Quantum continuous $\mathfrak{gl}_\infty$: Tensor products of Fock modules and $\mathcal{W}_n$-characters

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To the memory of the late Professor Hiroshi Saito

Abstract We construct a family of irreducible representations of the quantum continuous $\mathfrak{gl}_\infty$ whose characters coincide with the characters of representations in the minimal models of the $\mathcal{W}_n$-algebras of $\mathfrak{gl}_n$ type. In particular, we obtain a simple combinatorial model for all representations of the $\mathcal{W}_n$-algebras appearing in the minimal models in terms of n interrelating partitions.

1. Introduction

In [FFJ+] we initiated a study of representations of the algebra which we denote by $\mathcal{E}$ and call quantum continuous $\mathfrak{gl}_\infty$. This algebra depends on two parameters q_1, q_2 and is closely related to the Ding-Iohara algebra introduced in [DI] and then considered in [FHH+] and [FT]. Conjecturally, the algebra $\mathcal{E}$ is isomorphic to the infinite spherical double affine Hecke algebra constructed in [ScV1] and [ScV2] (see also [BS], [S]).

We argued that the representation theory of the algebra $\mathcal{E}$ has many features similar to the representation theory of $\mathfrak{gl}_\infty$. In particular, it has a family of vector representations. The algebra $\mathcal{E}$ is not a Hopf algebra. However there is a "comultiplication rule" which, under some conditions, defines an action of $\mathcal{E}$ on a tensor product of $\mathcal{E}$-modules. We used the comultiplication rule and the vector representations to construct Fock modules by the standard semiinfinite

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construction (which can also be called an inductive limit construction or semi-infinite wedge representation). Similarly to the case of $\mathfrak{gl}_\infty$, the Fock modules have a natural basis labeled by partitions.

On the other hand, due to the quantum nature of the algebra $\mathcal{E}$, the representation theory of $\mathcal{E}$ is richer than that of $\mathfrak{gl}_\infty$. In [FFJ+], we used the semiinfinite construction to define another family of modules for the case when $q_1^{1-r}q_3^{k+1}=1$ with $r\in\mathbb{Z}_{>1}$, $k\in\mathbb{Z}_{>0}$. These modules do not have $\mathfrak{gl}_\infty$-analogues. Their bases are given by the so-called (k,r)-admissible partitions λ satisfying $\lambda_i-\lambda_{i+k}\geq r$.

In [FJMM] it was shown that when the parameters q,t satisfy $q^{1-r}t^{k+1}=1$, the Macdonald polynomials $P_\lambda(q,t)$ with (k,r)-admissible partitions λ form a basis of the smallest ideal in the space of symmetric polynomials stable under the Macdonald operators (see also [K], [SV]). The $\mathcal{E}$-module we constructed is an inductive limit of these ideals in an appropriate sense.

In this paper we continue the study of representations of $\mathcal{E}$ started in [FFJ+]. We find that this algebra has a large class of modules which are tame in the following sense. The algebra $\mathcal{E}$ has a commutative subalgebra generated by ψ_i^+,ψ_{-i}^-, where $i\in\mathbb{Z}_{\geq 0}$, which serves as an analogue of a Cartan subalgebra. We call a module tame if this subalgebra acts diagonally with the simple joint spectrum.

In contrast to [FFJ+], to construct new modules we do not use the semi-infinite constructions but use the Fock spaces as building blocks. Namely, we consider subquotients of the tensor product of several Fock spaces.

The Fock space depends on a continuous parameter u and has a basis labeled by partitions. Therefore the tensor product of n Fock modules depends on n complex parameters u_i, where $i=1,\dots,n$, and has a basis labeled by n-tuples of partitions. The subalgebra of $\psi_i^\pm$ acts diagonally in this basis. Remarkably, all submodules and quotient modules we study also have a basis of n-tuples of partitions with some conditions. It means that we never have to deal with linear combinations of the basic vectors and all considerations are purely combinatorial.

For generic parameters q_1,q_2,u_i, where $i=1,\dots,n$, the tensor products of Fock modules are irreducible. We consider special values which we call resonances. In the case of the resonances, the $\mathcal{E}$-action on the tensor product of the Fock spaces is not defined. However we find a "subquotient" for which it is well defined. This subquotient is an irreducible $\mathcal{E}$-module with a basis labeled by n partitions $\lambda^{(1)},\dots,\lambda^{(n)}$ with conditions

$$\lambda_s^{(i)}\geq\lambda_{s+b_i}^{(i+1)}-a_i,\quad\text{where } i=1,\dots,n,\ s\in\mathbb{Z}_{>0}.$$

Here $\lambda^{(n+1)}=\lambda^{(1)}$ and $a_i,b_i\in\mathbb{Z}_{\geq 0}$ are the parameters of the module. The conditions with $i=1,\dots,n-1$ correspond to the submodules appearing under resonances $u_i/u_{i+1}=q_1^{a_i+1}q_3^{b_i+1}$, where $q_3=(q_1q_2)^{-1}$. The $i=n$ condition corresponds to the quotient module appearing under the resonance $q_1^{p'}q_3^{p}=1$, where $p'=\sum_{i=1}^n(a_i+1)$, $p=\sum_{i=1}^n(b_i+1)$.

We use a recursion to compute the graded character of the subquotient and find that it coincides with the character of the $\mathcal{W}_n$-module from the minimal (p',p)-theory labeled by the $\widehat{\mathfrak{sl}}_n$-weights $\boldsymbol{\eta}=\sum_{i=1}^n a_i\omega_i$ and $\boldsymbol{\xi}=\sum_{i=1}^n b_i\omega_i$ (see

Theorem 4.5). The recursion in the case $n=2$ is similar to the recursion in [ABB+].

