QUANTUM COHOMOLOGY OF ORTHOGONAL GRASSMANNIANS

ANDREW KRESCH AND HARRY TAMVAKIS

Abstract. Let V be a vector space with a nondegenerate symmetric form and OG be the orthogonal Grassmannian which parametrizes maximal isotropic subspaces in V. We give a presentation for the (small) quantum cohomology ring $QH^*(OG)$ and show that its product structure is determined by the ring of $\widetilde{P}$-polynomials. A 'quantum Schubert calculus' is formulated, which includes quantum Pieri and Giambelli formulas, as well as algorithms for computing Gromov–Witten invariants. As an application, we show that the table of 3-point, genus zero Gromov–Witten invariants for OG coincides with that for a corresponding Lagrangian Grassmannian LG, up to an involution.

1. Introduction

Consider a complex vector space V together with a nondegenerate symmetric form. Our aim is to study the structure of the small quantum cohomology ring of the orthogonal Grassmannian of maximal isotropic subspaces in V. In a companion paper to this one [KT2], we provide a similar analysis in type C, i.e., for the Lagrangian Grassmannian, and the reader is referred there and to [FP] [LT] for further background material. The story in the orthogonal case is similar, but with significant differences, both in the results and in their proofs.

Assuming the dimension of V is even and equals $2n+2$ for some natural number n, then the space of maximal isotropic subspaces of V has two connected components, each isomorphic to the *even orthogonal Grassmannian* or *spinor variety* $OG = OG(n+1, 2n+2) = SO_{2n+2}/P_{n+1}$. Here P_{n+1} is the maximal parabolic subroup of SO_{2n+2} associated to a 'right end root' in the Dynkin diagram of type D_{n+1}. We note that $OG(n+1, 2n+2)$ is isomorphic (in fact projectively equivalent) to the odd orthogonal Grassmannian $OG(n, 2n+1) = SO_{2n+1}/P_n$. Therefore, it suffices to only work with the even orthogonal example and we do so throughout this paper. We agree that a class α in the cohomology $H^{2k}(\mathfrak{X}, \mathbb{Z})$ of a complex variety $\mathfrak{X}$ has degree k to avoid doubling all degrees.

The cohomology ring $H^*(OG, \mathbb{Z})$ has a $\mathbb{Z}$-basis of Schubert classes τ_λ, one for each strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$ with $\lambda_1 \leqslant n$. Their multiplication can be described using the $\widetilde{P}$-*polynomials* of Pragacz and Ratajski [PR]. Let $X = (x_1, \ldots, x_n)$ be an n-tuple of variables and define $\widetilde{P}_0(X) = 1$ and $\widetilde{P}_i(X) = e_i(X)/2$ for each $i > 0$, where $e_i(X)$ denotes the i-th elementary symmetric polynomial in

Date: March 10, 2003.
2000 *Mathematics Subject Classification.* 14M15; 05E15.

X. For nonnegative integers i, j with $i \geqslant j$, set

$$\widetilde{P}_{i,j}(X) = \widetilde{P}_i(X)\widetilde{P}_j(X) + 2\sum_{k=1}^{j-1}(-1)^k \widetilde{P}_{i+k}(X)\widetilde{P}_{j-k}(X) + (-1)^j \widetilde{P}_{i+j}(X), \tag{1}$$

and for any partition λ of length $\ell = \ell(\lambda)$, not necessarily strict, define

$$\widetilde{P}_\lambda(X) = \text{Pfaffian}[\widetilde{P}_{\lambda_i,\lambda_j}(X)]_{1\leqslant i<j\leqslant r}, \tag{2}$$

where $r = 2\lfloor(\ell+1)/2\rfloor$. Let $\mathcal{D}_n$ be the set of strict partitions λ with $\lambda_1 \leqslant n$.

Let Λ'_n denote the $\mathbb{Z}$-algebra generated by the polynomials $\widetilde{P}_\lambda(X)$ for all $\lambda \in \mathcal{D}_n$; Λ'_n is isomorphic to the ring $\mathbb{Z}[X]^{S_n}$ of symmetric polynomials in X. By results of [P, Sect. 6] and [PR] we have that the map sending $\widetilde{P}_\lambda(X)$ to τ_λ for all $\lambda \in \mathcal{D}_n$ extends to a surjective *ring* homomorphism $\phi : \Lambda'_n \to H^*(OG, \mathbb{Z})$ with kernel generated by the relations $\widetilde{P}_{i,i}(X) = 0$ for $1 \leqslant i \leqslant n$. The map ϕ can be realized as evaluation on the Chern roots of the tautological quotient vector bundle Q over OG (note that the top Chern class of Q vanishes). In this way we obtain a presentation for the cohomology ring of OG, and equations (1) and (2) become Giambelli-type formulas, which express the Schubert classes in terms of the special ones.

We present an extension of these results to the (small) quantum cohomology ring of OG, denoted $QH^*(OG)$. This is an algebra over $\mathbb{Z}[q]$, where q is a formal variable of degree $2n$ (the classical formulas are recovered by setting $q = 0$).

Theorem 1. *The map which sends $\widetilde{P}_\lambda(X)$ to τ_λ for all $\lambda \in \mathcal{D}_n$ and $\widetilde{P}_{n,n}(X)$ to q extends to a surjective* ring *homomorphism $\Lambda'_n \to QH^*(OG)$ with kernel generated by the relations $\widetilde{P}_{i,i}(X) = 0$ for $1 \leqslant i \leqslant n-1$. The ring $QH^*(OG)$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \ldots, \tau_n, q]$ modulo the relations*

$$\tau_i^2 + 2\sum_{k=1}^{i-1}(-1)^k \tau_{i+k}\tau_{i-k} + (-1)^i \tau_{2i} = 0 \tag{3}$$

for all $i < n$, together with the quantum relation

$$\tau_n^2 = q \tag{4}$$

(it is understood that $\tau_j = 0$ for $j > n$). The Schubert class τ_λ in this presentation is given by the Giambelli formulas

$$\tau_{i,j} = \tau_i\tau_j + 2\sum_{k=1}^{j-1}(-1)^k \tau_{i+k}\tau_{j-k} + (-1)^j \tau_{i+j} \tag{5}$$

for $i > j > 0$, and

$$\tau_\lambda = \text{Pfaffian}[\tau_{\lambda_i,\lambda_j}]_{1\leqslant i<j\leqslant r}, \tag{6}$$

where quantum multiplication is employed throughout. In other words, classical Giambelli and quantum Giambelli coincide for OG.

We remark that the statements in Theorem 1 are direct analogues of the corresponding facts for SL_N-Grassmannians [Be]. However, these results stand in contrast to the case of the Lagrangian Grassmannian $LG(n, 2n)$, where quantum Giambelli does not coincide with classical Giambelli on $LG(n, 2n)$ (see [KT2] for more details).

Our proof of Theorem 1 follows the scheme of [KT2], with two main differences. We require a Pfaffian identity for type D Schubert polynomials [KT1, §3.3], which gives a key relation in the Chow group of a certain *orthogonal Quot scheme* OQ_d. The latter scheme compactifies the moduli space of degree d maps $\mathbb{P}^1 \to OG$; however our definition of OQ_d differs from that in the Lagrangian case of [KT2], as the direct analogue of the Grothendieck Quot scheme [G1] here is not suitable for doing computations.

In $QH^*(OG)$ there are formulas

$$\tau_\lambda \cdot \tau_\mu = \sum \langle \tau_\lambda, \tau_\mu, \tau_{\widehat{\nu}} \rangle_d \, \tau_\nu \, q^d,$$

where the sum is over $d \geqslant 0$ and strict partitions ν with $|\nu| = |\lambda| + |\mu| - 2nd$, and $\widehat{\nu}$ is the dual partition of ν, whose parts complement the parts of ν in the set $\{1, \ldots, n\}$. Each quantum structure constant $\langle \tau_\lambda, \tau_\mu, \tau_{\widehat{\nu}} \rangle_d$ is a genus zero Gromov–Witten invariant for OG, and is a nonnegative integer. We present explicit formulas and algorithms to compute these numbers. This includes a quantum Pieri rule, which extends the classical result of Hiller and Boe [HB]. As an application, we show that there is a direct identification between the 3-point, genus zero Gromov–Witten invariants on OG with corresponding ones for the Lagrangian Grassmannian $LG(n-1, 2n-2)$ (Theorem 6).

This paper is organized as follows. In Section 2 we study the $\widetilde{P}$-polynomials and type D Schubert polynomials, and prove a remarkable Pfaffian identity for the latter. The orthogonal Grassmannians are introduced in Section 3, which includes a proof of the presentation for $QH^*(OG)$. The proof of the quantum Giambelli formula (6) of Theorem 1 is done in Sections 4 and 5, by studying intersections on the orthogonal Quot scheme. In Section 6 we formulate a 'quantum Schubert calculus' for OG. Finally, the Appendix establishes an identity for $\widetilde{P}$-polynomials which is used in [KT1].

The main results of this article and its companion paper [KT2] were announced at the Bonn Mathematische Arbeitstagung 2001 [T]. The authors thank the Max-Planck-Institute für Mathematik for its hospitality during the preparation of this paper. We also thank Anders Buch and Bill Fulton for useful correspondence. Both authors were supported in part by National Science Foundation post-doctoral research fellowships.

2. $\widetilde{P}$-polynomials and type D Schubert polynomials

2.1. **Basic definitions.** All the notational conventions used in this section follow [KT1] and [KT2]. In particular, for strict partitions λ and μ, the difference $\lambda \smallsetminus \mu$ denotes the partition with parts given by the parts of λ which are not parts of μ. A *composition* is a sequence of nonnegative integers with only finitely many nonzero parts. The $\widetilde{P}$-polynomials make sense when indexed by any composition ν, and satisfy Pfaffian relations

$$\widetilde{P}_\nu(X) = \sum_{j=1}^{g-1} (-1)^{j-1} \widetilde{P}_{\nu_j, \nu_g}(X) \cdot \widetilde{P}_{\nu \smallsetminus \{\nu_j, \nu_g\}}(X), \tag{7}$$

where g is an even number such that $\nu_i = 0$ for $i > g$. Define also the $\widetilde{Q}$-polynomial $\widetilde{Q}_\nu(X) = 2^\ell \, \widetilde{P}_\nu(X)$ for each composition ν with ℓ nonzero parts. The $\widetilde{Q}$-polynomials have integer coefficients, and span the ring $\mathbb{Z}[X]^{S_n}$ of symmetric functions in n variables.

