $\mathbb{Z}[e_1, \dots, e_\ell]$. Their (ℓ, n) -Littlewood-Richardson coefficients $d_{\lambda\mu}^{\nu'}$ are thus our $a_{\lambda'\mu'}^{\nu'}$, for $\lambda', \mu', \nu' \in \Pi^{\ell n}$.

To provide some insight into how this connection arose, consider the special case of Eq. (4.1) with $\lambda = (1)$:

$$(4.2) \quad s_1 s_\mu = \sum_{\substack{\nu: \mu \subset \nu \in \Pi^{\ell n} \\ |\nu| = |\mu| + 1}} s_\nu \pmod{I^{\ell n}},$$

and define a poset by letting $\mu < \nu$ for all ν in the summand. Frank Sottile brought this poset to our attention and asked if it was related to our study [22] of the k -Young lattice Y^k . Y^k is defined by the k -Pieri rule, where $\mu < \nu$ when $\nu \in H_{\mu,1}^{(k)}$. Investigating his question, we discovered the posets can be connected through the principal order ideal $L^k(\ell, m)$ generated by an $\ell \times m$ rectangle in Y^k . In [21], we found that the vertices of $L^k(\ell, m)$ are the partitions contained in an $\ell \times m$ rectangle with no more than $k - \ell + 1$ rows shorter than k , and that μ covers λ in this poset if and only if $\lambda \subseteq \mu$ and $|\lambda| + 1 = |\mu|$. Therefore, the elements of $L^k(\ell, \infty)$ are precisely those of $\Pi^{\ell n}$ (given $k = n - 1$). Since the k -Young lattice was defined by multiplication by s_1 , we have

$$(4.3) \quad s_1 s_\lambda^{(k)} = \sum_{\substack{\mu: \lambda \subset \mu \in \Pi^{\ell n} \\ |\mu| = |\lambda| + 1}} s_\mu^{(k)} + \text{other terms},$$

where ‘‘other terms’’ are k -Schur functions indexed by $\mu \notin \Pi^{\ell n}$. The likeness of (4.2) and (4.3) led us to surmise the following result:

Theorem 4.3. *For any partition $\lambda \in \mathcal{P}^{n-1}$,*

$$(4.4) \quad s_\lambda^{(n-1)} \pmod{I^{\ell n}} = \begin{cases} s_\lambda & \text{if } \lambda \in \Pi^{\ell n}, \\ 0 & \text{otherwise.} \end{cases}$$

Before proving this theorem, we mention several implications. Since all partitions in $\Pi^{\ell n}$ are $(n - 1)$ -bounded ($\lambda_1 \leq \ell \leq n - 1$), the set of k -Schur functions indexed by partitions in $\Pi^{\ell n}$ forms a natural basis for the quotient $R^{\ell n}$. Computation modulo the ideal $I^{\ell n}$ is trivial in this basis. In particular, the structure constants under consideration are simply certain k -Littlewood Richardson coefficients in (3.9).

Corollary 4.4. *For all $\lambda, \mu, \nu \in \Pi^{\ell n}$,*

$$a_{\lambda\mu}^\nu = c_{\lambda\mu}^{\nu, n-1}.$$

Another consequence of our theorem produces a tableau interpretation for the dimension of the representations $\pi_\lambda^{(\ell, n)}$, for $\lambda' \in \Pi^{\ell n}$, of the Hecke algebras $H_\infty(q)$, when q is an n th root of unity (see [10] for details on these representations).

Corollary 4.5. For $\lambda' \in \Pi^{\ell n}$, the dimension of the representation $\pi_\lambda^{(\ell, n)}$ is the number of standard $(n - 1)$ -tableaux of shape $\mathfrak{c}(\lambda')$.²

Proof. Let $m = |\lambda|$, and $k = n - 1$. In [10], it is shown that the dimension of $\pi_\lambda^{(\ell, n)}$ is the coefficient of $s_{\lambda'}$ in $s_1^m \pmod{I^{\ell n}}$. By Theorem 4.3, this is the coefficient of $s_{\lambda'}^{(k)}$ in the k -Schur expansion of $s_1^m = h_{1^m}$. Using Definition 3.6 for k -Schur functions, this coefficient is $K_{\lambda'1^m}^{(k)}$ or the number of standard k -tableaux of shape $\mathfrak{c}(\lambda')$. \square

The Verlinde (fusion) algebra of the Wess-Zumino-Witten model associated to $\widehat{su}(\ell)$ at level $n - \ell$ is isomorphic to the quotient of $R^{\ell n}$ modulo the single relation $s_\ell \equiv 1$ [12], [32], [10]. The fusion coefficient $\mathcal{N}_{\lambda\mu}^\nu$ is defined for $\lambda', \mu', \nu' \in \mathcal{P}^{\ell-1, n-1}$ by

$$L(\lambda) \otimes_{n-\ell} L(\mu) = \bigoplus \mathcal{N}_{\lambda\mu}^\nu L(\nu),$$

where the fusion product $\otimes_{n-\ell}$ is the reduction of the tensor product of integrable representations with highest weight λ and μ via the representation at level $n - \ell$ of $\widehat{su}(\ell)$. Thus, our results imply that

Corollary 4.6. For all λ, μ, ν inside an $(n - \ell) \times (\ell - 1)$ rectangle,

$$\mathcal{N}_{\lambda\mu}^\nu = \hat{c}_{\lambda'\mu'}^{\hat{\nu}, n-1},$$

where $\hat{\nu} = (\ell(|\lambda| + |\mu| - |\nu|)/\ell, \nu')$.