For the information on $\mathcal{W}_n$ and its representations we refer to [FKW]. The $\mathcal{W}_n$-modules are not well understood apart from the Virasoro $n=2$ case, and we hope that such a relatively simple combinatorial description sheds new light on the subject. Moreover, there are many indications that the relation between $\mathcal{E}$ and $\mathcal{W}_n$-algebras is much deeper than the one described in this paper.

The paper is constructed as follows. In Section 2 we recall the main definitions and constructions from [FFJ+]. In Section 3 we construct the subquotients of the tensor products of Fock modules and prove that they are tame and irreducible. In Section 4 we describe the recursion for the set of n-tuples of partitions with conditions, give a solution of the recursion in a bosonic form, and compare it to the characters of $\mathcal{W}_n$-algebras. In Section 5 we show several isomorphisms of $\mathcal{E}$-modules, in particular, the isomorphism of the subquotients in the case $p=n+1$ to the $\mathcal{E}$-modules with bases of (k,r)-admissible partitions of [FFJ+].

2. Quantum continuous $\mathfrak{gl}_\infty$

In this section we recall the basic definitions about quantum continuous $\mathfrak{gl}_\infty$ and its representations following [FFJ+].

2.1. Algebra

Let $q_1, q_2, q_3 \in \mathbb{C}$ be complex parameters satisfying the relation $q_1q_2q_3=1$. In this paper we assume that neither q_i is a root of unity. Let

$$g(z,w)=(z-q_1w)(z-q_2w)(z-q_3w).$$

The *quantum continuous* $\mathfrak{gl}_\infty$ is an associative algebra $\mathcal{E}$ with generators e_i, f_i, where $i\in\mathbb{Z}$, ψ_i^+, ψ_{-i}^-, where $i\in\mathbb{Z}_{\geq 0}$, and $(\psi_0^\pm)^{-1}$ satisfying the following defining relations:

$$\begin{aligned} g(z,w)e(z)e(w) &= -g(w,z)e(w)e(z), \\ g(w,z)f(z)f(w) &= -g(z,w)f(w)f(z), \end{aligned} \tag{2.1}$$

$$\begin{aligned} g(z,w)\psi^\pm(z)e(w) &= -g(w,z)e(w)\psi^\pm(z), \\ g(w,z)\psi^\pm(z)f(w) &= -g(z,w)f(w)\psi^\pm(z), \end{aligned} \tag{2.2}$$

$$[e(z),f(w)]=\frac{\delta(z/w)}{g(1,1)}\big(\psi^+(z)-\psi^-(z)\big), \tag{2.3}$$

$$[\psi_i^\pm,\psi_j^\pm]=0, \qquad [\psi_i^\pm,\psi_j^\mp]=0, \tag{2.4}$$

$$\psi_0^\pm(\psi_0^\pm)^{-1}=(\psi_0^\pm)^{-1}\psi_0^\pm=1, \tag{2.5}$$

$$\big[e_0,[e_1,e_{-1}]\big]=0, \qquad \big[f_0,[f_1,f_{-1}]\big]=0. \tag{2.6}$$

Here $\delta(z)=\sum_{n\in\mathbb{Z}} z^n$ denotes the formal delta function, and the generating series of the generators of $\mathcal{E}$ are given by

$$e(z)=\sum_{i\in\mathbb{Z}} e_i z^{-i}, \qquad f(z)=\sum_{i\in\mathbb{Z}} f_i z^{-i}, \qquad \psi^{\pm}(z)=\sum_{\pm i\geq 0} \psi_i^{\pm} z^{-i}.$$

Note that the algebra $\mathcal{E}$ depends on the *unordered* set of parameters $\{q_1,q_2,q_3\}$ as all q_i enter the relations symmetrically through the function $g(z,w)$.

The algebra generated by $\{\psi_i^+,\psi_{-i}^-\}_{i\in\mathbb{Z}_{\geq 0}}$ is commutative. We call an $\mathcal{E}$-module V *tame* if the operators $\{\psi_i^+,\psi_{-i}^-\}_{i\in\mathbb{Z}_{\geq 0}}$ act by diagonalizable operators with simple joint spectrum.

The elements $\psi_0^{\pm}\in\mathcal{E}$ are central and invertible. For $x,y\in\mathbb{C}^{\times}$, we say that an $\mathcal{E}$-module V is of *level* (x,y) if ψ_0^+ acts by x and ψ_0^- acts by y.

Let $\phi_{\emptyset}^{\pm}(z)\in\mathbb{C}[[z^{\mp 1}]]$ be a formal series in $z^{\mp 1}$. We call an $\mathcal{E}$-module V *highest weight module with highest vector $v\in V$ and highest weight $\phi_{\emptyset}^{\pm}(z)$* if v generates V and

$$f(z)v=0, \qquad \psi^+(z)v=\phi_{\emptyset}^+(z)v, \qquad \psi^-(z)v=\phi_{\emptyset}^-(z)v.$$

The comultiplication rule (see [FFJ+], [DI]) is given by

$$\Delta e(z)=e(z)\otimes 1+\psi^-(z)\otimes e(z), \tag{2.7}$$

$$\Delta f(z)=f(z)\otimes \psi^+(z)+1\otimes f(z), \tag{2.8}$$

$$\Delta \psi^{\pm}(z)=\psi^{\pm}(z)\otimes \psi^{\pm}(z). \tag{2.9}$$

These formulas do not define a comultiplication in the usual sense since the right-hand sides are not elements of $\mathcal{E}\otimes\mathcal{E}$ as they contain infinite sums. However, under some conditions the comultiplication rule does give an $\mathcal{E}$-module structure on $V_1\otimes V_2$. We will discuss these conditions somewhere else. In [FFJ+] and in this paper we use the comultiplication rule as a way to obtain the formulas for the $\mathcal{E}$-action, and then we check directly that the answer is consistent with the relations in the algebra $\mathcal{E}$.