Let $\widetilde{W}_n$ be the Weyl group for the root system D_n, whose elements are denoted as barred permutations. Recall that W_n is generated by the elements $s_\Box, s_1, \ldots, s_{n-1}$: for $i > 0$, s_i is the transposition interchanging i and $i+1$, and $s_\Box$ is defined by

$$(u_1, u_2, u_3, \ldots, u_n)s_\Box = (\overline{u}_2, \overline{u}_1, u_3, \ldots, u_n).$$

Let $\widetilde{w}_0$ denote the element of maximal length in $\widetilde{W}_n$. For each $\lambda \in \mathcal{D}_{n-1}$ we have a *maximal Grassmannian element* w_λ of $\widetilde{W}_n$, defined as in [KT1, §3.2].

Each generator s_i acts naturally on the polynomial ring $A[X]$, where $A = \mathbb{Z}[\frac{1}{2}]$; for $i > 0$, s_i interchanges x_i and x_{i+1}, while $s_\Box$ sends (x_1, x_2) to $(-x_2, -x_1)$; all other variables remain fixed. There are divided difference operators ∂'_i and $\partial_\Box$ on $A[X]$; for $i > 0$ they are defined by

$$\partial'_i(f) = (f - s_i f)/(x_{i+1} - x_i)$$

while

$$\partial_\Box(f) = (f - s_\Box f)/(x_1 + x_2),$$

for all $f \in A[X]$. These give rise to operators $\partial'_w : A[X] \to A[X]$ for each element $w \in \widetilde{W}_n$, as in [KT1, §3.2].

For all $w \in \widetilde{W}_n$ we have a *type D Schubert polynomial* $\mathfrak{D}_w(X) \in A[X]$ defined by

$$\mathfrak{D}_w(X) = (-1)^{n(n-1)/2} \partial'_{w^{-1}\widetilde{w}_0}\Big(x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \widetilde{P}_{n-1}(X)\Big).$$

These type D polynomials were defined in [KT1, §3.3]; they agree with the orthogonal Schubert polynomials of [LP] up to a sign, which depends on the degree. The polynomial $\mathfrak{D}_w(X)$ represents the Schubert class associated to w in the cohomology ring of the flag manifold SO_{2n}/B. Let us define $\mathfrak{D}'_\lambda(X) = \mathfrak{D}_{w_\lambda s_\Box}(X)$. It follows from the definitions and [KT1, Theorem 7] that $\mathfrak{D}'_\lambda(X) = \partial_\Box(\widetilde{P}_\lambda(X))$, for all non-zero partitions $\lambda \in \mathcal{D}_{n-1}$.

2.2. **A Pfaffian identity.** We require the identity in the following theorem for our proof of the quantum Giambelli formula for $OG(n+1, 2n+2)$.

Theorem 2. *Fix $\lambda \in \mathcal{D}_n$ of length $\ell \geqslant 3$, and set $r = 2\lfloor(\ell+1)/2\rfloor$. Then*

$$\sum_{j=1}^{r-1} (-1)^{j-1}\, \mathfrak{D}'_{\lambda_j,\lambda_r}(X)\, \mathfrak{D}'_{\lambda \smallsetminus \{\lambda_j,\lambda_r\}}(X) = 0. \tag{8}$$

Proof. We first observe, using the homogeneity of the two sides, that (8) is equivalent to the identity

$$\sum_{j=1}^{r-1} (-1)^{j-1}\, \partial_\Box(\widetilde{Q}_{\lambda_j,\lambda_r}(X)) \cdot \partial_\Box(\widetilde{Q}_{\lambda \smallsetminus \{\lambda_j,\lambda_r\}}(X)) = 0 \tag{9}$$

for $\widetilde{Q}$-polynomials, which should hold for λ and r as in the theorem.

Let $X'' = (x_3, \ldots, x_n)$ and define

$$m_{r,s}(x_1, x_2) = \begin{cases} x_1^r x_2^s + x_1^s x_2^r & \text{if } r \neq s, \\ x_1^r x_2^r & \text{if } r = s \end{cases}$$

to be the monomial symmetric function in x_1 and x_2. For any partition λ and nonnegative integers a and b, let $C(\lambda, a, b)$ denote the set of compositions μ with $\lambda_i - \mu_i \in \{0, 1, 2\}$ for all i and $\lambda_i - \mu_i = 1$ (resp. $\lambda_i - \mu_i = 2$) for exactly a (resp. b) values of i.

Proposition 1. *For any nonzero strict partition λ, we have*

$$\partial_\Box(\widetilde{Q}_\lambda(X)) = 2 \sum_{\substack{0\leqslant s\leqslant r\leqslant \ell \\ r+s \text{ even}}} m_{r,s}(x_1,x_2) \sum_{\substack{a+2b=r+s+1 \\ 0\leqslant b\leqslant s}} \binom{a-1}{s-b} \sum_{\mu\in C(\lambda,a,b)} \widetilde{Q}_\mu(X''). \tag{10}$$

Proof. Let $X' = (x_2, \ldots, x_n)$. According to [KT2, Prop. 1], for any partition λ of length ℓ (not necessarily strict), we have

$$\widetilde{Q}_\lambda(X) = \sum_{k=0}^{\ell} x_1^k \sum_{\mu\in B(\lambda,k)} \widetilde{Q}_\mu(X'), \tag{11}$$

where $B(\lambda,k)$ is defined to be the set of all compositions μ such that $|\lambda|-|\mu| = k$ and $\lambda_i - \mu_i \in \{0,1\}$ for each i. By applying (11) twice we obtain

$$\widetilde{Q}_\lambda(X) = \sum_{0\leqslant s\leqslant r\leqslant \ell} m_{r,s}(x_1,x_2) \sum_{\substack{j+2k=r+s \\ 0\leqslant k\leqslant s}} \binom{j}{s-k} \sum_{\mu\in C(\lambda,j,k)} \widetilde{Q}_\mu(X''). \tag{12}$$

Suppose that $r \geqslant s \geqslant 0$. If $r+s$ is even, then $\partial_\Box(m_{r,s}(x_1,x_2)) = 0$. If $r+s$ is odd, we have

$$\partial_\Box(m_{r,s}(x_1,x_2)) = 2 \sum_{\substack{c+d=r+s-1 \\ c,d\geqslant s}} (-1)^{c-s} x_1^c x_2^d.$$

We now apply this to (12) and gather terms to obtain (10). □

Example. For all a, b with $a > b \geqslant 0$, we have

$$\begin{aligned} \partial_\Box(\widetilde{Q}_{a,b}(X)) &= 2\left(\widetilde{Q}_{a-1,b}(X'') + \widetilde{Q}_{a,b-1}(X'')\right) \\ &\quad + 2\,x_1x_2\left(\widetilde{Q}_{a-2,b-1}(X'') + \widetilde{Q}_{a-1,b-2}(X'')\right). \end{aligned} \tag{13}$$

In the equation (13) and later on we agree that $\widetilde{Q}_\mu(X'') = 0$ if any of the components of μ are negative.

As in [KT2, §2.3], the rest of the argument can be expressed using only the partitions which index the polynomials involved. We thus begin by defining a commutative $\mathbb{Z}$-algebra $\mathcal{B}$ with formal variables which represent these indices. The algebra $\mathcal{B}$ is generated by symbols $(a_1, a_2, \ldots)$, where the entries a_i are barred integers; each a_i can have up to two bars. The symbol $(a_1, a_2, \ldots)$ corresponds to the polynomial $\widetilde{Q}_\mu(X'')$, where μ is the composition with μ_i equal to the integer a_i minus the number of bars over a_i. We identify $(a,0)$ with (a).

Let μ be a barred partition, that is, a partition in which bars have been added to some of the entries. For $\ell(\mu) \geqslant 3$, we impose the Pfaffian relation

$$(\mu) = \sum_{j=1}^{m-1} (-1)^{j-1} (\mu_j, \mu_m) \cdot (\mu \smallsetminus \{\mu_j, \mu_m\}), \tag{14}$$

which corresponds to (7) for $\nu = \mu$ (here $m = 2\lfloor(\ell(\mu)+1)/2\rfloor$, as usual). Iterating this gives

$$(\mu) = \sum \epsilon(\mu,\nu)(\nu_1,\nu_2)\cdots(\nu_{m-1},\nu_m), \tag{15}$$

where the sum is over all $(m-1)(m-3)\cdots(1)$ ways to write the set $\{\mu_1, \ldots, \mu_m\}$ as a union of pairs $\{\nu_1,\nu_2\} \cup \cdots \cup \{\nu_{m-1},\nu_m\}$, and where $\epsilon(\mu,\nu)$ is the sign of the permutation that takes $(\mu_1,\ldots,\mu_m)$ into $(\nu_1,\ldots,\nu_m)$; we adopt the convention that $\nu_{2i-1}\geqslant\nu_{2i}$.

We also define the square bracket symbols $[a]=(\overline{a})$ and $[a,b]=(\overline{a},b)+(a,\overline{b})$, where a and b are integers, each with up to one bar. For example, the right hand side of equation (13) corresponds to the sum $2\,[a,b]+2\,x_1x_2\,[\overline{a},\overline{b}]$ in $\mathcal{B}[x_1,x_2]$. Finally, we impose the relations

$$[a,b]=(\overline{a})(b)-(a)(\overline{b}) \tag{16}$$

for integers a, b, with up to one bar each; this agrees with a corresponding identity

$$\widetilde{Q}_{a-1,b}+\widetilde{Q}_{a,b-1}=\widetilde{Q}_{a-1}\widetilde{Q}_b-\widetilde{Q}_a\widetilde{Q}_{b-1}$$

of $\widetilde{Q}$-polynomials.

Using these conventions and equations (10) and (13), we are reduced to showing that $S_1+S_2=0$, where

$$S_1=\sum_{\substack{a+2b=r+s+1\\0\leqslant b\leqslant s}}\binom{a-1}{s-b}\sum_{j=1}^{r-1}(-1)^{j-1}\,[\lambda_j,\lambda_r]\sum_{\mu\in C(\lambda\smallsetminus\{\lambda_j,\lambda_r\},a,b)}(\mu),$$

$$S_2=\sum_{\substack{a'+2b'=r+s-1\\0\leqslant b'\leqslant s-1}}\binom{a'-1}{s-b'-1}\sum_{j=1}^{r-1}(-1)^{j-1}\,[\overline{\lambda}_j,\overline{\lambda}_r]\sum_{\mu\in C(\lambda\smallsetminus\{\lambda_j,\lambda_r\},a',b')}(\mu),$$

and $r\geqslant s\geqslant 0$ are fixed integers with $r+s$ even. The proof of this is rather similar to the proofs of Theorems 2 and 3 of [KT2], and we will point out only the main difference here.