4.2. Proof of connection. Throughout this section, k stands for $n - 1$. Rather than working with $\Lambda^\ell/I^{\ell n}$, it is more convenient to work with Λ/\mathcal{I} , where the ideal \mathcal{I} in Λ is generated by the Schur functions that have exactly $n - \ell + 1$ parts smaller than ℓ and by the h_i with $i > \ell$. That is,

$$\mathcal{I} = \langle A \cup B \rangle,$$

where

$$A = \{s_\lambda \mid \#\{j \mid \lambda_j < \ell\} = n - \ell + 1\} \quad \text{and} \quad B = \{h_i \mid i > \ell\}.$$

Note that $\Lambda^\ell/I^{\ell n} \simeq \Lambda/\mathcal{I}$ since the Schur functions in $I^{\ell n}$ can be considered as expressions in Λ^ℓ (using the Jacobi-Trudi determinant for instance to express them in the homogeneous basis and then sending $h_i \rightarrow 0$ if $i > \ell$). Also, observe that a k -Schur function can be considered as an element of Λ under the natural inclusion $\Lambda^\ell \rightarrow \Lambda$.

The proof of Theorem 4.3 relies on two preliminary properties.

Property 4.7. For any k -bounded partition λ and $\ell \leq k$, $s_\lambda^{(k)} \equiv_{\mathcal{I}} 0$ when $\lambda_1 > \ell$.

Proof. Since $\mu \geq \lambda$ implies that $\mu_1 \geq \lambda_1$, the unitriangular relation between $\{s_\lambda^{(k)}\}$ and $\{h_\lambda\}$ implies

$$s_\lambda^{(k)} = \sum_{\mu: \mu_1 > \ell} * h_\mu.$$

The claim thus follows since $h_\mu \in \mathcal{I}$ when $\mu_1 > \ell$. \square

Property 4.8. For any k -bounded partition λ with $\lambda_1 \leq \ell$,

$$(4.5) \quad s_\lambda^{(k)} \equiv_{\mathcal{I}} 0 \implies s_{(\ell^m, \lambda)}^{(k)} \equiv_{\mathcal{I}} 0 \quad \text{for all } m \geq 0.$$

²Equivalently, this is the number of reduced words for a certain affine permutation $\sigma_{\lambda'} \in \hat{S}_n/S_n$. See [22] for the precise correspondence.

Proof. The k -Pieri rule (3.8) implies, in particular, that any k -Schur occurring in the expansion of $h_\ell s_\nu^{(k)}$ is indexed by a partition obtained by adding a horizontal ℓ -strip to ν . Thus, when $\ell \geq \nu_1$, we have

$$(4.6) \quad h_\ell s_\nu^{(k)} = s_{(\ell, \nu)}^{(k)} + \sum_{\substack{\mu: \mu_1 > \ell \\ \mu \in H_{\nu, \ell}^k}} s_\mu^{(k)}.$$

Starting from $s_\lambda^{(k)} \equiv_{\mathcal{I}} 0$, and assuming by induction that $s_{(\ell^{m-1}, \lambda)}^{(k)} \equiv_{\mathcal{I}} 0$, the claim follows from Property 4.7 and the previous expression (4.6),

$$0 \equiv_{\mathcal{I}} h_\ell s_{(\ell^{m-1}, \lambda)}^{(k)} = s_{(\ell^m, \lambda)}^{(k)} + \sum_{\gamma: \gamma_1 > \ell} * s_\gamma^{(k)} \equiv_{\mathcal{I}} s_{(\ell^m, \lambda)}^{(k)}.$$

□

4.3. Proof of Theorem 4.3. Recall $n = k + 1$, and that $\lambda \in \Pi^{\ell, k+1}$ has the form $\lambda = (\ell^m, \mu)$ for some $\mu \in \mathcal{P}^{\ell-1, k}$. First, by induction on m we prove that $s_\lambda^{(k)} \equiv_{\mathcal{I}} s_\lambda$ for each such λ . Since $h_{(1,1)}(\lambda) \leq k$ when $m = 0$, $s_\lambda^{(k)} = s_\lambda$ by (3.7). By induction, assuming $s_{(\ell^m, \mu)}^{(k)} \equiv_{\mathcal{I}} s_{(\ell^m, \mu)}$, we have $h_\ell s_{(\ell^m, \mu)}^{(k)} \equiv_{\mathcal{I}} h_\ell s_{(\ell^m, \mu)}$. On the other hand, since $s_\gamma \equiv_{\mathcal{I}} 0$ when $\gamma_1 > \ell$, identity (4.6) implies $h_\ell s_{(\ell^m, \mu)}^{(k)} \equiv_{\mathcal{I}} s_{(\ell^{m+1}, \mu)}^{(k)}$. Therefore,

$$h_\ell s_{(\ell^m, \mu)}^{(k)} \equiv_{\mathcal{I}} s_{(\ell^{m+1}, \mu)}^{(k)}.$$

The claim then follows by noting that the Pieri rule gives an expansion similar to (4.6) for $h_\ell s_{\ell^m, \mu}$, implying that $h_\ell s_{(\ell^m, \mu)} \equiv_{\mathcal{I}} s_{(\ell^{m+1}, \mu)}$.

It remains to prove that $s_\eta^{(k)} \equiv_{\mathcal{I}} 0$ when $\eta \notin \Pi^{\ell, k+1}$. Since Property 4.7 proves the case when $\eta_1 > \ell$, we must show $s_\eta^{(k)} \equiv_{\mathcal{I}} 0$ for any η in the set:

$$\mathcal{Q} = \{(\ell^m, \beta) \in \mathcal{P} : \beta_1 < \ell \text{ and } \ell(\beta) \geq k - \ell + 2\}.$$

Our proof is inductive, using an order defined on \mathcal{Q} as follows: $\eta = (\ell^a, \beta) \preceq (\ell^b, \alpha) = \mu$ if $\ell(\beta) < \ell(\alpha)$ or if $\ell(\beta) = \ell(\alpha)$ and $\eta \supseteq \mu$ (this is a well-ordering if we restrict ourselves to $|\mu| = |\eta|$). Our base case includes partitions $\eta = (\ell^a, \beta)$ with $\beta_1 < \ell$ and $\ell(\beta) = k - \ell + 2$. In this case, $h(\beta) \leq k$ implies $s_\beta^{(k)} = s_\beta$ from (3.7), and since $s_\beta \in \mathcal{I}$ when β has $k - \ell + 2$ parts smaller than ℓ , we have $s_\beta^{(k)} \equiv_{\mathcal{I}} 0$. Property 4.8 then proves $s_\eta^{(k)} \equiv_{\mathcal{I}} 0$ in this case.