Note that this comultiplication rule is compatible with permutations of q_i.

We say that a set of complex parameters $v_1,\dots,v_n$ is *generic* if no monomial in v_i is equal to 1: $\prod_{i=1}^n v_i^{j_i}=1$ with $j_i\in\mathbb{Z}$ implies that $j_1=\cdots=j_n=0$.

Clearly, q_1,q_2 are generic if and only if q_1,q_3 are generic, and q_1,q_2 are generic if and only if q_2,q_3 are generic. If a set of parameters is generic, then any subset of that set is also generic. Throughout the paper we assume that neither q_i is a root of unity. That is, we always assume that q_1 is generic, and q_2 is generic, and q_3 is generic.

The algebra $\mathcal{E}$ is $\mathbb{Z}$-graded. The degrees of generators are given by

$$\deg e_i=1, \qquad \deg f_i=-1, \qquad \deg \psi_i^{\pm}=0. \tag{2.10}$$

In fact, $\mathcal{E}$ is $\mathbb{Z}^2$-graded (see [FFJ+, Lemma 2.4]), but we do not need the second component of the grading in this paper.

2.2. Vector representations

For $u \in \mathbb{C}^\times$, let $V(u)$ be a complex vector space spanned by the basis $|i\rangle_u$, where $i \in \mathbb{Z}$.

We also use the notation ${}_u\langle i|$ with $i \in \mathbb{Z}$ for the dual basis. When it is clear, we skip the subscript u and write simply $|i\rangle$, $\langle i|$.

LEMMA 2.1

The formulas

$$\begin{aligned}
&(1-q_1)e(z)|i\rangle = \delta(q_1^i u/z)|i+1\rangle,\\
&(q_1^{-1}-1)f(z)|i\rangle = \delta(q_1^{i-1}u/z)|i-1\rangle,\\
&\psi^+(z)|i\rangle = \frac{(1-q_1^i q_3 u/z)(1-q_1^i q_2 u/z)}{(1-q_1^i u/z)(1-q_1^{i-1}u/z)}|i\rangle,\\
&\psi^-(z)|i\rangle = \frac{(1-q_1^{-i}q_3^{-1}z/u)(1-q_1^{-i}q_2^{-1}z/u)}{(1-q_1^{-i}z/u)(1-q_1^{-i+1}z/u)}|i\rangle
\end{aligned}$$

define a structure of an irreducible tame $\mathcal{E}$-module on $V(u)$ of level $(1,1)$.

Proof

These formulas define an $\mathcal{E}$-module by [FFJ+] and [FHH+]. It is easy to check that it is tame and irreducible. □

We call the $\mathcal{E}$-module $V(u)$ the *vector representation*. Note that q_1 plays a special role in the definition of $V(u)$, while q_2 and q_3 participate symmetrically. Therefore there are two other vector representations obtained from $V(u)$ by switching roles of q_i. We do not use these other two modules in this paper.

Note that the vector representation is not a highest weight representation.

The comultiplication rule defines an $\mathcal{E}$-module structure on the tensor product of vector representations $V(u_1)\otimes\cdots\otimes V(u_n)$ for generic $q_1, q_2, u_1, \ldots, u_n$. This is again an irreducible representation of level $(1,1)$.

If these parameters are not generic, the comultiplication rule may fail to define the action of series $e(z)$ and $f(z)$. In general, some matrix coefficients are well defined and some are not. Also, some matrix coefficients which are generically nonzero become zero for special values of parameters.

The study of this phenomena is reduced to the case $n=2$, and the following simple lemma about tensor products of two vector representations of $\mathcal{E}$ describes the situation.

LEMMA 2.2

In $V(u)\otimes V(v)$, all matrix coefficients of operators $\psi^\pm(z)$, $e(z)$, and $f(z)$ are well defined except possibly for

$$\langle i|\otimes\langle j|e(z)|i\rangle\otimes|j-1\rangle, \qquad \langle i|\otimes\langle j|f(z)|i+1\rangle\otimes|j\rangle.$$

Each of these matrix coefficients is undefined if and only if

$$v/u = q_1^{i-j} \qquad or \qquad v/u = q_1^{i-j+1}$$

and is equal to zero if and only if

$$v/u = q_1^{i-j} q_2^{-1} \qquad or \qquad v/u = q_1^{i-j} q_3^{-1}.$$

Moreover, the matrix coefficients

$$\langle i+1| \otimes \langle j|e(z)|i\rangle \otimes |j\rangle, \qquad \langle i| \otimes \langle j|f(z)|i\rangle \otimes |j+1\rangle$$

are always nonzero.

Proof
The proof is straightforward. □

2.3. Fock modules

A *partition* λ is the sequence $(\lambda_s)_{s\in\mathbb{Z}_{\geq 1}}$, such that $\lambda_s \in \mathbb{Z}$ and $\lambda_s \geq \lambda_{s+1}$ for all s. Let $\mathcal{P}$ be the set of all partitions. Let $\mathcal{P}^+ \subset \mathcal{P}$ be the set of finite partitions: $\lambda \in \mathcal{P}$ is in $\mathcal{P}^+$ if and only if only finitely many λ_s are nonzero.

For $s \in \mathbb{Z}_{\geq 1}$, denote $\mathbf{1}_s = (\delta_{sm})_{m\in\mathbb{Z}_{\geq 1}}$. For example, we have $\lambda + \mathbf{1}_s = (\lambda_1, \ldots, \lambda_{s-1}, \lambda_s + 1, \lambda_{s+1}, \ldots)$.

For $u \in \mathbb{C}$, let $\mathcal{F}(u)$ be a complex vector space spanned by $|\lambda\rangle_u$, where $\lambda \in \mathcal{P}^+$.