We first apply (15) to expand the terms (μ) in both S_1 and S_2. The cancellation technique of [KT2, §2.3], notably, the identity

$$[a,b][c,d]-[a,c][b,d]+[a,d][b,c]=0, \tag{17}$$

implies the vanishing of the sum of those summands in S_1 which contain a pair with exactly one bar, or at least two pairs with exactly three bars. The remainder is a sum S_1' consisting of those summands in S_1 with a unique pair containing three bars, and no pair with only one bar. In the same way, one checks the vanishing of the sum of those summands in S_2 which contain a pair with exactly three bars, or at least two pairs with exactly one bar. There remains a sum S_2' consisting of those summands in S_2 with a unique pair containing only one bar, and no pair with exactly three bars. Hence, it is enough to show that $S_1'+S_2'=0$.

There is an obvious bijection between the summands in S_1' and S_2', obtained by adding two bars to the unbarred part of the pair in S_2' which contains only one bar (note that the corresponding binomial coefficients agree, as $(a,b)=(a',b'+1)$ for these two summands). To prove that the sum of all corresponding terms is zero, it suffices to show that the expression

$$\Big([a,b][\overline{c},\overline{d}]-[a,c][\overline{b},\overline{d}]+[a,d][\overline{b},\overline{c}]\Big)+\Big([\overline{a},\overline{b}][c,d]-[\overline{a},\overline{c}][b,d]+[\overline{a},\overline{d}][b,c]\Big) \tag{18}$$

vanishes identically in $\mathcal{B}$ (we then apply this with $a=\lambda_r$, always). To check this, begin from the basic identities

$$[a,b][\overline{c},\overline{d}]-[a,\overline{c}][b,\overline{d}]+[a,\overline{d}][b,\overline{c}]=0 \tag{19}$$

and

$$[\overline{a},\overline{b}][c,d]-[\overline{a},c][\overline{b},d]+[\overline{a},d][\overline{b},c]=0 \tag{20}$$

which are easily shown using (16). Let $\langle x,y\rangle=[\overline{x},y]+[x,\overline{y}]$ and note that

$$\langle a,b\rangle\,\langle c,d\rangle-\langle a,c\rangle\,\langle b,d\rangle+\langle a,d\rangle\,\langle b,c\rangle=0, \tag{21}$$

which is shown using $\langle x,y\rangle=(\overline{\overline{a}})(b)-(a)(\overline{\overline{b}})$ (another consequence of (16)). The vanishing of (18) follows by combining (19), (20) and (21). □

3. Orthogonal Grassmannians

3.1. **Schubert varieties and incidence loci.** Let V be a fixed $(2n{+}2)$-dimensional complex vector space equipped with a nondegenerate symmetric bilinear form on V. The principal object of study is the orthogonal Grassmannian $OG(n{+}1,2n{+}2)$ which is one component of the parameter space of $(n+1)$-dimensional isotropic subspaces of V. When n is fixed, we write OG for $OG(n+1,2n+2)$. We have $\dim_{\mathbb{C}} OG=n(n+1)/2$. The identities in cohomology that we establish in this section remain valid if we work over an arbitrary base field, and use Chow rings in place of cohomology.

Let $F_\bullet$ be a fixed complete isotropic flag of subspaces of V. By convention, then, OG parametrizes maximal isotropic spaces $\Sigma\subset V$ such that $\Sigma\cap F_{n+1}$ has even codimension in F_{n+1}. We define the alternative flag $\widetilde{F}_\bullet$ to be the flag $F_1\subset\cdots\subset F_n\subset\widetilde{F}_{n+1}$, where $\widetilde{F}_{n+1}$ is the unique maximal isotropic space containing F_n but not equal to F_{n+1}. We let

$$F^{(i)}_\bullet=\begin{cases}F_\bullet & \text{if } i\equiv(n+1) \bmod 2,\\ \widetilde{F}_\bullet & \text{otherwise.}\end{cases} \tag{22}$$

The Schubert varieties $\mathfrak{X}_\lambda\subset OG$ are indexed by partitions $\lambda\in\mathcal{D}_n$. We record two ways to write the conditions which define the Schubert variety $\mathfrak{X}_\lambda$:

$$\mathfrak{X}_\lambda=\{\,\Sigma\in OG\mid \operatorname{rk}(\Sigma\to V/F_{n+1-\lambda_i})\leqslant n+1-i,\ i=1,\ldots,\ell(\lambda)\,\} \tag{23}$$

$$=\{\,\Sigma\in OG\mid \operatorname{rk}(\Sigma\to V/F^{(i)\perp}_{n+1-\lambda_i})\leqslant n+1-i-\lambda_i,\ i=1,\ldots,\ell(\lambda)+1\,\}. \tag{24}$$

Let τ_λ be the class of $\mathfrak{X}_\lambda$ in $H^*(OG,\mathbb{Z})$. The classical Giambelli formula (6) for OG is equivalent to the following identity in $H^*(OG,\mathbb{Z})$:

$$\tau_\lambda=\sum_{j=1}^{r-1}(-1)^{j-1}\tau_{\lambda_j,\lambda_r}\cdot\tau_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}}, \tag{25}$$

for $r=2\lfloor(\ell(\lambda)+1)/2\rfloor$. Let $\rho_n=(n,n-1,\ldots,1)$ and for $\mu\in\mathcal{D}_n$, denote by $\widehat{\mu}=\rho_n\smallsetminus\mu$, the dual partition. The Poincaré duality pairing on OG satisfies

$$\int_{OG}\tau_\lambda\,\tau_\mu=\delta_{\lambda\widehat{\mu}}.$$

Given an isotropic space $A\subset V$ of dimension $n{-}k$ $(k\geqslant 0)$, the variety of maximal isotropic spaces containing A is a translate of the Schubert variety $\mathfrak{X}_{n,n-1,\ldots,k+1}$. We have the following result on intersections of such varieties with the Schubert varieties $\mathfrak{X}_\lambda$; this is analogous to a similar result in type C ([KT2, Prop. 3]).

Proposition 2. *Let $k \geqslant 0$ and $\lambda \in \mathcal{D}_n$. Let A be an isotropic subspace of V of dimension $n-k$, and let $Y \subset OG$ be the subvariety of maximal isotropic subspaces of V which contain A. Then $\mathfrak{X}_\lambda \cap Y$ is a Schubert variety in $Y \simeq OG(k+1, 2k+2)$. Moreover, if $\ell(\lambda) < k$ then the intersection, if nonempty, has positive dimension.*

Proof. As in [KT2], the intersection is defined by the attitude of Σ/A with respect to $F'_\bullet$, where $F'_i = ((F_i + A) \cap A^\perp)/A$. For the intersection to be a point would require at least k rank conditions, and hence $\ell(\lambda) \geqslant k$. □

The space $OG(n-1, 2n+2)$ is the parameter space of lines on OG. For a nonempty partition λ, the variety of lines incident to $\mathfrak{X}_\lambda$ is the Schubert variety $\mathfrak{Y}_\lambda$, consisting of those $\Sigma' \in OG(n-1, 2n+2)$ such that

$$\operatorname{rk}(\Sigma' \to V/F^{(i)\perp}_{n+1-\lambda_i}) \leqslant n+1-i-\lambda_i, \quad \text{for} \quad i = 1, \ldots, \ell+1. \tag{26}$$

The codimension of $\mathfrak{Y}_\lambda$ is $|\lambda|-1$. Note that (i) the rank conditions (26) are identical to those in (24); (ii) the rank condition corresponding to $i = \ell(\lambda)+1$, which was redundant in defining the Schubert varieties in OG, is necessary here.

3.2. **A Pfaffian identity on $OG(n-1, 2n+2)$.** Let $F = F_{SO}(V)$ denote the variety of complete isotropic flags in $V = \mathbb{C}^{2n+2}$. There is a natural projection map from F to the orthogonal Grassmannian $OG(n-1, 2n+2)$, inducing an injective pullback morphism on cohomology. Introduce an extra variable x_{n+1} and let $X^+ = (x_1, \ldots, x_{n+1})$. Referring to [KT1, §2.4 and Sect. 3], we check that the Schubert class $[\mathfrak{Y}_\lambda]$ in $H^*(OG(n-1, 2n+2))$ pulls back to the class represented by $\mathfrak{D}'_\lambda(X^+)$ in $H^*(F)$, for each $\lambda \in \mathcal{D}_{n-1}$. Here X^+ corresponds to the vector of Chern roots of the dual to the tautological rank $n+1$ vector bundle over F, ordered as in [KT1, Sect. 2]. Theorem 2 remains true with X^+ in place of X, and gives

Corollary 1. *For every $\lambda \in \mathcal{D}_n$ of length $\ell \geqslant 3$ and $r = 2\lfloor(\ell+1)/2\rfloor$ we have*

$$\sum_{j=1}^{r-1} (-1)^{j-1} \, [\mathfrak{Y}_{\lambda_j, \lambda_r}] \, [\mathfrak{Y}_{\lambda \smallsetminus \{\lambda_j, \lambda_r\}}] = 0 \tag{27}$$

in $H^*(OG(n-1, 2n+2), \mathbb{Z})$.

3.3. **Quantum relations and two-condition Giambelli.** Recall that in $QH(OG)$, the degree of q is

$$\int_{OG} c_1(T_{OG}) \cdot \tau_{\hat{1}} = 2n.$$

It follows, for degree reasons, that the relations in cohomology (3) and the quantum Giambelli formula for the two-condition Schubert classes (5) – which we know to hold classically – hold in $QH(OG)$. The degree $2n$ quantum relation (4) follows from the elementary enumerative fact that there is a unique line on OG through a given point, incident to two general translates of $\mathfrak{X}_n$. Arguing as in [ST], now, we obtain a presentation of $QH^*(OG)$ as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \ldots, \tau_n, q]$ modulo the relations (3) and (4) (see also [FP, Sect. 10]).

The proof of the more difficult quantum Giambelli formula (6) occupies Sections 4 and 5.

4. Orthogonal Quot schemes

4.1. **Overview.** In the next two sections, we define the orthogonal Quot scheme and establish an identity in its Chow group, from which identity (6) in $QH^*(OG)$ readily follows. We make use of type D degeneracy loci for isotropic morphisms of vector bundles [KT1] to define classes $[W_\lambda(p)]_k$ ($p \in \mathbb{P}^1$) of the appropriate dimension $k := n(n+1)/2 + 2nd - |\lambda|$ in the Chow group of the orthogonal Quot scheme OQ_d, which compactifies the space of degree-d maps $\mathbb{P}^1 \to OG$. Let $p' \in \mathbb{P}^1$ be distinct from p, and denote by W' the degeneracy locus defined by a general translate of the fixed isotropic flag $F_\bullet$. We produce a Pfaffian formula analogous to (25):

$$[W_\lambda(p)]_k = \sum_{j=1}^{r-1} (-1)^{j-1} [W_{\lambda_j,\lambda_r}(p) \cap W'_{\lambda \smallsetminus \{\lambda_j,\lambda_r\}}(p')]_k, \tag{28}$$

for any $\lambda \in \mathcal{D}_n$ with $\ell(\lambda) \geqslant 3$ and $r = 2\lfloor(\ell(\lambda)+1)/2\rfloor$.