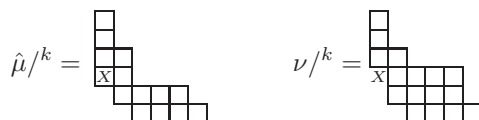
Now assume by induction that $s_\eta^{(k)} \equiv_{\mathcal{I}} 0$ for all $\eta \in \mathcal{Q}$ such that $\eta \prec \mu$, where $\mu = (\ell^b, \alpha)$ with $\alpha_1 < \ell$ and $\ell(\alpha) > k - \ell + 2$. With $r < \ell$ denoting the last part of μ (and thus also the last part of α), let $\mu = (\hat{\mu}, r) = (\ell^b, \hat{\alpha}, r)$ and note that $\hat{\mu} \prec \mu$. Thus, using the induction hypothesis and the k -Pieri rule, we have

$$0 \equiv_{\mathcal{I}} s_r s_{\hat{\mu}}^{(k)} = s_\mu^{(k)} + \sum_{\nu \in H_{\hat{\mu}, r}^{(k)} \setminus \{\mu\}} s_\nu^{(k)},$$

and it suffices to show that $s_\nu^{(k)} \equiv_{\mathcal{I}} 0$ for all $\nu \in H_{\hat{\mu}, r}^{(k)} \setminus \{\mu\}$. Property 4.7 proves this immediately for any ν with $\nu_1 > \ell$, and thus we shall consider only ℓ -bounded ν . Two properties of such ν follow since ν is obtained by adding a horizontal r -strip to $\hat{\mu} = (\ell^b, \hat{\alpha})$: $\nu \supseteq \mu$, and $\nu = (\ell^b, \beta)$, where $\ell(\beta) \leq \ell(\hat{\alpha}) + 1 = \ell(\alpha)$. Thus, if these ν lie in \mathcal{Q} , then $\nu \prec \mu$ and our claim follows from the induction hypothesis. Since each such ν is obtained by adding a horizontal strip to $\hat{\mu} = (\ell^b, \hat{\alpha})$, and $\ell(\alpha) > k - \ell + 2$, we have $\ell(\beta) \geq \ell(\hat{\alpha}) \geq k - \ell + 2$. Thus, these $\nu = (\ell^b, \beta)$ all lie in \mathcal{Q} except in the

case that $\ell(\beta) = \ell(\hat{\alpha}) = k - \ell + 2$ and $\beta_1 = \ell$. The following paragraph explains why, in this case, $\nu \notin H_{\hat{\mu},r}^{(k)}$, and thus never arises.

Given $\ell(\beta) = k - \ell + 2$ and $\beta_1 = \ell$, $h(\beta) > k$ implies there is no cell in position $X = (1, \ell(\nu) - \ell(\beta))$ of ν/k . Assume by contradiction that $\nu \in H_{\hat{\mu},r}^{(k)}$ – hence, in particular, that $\nu^{\omega_k}/\hat{\mu}^{\omega_k}$ is a vertical strip. Since $\ell(\hat{\alpha}) = \ell(\beta) = k - \ell + 2$ and $\hat{\alpha}_1 < \ell$ imply $h(\hat{\alpha}) \leq k$, there is a cell in $\hat{\mu}/k$ in position $(1, \ell(\mu) - \ell(\hat{\alpha})) = X$. Since the height of ν/k and $\hat{\mu}/k$ are equal, but position X is empty in ν/k and filled in $\hat{\mu}/k$, the first column of ν/k is shorter than that of $\hat{\mu}/k$, implying $\nu^{\omega_k}/\hat{\mu}^{\omega_k}$ is not a vertical strip. By contradiction, $\nu \notin H_{\hat{\mu},r}^{(k)}$ as claimed. Here is an example with $\hat{\mu} = (4, 4, 2, 2, 1, 1)$, $\nu = (4, 4, 4, 2, 1, 1)$, $n = 4$ and $k = 6$:



□

5. QUANTUM COHOMOLOGY

Witten [33] proved that the Verlinde algebra of $\widehat{u}(\ell)$ at level $n - \ell$ and the quantum cohomology of the Grassmannian $Gr_{\ell n}$ are isomorphic (see also [1]). Since $u(\ell) = su(\ell) \times u(1)$, the connection between k -Schur functions and the fusion coefficients of $\widehat{su}(\ell)$ at level $n - \ell$ given in the last section implies that there is also a connection between k -Schur functions and the quantum cohomology of the Grassmannian. We now set out to make this connection explicit.

Recall from the Introduction that the quantum structure constants, or 3-point Gromov-Witten invariants $C_{\lambda\mu}^{\nu,d}$, arise in the expansion, for $\lambda, \mu \in \mathcal{P}^{\ell n}$,

$$(5.1) \quad s_\lambda s_\mu = \sum_{\substack{d \geq 0, \nu \in \mathcal{P}^{\ell n} \\ |\nu| = |\lambda| + |\mu| - dn}} q^d C_{\lambda\mu}^{\nu,d} s_\nu \pmod{J_q^{\ell n}},$$

where

$$J_q^{\ell n} = \langle e_{n-\ell+1}, \dots, e_{n-1}, e_n + (-1)^\ell q \rangle.$$

Our main goal is to prove that the k -Schur function basis gives a direct route to these constants. In particular, by determining the value of a k -Schur function modulo this ideal, we will see that the Gromov-Witten invariants arise as special cases of the k -Littlewood-Richardson coefficients.

Instead of working in $(\Lambda^\ell \otimes \mathbb{Z}[q])/J_q^{\ell n}$, we can instead (by reasoning similar to that in Section 4.2) work in $(\Lambda \otimes \mathbb{Z}[q])/J_q$, where J_q is the ideal generated by

$$J_q = \langle e_{n-\ell+1}, \dots, e_{n-1}, e_n + (-1)^\ell q, h_{\ell+1}, h_{\ell+2}, \dots \rangle.$$

Theorem 4.3 reveals that a k -Schur function modulo the ideal \mathcal{I} is a Schur function when $\lambda \in \Pi^{\ell n}$ and is otherwise zero. By showing that \mathcal{I} is a subideal of J_q , our task to determine a k -Schur function mod J_q is thus reduced to examining what happens to a usual Schur function $s_\lambda \pmod{J_q}$ in the special case that $\lambda \in \Pi^{\ell n}$.