We use the notation ${}_u\langle\lambda|$ with $\lambda \in \mathcal{P}^+$ for the dual basis. When it is clear we omit the subscript u and write simply $\langle\lambda|$ and $|\lambda\rangle$.

For a partition λ we denote by $\langle\lambda|\psi^{\pm}(z)_i|\lambda\rangle$ the eigenvalue of the series $\psi^{\pm}(z)$ on the vector $|\lambda_i - i + 1\rangle_{uq_2^{-i+1}} \in V(uq_2^{-i+1})$, that is,

$$\psi^{\pm}(z)|\lambda_i - i + 1\rangle_{uq_2^{-i+1}} = \langle\lambda|\psi^{\pm}(z)_i|\lambda\rangle|\lambda_i - i + 1\rangle_{uq_2^{-i+1}}.$$

The subscript i in $\psi^{\pm}(z)_i$ indicates the ith component λ_i of the partition λ and at the same time the shifts

$$\lambda_i \mapsto \lambda_i - i + 1, \qquad u \mapsto uq_2^{-i+1}$$

in the vector $|\lambda_i - i + 1\rangle_{uq_2^{-i+1}}$.

Similarly, we introduce the matrix coefficients $\langle\lambda + \mathbf{1}_i|e(z)_i|\lambda\rangle$ and $\langle\lambda|f(z)_i|\lambda + \mathbf{1}_i\rangle$:

$$\langle\lambda + \mathbf{1}_i|e(z)_i|\lambda\rangle = \frac{\delta(q_1^{\lambda_i} q_3^{i-1} u/z)}{1 - q_1}, \qquad \langle\lambda|f(z)_i|\lambda + \mathbf{1}_i\rangle = \frac{q_1\delta(q_1^{\lambda_i} q_3^{i-1} u/z)}{1 - q_1}.$$

In what follows we multiply these delta functions by rational functions $\langle\lambda|\psi^{\pm}(z)_j|\lambda\rangle$. We mean that they are multiplied by the values of the rational functions at the support of the delta function $F(z)\delta(v/z) = F(v)\delta(v/z)$.

Let $\emptyset$ be the empty partition, $\emptyset_i = 0$, $i \in \mathbb{Z}_{\geq 1}$. Introduce the series $\psi^{\pm}_{\emptyset}(z) \in \mathbb{C}[[z^{\pm 1}]]$ by

$$\psi^{+}_{\emptyset}(z) = \frac{1 - q_2 z}{1 - z}, \qquad \psi^{-}_{\emptyset}(z) = q_2\frac{1 - q_2^{-1} z^{-1}}{1 - z^{-1}}.$$

Define the action of $e(z)$ on $\mathcal{F}(u)$ by

$$\langle \lambda + \mathbf{1}_i | e(z) | \lambda \rangle = \langle \lambda + \mathbf{1}_i | e(z)_i | \lambda \rangle \prod_{j=1}^{i-1} \langle \lambda | \psi^-(z)_j | \lambda \rangle, \tag{2.11}$$

and set all other matrix coefficients to be zero. Define the action of $f(z)$ on $\mathcal{F}(u)$ by

$$\langle \lambda | f(z) | \lambda + \mathbf{1}_i \rangle = \langle \lambda | f(z)_i | \lambda + \mathbf{1}_i \rangle \psi^+_\emptyset (u q_3^i / z) \prod_{j=i+1}^{\infty} \frac{\langle \lambda | \psi^+(z)_j | \lambda \rangle}{\langle \emptyset | \psi^+(z)_j | \emptyset \rangle}, \tag{2.12}$$

and set all other matrix coefficients to be zero. Define the action of $\psi^\pm(z)$ on $\mathcal{F}(u)$ by

$$\langle \lambda | \psi^\pm(z) | \lambda \rangle = \psi^\pm_\emptyset (u/z) \prod_{j=1}^{\infty} \frac{\langle \lambda | \psi^\pm(z)_j | \lambda \rangle}{\langle \emptyset | \psi^\pm(z)_j | \emptyset \rangle}, \tag{2.13}$$

and set all other matrix coefficients to be zero.

Note that although the formulas are written using infinite products, each product in fact is finite since $\lambda_j = 0 = \emptyset_j$ for all but finitely many indices j.

For example, explicitly we have

$$\begin{aligned} &\psi^+(z)|\lambda\rangle = \psi_\lambda(u/z)|\lambda\rangle, \\ &\psi_\lambda(u/z) = \frac{1 - q_1^{\lambda_1 - 1} q_3^{-1} u/z}{1 - q_1^{\lambda_1} u/z} \prod_{i=1}^{\infty} \frac{(1 - q_1^{\lambda_i} q_3^i u/z)(1 - q_1^{\lambda_{i+1} - 1} q_3^{i-1} u/z)}{(1 - q_1^{\lambda_{i+1}} q_3^i u/z)(1 - q_1^{\lambda_i - 1} q_3^{i-1} u/z)}. \end{aligned} \tag{2.14}$$

LEMMA 2.3

Formulas (2.11), (2.12), and (2.13) define a structure of an irreducible tame $\mathcal{E}$-module on $\mathcal{F}(u)$ of level $(1, q_2)$. It is a highest weight module with highest vector $|\emptyset\rangle$ and highest weight $\psi^\pm_\emptyset(u/z)$.

Proof

These formulas define an $\mathcal{E}$-module by [FFJ+] (see also [FT]). It is easy to check that it is tame and irreducible (see also Theorem 3.4 below). The highest weight conditions are obvious. □

We call $\mathcal{F}(u)$ the Fock module. It is an analogue of a Fock module for $\mathfrak{gl}_\infty$. This module appeared in [FT] from geometric considerations.