As in [KT2], we need the cycles in (28) to remain rationally equivalent under further intersection with some (general translate of) $W_\mu(p'')$, for $\mu \in \mathcal{D}_n$ and $p'' \in \mathbb{P}^1$ distinct from p, p', Also, as in loc. cit., we accomplish this by working on a modification $OQ_d(p'')$, on which the evaluation-at-p'' map is globally defined, and employing refined intersection operation from OG.

The rational equivalences that we produce — (28) and a similar equivalence on $OQ_d(p'')$ — come by combining equivalences of the following types: (i) the classical Pfaffian formulas on OG (25); (ii) the Pfaffian identities (27) on $OG(n-1, 2n+2)$; (iii) rational equivalences $\{p\} \sim \{p'\}$ on $\mathbb{P}^1$. Indeed, the essence of (iii) is that we can replace p' with p in (28); the intersection $W_{\lambda_j,\lambda_r}(p) \cap W'_{\lambda \smallsetminus \{\lambda_j,\lambda_r\}}(p)$ now has k-dimension components supported in the boundary of the Quot scheme. The cancellation of these contributions in the Chow group is precisely equation (27).

4.2. **Definition of OQ_d.** Let V be a complex vector space V of dimension $N = r + s$ and fix $d \geqslant 0$. Following Grothendieck [G1], there is a smooth projective variety Q_d, the *Quot scheme*, which parametrizes flat families of quotient sheaves of $\mathcal{O}_{\mathbb{P}^1} \otimes V$ with Hilbert polynomial $p(t) = st + s + d$. This variety compactifies the space of parametrized degree-d maps from $\mathbb{P}^1$ to the Grassmannian of r-dimensional subspaces of V. On $\mathbb{P}^1 \times Q_d$ there is a universal exact sequence of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O} \otimes V \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{29}$$

with $\mathcal{E}$ locally free of rank r. From now on, we fix V as in Section 3 and $r = s = n+1$.

Definition 1. Let d be a nonnegative integer. The *isotropic locus* Q_d^{iso} is the closed subscheme of Q_d which is defined by the vanishing of the composite

$$\mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}^1} \otimes V \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^1} \otimes V^* \longrightarrow \mathcal{E}^*$$

where α is the isomorphism defined by the given bilinear form on V.

The embedding of OG in the Grassmannian $G(n+1, 2n+2)$ of $(n+1)$-dimensional subspaces of V is degree-doubling, that is, in the sheaf sequence (29) corresponding to degree-d maps $\mathbb{P}^1 \to OG$, the sheaf $\mathcal{Q}$ has degree $2d$. For any d, Q_{2d}^{iso} contains an open subscheme isomorphic to the moduli space $M_{0,3}(OG, d)$:

Definition 2. Let d be a nonnegative integer. Then OM_d is the open subscheme of Q_{2d}^{iso} defined by the conditions (i) $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1} \otimes V$ has everywhere full rank; (ii) the image of $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^1} \otimes V$ at any point has intersection with F_{n+1} of dimension congruent to $(n+1)$ mod 2.

Unfortunately, Q_{2d}^{iso} generally has components of dimension larger than the dimension of OM_d. The remedy is to throw away any point of (29) where the rank of $\mathcal{E} \to \mathcal{O} \otimes V$ drops by just 1 at some point of $\mathbb{P}^1$. We can do this, and still be left with a closed subscheme of Q_{2d}^{iso}, because in any degeneration situation in which the rank of $\mathcal{E} \to \mathcal{O} \otimes V$ drops from full to less than full, the drop is by at least 2.

Definition 3. For $d \in (1/2)\mathbb{Z}$, the *orthogonal Quot scheme* OQ_d is the subset of Q_{2d}^{iso} consisting of points whose sheaf sequence (29) satisfies $\mathrm{rk}(\mathcal{E}_p \to V) \neq n$ for all $p \in \mathbb{P}^1$, and such that where it has full rank, the image has intersection with F_{n+1} of even codimension in F_{n+1}. This subset, evidently constructible and closed by virtue of Proposition 3, below, is given the reduced scheme structure.

Lemma 1. *Let $\psi\colon C_0 \to G(n+1, 2n+2)$ be a morphism, with $C_0 \cong \mathbb{P}^1$, and let C be a tree of $\mathbb{P}^1$'s containing C_0 and $\varphi\colon C \to G(n+1, 2n+2)$ a map which restricts to ψ on C_0. Let*

$$\widetilde{C} := C_1 \cup C_2 \cup \cdots \cup C_m$$

$(m \geqslant 1)$ denote a chain of components in C, with $C_i \neq C_0$ for all $i \geqslant 1$, and assume C_1 meets C_0 at the point p and C_i is collapsed by φ for all i with $1 \leqslant i \leqslant m-1$. Let $\pi\colon C \to C_0$ denote the morphism which collapses all components of C except C_0. Let

$$0 \to \mathcal{E}_0 \to \mathcal{O} \otimes V \to \mathcal{Q}_0 \to 0$$

denote the pullback of the universal sequence via ψ, and let

$$0 \to \mathcal{E} \to \mathcal{O} \otimes V \to \mathcal{Q} \to 0$$

denote the pullback of the universal sequence via φ (so that $\mathcal{E}|_{C_0} \simeq \mathcal{E}_0$). Assume the restriction of $\mathcal{E}$ to C_m splits as

$$\mathcal{O}(-b_1) \oplus \cdots \oplus \mathcal{O}(-b_j) \oplus \mathcal{O}^{n+1-j}$$

with $b_1, \ldots, b_j \geqslant 1$. Then the morphism $\pi_*\mathcal{E} \to \pi_*(\mathcal{O} \otimes V) = \mathcal{O} \otimes V$ factors through $\mathcal{E}_0$, and the cokernel of $\pi_*\mathcal{E} \to \mathcal{E}_0$ is a torsion sheaf whose fiber at p has dimension at least j.

Proof. We may choose $n-j$ independent sections $s_1, \ldots, s_{n-j}$ of $\mathcal{E}|_{C_m}$. These extend uniquely to $n-j$ independent sections of $\mathcal{E}|_{\widetilde{C}}$, and hence span an $(n-j)$-dimensional subspace Σ of the fiber of $\mathcal{E}$ at the point p. The map $(\pi_*\mathcal{E})_p \to (\mathcal{E}_0)_p$ on fibers at p has image contained in Σ. Hence the dimension of the fiber at p of the cokernel of $\pi_*\mathcal{E} \to \mathcal{E}_0$ is at least j. □

Proposition 3. *For any $d \in (1/2)\mathbb{Z}$, the subset $OQ_d \subset Q_{2d}^{\mathrm{iso}}$ is closed under specialization.*

Proof. Suppose $x_1 \in OQ_d$ specializes to $x_0 \in Q_{2d}$. Then there is a discrete valuation ring R and a morphism $\varphi\colon \operatorname{Spec} R \to Q_{2d}$ such that the generic point maps to x_1 and the special point maps to x_0.

Denote the fraction field of R by K and the residue field by k. It suffices to consider the case where x_0 is a closed point, hence $k = \mathbb{C}$ is algebraically closed. We show that given the exact sequence of coherent sheaves at the generic point

$$0 \to \mathcal{E} \to \mathcal{O} \otimes V \to \mathcal{Q} \to 0 \tag{30}$$

on $\mathbb{P}^1_K$, we can reconstruct the map φ and hence the sheaf sequence at the special point (possibly replacing R by its integral closure in a finite extension of K). Then, we note that the torsion of the quotient sheaf at the special point cannot have rank 1 at any point of $\mathbb{P}^1_k$.

Let the sequence (30) be given. The support of $\mathcal{Q}^{\text{tors}}$ specializes to a well-defined closed subset $Z \subset \mathbb{P}^1_k$; we let $Y = \text{Supp}(\mathcal{Q}^{\text{tors}}) \cup Z$. Now consider:

$$0 \to \mathcal{E}' \to \mathcal{O} \otimes V \to \mathcal{Q}/\mathcal{Q}^{\text{tors}} \to 0 \tag{31}$$

on $\mathbb{P}^1_K$. This corresponds to a morphism $\mathbb{P}^1_K \to OG$ (the actual map to the orthogonal Grassmannian underlying the sheaf sequence (30)). By replacing K by a finite extension and R by its integral closure in the extension, if necessary, then there exists, by semistable reduction, a modification

$$\pi \colon S \to \mathbb{P}^1_R$$

with exceptional divisor a tree of $\mathbb{P}^1$'s, and a morphism $S \to OG$, such that π restricts to the given morphism $\mathbb{P}^1_K \to OG$. We consider the pullback of the universal exact sequence

$$0 \to \widetilde{\mathcal{E}} \to \mathcal{O} \otimes V \to \widetilde{\mathcal{Q}} \to 0$$

on S. Pushing forward the map $\mathcal{E} \to \mathcal{O} \otimes V$ by π yields an exact sequence

$$0 \to \pi_*\widetilde{\mathcal{E}} \to \mathcal{O} \otimes V \to \mathcal{C} \to 0 \tag{32}$$

The cokernel $\mathcal{C}$, being a subsheaf of $\pi_*\widetilde{\mathcal{Q}}$, is torsion-free over $\operatorname{Spec} R$, and hence flat: (32) corresponds to the map from $\operatorname{Spec} R$ to the (possibly smaller degree) Quot scheme determined by (31).

We extend (30) to all of $\mathbb{P}^1_R$ by patching and pushing forward. The sequences (30) on $\mathbb{P}^1_K$ and (32) on $\mathbb{P}^1_R \smallsetminus Y$ patch to give the sequence

$$0 \to \widehat{\mathcal{E}} \to \mathcal{O} \otimes V \to \widehat{\mathcal{Q}} \to 0$$

on $\mathbb{P}^1_R \smallsetminus Z$. Pushing forward via $i \colon \mathbb{P}^1_R \smallsetminus Z \to \mathbb{P}^1_R$ gives

$$0 \to i_*\widehat{\mathcal{E}} \to \mathcal{O} \otimes V \to \mathcal{D} \to 0, \tag{33}$$

(where $\mathcal{D}$ is the indicated cokernel), flat over $\mathbb{P}^1_R$ since $i_*\widehat{\mathcal{E}}$ is locally free. This gives the morphism $\varphi\colon \operatorname{Spec} R \to Q_{2d}$ that we started with.