Proposition 5.1. *If $f \in \mathcal{I}$, then $f \in J_q$.*

Proof. It suffices to prove that $s_\lambda \in \mathcal{J}_q$ when $\lambda = (\ell^m, \alpha)$, for some m and partition α such that $\alpha_1 < \ell$ and $\ell(\alpha) = n - \ell + 1$. When $m = 0$, the result follows from the Jacobi-Trudi determinantal formula since the first row of the determinant of s_α has entries $e_{n-\ell+1}, \dots, e_{n+\alpha_1-\ell} \in \mathcal{J}_q$ given $\alpha_1 < \ell$. Assuming by induction that $s_{(\ell^m, \alpha)} \in \mathcal{J}_q$, since the Pieri rule implies $h_\ell s_{(\ell^m, \alpha)} = s_{(\ell^{m+1}, \alpha)} \pmod{\langle h_{\ell+1}, h_{\ell+2}, \dots \rangle}$, the result follows by induction. \square

Now to determine the value of a usual Schur function $s_\lambda \pmod{\mathcal{J}_q}$ for partitions in $\Pi^{\ell n}$, we shall use an important result from [5], where the theory of rim-hooks was used to study the Schur functions modulo \mathcal{J}_q . To state their result, we first recall the necessary definitions. An “ n -rim hook” is a connected skew diagram of size n that contains no 2×2 rectangle. “ $\tau(\lambda)$ ” denotes the n -core of λ , obtained by removing as many n -rim hooks as possible from the diagram of λ (this is well-defined since the order in which rims are removed is known to be irrelevant [16]). The width of a rim hook is the number of columns it occupies minus one. Given a partition λ , let “ d_λ ” be the number of n -rim hooks that are removed to obtain $\tau(\lambda)$. Also, let “ ϵ_λ ” equal $d_\lambda(\ell - 1)$ minus the sum of the widths of these rim hooks.

Theorem 5.2 ([5]). *For $\lambda \in \mathcal{P}^\ell$,*

$$(5.2) \quad s_\lambda \pmod{\mathcal{J}_q^{\ell n}} = \begin{cases} (-1)^{\epsilon_\lambda} q^{d_\lambda} s_{\tau(\lambda)} & \text{if } \tau(\lambda) \in \mathcal{P}^{\ell n}, \\ 0 & \text{otherwise.} \end{cases}$$

This result helps us prove that in the special case that $\lambda \in \Pi^{\ell n}$, $s_\lambda \equiv_{\mathcal{J}_q} q^{d_\lambda} s_\nu$ for a partition ν obtained using the following operators:

Definition 5.3. For $\lambda \in \Pi^{\ell n}$, $\blacktriangle(\lambda)$ is the partition obtained by adding an n -rim hook to λ starting in column ℓ and ending in the first column. For $\lambda \in \Pi^{\ell n}$ that is not an n -core, $\blacktriangledown(\lambda)$ is the partition obtained by removing an n -rim hook from λ starting in the first column of λ .

Note that \blacktriangledown is well-defined since when $\lambda \in \Pi^{\ell n}$ is not an n -core, $\ell(\lambda) > n - \ell$. Thus for $r = \ell(\lambda) - (n - \ell)$, $\lambda_r = \ell$ and $h_{(r,1)}(\lambda) = n$ implying an n -rim hook can be removed starting in the first column of λ and ending in the last column ℓ . Since the difference between the heights of the starting point and the ending point is $n - \ell$,

$$(5.3) \quad \blacktriangledown : \{ \lambda \mid \lambda \in \Pi^{\ell n} \ \& \ \lambda \neq n\text{-core} \} \rightarrow \Pi^{\ell n} .$$

Similarly, for any $\lambda \in \Pi^{\ell n}$, the difference between the heights of the first column and column ℓ is at most $n - \ell$. Thus, an n -rim hook can be added to λ starting from column ℓ and ending in the first column. Since the difference in heights of the starting point and ending point of the added n -rim hook is $n - \ell$, we have that

$$(5.4) \quad \blacktriangle : \Pi^{\ell n} \rightarrow \Pi^{\ell n} .$$

By construction, as long as $\blacktriangledown(\lambda)$ is defined, we have

$$(5.5) \quad \blacktriangledown(\blacktriangle(\lambda)) = \lambda \quad \text{and} \quad \blacktriangle(\blacktriangledown(\lambda)) = \lambda .$$

Proposition 5.4. *For $\lambda \in \Pi^{\ell n}$,*

$$(5.6) \quad s_\lambda \equiv q^{d_\lambda} s_\nu \pmod{\mathcal{J}_q},$$

where $\nu = \blacktriangledown^{d_\lambda}(\lambda) \in \mathcal{P}^{\ell n}$.

Proof. We have $s_\lambda \equiv \mathcal{J}_q(-1)^{\epsilon_\lambda} q^{d_\lambda} s_{\tau(\lambda)}$ by (5.2). When λ is an n -core, then $\lambda \in \Pi^{\ell n}$ implies $\lambda \in \mathcal{P}^{\ell n}$. Thus $\tau(\lambda) = \lambda$ and (5.6) holds with $d_\lambda = 0$. Otherwise, $\tau(\lambda)$ is obtained by removing d_λ n -rim hooks in any order. Thus, by successively applying \blacktriangledown , we obtain $\tau(\lambda) = \blacktriangledown^{d_\lambda}(\lambda)$. Since \blacktriangledown preserves $\Pi^{\ell n}$ by (5.3), $\tau(\lambda) \in \mathcal{P}^{\ell n}$. Further, $\epsilon_\lambda = d_\lambda(\ell - 1) - d_\lambda(\ell - 1) = 0$ since each removed n -rim hook has width $\ell - 1$. \square

In this notation, we can now determine the value of a k -Schur function mod \mathcal{J}_q .