2.4. The module $G^{k,r}_{\mathbf{a}}$

Assume $q_1^{1-r} q_3^{k+1} = 1$, where $r, k+1 \in \mathbb{Z}_{\geq 2}$. More precisely, we mean that $q_1^x q_3^y = 1$ if and only if $x = (1-r)\kappa$, $y = (k+1)\kappa$ for some $\kappa \in \mathbb{Z}$.

Fix a sequence of nonnegative integers $\mathbf{a} = (a_1, \dots, a_k)$ satisfying $\sum_{i=1}^k a_i = r$, and set $c_j = \sum_{i=1}^j a_i$. Define the vacuum partition $\Lambda^0 \in \mathcal{P}$ by

$$\Lambda^0_{\nu k + i + 1} = -\nu r - c_i, \qquad \text{where } \nu \in \mathbb{Z}_{\geq 0},\ i = 0, \dots, k-1.$$

We define the sets of (k,r)-admissible partitions

$\mathcal{P}^{k,r}_{\mathbf{a}} = \{\Lambda \in \mathcal{P} | \Lambda_j - \Lambda_{j+k} \geq r \quad \text{for } j \in \mathbb{Z}_{\geq 1}; \quad \Lambda_i = \Lambda^0_i \text{ for all sufficiently large } i\}.$

We recall the semiinfinite construction of $\mathcal{E}$-modules given in [FFJ+]. Let $W^{k,r}_{\mathbf{a}}(u)$ be the space spanned by $|\Lambda\rangle$, where $\Lambda \in \mathcal{P}^{k,r}_{\mathbf{a}}$.

REMARK 2.4
Note that our notation is different from the one used in [FFJ+]. The module $W^{k,r}_{\mathbf{c}}(u)$ in [FFJ+] is denoted by $W^{k,r}_{\mathbf{a}}(u)$ here, where $c_j = \sum_{i=1}^{j} a_i$. The formulas for the action are shorter in terms of c_i; on the other hand, a_i are natural parameters for many related objects appearing in this paper.

Define series $\varphi^{\pm}_{\emptyset}(z) \in \mathbb{C}[[z^{\pm 1}]]$ by

$$\varphi^{+}_{\emptyset}(z) = \frac{1 - q_3 z}{1 - z}, \qquad \varphi^{-}_{\emptyset}(z) = q_3 \frac{1 - q_3^{-1} z^{-1}}{1 - z^{-1}}.$$

Define the action of $e(z)$ on $W^{k,r}_{\mathbf{a}}(u)$ by

$$\langle \Lambda + \mathbf{1}_i | e(z) | \Lambda \rangle = \langle \Lambda + \mathbf{1}_i | e(z)_i | \Lambda \rangle \prod_{j=1}^{i-1} \langle \Lambda | \psi^{-}(z)_j | \Lambda \rangle, \tag{2.15}$$

and set all other matrix coefficients to be zero. Define the action of $f(z)$ on $W^{k,r}_{\mathbf{a}}(u)$ by

$$\begin{aligned} &\langle \Lambda | f(z) | \Lambda + \mathbf{1}_i \rangle \\ &\quad = \langle \Lambda | f(z)_i | \Lambda + \mathbf{1}_i \rangle \prod_{j=i+1}^{\infty} \frac{\langle \Lambda | \psi^{+}(z)_j | \Lambda \rangle}{\langle \Lambda^0 | \psi^{+}(z)_j | \Lambda^0 \rangle} \prod_{j=0}^{k-1} \varphi^{+}_{\emptyset}(u q_1^{\Lambda^0_{i+j+1}} q_3^{i+j}/z)), \end{aligned} \tag{2.16}$$

and set all other matrix coefficients to be zero. Define the action of $\psi^{\pm}(z)$ on $W^{k,r}_{\mathbf{a}}(u)$ by

$$\langle \Lambda | \psi^{\pm}(z) | \Lambda \rangle = \prod_{i=0}^{k-1} \varphi^{\pm}_{\emptyset}(u q_1^{-a_i} q_3^{i}/z)) \prod_{i \geq 1} \frac{\langle \Lambda | \psi^{\pm}(z)_i | \Lambda \rangle}{\langle \Lambda^0 | \psi^{\pm}(z)_i | \Lambda^0 \rangle}, \tag{2.17}$$

and set all other matrix coefficients to be zero.

LEMMA 2.5
Suppose $q_1^{1-r} q_3^{k+1} = 1$ *with* $r, k+1 \in \mathbb{Z}_{\geq 2}$. *Then formulas (2.15), (2.16), and (2.17) define a structure of an irreducible tame* $\mathcal{E}$*-module on* $W^{k,r}_{\mathbf{a}}(u)$ *of level* $(1, q_3^k)$. *It is a highest weight module with highest vector* $|\Lambda^0\rangle$ *and highest weight* $\prod_{i=0}^{k-1} \varphi^{\pm}_{\emptyset}(u q_1^{-a_i} q_3^{i}/z)$.

Proof
These formulas define an $\mathcal{E}$-module by [FFJ+]. It is easy to check that it is tame and irreducible. (It also follows from Theorem 5.3 of this paper.) The highest weight conditions are obvious. □

Let $q_1^{p'} q_3^p = 1$, and let $p = k+1$, $p' = k+r$, where as above, $r, k+1 \in \mathbb{Z}_{\geq 2}$. This is equivalent to $q_1^{1-r} q_2^{k+1} = 1$. Again, by that we mean that $q_1^x q_3^y = 1$ if and only if $x = (k+r)\kappa$, $y = (k+1)\kappa$ for some $\kappa \in \mathbb{Z}$.