We now consider the restriction of (33) to the special fiber:

$$0 \to (i_*\widehat{\mathcal{E}})_k \to \mathcal{O} \otimes V \to \mathcal{D}_k \to 0,$$

and verify it satisfies the rank conditions. By semicontinuity, the dimension of the fiber of $\mathcal{D}_k^{\text{tors}}$ is $\geqslant 2$ at every point of Z. Suppose p is a point in $\mathbb{P}^1_k \smallsetminus Z$. Then $\mathcal{D}_k$, on a neighborhood of p, is isomorphic to $\mathcal{C}_k := \mathcal{C} \otimes_R k$, so it suffices to show every nonzero fiber of $\mathcal{C}_k^{\text{tors}}$ has dimension $\geqslant 2$. Letting $(\,)_k$ denote restriction to the special fiber, we have: $(\pi_*\widetilde{\mathcal{E}})_k \to \mathcal{O} \otimes V$ factors through $(\pi_k)_*(\widetilde{\mathcal{E}}_k) \to \mathcal{O} \otimes V$, which in turn factors through a vector subbundle $[(\pi_k)_*(\widetilde{\mathcal{E}}_k)]'$ of $\mathcal{O} \otimes V$ (the pullback of the universal subbundle by the actual map $\mathbb{P}^1_k \to OG$ at the special fiber), and $\dim \mathcal{C}_k^{\text{tors}} \otimes \mathcal{O}_p$ is greater than or equal to the dimension of the fiber at p of $[(\pi_k)_*(\widetilde{\mathcal{E}}_k)]'/(\pi_k)_*(\widetilde{\mathcal{E}}_k)$. But now we are in the situation of Lemma 1: this dimension is at least the number of negative line bundles in the direct sum decomposition of the pullback of the universal subbundle of OG under some positive-degree map from a copy of $\mathbb{P}^1_k$ to OG, and this must be at least 2. □

4.3. **Degeneracy loci.** Degeneracy loci for vector bundles in type D were defined using rank inequalities in [KT1].

Definition 4. The degeneracy loci W_λ and $W_\lambda(p)$ ($\lambda \in \mathcal{D}_n$, with $\ell = \ell(\lambda)$, and $p \in \mathbb{P}^1$) are the following subschemes of $\mathbb{P}^1 \times OQ_d$:

$$W_\lambda = \{\, x \in \mathbb{P}^1 \times OQ_d \mid \mathrm{rk}(\mathcal{E} \to \mathcal{O} \otimes V/F^{(i)\perp}_{n+1-\lambda_i})_x \leqslant n+1-i-\lambda_i, i = 1, \dots, \ell+1 \,\},$$
$$W_\lambda(p) = W_\lambda \cap (\{p\} \times OQ_d)$$

Define also
$$h(n,d) = n(n+1)/2 + 2nd,$$
which is the dimension of the orthogonal Quot scheme OQ_d when d is a nonnegative integer. As in types A and C, we establish a Moving Lemma, and deduce from this that all three-term Gromov–Witten invariants on OG count points in intersections of degeneracy loci on OQ_d.

Moving Lemma. *Let k be a positive integer, and let $p_1, \dots, p_k$ be distinct points on $\mathbb{P}^1$. Let $\lambda^1, \dots, \lambda^k$ be partitions in $\mathcal{D}_n$, and let us take the degeneracy loci $W_{\lambda^1}(p_1), \dots, W_{\lambda^k}(p_k)$ to be defined by isotropic flags of vector spaces in general position. Consider the intersection*
$$Z := W_{\lambda^1}(p_1) \cap \cdots \cap W_{\lambda^k}(p_k).$$
Then Z has dimension at most $h(n,d) - \sum_{i=1}^k |\lambda^i|$. Moreover, $Z \cap OM_d$ is either empty or generically reduced and of pure dimension $h(n,d) - \sum_i |\lambda^i|$; also, $Z \cap (OQ_d \smallsetminus OM_d)$ has dimension at most $h(n,d) - \sum_{i=1}^k |\lambda^i| - 1$.

The following are immediate consequences of the Moving Lemma.

Corollary 2. *Let p, p', $p'' \in \mathbb{P}^1$ be distinct points. Suppose λ, μ, $\nu \in \mathcal{D}_n$ satisfy $|\lambda| + |\mu| + |\nu| = h(n,d)$. With degeneracy loci defined with respect to isotropic flags in general position, the intersection $W_\lambda(p) \cap W_\mu(p') \cap W_\nu(p'')$ consists of finitely many reduced points, all contained in OM_d, and the corresponding Gromov–Witten invariant on OG satisfies*
$$\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d = \#\big(W_\lambda(p) \cap W_\mu(p') \cap W_\nu(p'')\big).$$

Corollary 3. *If p and p' are distinct points of $\mathbb{P}^1$ and if $|\lambda| + |\mu| = h(n,d)$, then $W_\lambda(p) \cap W'_\mu(p') = \emptyset$ for a general translate $W'_\mu(p')$ of $W_\mu(p')$.*

The Moving Lemma itself is proved using an analysis of the boundary of OQ_d. As in [Be] and [KT2], this boundary is covered by Grassmann bundles over smaller Quot schemes.

Definition 5. For $c \in (1/2)\mathbb{Z}$, with $c \geqslant 1$, we let $\pi_c \colon G_c \to \mathbb{P}^1 \times OQ_{d-c}$ denote the Grassmann bundle of $(2c)$-dimensional quotients of the universal bundle $\mathcal{E}$ on $\mathbb{P}^1 \times OQ_{d-c}$. The morphism $\beta_c \colon G_c \to OQ_d$ is given by the modification of the sheaf sequence $\mathcal{E} \to \mathcal{O} \otimes V$ along the graph of the projection to $\mathbb{P}^1$. Precisely: let $\mathcal{F}_c$ denote the universal quotient bundle on G_c; if i_c denotes the morphism $G_c \to \mathbb{P}^1 \times G_c$ given by $(\mathrm{pr}_1 \circ \pi_c, \mathrm{id})$, then $\mathcal{E}_c$ is defined as the kernel of the natural morphism of sheaves $(\mathrm{id} \times (\mathrm{pr}_2 \circ \pi_c))^*\mathcal{E} \to i_{c*}\pi_c^*\mathcal{E}$ composed with i_{c*} applied to the morphism to $\mathcal{F}_c$.

We also consider degeneracy loci with respect to the bundles $\mathcal{E}_c$.

Definition 6. We define $\widehat{W}_{c,\lambda}$ and $\widehat{W}_{c,\lambda}(p)$ to be the following subschemes of G_c:

$$\widehat{W}_{c,\lambda} = \{\, x \in G_c \mid \operatorname{rk}(\mathcal{E}_c \to \mathcal{O} \otimes V/F^{\perp}_{n+1-\lambda_i})_x \leqslant n+1-i-\lambda_i,\ i = 1, \ldots, \ell+1 \,\},$$
$$\widehat{W}_{c,\lambda}(p) = \widehat{W}_{c,\lambda}(p) \cap \pi_c^{-1}(\{p\} \times OQ_{d-c})$$

4.4. **Boundary structure of OQ_d.** The boundary of OQ_d is made up of points where $\mathcal{E} \to \mathcal{O} \otimes V$ drops rank at one or more points of $\mathbb{P}^1$; note that wherever it drops rank, it does so by at least two (by our definition of the Quot scheme).

Theorem 3. *For any $d \in (1/2)\mathbb{Z}$, with $d \geqslant 0$ and $d \neq 1/2$, we have*

$$\dim OQ_d = \begin{cases} h(n,d) & \text{if } d \in \mathbb{Z}, \\ h(n,d) - 5 & \text{otherwise.} \end{cases}$$

Furthermore, for $c \in (1/2)\mathbb{Z}$, $c \geqslant 1$, the map $\beta_c \colon G_c \to OQ_d$ satisfies
(i) *Given $x \in OQ_d$, if $\mathcal{Q}_x$ has rank at least $n+1+c$ at $p \in \mathbb{P}^1$, then x lies in the image of β_c.*
(ii) *The restriction of β_c to $\pi_c^{-1}(\mathbb{P}^1 \times OM_{d-c})$ is a locally closed immersion.*
(iii) *We have*

$$\beta_c^{-1}(W_\lambda(p)) = \pi_c^{-1}(\mathbb{P}^1 \times W_\lambda(p)) \cup \widehat{W}_{c,\lambda}(p)$$

where on the right, $W_\lambda(p)$ denotes the degeneracy locus in OQ_{d-c}.

The proof of Theorem 3, as well as that of the Moving Lemma (which uses Theorem 3), is similar to that of the corresponding results in [Be] and [KT2]. Details are left to the reader.

5. Intersection Theory on OQ_d

The Chow group of algebraic cycles modulo rational equivalence of a scheme $\mathfrak{X}$ is denoted $A_*\mathfrak{X}$. We also employ the following notation.

Definition 7. Let p denote a point of $\mathbb{P}^1$.
(i) $\mathrm{ev}^p \colon OM_d \to OG$ is the evaluation at p morphism;
(ii) $\tau(p) \colon OQ_d(p) \to OQ_d$ is the projection from the *relative orthogonal Grassmannian* $OQ_d(p) := OG_{n+1}(\mathcal{Q}|_{\{p\}\times OQ_d})$, that is, the closed subscheme of the Grassmannian Grass_{n+1} of rank-$(n+1)$ quotients [G2] of the indicated coherent sheaf, defined by isotropicity and parity conditions on the kernel of the composite morphism from $\mathcal{O}_{\mathrm{Grass}_{n+1}} \otimes V$ to the universal quotient bundle of the relative Grassmannian;
(iii) $\mathrm{ev}(p) \colon OQ_d(p) \to LG$ is the evaluation morphism on the relative orthogonal Grassmannian;
(iv) $\mathrm{ev}_c^p \colon \pi_c^{-1}(\{p\} \times OM_{d-c}) \to OG(n+1-2c, 2n+2)$ is evaluation at p.

Lemma 2 ([KT2])**.** *Let T be a projective variety which is a homogenous space for an algebraic group G. Let $\mathfrak{X}$ be a scheme, equipped with an action of the group G. Let U be a G-invariant integral open subscheme of $\mathfrak{X}$, and let $f \colon U \to T$ be a G-equivariant morphism. Then the map on algebraic cycles*

$$[V] \mapsto \left[\, f^{-1}(V)^{-} \right]$$

respects rational equivalence, and hence induces a map on Chow groups $A_*T \to A_*\mathfrak{X}$.