Theorem 5.5. *For any k -bounded partition λ ,*

$$s_\lambda^{(n-1)} \text{ mod } \mathcal{J}_q = \begin{cases} q^{d_\lambda} s_\nu & \text{if } \lambda \in \Pi^{\ell n} \\ 0 & \text{otherwise} \end{cases},$$

where $\nu = \tau(\lambda) = \blacktriangledown^{d_\lambda}(\lambda) \in \mathcal{P}^{\ell n}$.

Proof. Proposition 5.1 gives that \mathcal{I} is a subideal of \mathcal{J}_q , implying

$$s_\lambda^{(n-1)} \text{ mod } \mathcal{J}_q = \left(s_\lambda^{(n-1)} \text{ mod } \mathcal{I} \right) \text{ mod } \mathcal{J}_q.$$

For $\lambda \notin \Pi^{\ell n}$, $s_\lambda^{(n-1)} \text{ mod } \mathcal{I} = 0$ by Theorem 4.3. For $\lambda \in \Pi^{\ell n}$, Theorem 4.3 implies that $s_\lambda^{(n-1)} \text{ mod } \mathcal{I} = s_\lambda$, and the claim then follows by further modding out by \mathcal{J}_q according to Proposition 5.4. \square

This theorem enables us to connect the quantum product to the product of k -Schur functions.

Theorem 5.6. *For $\lambda, \mu, \nu \in \mathcal{P}^{\ell n}$, the 3-point Gromov-Witten invariants $C_{\lambda\mu}^{\nu,d}$ are*

$$(5.7) \quad C_{\lambda\mu}^{\nu,d} = c_{\lambda\mu}^{\hat{\nu},n-1},$$

where $\hat{\nu} = \blacktriangle^d(\nu)$, and where $c_{\lambda\mu}^{\hat{\nu},n-1}$ is a k -Littlewood-Richardson coefficient.

Proof. For $\lambda, \mu \in \mathcal{P}^{\ell n}$, (5.1) shows that $C_{\lambda\mu}^{\nu,d}$ arise in the expansion

$$(5.8) \quad s_\lambda s_\mu \equiv \sum_{\substack{d \geq 0, \nu \in \mathcal{P}^{\ell n} \\ |\nu| = |\lambda| + |\mu| - dn}} C_{\lambda\mu}^{\nu,d} q^d s_\nu \text{ mod } \mathcal{J}_q.$$

On the other hand, since $\lambda, \mu \in \mathcal{P}^{\ell n}$ have hook-length smaller than n , (3.7) implies that $s_\lambda^{(n-1)} s_\mu^{(n-1)} = s_\lambda s_\mu$. Therefore, applying Proposition 5.4 to the k -Schur expansion of this product gives

$$(5.9) \quad s_\lambda s_\mu = \sum_{\gamma: |\gamma| = |\lambda| + |\mu|} c_{\lambda\mu}^{\gamma,n-1} s_\gamma^{(n-1)} \equiv \sum_{\gamma \in \Pi^{\ell n}} c_{\lambda\mu}^{\gamma,n-1} q^{d_\gamma} s_\beta \text{ mod } \mathcal{J}_q,$$

where $\beta = \blacktriangledown^{d_\gamma}(\gamma)$. Taking the coefficient of $q^d s_\nu$ in (5.8) and (5.9) implies

$$C_{\lambda\mu}^{\nu,d} = \sum_{\substack{\gamma \in \Pi^{\ell n} \\ \gamma: \nu = \blacktriangledown^d(\gamma)}} c_{\lambda\mu}^{\gamma,n-1}.$$

Since $\nu = \blacktriangledown^d(\gamma) \in \mathcal{P}^{\ell n} \subseteq \Pi^{\ell n}$, we can apply \blacktriangle to find there is a unique γ in the right summand. That is, $\blacktriangle^d(\nu) = \gamma$ by (5.5). \square

6. DUAL k -SCHUR FUNCTIONS

The quantum structure constants $C_{\lambda\mu}^{\nu,d}$ are indexed by $\lambda, \mu, \nu \in \mathcal{P}^{\ell n}$ and we have now seen that these numbers are precisely k -Littlewood-Richardson coefficients in the relevant cases. However, since there are far more k -Littlewood-Richardson coefficients than Gromov-Witten invariants we naturally sought the larger geometric picture that would be explained by the complete set of k -Littlewood Richardson coefficients. As mentioned in the Introduction, Mark Shimozono conjectured that the k -Schur functions form the Schubert basis for the homology of affine (loop) Grassmannian of GL_{k+1} , and that the expansion coefficients of the coproduct of k -Schur functions in terms of k -Schur functions give the integral cohomology of loop Grassmannian (see e.g. [8], [13] for more details on the loop Grassmannian). Theorem 5.6 provides further evidence for this assertion based on the existence [26] of a surjective ring homomorphism from the homology of the loop Grassmannian onto the quantum cohomology of the Grassmannian at $q = 1$.³

While the homology of the loop Grassmannian is isomorphic to Λ^k , the cohomology is isomorphic to $\Lambda/\mathfrak{J}^{(k)}$ for the ideal

$$\mathfrak{J}^{(k)} = \langle m_\lambda : \lambda_1 > k \rangle.$$

See [11] for details on this identification. We will show that there is a duality between Λ^k and $\Lambda/\mathfrak{J}^{(k)}$ that implies that the coproduct in Λ^k amounts to a product in $\Lambda/\mathfrak{J}^{(k)}$. Under the aforementioned isomorphisms, Shimozono's conjectures thus imply that the Schubert classes of the integral homology and cohomology of the loop Grassmannian of GL_{k+1} are respectively sent to k -Schur functions and to symmetric functions dual to the k -Schur functions. This suggests that there is a fundamental basis for $\Lambda/\mathfrak{J}^{(k)}$ that is closely tied to the k -Schur basis. Here, we introduce a family of functions defined by the k -weight of k -tableaux and derive a number of properties including a duality relation to the k -Schur functions. In particular, it will develop that if the coproduct of k -Schur functions in terms of k -Schur functions indeed gives the integral cohomology of the loop Grassmannian, then these functions are the Schubert basis for the cohomology of the loop Grassmannian.