Let $G_{\mathbf{a}}^{k,r}$ be the space spanned by $|\lambda\rangle$, where $\lambda \in \mathcal{P}_{\mathbf{a}}^{k,r}$. Define the action of operators $e(z), f(z), \psi^{\pm}(z)$ on $G_{\mathbf{a}}^{k,r}$ by formulas (2.15), (2.16), and (2.17), where q_2 is replaced with q_3 and q_3 is replaced with q_2.

LEMMA 2.6

Suppose $q_1^{p'} q_3^p = 1$, and suppose $p = k+1$, $p' = k+r$, where $k+1, r \in \mathbb{Z}_{\geq 2}$. Then formulas (2.15), (2.16), and (2.17), where q_2 is replaced with q_3 and q_3 is replaced with q_2, define a structure of an irreducible tame $\mathcal{E}$-module on $G_{\mathbf{a}}^{k,r}$ of level $(1, q_2^k)$. It is a highest weight module with highest vector $|\Lambda^0\rangle$ and highest weight $\prod_{i=0}^{k-1} \psi_\emptyset^{\pm}(u q_1^{-a_i} q_2^i / z))$.

Proof

The lemma follows from Lemma 2.5 and the symmetry of the algebra $\mathcal{E}$ with respect to permutations of parameters q_1, q_2, q_3. □

3. Construction of $\mathcal{E}$-modules

In this section we construct $\mathcal{E}$-modules $\mathcal{M}_{\mathbf{a},\mathbf{b}}^{p',p}(u)$ as subquotients of tensor products of Fock modules.

3.1. Generic tensor products

Consider a tensor product of n Fock modules $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$ with $n \geq 2$. In this section we assume that $q_1, q_2, u_1, \ldots, u_n$ are generic.

A basis of $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$ is given by $|\lambda^{(1)}\rangle_{u_1} \otimes \cdots \otimes |\lambda^{(n)}\rangle_{u_n}$, where $\lambda^{(i)} \in \mathcal{P}^+$ for $i = 1, \ldots, n$.

We use the following notation. We write basic vectors in $\mathcal{F}(u_i)$ with upper index i and skip the index u_i: we write simply $|\lambda^{(i)}\rangle$ instead of $|\lambda^{(i)}\rangle_{u_i}$. Moreover, we use the notation $|\lambda^{(1)}, \ldots, \lambda^{(n)}\rangle$ and $\langle \lambda^{(1)}, \ldots, \lambda^{(n)}|$ for $|\lambda^{(1)}\rangle \otimes \cdots \otimes |\lambda^{(n)}\rangle$ and $\langle \lambda^{(1)}| \otimes \cdots \otimes \langle \lambda^{(n)}|$. Sometimes we use the bold font notation $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)})$, and then we denote $\mathbf{1}_s$ in the ith place by $\mathbf{1}_s^{(i)}$:

$$\boldsymbol{\lambda} + \mathbf{1}_s^{(i)} = (\lambda^{(1)}, \ldots, \lambda^{(i)} + \mathbf{1}_s, \ldots, \lambda^{(n)}).$$

Define a $\mathbb{Z}$-grading on $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$ by (cf. (2.10))

$$\deg(|\lambda^{(1)}\rangle \otimes \cdots \otimes |\lambda^{(n)}\rangle) = \sum_{i=1}^n |\lambda^{(i)}| = \sum_{i=1}^n \sum_{s=1}^\infty \lambda_s^{(i)}.$$

LEMMA 3.1

Assume that $q_1, q_2, u_1, \ldots, u_n$ are generic. The comultiplication rule defines on $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$ a structure of an irreducible graded tame $\mathcal{E}$-module of level $(1, q_2^n)$.

The module $\mathcal{F}(u_1)\otimes\cdots\otimes\mathcal{F}(u_n)$ is a highest weight module with highest vector $|\emptyset^{(1)}\rangle\otimes\cdots\otimes|\emptyset^{(n)}\rangle$ and highest weight $\prod_{i=1}^n\psi^{\pm}_{\emptyset}(u_i/z)$.

Proof
To check that the comultiplication rule gives well-defined formulas, it is sufficient to consider the case $n=2$. In the case $n=2$, the well-definedness is obvious due to Lemma 2.2.

These formulas give a well-defined action of $\mathcal{E}$; that is, the relations in $\mathcal{E}$ are respected. Indeed, the check is reduced to the case of a tensor product of vector representation, which is done in [FFJ+, Lemma 2.5].

Let us prove the simplicity of the spectrum of $\psi(z)$. Recall the eigenvalue $\psi_\lambda(u/z)$ of $\psi^+(z)$ on $|\lambda\rangle$ in $\mathcal{F}(u)$ (see (2.14)). Assume that

$$\prod_{i=1}^n\psi_{\lambda^{(i)}}(u_i/z)=\prod_{i=1}^n\psi_{\mu^{(i)}}(u_i/z).$$

We need to show that this implies $\lambda^{(i)}=\mu^{(i)}$ for $i=1,\dots,n$.

Note that $\psi_{\lambda^{(i)}}(u_i/z)$ has a pole at $z=q_1^{\lambda_1^{(i)}}u_i$. Since $q_1,q_2,u_1,\dots,u_n$ are generic, this pole can be canceled only by the poles $z=q_1^{\mu_1^{(i)}}u_i$ or $z=q_1^{\mu_1^{(i)}-1}u_i$. The latter is impossible because in such a case $\lambda_1^{(i)}=\mu_1^{(i)}-1$, and the pole $z=q_1^{\mu_1^{(i)}}u_i$ is not canceled. Therefore we obtain $\lambda_1^{(i)}=\mu_1^{(i)}$ for $i=1,\dots,n$. Cancel the terms with $\lambda_1^{(i)}$ and $\mu_1^{(i)}$, and replace u_i with u_i/q_3. Then the same argument gives $\lambda_2^{(i)}=\mu_2^{(i)}$ for $i=1,\dots,n$. Repeating the process, we obtain $\lambda^{(i)}=\mu^{(i)}$ for $i=1,\dots,n$.