Corollary 4. *Fix distinct points p, $p' \in \mathbb{P}^1$. For any $\lambda \in \mathcal{D}_n$ of length $\ell = \ell(\lambda) \geqslant 3$, the following cycles are rationally equivalent to zero on OQ_d and on $OQ_d(p')$:*
(i) $\big[\,(\mathrm{ev}^p)^{-1}(\mathfrak{X}_\lambda)^-\,\big] - \sum_{j=1}^{r-1}(-1)^{j-1}\big[\,(\mathrm{ev}^p)^{-1}(\mathfrak{X}_{\lambda_j,\lambda_r} \cap \mathfrak{X}'_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}})^-\,\big]$.
(ii) $\sum_{j=1}^{r-1}(-1)^{j-1}\big[\,\beta_1\big((\mathrm{ev}_1^p)^{-1}(\mathfrak{Y}_{\lambda_j,\lambda_r} \cap \mathfrak{Y}'_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}})\big)^-\,\big]$.

Here, and in the sequel, $\mathfrak{X}'_\mu$ and $\mathfrak{Y}'_\mu$ denote the translates of $\mathfrak{X}_\mu$ and $\mathfrak{Y}_\mu$ by a general element of the group SO_{2n+2}.

As is standard, for any closed subscheme Z of a scheme $\mathfrak{X}$, $[Z] \in A_*\mathfrak{X}$ denotes the class in the Chow group of the cycle associated to Z; we let $[Z]_k$ be the dimension k component of $[Z]$.

Proposition 4. (a) *Suppose λ and μ are in $\mathcal{D}_n$, and let p, p', p'' be distinct points in $\mathbb{P}^1$. Assume that $\ell(\lambda)$ equals 1 or 2 and μ has even length $\geqslant 2$. Let $k = h(n,d) - |\lambda| - |\mu|$. Then*

$$\big[W_\lambda(p) \cap W'_\mu(p')\big]_k = \big[W_\lambda(p) \cap W'_\mu(p)\big]_k \text{ in } A_*OQ_d,$$
$$\big[\tau(p'')^{-1}\big(W_\lambda(p) \cap W'_\mu(p')\big)\big]_k = \big[\tau(p'')^{-1}\big(W_\lambda(p) \cap W'_\mu(p)\big)\big]_k \text{ in } A_*OQ_d(p''),$$

where $W'_\mu(p)$ denotes degeneracy locus with respect to a general translate of the isotropic flag of subspaces.

(b) *In A_*OQ_d, we have*

$$\big[W_\lambda(p) \cap W'_\mu(p)\big]_k = \big[\,(\mathrm{ev}^p)^{-1}(\mathfrak{X}_\lambda \cap \mathfrak{X}'_\mu)^-\,\big] + \big[\,\beta_1\big((\mathrm{ev}_1^p)^{-1}(\mathfrak{Y}_\lambda \cap \mathfrak{Y}'_\mu)\big)^-\,\big] \tag{34}$$

and in $A_*OQ_d(p'')$, the cycle class $\big[\tau(p'')^{-1}\big(W_\lambda(p) \cap W'_\mu(p)\big)\big]_k$ is equal to the right-hand side of (34).

Proof. By a dimension count which uses Proposition 2, the irreducible components of dimension k in $W_\lambda(p) \cap W'_\mu(p)$ are the ones indicated on the right-hand side of (34). As in [KT2], now, the result follows from the rational equivalence $\{p\} \sim \{p'\}$ on $\mathbb{P}^1$, pulled back to $Y := (\mathbb{P}^1 \times W_\lambda(p)) \cap W'_\mu$ (or further pulled back to $OQ_d(p'')$), once we know that the irreducible components of $W_\lambda(p) \cap W'_\mu(p)$ of dimension k are generically smooth and in the closure of the complement of the fiber of Y over p (and that this remains true after pullback by $\tau(p'')$). The 'in the closure' portion of the claim follows by an argument involving the Kontsevich compactification of OM_d, as in op. cit. Generic smoothness is clear for $(\mathrm{ev}^p)^{-1}(\mathfrak{X}_\lambda \cap \mathfrak{X}'_\mu)$. Transverality of a general translate also establishes generic smoothness for the other component, once we notice that any point x in a dense open subset of $\beta_1((\mathrm{ev}_1^p)^{-1}(\mathfrak{Y}_\lambda \cap \mathfrak{Y}'_\mu))$ has the property that for any local $\mathbb{C}$-algebra R with residue field $R/\mathfrak{m} \simeq \mathbb{C}$ and any $\psi\colon R \to W_\lambda(p) \cap W'_\mu(p)$ with closed point mapping to x, the map ψ factors through the restriction of β_1 to $\pi_1^{-1}(\{p\} \times OM_{d-1})$.

This assertion follows from elementary linear algebra, but because of some tricky cases involving parity, we give a sketch of the argument. Fix a basis $\{v_i\}$ of V so that the symmetric form is given by $\langle v_i, v_j\rangle = \delta_{i+j,2n+3}$. Without loss of generality, the two general-position flags are

$$F_i = \mathrm{Span}(v_1, \ldots, v_i)$$

and

$$G_i^{(0)} = \mathrm{Span}(v_{2n+3-i}, \ldots, v_{2n+2}),$$

where the latter specifies G_{n+1} or $\widetilde{G}_{n+1}$ equal to $\mathrm{Span}(v_{n+2}, \ldots, v_{2n+2})$ according to parity; see (22). We will show that the condition on x holds whenever x is in the preimage of the intersection of the Schubert *cells* corresponding to $\mathfrak{Y}_\lambda$ and $\mathfrak{Y}'_\mu$, subject to the further condition that the line on OG parametrized by the point in $OG(n-1,2n+2)$ is incident to $\mathfrak{X}_\lambda$ and $\mathfrak{X}'_\mu$ at two *distinct* points.

Consider first the case $\ell(\lambda)=1$. Let x correspond to $(n-1)$-dimensional $A\subset V$ at the point p. The condition to be in the Schubert cell for $\mathfrak{Y}_\lambda$ implies that $A\cap F_n^\perp=0$, so $\mathrm{rk}(A\to V/F^{(i)}_{n+1})=n-1$ for any i. By Definition 4, the sheaf sequence corresponding to ψ satisfies the rank condition

$$\mathrm{rk}(\mathcal{E}\to\mathcal{O}\otimes V/F^{(0)}_{n+1})\leqslant n-1. \tag{35}$$

Turning to the conditions coming from μ, we have $\mathrm{rk}(A\cap G^{(1)}_{n+1})=n-\ell$, from membership in the Schubert cell. Suppose n is even, so that $F^{(0)}_{n+1}=\widetilde{F}_{n+1}$ and $G^{(1)}=G_{n+1}$ are disjoint. Note that in this case Definition 4 imposes the condition

$$\mathrm{rk}(\mathcal{E}\to\mathcal{O}\otimes V/G_{n+1})\leqslant n-\ell. \tag{36}$$

The following basic argument is used to show that ψ factors through the restriction of β_1 to $\pi_1^{-1}(\{p\}\times OM_{d-1})$. We have a sheaf sequence on $\mathbb{P}^1_R$; after restricting to $\mathbb{A}^1_R$ the sheaf $\mathcal{E}$ can be trivialized, so let us assume the map to $\mathcal{O}\otimes V$ is given by the $(2n+2)\times(n+1)$ matrix L with values in $R[t]$, with coordinates assigned so the top half of the matrix corresponds to $\widetilde{F}_{n+1}$ and the bottom half corresponds to G_{n+1}. We may assume $t=0$ defines p, and also assume that mod $\mathfrak{m}$, the rightmost two columns of L vanish at $t=0$. We localize at $\mathfrak{m}+tR[t]$. It suffices to show that conditions (35) and (36) imply, after column operations, that the rightmost two columns of L have values in the ideal generated by t. We have $\mathrm{rk}(A\to V/F_{n+1})=n-1$, that is, some $(n-1)\times(n-1)$ minor in the bottom half of L has full rank. Now by performing column operations and invoking (35) we have all the entries in the bottom right $(n+1)\times 2$ submatrix of L lying in the ideal (t). Let L' denote the top right $(n+1)\times 2$ submatrix of L. The remaining isotropicity and rank conditions amount to $UL'=0 \bmod t$ for some matrix U, whose entries are polynomial functions of the entries of L in the first $n-1$ columns. The condition that the line corresponding to A meets the Schubert varieties in distinct points implies that the nullspace of U is trivial, and hence L' has entries in (t) as well.

If, instead, n is odd, we use the fact that $\mathrm{rk}(A\cap G_{n+1})=n+1-\ell$ (also a condition to be in the Schubert cell). From Definition 4,

$$\mathrm{rk}(\mathcal{E}\to\mathcal{O}\otimes V/G_{n+1})\leqslant \mathrm{rk}(\mathcal{E}\to\mathcal{O}\otimes V/G_n^\perp)\leqslant n+1-\ell. \tag{37}$$

Now $F^{(0)}_{n+1}=F_{n+1}$ and G_{n+1} are disjoint, and the basic argument applies, using (35) and (37).

In case $\ell(\lambda)=2$, we have $A\cap F^{(0)}_{n+1}=0$ and (35) still holds, so the argument is the same. □

We now establish the rational equivalences on OQ_d — and on $OQ_d(p'')$ — which directly imply the quantum Giambelli formula of Theorem 1.

Proposition 5. *Fix $\lambda\in\mathcal{D}_n$ with $\ell=\ell(\lambda)\geqslant 3$. Set $r=2\lfloor(\ell+1)/2\rfloor$. Let p, p', p'' denote distinct points in $\mathbb{P}^1$. Then we have the following identity of cycle classes*

$$\big[\,(\mathrm{ev}^p)^{-1}(\mathfrak{X}_\lambda)^-\,\big]=\sum_{j=1}^{r-1}(-1)^{j-1}\big[\,\big((\mathrm{ev}^p)^{-1}(\mathfrak{X}_{\lambda_j,\lambda_r})\cap(\mathrm{ev}^{p'})^{-1}(\mathfrak{X}'_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}})\big)^-\,\big], \tag{38}$$

both on OQ_d and on $OQ_d(p'')$, where $\mathfrak{X}'_\mu$ denotes the translate of $\mathfrak{X}_\mu$ by a generally chosen element of the group SO_{2n+2}.