Recall that a Schur function can be defined as

$$(6.1) \quad s_\lambda = \sum_T x^T,$$

where the sum is over all tableaux of shape λ . We extend this idea by considering the family of functions that arises by summing over all k -tableaux (defined in § 3 with k -weights).

Definition 6.1. For any $\lambda \in \mathcal{P}^k$, the “dual k -Schur function” is defined by

$$(6.2) \quad \mathfrak{S}_\lambda^{(k)} = \sum_T x^T,$$

where the sum is over all k -tableaux of shape $\mathfrak{c}(\lambda)$, and $x^T = x^{k\text{-weight}(T)}$.

Using a generalization of the Bender-Knuth involution [6], it was shown in [23] that the number of k -tableaux of shape $\mathfrak{c}(\lambda)$ and k -weight γ equals the number of k -tableaux of shape $\mathfrak{c}(\lambda)$ and k -weight α , for α any rearrangement of the composition

³These conjectures have recently been proven by Thomas Lam. See [18] for the proof as well as further details on the surjection of [26] and Shimozono's conjectures.

γ . The symmetry of the dual k -Schur functions then follows since the coefficient of x^γ in $\mathfrak{S}_\lambda^{(k)}$ is the number of k -tableaux of k -weight γ .

Proposition 6.2. *For any k -bounded partition λ , $\mathfrak{S}_\lambda^{(k)}$ is a symmetric function.*

Since the k -Kostka number $K_{\lambda\alpha}^{(k)}$ denotes the number of k -tableaux of shape $\mathfrak{c}(\lambda)$ and k -weight α , the symmetry of dual k -Schur functions and the unitriangularity of k -Kostka numbers (3.2) imply the following alternative characterization for dual k -Schur functions:

Proposition 6.3. *For any k -bounded partition λ ,*

$$(6.3) \quad \mathfrak{S}_\lambda^{(k)} = m_\lambda + \sum_{\mu \triangleleft \lambda} K_{\lambda\mu}^{(k)} m_\mu.$$

From this, we see that the dual k -Schur functions are a basis for the quotient of the symmetric function space by the ideal $\mathfrak{J}^{(k)}$:

Proposition 6.4. *The dual k -Schur functions form a basis of $\Lambda/\mathfrak{J}^{(k)}$.*

Recall that the k -Schur functions form a basis for $\Lambda/\langle h_i \mid i > k \rangle$. The ideal $\mathfrak{J}^{(k)}$ is dual to $\langle h_i \mid i > k \rangle$ with respect to the scalar product defined on Λ by

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}.$$

Since the definition of the k -Schur function, $s_\lambda^{(k)} = \sum_\nu \bar{K}_{\nu\lambda}^{(k)} h_\nu$, implies that

$$\langle s_\lambda^{(k)}, \mathfrak{S}_\mu^{(k)} \rangle = \left\langle \sum_\alpha \bar{K}_{\alpha\lambda}^{(k)} h_\alpha, \sum_\beta K_{\mu\beta}^{(k)} m_\beta \right\rangle = \sum_\alpha K_{\mu\alpha}^{(k)} \bar{K}_{\alpha\lambda}^{(k)} = \delta_{\lambda\mu},$$

as suggested by their name, the dual k -Schur basis is dual to the k -Schur basis.

Proposition 6.5. *Let λ and μ be a k -bounded partition. Then,*

$$\langle s_\lambda^{(k)}, \mathfrak{S}_\mu^{(k)} \rangle = \delta_{\lambda\mu}.$$

We can extract several combinatorial properties for dual k -Schur functions from the k -Schur function properties using duality and the following lemma.

Lemma 6.6. *Let $f \in \Lambda^k$. Then, for $g \in \Lambda$, we have*

$$\langle f, g \rangle = \langle f, g \pmod{\mathfrak{J}^{(k)}} \rangle.$$

Proof. It suffices to consider $f = h_\lambda$, with $\lambda \in \mathcal{P}^k$. If $A \in \mathfrak{J}^{(k)}$, then $A = \sum_\mu a_\mu m_\mu$ summing over $\mu \notin \mathcal{P}^k$. Thus, $\langle h_\lambda, A \rangle = 0$ and the claim follows. \square

Since the ω -involution is an isometry with respect to $\langle \cdot, \cdot \rangle$, we discover from the action $\omega s_\mu^{(k)} = s_{\mu^{\omega_k}}^{(k)}$ (3.6) that ω acts naturally on the dual k -Schur functions.

Proposition 6.7. *Let λ be a k -bounded partition. Then*

$$\omega \left(\mathfrak{S}_\lambda^{(k)} \right) \pmod{\mathfrak{J}^{(k)}} = \mathfrak{S}_{\lambda^{\omega_k}}^{(k)}.$$

As with the Schur functions, the definition of $\mathfrak{S}_\lambda^{(k)}$ makes sense if λ is replaced by a skew diagram.

Definition 6.8. For k -bounded partitions $\mu \subseteq \nu$, the “dual skew k -Schur function” is defined by

$$(6.4) \quad \mathfrak{S}_{\nu/\mu}^{(k)} = \sum_{\mathcal{T}} x^{k\text{-weight}(\mathcal{T})},$$

where the sum is over all skew k -tableaux of shape $\mathfrak{c}(\nu)/\mathfrak{c}(\mu)$.

The skew k -tableaux are well-defined since $\mu \subseteq \nu$ implies that $\mathfrak{c}(\mu) \subseteq \mathfrak{c}(\nu)$ (e.g. [22], Prop. 14). The involution defined in [23] permuting the weight of k -tableaux does the same on skew k -tableaux. Thus, $\mathfrak{S}_{\nu/\mu}^{(k)}$ is also a symmetric function by the same reasoning that implies Proposition 6.2. Since $K_{\nu/\mu,\lambda}^{(k)}$ denotes the number of skew k -tableaux of k -weight λ and shape $\mathfrak{c}(\nu)/\mathfrak{c}(\lambda)$, and since a given letter cannot have k -weight larger than k , we have the expansion:

$$(6.5) \quad \mathfrak{S}_{\nu/\mu}^{(k)} = \sum_{\lambda; \lambda_1 \leq k} K_{\nu/\mu,\lambda}^{(k)} m_\lambda.$$

This form of the skew dual k -Schur function makes it clear that the “skew affine Schur functions” of [17] are the same functions.