Since the representation is tame, to show that it is irreducible it is sufficient to check that the matrix coefficients $\langle\boldsymbol{\lambda}+\mathbf{1}_s^{(i)}|e(z)|\boldsymbol{\lambda}\rangle$ and $\langle\boldsymbol{\lambda}|f(z)|\boldsymbol{\lambda}+\mathbf{1}_s^{(i)}\rangle$ are nonzero for all i,s. This is reduced to the $n=2$ case where it follows from Lemma 2.2. □

3.2. Resonance in $u_i/u_{i+1},q_1,q_3$

We turn to special values of parameters. In doing so we always keep in mind that the matrix coefficients of the considered modules are rational functions of parameters, sometimes multiplied by the delta functions. When we go to special values of parameters, we just take limits of these rational functions. In particular, we first cancel factors at generic values of parameters as much as possible and then simply substitute the special values.

Consider the tensor product of n Fock modules $\mathcal{F}(u_1)\otimes\cdots\otimes\mathcal{F}(u_n)$ with $n\geq 2$, and let

$$u_i=u_{i+1}q_1^{a_i+1}q_3^{b_i+1},\quad\text{where } a_i,b_i\in\mathbb{Z}_{\geq 0}\text{ and } i=1,\dots,n-1. \tag{3.1}$$

Let $\mathbf{a}=(a_1,\dots,a_{n-1})$, $\mathbf{b}=(b_1,\dots,b_{n-1})$, $u_1=u$, and let

$$\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)=\operatorname{span}\bigl\{|\,\lambda^{(1)},\dots,\lambda^{(n)}\rangle\;\big|\;\lambda_s^{(i)}\geq\lambda_{s+b_i}^{(i+1)}-a_i,\ \text{where } s\in\mathbb{Z}_{\geq 1},\ i=1,\dots,n-1\bigr\}.$$

Note that if a_i were negative for some i, then the space $\mathcal{M}_{\mathbf{a},\mathbf{b}}$ would be trivial.

The following lemma shows that the definition of $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ is in fact a superposition of $n=2$ conditions. Note that for $1 \le i < j \le n$, we have

$$u_i = u_j q_1^{a_{ij}+1} q_3^{b_{ij}+1}, \qquad a_{ij} = \sum_{l=i}^{j-1}(a_l+1)-1, \qquad b_{ij} = \sum_{l=i}^{j-1}(b_l+1)-1.$$

LEMMA 3.2

We have $|\lambda^{(1)}, \dots, \lambda^{(n)}\rangle \in \mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ *if and only if for all* i, j, $1 \le i < j \le n$, $|\lambda^{(i)}\rangle \otimes |\lambda^{(j)}\rangle \in \mathcal{M}_{a_{ij},b_{ij}}(u_i)$.

Proof

The lemma is straightforward. □

We have an obvious inclusion of vector spaces $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u) \to \mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$. In particular, the space $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ inherits the $\mathbb{Z}$-grading.

We define the action of operators $\psi^{\pm}(z), e(z), f(z)$ on $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ by using the action of $\mathcal{E}$ on the tensor product $\mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n)$. Namely, let the matrix coefficients of operators $\psi^{\pm}(z), e(z), f(z)$ acting on $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ in the basis $|\lambda^{(1)}, \dots, \lambda^{(n)}\rangle$ be the same as the corresponding matrix coefficients for the tensor action.

PROPOSITION 3.3

Assume that q_1, q_2, u *are generic. Then the action of operators* $\psi^{\pm}(z), e(z), f(z)$ *in* $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ *is well defined and gives a structure of a graded* $\mathcal{E}$*-module.*

Proof

Consider the case $n=2$. Let $a_1 = a, b_1 = b$.

It is sufficient to perform the following checks.

(i) If $|\lambda, \mu\rangle \in \mathcal{M}_{a,b}(u)$, then the matrix coefficients $\langle \lambda, \mu | e(z) | \lambda, \mu - \mathbf{1}_s\rangle$, $\langle \lambda, \mu | f(z) | \lambda + \mathbf{1}_s, \mu\rangle$ are well defined.

(ii) If $|\lambda, \mu\rangle \in \mathcal{M}_{a,b}(u)$, then

$$|\lambda, \mu + \mathbf{1}_s\rangle \notin \mathcal{M}_{a,b}(u) \Rightarrow \langle \lambda, \mu + \mathbf{1}_s | e(z) | \lambda, \mu\rangle = 0,$$

$$|\lambda - \mathbf{1}_s, \mu\rangle \notin \mathcal{M}_{a,b}(u) \Rightarrow \langle \lambda - \mathbf{1}_s, \mu | f(z) | \lambda, \mu\rangle = 0.$$

All the checks are straightforward using Lemma 2.2.

For example, let $|\lambda\rangle \otimes |\mu\rangle, |\lambda + \mathbf{1}_s\rangle \otimes |\mu\rangle \in \mathcal{M}_{a,b}(u)$, and consider $\langle \lambda| \otimes \langle \mu | f(z) | \lambda + \mathbf{1}_s\rangle \otimes |\mu\rangle$. By Lemma 2.2 (where $u = v q_1^{a+1} q_3^{b+1}$), the poles of this matrix coefficient happen if

$$u q_2^{1-s}/(v q_2^{1-l}) = q_1^{\mu_l - l - \lambda_s + s} \qquad \text{or} \qquad u q_2^{1-s}/(v q_2^{1-l}) = q_1^{\mu_l - l - \lambda_s + s - 1}$$

for some $l \in \mathbb{Z}_{\ge 1}$, which means $q_1^{\mu_l - \lambda_s - a - 1} = q_3^{s-l+b+1}$ or $q_1^{\mu_l - \lambda_s - a - 2} = q_3^{s-l+b+1}$. Equivalently, $l = s+b+1$, and $\lambda_s = \mu_{s+b+1} - a - 1$ or $\lambda_s = \mu_{s+b+1} - a - 2$. This is impossible because

$$\lambda_s \geq \mu_{s+b} - a \geq \mu_{s+b+1} - a > \mu_{s+b+1} - a - 1 > \mu_{s+b+1} - a - 2.$$