Proof. Combining parts (a) and (b) of Proposition 4 gives

$$\begin{aligned} &\big[\big((\mathrm{ev}^p)^{-1}(\mathfrak{X}_{\lambda_j,\lambda_r}) \cap (\mathrm{ev}^{p'})^{-1}(\mathfrak{X}'_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}})\big)^-\big] \\ &\quad = \big[(\mathrm{ev}^p)^{-1}(\mathfrak{X}_{\lambda_j,\lambda_r} \cap \mathfrak{X}'_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}})^-\big] + \big[\beta_1\big((\mathrm{ev}_1^p)^{-1}(\mathfrak{Y}_{\lambda_j,\lambda_r} \cap \mathfrak{Y}'_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}})\big)^-\big] \end{aligned}$$

for each j, with $1 \leqslant j \leqslant r-1$. Now (38) follows by summing and applying (i) and (ii) of Corollary 4. ☐

Theorem 4. *Suppose $\lambda \in \mathcal{D}_n$, with $\ell = \ell(\lambda) \geqslant 3$, and set $r = 2\lfloor(\ell+1)/2\rfloor$. Then we have the following identity in $QH^*(OG)$:*

$$\tau_\lambda = \sum_{j=1}^{r-1} (-1)^{j-1} \tau_{\lambda_j,\lambda_r} \tau_{\lambda\smallsetminus\{\lambda_j,\lambda_r\}}. \tag{39}$$

Proof. The classical component of (39) follows from the classical Giambelli formula for OG. To handle the remaining terms, apply a refined cap product operation [F, §8.1] along $\mathrm{ev}(p'')$ to general translates of $\mathfrak{X}_\mu$ for all $\mu \in \mathcal{D}_n$ with $|\mu| = h(n,d) - |\lambda|$, and invoke Corollaries 3 and 2 (as in the proof of [KT2, Thm. 5]). ☐

6. Quantum Schubert calculus

Our aim in this Section is to use Theorem 1 and the algebra of $\widetilde{P}$-polynomials to find combinatorial rules that compute some of the quantum structure constants that appear in the quantum product of two Schubert classes.

6.1. **Algebraic background.** Let $\mathcal{E}_n$ denote the set of all partitions λ with $\lambda_1 \leqslant n$. The main properties of $\widetilde{Q}$-polynomials that we need are collected in [KT2, §2.1 and §6.1]. They imply corresponding facts about the $\widetilde{P}$-polynomials, in particular, that the set $\{\widetilde{P}_\lambda(X) \mid \lambda \in \mathcal{E}_n\}$ is a free $\mathbb{Z}$-basis of the ring Λ'_n that they span. Hence, there exist integers $f(\lambda,\mu;\,\nu)$ such that

$$\widetilde{P}_\lambda(X)\,\widetilde{P}_\mu(X) = \sum_\nu f(\lambda,\mu;\,\nu)\,\widetilde{P}_\nu(X); \tag{40}$$

the constants $f(\lambda,\mu;\,\nu)$ are independent of n, and defined for any $\lambda,\mu,\nu \in \mathcal{E}_n$. The corresponding coefficients $e(\lambda,\mu;\,\nu)$ in the expansion of the product $\widetilde{Q}_\lambda(X)\,\widetilde{Q}_\mu(X)$ are related to these by the equation

$$e(\lambda,\mu;\,\nu) = 2^{\ell(\lambda)+\ell(\mu)-\ell(\nu)} f(\lambda,\mu;\,\nu). \tag{41}$$

There are explicit combinatorial rules (involving signs in general) for computing the integers $f(\lambda,\mu;\,\nu)$, which follow from corresponding formulas for decomposing products of Hall-Littlewood polynomials; for more details, see [KT2, §6.1]. Define the connected components of a skew Young diagram by specifying that two boxes are connected if they share a vertex or an edge. We then have the following Pieri type formula for λ *strict*:

$$\widetilde{P}_\lambda(X)\,\widetilde{P}_k(X) = \sum_\mu 2^{N'(\lambda,\mu)}\,\widetilde{P}_\mu(X), \tag{42}$$

where the sum is over all partitions $\mu \supset \lambda$ with $|\mu| = |\lambda| + k$ such that μ/λ is a horizontal strip, and $N'(\lambda, \mu)$ is one less than the number of connected components of μ/λ. In particular, we have $\widetilde{P}_\lambda(X)\widetilde{P}_n(X) = \widetilde{P}_{(n,\lambda)}(X)$ for all $\lambda \in \mathcal{D}_n$.

When λ, μ and ν are strict partitions, the $f(\lambda, \mu; \nu)$ are classical structure constants for $OG(n+1, 2n+2)$,

$$\tau_\lambda \tau_\mu = \sum_{\nu \in \mathcal{D}_n} f(\lambda, \mu; \nu)\, \tau_\nu,$$

and hence are nonnegative integers. In this case, Stembridge [St] has given a combinatorial rule for the numbers $f(\lambda, \mu; \nu)$, analogous to the usual Littlewood-Richardson rule in type A. Specifically, $f(\lambda, \mu; \nu)$ is equal to the number of marked tableaux of weight λ on the shifted skew shape $\mathcal{S}(\nu/\mu)$ satisfying certain conditions (see [St] and [P, Sect. 6] for more details).

6.2. **Quantum multiplication.** Recall from the Introduction that for any $\lambda, \mu \in \mathcal{D}_n$ there is a formula

$$\tau_\lambda \cdot \tau_\mu = \sum f^{\nu}_{\lambda\mu}(n)\, \tau_\nu\, q^d$$

in $QH^*(OG(n+1, 2n+2))$, with each $f^{\nu}_{\lambda\mu}(n)$ equal to a Gromov–Witten invariant $\langle \tau_\lambda, \tau_\mu, \tau_{\widehat{\nu}} \rangle_d$ (defined when $|\lambda| + |\mu| = |\nu| + 2nd$). The nonnegative integer $f^{\nu}_{\lambda\mu}(n)$ counts the number of degree-d rational maps $\psi : \mathbb{P}^1 \to OG$ such that $\psi(0) \in \mathfrak{X}_\lambda$, $\psi(1) \in \mathfrak{X}_\mu$ and $\psi(\infty) \in \mathfrak{X}_{\widehat{\nu}}$, when the three Schubert varieties $\mathfrak{X}_\lambda$, $\mathfrak{X}_\mu$ and $\mathfrak{X}_{\widehat{\nu}}$ are in general position.

We adopt the convention that $\tau_\lambda = 0$ for all non-strict partitions λ. Now Theorem 1 and the Pieri rule (42) give

Corollary 5 (Quantum Pieri Rule)**.** *For any $\lambda \in \mathcal{D}_n$ and $k \geqslant 0$ we have*

$$\tau_\lambda \tau_k = \sum_{\mu} 2^{N'(\lambda,\mu)} \tau_\mu + \sum_{\mu \supset (n,n)} 2^{N'(\lambda,\mu)} \tau_{\mu \smallsetminus (n,n)}\, q$$

where both sums are over $\mu \supset \lambda$ with $|\mu| = |\lambda| + k$ such that μ/λ is a horizontal strip, and the second sum is restricted to those μ with two parts equal to n.

In recent work with Buch [BKT], we give a more direct proof of the quantum Pieri rule for OG, and the corresponding rule for the Lagrangian Grassmannian.

For any $d, n \geqslant 0$ and partition ν, let (n^d, ν) denote the partition

$$(n, n, \ldots, n, \nu_1, \nu_2, \ldots),$$

where n appears d times before the first component ν_1 of ν. Theorem 1 now gives

Theorem 5. *For any $d \geqslant 0$ and strict partitions $\lambda, \mu, \nu \in \mathcal{D}_n$ with $|\nu| = |\lambda| + |\mu| - 2nd$, the quantum structure constant $f^{\nu}_{\lambda\mu}(n)$ satisfies $f^{\nu}_{\lambda\mu}(n) = f(\lambda, \mu; (n^{2d}, \nu))$.*

We deduce that for any strict partitions $\lambda, \mu, \nu \in \mathcal{D}_n$, the coefficient $f(\lambda, \mu; (n^d, \nu))$ is a nonnegative integer. The constants $f(\lambda, \mu; \nu)$ can be negative; for example

$$f(\rho_3, \rho_3; (4, 4, 2, 2)) = -1.$$

This follows from the Remark in [KT2, §6.2].

6.3. **The relation to $QH^*(LG(n-1,2n-2))$.** The quantum Pieri rule of Proposition 5 implies that

$$\tau_n\,\tau_\lambda = \begin{cases} \tau_{(n,\lambda)} & \text{if } \lambda_1 < n, \\ \tau_{\lambda\smallsetminus(n)}\,q & \text{if } \lambda_1 = n \end{cases}$$

in the quantum cohomology ring of $OG(n+1,2n+2)$. Therefore, to compute all the Gromov–Witten invariants for OG, it suffices to evaluate the $\langle\tau_\lambda,\tau_\mu,\tau_\nu\rangle_d$ for $\mu,\nu\in\mathcal{D}_{n-1}$. Define a map $*:\mathcal{D}_n\to\mathcal{D}_{n-1}$ by setting $\lambda^*=(n-\lambda_\ell,\dots,n-\lambda_1)$ for any partition λ of length ℓ, and $(0)^*=(0)$.

Partitions in $\mathcal{D}_{n-1}$ also parametrize the Schubert classes σ_λ in the (quantum) cohomology ring of the Lagrangian Grassmannian $LG(n-1,2n-2)$, which was studied in [KT2]. For the remainder of this paper, we let $':\mathcal{D}_{n-1}\to\mathcal{D}_{n-1}$ denote the duality involution for this space, so that the parts of λ' complement the parts of λ in the set $\{1,2,\dots,n-1\}$. Notice that the restriction of $*$ to $\mathcal{D}_{n-1}$ defines a second involution on this set, which was considered in [KT2, §6.3].

Theorem 6. *Suppose that $\lambda\in\mathcal{D}_n$ is a non-zero partition with $\ell(\lambda)=2d+e+1$ for some nonnegative integers d and e. For any $\mu,\nu\in\mathcal{D}_{n-1}$, we have an equality*

$$\langle\tau_\lambda,\tau_\mu,\tau_\nu\rangle_d = \langle\sigma_{\lambda^*},\sigma_{\mu'},\sigma_{\nu'}\rangle_e \tag{43}$$

of Gromov–Witten invariants for $OG(n+1,2n+2)$ and $LG(n-1,2n-2)$, respectively. If λ is zero or $\ell(\lambda)<2d+1$, then $\langle\tau_\lambda,\tau_\mu,\tau_\nu\rangle_d=0$.