In what follows, the methods of proof are straightforward extensions of results concerning skew-Schur functions that can be found for instance in [24]. The next theorem will illustrate these methods and is the only one for which we shall provide a proof.

Theorem 6.9. For any k -bounded partitions $\mu \subseteq \nu$,

$$\mathfrak{S}_{\nu/\mu}^{(k)} = \sum_{\lambda} c_{\mu\lambda}^{\nu,k} \mathfrak{S}_{\lambda}^{(k)}.$$

Proof. Since $\mathfrak{S}_{\nu/\mu}^{(k)}$ lies in $\Lambda/\mathfrak{J}^{(k)}$, for which the dual k -Schur functions form a basis,

$$\mathfrak{S}_{\nu/\mu}^{(k)} = \sum_{\lambda} A_{\mu\lambda}^{\nu,k} \mathfrak{S}_{\lambda}^{(k)},$$

for some $A_{\mu\lambda}^{\nu,k}$. On one hand consider:

$$\langle s_{\lambda}^{(k)}, \mathfrak{S}_{\nu/\mu}^{(k)} \rangle = \langle s_{\lambda}^{(k)}, \sum_{\alpha} A_{\mu\alpha}^{\nu,k} \mathfrak{S}_{\alpha}^{(k)} \rangle = A_{\mu\lambda}^{\nu,k}$$

and

$$\langle s_{\lambda}^{(k)}, \mathfrak{S}_{\nu/\mu}^{(k)} \rangle = \langle \sum_{\alpha} \bar{K}_{\alpha\lambda}^{(k)} h_{\alpha}, \sum_{\beta} K_{\nu/\mu,\beta}^{(k)} m_{\beta} \rangle = \sum_{\alpha} \bar{K}_{\alpha\lambda}^{(k)} K_{\nu/\mu,\alpha}^{(k)}.$$

On the other hand, since (3.5) tells us $h_{\lambda} s_{\mu}^{(k)} = \sum_{\nu} K_{\nu/\mu,\lambda}^{(k)} s_{\nu}^{(k)}$, we have

$$\begin{aligned} \langle s_{\mu}^{(k)} s_{\lambda}^{(k)}, \mathfrak{S}_{\nu}^{(k)} \rangle &= \langle \sum_{\alpha} \bar{K}_{\alpha\lambda}^{(k)} h_{\alpha} s_{\mu}^{(k)}, \mathfrak{S}_{\nu}^{(k)} \rangle = \langle \sum_{\alpha} \bar{K}_{\alpha\lambda}^{(k)} \sum_{\beta} K_{\beta/\mu,\alpha}^{(k)} s_{\beta}^{(k)}, \mathfrak{S}_{\nu}^{(k)} \rangle \\ &= \sum_{\alpha} \bar{K}_{\alpha\lambda}^{(k)} K_{\nu/\mu,\alpha}^{(k)} \end{aligned}$$

and

$$\langle s_{\mu}^{(k)} s_{\lambda}^{(k)}, \mathfrak{S}_{\nu}^{(k)} \rangle = \langle \sum_{\alpha} c_{\lambda\mu}^{\alpha,k} s_{\alpha}^{(k)}, \mathfrak{S}_{\nu}^{(k)} \rangle = c_{\lambda\mu}^{\nu,k}.$$

Therefore the result follows from

$$A_{\lambda\mu}^{\nu,k} = \sum_{\alpha} \bar{K}_{\alpha\lambda}^{(k)} K_{\nu/\mu,\alpha}^{(k)} = c_{\lambda\mu}^{\nu,k}.$$

□

Given the duality between Λ^k and $\Lambda/\mathfrak{J}^{(k)}$, it is natural to also consider a skew k -Schur function. The previous proposition, exposing k -Littlewood-Richardson coefficients as the expansion coefficients for a the dual skew k -Schur function in terms of dual k -Schur functions, leads us to also consider the coefficients in

$$\mathfrak{S}_{\lambda}^{(k)} \mathfrak{S}_{\mu}^{(k)} = \sum_{\nu} \mathfrak{d}_{\lambda\mu}^{\nu,k} \mathfrak{S}_{\nu}^{(k)} \pmod{\mathfrak{J}^{(k)}}.$$

Similar to the k -Littlewood-Richardson coefficients, Proposition 6.7 implies a symmetry satisfied by these coefficients:

$$(6.6) \quad \mathfrak{d}_{\lambda\mu}^{\nu,k} = \mathfrak{d}_{\lambda^{\omega_k} \mu^{\omega_k}}^{\nu^{\omega_k},k}.$$

We also note that:

$$\langle s_{\nu}^{(k)}, \mathfrak{S}_{\lambda}^{(k)} \mathfrak{S}_{\mu}^{(k)} \rangle = \langle s_{\nu}^{(k)}, \mathfrak{S}_{\lambda}^{(k)} \mathfrak{S}_{\mu}^{(k)} \pmod{\mathfrak{J}^{(k)}} \rangle = \mathfrak{d}_{\lambda\mu}^{\nu,k}.$$

We can now introduce the skew k -Schur function and discuss several identities regarding the relations between these functions and their dual.