Proof. Assume first that $\lambda_1<n$, so $\lambda\in\mathcal{D}_{n-1}$. We then have

$$\begin{aligned}\langle\tau_\lambda,\tau_\mu,\tau_\nu\rangle_d &= f(\lambda,\mu;\,(n^{2d+1},\nu')) \\ &= 2^{n+2d-\ell(\lambda)-\ell(\mu)-\ell(\nu)}\,e(\lambda,\mu;\,(n^{2d+1},\nu')) \\ &= 2^{n+4d+1-\ell(\lambda)-\ell(\mu)-\ell(\nu)}\,\langle\sigma_\lambda,\sigma_\mu,\sigma_\nu\rangle_{2d+1}\end{aligned}$$

where the last equality comes from [KT2, Thm. 6]. The result now follows by applying the eight-fold symmetry [KT2, Thm. 7] for $QH^*(LG(n-1,2n-2))$, which dictates

$$2^{n+2d}\,\langle\sigma_\lambda,\sigma_\mu,\sigma_\nu\rangle_{2d+1} = 2^{\ell(\mu)+\ell(\nu)+e}\,\langle\sigma_{\lambda^*},\sigma_{\mu'},\sigma_{\nu'}\rangle_e. \tag{44}$$

If $\lambda_1=n$, then

$$\langle\tau_\lambda,\tau_\mu,\tau_\nu\rangle_d = \langle\tau_{\lambda\smallsetminus(n)},\tau_\mu,\tau_{(n,\nu)}\rangle_d = f(\lambda\smallsetminus(n),\mu;\,(n^{2d},\nu')),$$

and the previous analysis applies, since $\lambda^*=(\lambda\smallsetminus(n))^*$. □

Of course this theorem also provides an equality of Gromov–Witten invariants going the other way. For any $\lambda,\mu,\nu\in\mathcal{D}_{n-1}$, we have

$$\langle\sigma_\lambda,\sigma_\mu,\sigma_\nu\rangle_e = \begin{cases}\langle\tau_{\lambda^*},\tau_{\mu'},\tau_{\nu'}\rangle_d & \text{if } \ell(\lambda)-e=2d+1 \text{ is odd,} \\ \langle\tau_{(n,\lambda^*)},\tau_{\mu'},\tau_{\nu'}\rangle_d & \text{if } \ell(\lambda)-e=2d \text{ is even.}\end{cases}$$

The $(\mathbb{Z}/2\mathbb{Z})^3$-symmetry (44) enjoyed by the Gromov–Witten invariants for $LG(n-1,2n-2)$ implies a similar one for $QH^*(OG)$.

Proposition 6. *Let $\lambda\in\mathcal{D}_n$ be non-zero and $\mu,\nu\in\mathcal{D}_{n-1}$. For any $d,e\geqslant 0$ with $2d+e+1=\ell(\lambda)$, we have*

$$2^{\ell(\mu)+\ell(\nu)+e+\delta}\,\langle\tau_\lambda,\tau_\mu,\tau_\nu\rangle_d = 2^{n+2d}\begin{cases}\langle\tau_{\lambda^*},\tau_{\mu'},\tau_{\nu'}\rangle_g & \textit{if } e=2g+1 \textit{ is odd,} \\ \langle\tau_{(n,\lambda^*)},\tau_{\mu'},\tau_{\nu'}\rangle_g & \textit{if } e=2g \textit{ is even,}\end{cases}$$

where $\delta = \delta_{\lambda_1,n}$ is the Kronecker symbol.

We now obtain orthogonal analogues of [KT2, Prop. 10] and [KT2, Cor. 8].

Corollary 6. *Let λ, μ, ν and δ be as in Proposition 6. Then the inequalities*

$$\ell(\mu) + \ell(\nu) - n + \delta \leqslant 2d \leqslant \ell(\lambda) + \ell(\mu) + \ell(\nu) - n \tag{45}$$

are necessary conditions for the Gromov–Witten invariant $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d$ to be nonzero. Moreover, if the two sides of either of the inequalities in (45) differ by 0 or 1, then $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d$ is related by the eight-fold symmetry to a classical structure constant.

Corollary 7. *For any $\lambda \in \mathcal{D}_n$, we have*

$$\tau_\lambda \cdot \tau_{\rho_{n-1}} = \begin{cases} \tau_{\lambda^{*\prime}}\, q^d & \text{if } \ell(\lambda) = 2d \text{ is even,} \\ \tau_{(n,\lambda^{*\prime})}\, q^d & \text{if } \ell(\lambda) = 2d+1 \text{ is odd.} \end{cases}$$

in $QH^*(OG)$. In particular,

$$\tau_{\rho_n} \cdot \tau_{\rho_n} = \begin{cases} \tau_n\, q^{n/2} & \text{if } n \text{ is even,} \\ q^{(n+1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

7. Appendix: An identity in $\widetilde{P}$-polynomials

We give a proof of the following identity, which is used to simplify a formula for degeneracy loci in type D [KT1]. The proof uses the algebraic formalism of §2.2.

Proposition 7. *Let $X = (x_1, \ldots, x_n)$ be an n-tuple of variables, and consider also $\widetilde{X} = (-x_1, x_2, \ldots, x_n)$ and $X' = (x_2, \ldots, x_n)$. Then, for any $\lambda \in \mathcal{E}_n$ of length $\ell \geqslant 1$ we have*

$$\sum_{i=1}^{\ell} (-1)^{i-1} \widetilde{P}_{\lambda \smallsetminus \{\lambda_i\}}(X) e_{\lambda_i}(X') = \widetilde{P}_\lambda(\widetilde{X}) + (-1)^{\ell+1} \widetilde{P}_\lambda(X). \tag{46}$$

Proof. By homogeneity, (46) is equivalent to the identity

$$\sum_{i=1}^{\ell} (-1)^{i-1} \widetilde{Q}_{\lambda \smallsetminus \{\lambda_i\}}(X) \widetilde{Q}_{\lambda_i}(X') = \frac{1}{2}(\widetilde{Q}_\lambda(\widetilde{X}) + (-1)^{\ell+1} \widetilde{Q}_\lambda(X)). \tag{47}$$

To establish (47), we use identity (11) and are reduced to

$$\sum_{i=1}^{\ell} (-1)^{i-1} \widetilde{Q}_{\lambda_i}(X') \sum_{\mu \in B(\lambda \smallsetminus \{\lambda_i\}, k)} \widetilde{Q}_\mu(X') = \begin{cases} \sum_{\mu \in B(\lambda,k)} \widetilde{Q}_\mu(X'), & \text{if } k \neq \ell \bmod 2, \\ 0 & \text{if } k = \ell \bmod 2, \end{cases}$$

for all integers k, where $B(\lambda, k)$ is defined as in the proof of Proposition 1. This corresponds to an identity in the algebra $\mathcal{A}$ of formal variables with imposed relations of [KT2, §2.3], which is similar to the algebra $\mathcal{B}$ of §2.2, except that only single bars appear.

Using the equalities

$$[a,b](c) - [a,c](b) + [b,c](a) = 0 \tag{48}$$

and

$$[a,b](\overline{c}) - [a,c](\overline{b}) + [b,c](\overline{a}) = 0 \tag{49}$$

in $\mathcal{A}$, one can verify, for each combination of parities of k and ℓ, that the corresponding identity in $\mathcal{A}$ is true (one case, that of k odd, ℓ even, uses also the identity (17)). For example, when k is even and ℓ is odd, we need to show that

$$\sum_{i=1}^{\ell}(-1)^{i-1}(\lambda_i)\sum_{\mu\in B(\lambda\smallsetminus\{\lambda_i\},k)}\sum \epsilon(\mu,\nu)(\nu_1,\nu_2)\cdots(\nu_{\ell-2},\nu_{\ell-1}) = \sum_{\nu\in B(\lambda,k)}(\nu) \tag{50}$$

where the innermost sum on the left is over all $(\ell-2)(\ell-4)\cdots(1)$ ways to write the set of entries of μ as a union of pairs $\{\nu_1,\nu_2\}\cup\cdots\cup\{\nu_{\ell-2},\nu_{\ell-1}\}$. Using (48), the sum of the terms on the left hand side which contain a pair with exactly one bar vanishes. The remaining terms are seen, using (48) and (49), to be equal to the Pfaffian expansion of the right-hand side of (50). □

References

[Be] A. Bertram : *Quantum Schubert calculus*, Adv. Math. **128** (1997), no. 2, 289–305.

[BKT] A. Buch, A. Kresch and H. Tamvakis : *Gromov–Witten invariants on Grassmannians*, J. Amer. Math. Soc., to appear.

[F] W. Fulton : *Intersection Theory*, Second edition, Ergebnisse der Math. **2**, Springer-Verlag, Berlin, 1998.

[FP] W. Fulton and R. Pandharipande : *Notes on stable maps and quantum cohomology*, in Algebraic Geometry (Santa Cruz, 1995), 45–96, Proc. Sympos. Pure Math. **62**, Part 2, Amer. Math. Soc., Providence, 1997.

[G1] A. Grothendieck : *Techniques de construction et théorèmes d'existence en géométrie algébrique IV: Les schémas de Hilbert*, Séminaire Bourbaki, Vol. 6 (1960/61), Exp. 221, 249–276, Soc. Math. France, Paris, 1995.

[G2] A. Grothendieck : *Techniques de construction en géométrie analytique V: Fibrés vectoriels, fibrés projectifs, fibrés en drapeaux*, in Familles d'espaces complexes et fondements de la géométrie analytique, Séminaire Henri Cartan, Vol. 13 (1960/61), Exp. 12, Secrétariat mathématique, Paris, 1962.

[HB] H. Hiller and B. Boe: *Pieri formula for SO_{2n+1}/U_n and Sp_n/U_n*, Adv. in Math. **62** (1986), 49–67.

[KT1] A. Kresch and H. Tamvakis : *Double Schubert polynomials and degeneracy loci for the classical groups*, Ann. Inst. Fourier (Grenoble) **52** (2002), 1681–1727.

[KT2] A. Kresch and H. Tamvakis : *Quantum cohomology of the Lagrangian Grassmannian*, J. Algebraic Geom., to appear.

[LP] A. Lascoux and P. Pragacz : *Orthogonal divided differences and Schubert polynomials, $\widetilde{P}$-functions, and vertex operators*, Michigan Math. J. **48** (2000), 417–441.

[LT] J. Li and G. Tian : *The quantum cohomology of homogeneous varieties*, J. Algebraic Geom. **6** (1997), 269–305.

[P] P. Pragacz : *Algebro-geometric applications of Schur S- and Q-polynomials*, Séminare d'Algèbre Dubreil-Malliavin 1989-1990, Lecture Notes in Math. **1478** (1991), 130–191, Springer-Verlag, Berlin, 1991.

[PR] P. Pragacz and J. Ratajski : *Formulas for Lagrangian and orthogonal degeneracy loci; $\widetilde{Q}$-polynomial approach*, Compositio Math. **107** (1997), no. 1, 11–87.

[ST] B. Siebert and G. Tian : *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, Asian J. Math. **1** (1997), no. 4, 679–695.

[St] J. R. Stembridge : *Shifted tableaux and the projective representations of symmetric groups*, Adv. in Math. **74** (1989), 87–134.

[T] H. Tamvakis : *Quantum cohomology of Lagrangian and orthogonal Grassmannians*, Mathematische Arbeitstagung, Bonn, 2001, MPIM-Preprint 2001-50.

Department of Mathematics, University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104-6395, USA

E-mail address: `kresch@math.upenn.edu`

Department of Mathematics, Brandeis University, MS 050, P. O. Box 9110, Waltham, MA 02454-9110, USA
E-mail address: harryt@brandeis.edu