Definition 6.10. For any k -bounded partitions $\mu \subseteq \nu$, the “skew k -Schur function” is defined by

$$s_{\nu/\mu}^{(k)} = \sum_{\lambda} \mathfrak{d}_{\mu\lambda}^{\nu,k} s_{\lambda}^{(k)}.$$

This given, our first property is:

Proposition 6.11. For any $f \in \Lambda$,

$$\langle s_{\nu/\mu}^{(k)}, f \rangle = \langle s_{\nu}^{(k)}, f \mathfrak{S}_{\mu}^{(k)} \rangle,$$

and for any $f \in \Lambda^k$,

$$\langle f, \mathfrak{S}_{\nu/\mu}^{(k)} \rangle = \langle f s_{\mu}^{(k)}, \mathfrak{S}_{\nu}^{(k)} \rangle.$$

The ω -involution again has a natural role in our study. Given its action on k -Schur functions and their dual, with the symmetries (3.10) and (6.6), we find

Proposition 6.12.

$$\omega \left(\mathfrak{S}_{\nu/\mu}^{(k)} \right) \pmod{\mathfrak{J}^{(k)}} = \mathfrak{S}_{\nu^{\omega_k}/\mu^{\omega_k}}^{(k)} \quad \text{and} \quad \omega \left(s_{\nu/\mu}^{(k)} \right) = s_{\nu^{\omega_k}/\mu^{\omega_k}}^{(k)}.$$

The corollary of the next proposition explains why the coproduct of k -Schur functions in terms of k -Schur functions has the dual k -Littlewood-Richardson coefficients as expansion coefficients, and thus connects the dual k -Schur functions with the cohomology of the loop Grassmannian based on the conjecture of Shimozono. Recall (e.g. [24]) that from the coproduct, $\Delta : \Lambda \rightarrow \Lambda(x) \otimes \Lambda(y)$ by $\Delta f = f(x, y)$, a bialgebra structure is imposed:

$$\langle \Delta f, g(x) h(y) \rangle = \langle f, gh \rangle,$$

where the first scalar product is in $\Lambda(x) \otimes \Lambda(y)$.

Proposition 6.13. *For any $\lambda \in \mathcal{P}^k$ and two sets of indeterminants, x and y ,*

$$s_{\lambda/\mu}^{(k)}(x, y) = \sum_{\nu} s_{\lambda/\nu}^{(k)}(x) s_{\nu/\mu}^{(k)}(y)$$

and

$$\mathfrak{S}_{\lambda/\mu}^{(k)}(x, y) = \sum_{\nu} \mathfrak{S}_{\lambda/\nu}^{(k)}(x) \mathfrak{S}_{\nu/\mu}^{(k)}(y).$$

Corollary 6.14. *For any $\lambda \in \mathcal{P}^k$ and two sets of indeterminants, x and y ,*

$$\Delta s_{\lambda}^{(k)} = s_{\lambda}^{(k)}(x, y) = \sum_{\mu, \nu} \mathfrak{d}_{\mu\nu}^{\lambda, k} s_{\mu}^{(k)}(x) s_{\nu}^{(k)}(y)$$

and

$$\Delta \mathfrak{S}_{\lambda}^{(k)} = \mathfrak{S}_{\lambda}^{(k)}(x, y) = \sum_{\mu, \nu} c_{\mu\nu}^{\lambda, k} \mathfrak{S}_{\mu}^{(k)}(x) \mathfrak{S}_{\nu}^{(k)}(y).$$

We conclude our exploration of the k -Schurs and their dual by mentioning that they give rise to a refined Cauchy formula. Letting $\mathfrak{J}^{(k)}(y)$ stand for $\mathfrak{J}^{(k)}$ in the y variables, we have:

Theorem 6.15. *Consider two bases of homogeneous symmetric functions, $\{a_{\lambda}\}_{\lambda \in \mathcal{P}^k}$ and $\{b_{\lambda}\}_{\lambda \in \mathcal{P}^k}$ for Λ^k and $\Lambda/\mathfrak{J}^{(k)}$, respectively.*

$$\prod_{i, j} (1 - x_i y_j)^{-1} \text{ mod } \mathfrak{J}^{(k)}(y) = \sum_{\lambda} a_{\lambda}(x) b_{\lambda}(y)$$

iff $\langle a_{\lambda}, b_{\mu} \rangle = \delta_{\lambda\mu}$ for all k -bounded partitions λ and μ . Note that modding out by $\mathfrak{J}^{(k)}(y)$ simply amounts to setting $y_j^n = 0$ whenever $n > k$ in the series expansion of the Cauchy kernel.

Proof. The proof is similar to that of [24], I(4.6). □

Corollary 6.16.

$$\prod_{i, j} (1 - x_i y_j)^{-1} \text{ mod } \mathfrak{J}^{(k)}(y) = \sum_{\lambda_1 \leq k} h_{\lambda}(x) m_{\lambda}(y) = \sum_{\lambda_1 \leq k} s_{\lambda}^{(k)}(x) \mathfrak{S}_{\lambda}^{(k)}(y).$$

7. FURTHER WORK

As detailed in the Introduction, the Schur functions provide a vehicle to directly reach the structure constants for multiplication in the cohomology of the Grassmannian from the Littlewood-Richardson coefficients. More generally, Theorem 5.6 implies that the k -Schur functions provide the analogous link between the quantum structure constants (or 3-point Gromov-Witten invariants) and the k -Littlewood-Richardson coefficients. There are beautiful combinatorial methods for computing the Littlewood-Richardson coefficients that use, for example, skew tableaux or reduced words for permutations. Although there has been progress in certain cases [2], [3], [4], [14], [15], [27], [28], [30], a combinatorial interpretation for the 3-point Gromov-Witten invariants in complete generality remains an open problem. The theory of k -Schur functions suggests a number of natural approaches to this problem, with an extended notion of skew tableaux to k -skew tableaux, and the revelation [22] that affine permutations are the appropriate extended notion of permutations in this study.

The conjecture that the (dual) k -Schur functions are the Schubert basis for the (cohomology) homology of the loop Grassmannian remains to be proven. More

generally, this supports the idea that there exists an affine version of Schubert polynomials related to k -Schur functions. In particular, the (dual) k -Schur functions are indexed by k -bounded partitions, which are in bijection with affine permutations in the quotient [7]. Such affine permutations can be considered as the Grassmannian version of affine permutations. The results here, with the conjectures that the (dual) k -Schur basis is related to the Schubert basis for the (cohomology) homology of the loop Grassmannian, suggest that these functions provide the symmetric component of (dual) affine Schubert polynomials. The first step in this direction is being developed by Thomas Lam in a forthcoming paper [17] on affine Stanley symmetric functions.

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