Similarly, let $|\lambda\rangle \otimes |\mu\rangle \in \mathcal{M}_{a,b}(u)$, and let $|\lambda\rangle \otimes |\mu + \mathbf{1}_s\rangle \notin \mathcal{M}_{a,b}(u)$. Then we have $s - b \geq 1$ and $\lambda_{s-b} = \mu_s - a$. It follows that the coefficient $\langle\lambda| \otimes \langle\mu + \mathbf{1}_s|e(z)|\lambda\rangle \otimes |\mu\rangle$ vanishes by Lemma 2.2 since $vq_2^{1-s}/(uq_2^{1-(s-b)}) = q_1^{b-a}q_2 = q_1^{\lambda_{s-b}-\mu_s+b-1}q_3^{-1}$.

We omit further details.

Since all the necessary checks reduce to the case of $n = 2$ due to Lemma 3.2, the general case of Proposition 3.3 follows. □

THEOREM 3.4

Assume that q_1, q_2, and u are generic. Then $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ is an irreducible, tame, highest-weight $\mathcal{E}$-module with highest weight $\prod_{i=1}^n \psi_\emptyset^\pm(u_i/z)$.

Proof

First, let us show that the module is tame. Assume that for some $|\lambda^{(1)}, \ldots, \lambda^{(n)}\rangle$, $|\mu^{(1)}, \ldots, \mu^{(n)}\rangle \in \mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$, we have

$$\prod_{i=1}^n \psi_{\lambda^{(i)}}(u_i/z) = \prod_{i=1}^n \psi_{\mu^{(i)}}(u_i/z).$$

Recall that $\psi_{\lambda^{(i)}}(u_i/z)$ has a pole at $z = q_1^{\lambda_1^{(i)}} u_i$ (see (2.14)). We show that for $i = 1, \ldots, n$, $\lambda_1^{(i)} = \mu_1^{(i)}$ by showing that the pole $z = q_1^{\lambda_1^{(i)}} u_i$ in the left-hand side is canceled by the pole $z = q_1^{\mu_1^{(i)}} u_i$ in the right-hand side.

Suppose that the pole $z = q_1^{\lambda_1^{(i)}} u_i$ is canceled by zeros of $\psi_{\lambda^{(j)}}(u_j/z)$. There are two possible cases: $q_1^{\lambda_1^{(i)}} u_i = q_1^{\lambda_s^{(j)}} q_3^s u_j$, where $s \in \mathbb{Z}_{\geq 1}$, and $q_1^{\lambda_1^{(i)}} u_i = q_1^{\lambda_{s+1}^{(j)}-1} \times q_3^{s-1} u_j$, where $s \in \mathbb{Z}_{\geq 0}$.

Suppose $q_1^{\lambda_1^{(i)}} u_i = q_1^{\lambda_s^{(j)}} q_3^s u_j$, where $s \in \mathbb{Z}_{\geq 1}$. Since $s \geq 1$ and (3.1) holds, it implies that $j > i$ and

$$\lambda_1^{(i)} = \lambda_s^{(j)} - \sum_{l=i}^{j-1}(a_l + 1), \quad s = \sum_{l=i}^{j-1}(b_l + 1).$$

But then we have

$$\lambda_1^{(i)} \geq \lambda^{(j)}_{1+\sum_{l=i}^{j-1} b_l} - \sum_{l=i}^{j-1} a_l > \lambda^{(j)}_{\sum_{l=i}^{j-1}(b_l+1)} - \sum_{l=i}^{j-1}(a_l + 1) = \lambda_1^{(i)},$$

which is a contradiction.

Suppose $q_1^{\lambda_1^{(i)}} u_i = q_1^{\lambda_{s+1}^{(j)}-1} q_3^{s-1} u_j$, where $s \in \mathbb{Z}_{\geq 0}$. By a similar argument, we show that it is possible only if $s = 0, j = i - 1, b_{i-1} = 0$, and $\lambda_1^{(i)} = \lambda_1^{(i-1)} - a_{i-1}$.

Now suppose that the pole $z = q_1^{\lambda_1^{(i)}} u_i$ is canceled by poles of $\psi_{\mu^{(j)}}(u_j/z)$. We again have two cases. For example, suppose $q_1^{\lambda_1^{(i)}} u_i = q_1^{\mu_{s+1}^{(j)}} q_3^s u_j$ and $(i, 1) \neq$

$(j,s+1)$. Then we necessarily have $i<j$ and

$$\lambda_1^{(i)}=\mu_{s+1}^{(j)}-\sum_{l=i}^{j-1}(a_l+1),\quad s=\sum_{l=i}^{j-1}(b_l+1).$$

It implies that

$$\mu_1^{(i)}\geq\mu^{(j)}_{1+\sum_{l=i}^{j-1}b_l}-\sum_{l=i}^{j-1}a_l>\mu^{(j)}_{\sum_{l=i}^{j-1}(b_l+1)}-\sum_{l=i}^{j-1}(a_l+1)=\lambda_1^{(i)}.$$

Similarly, we obtain $\mu_1^{(i)}>\lambda_1^{(i)}$ in the other case of $q_1^{\lambda_1^{(i)}}u_i=q_1^{\mu_s^{(j)}-1}q_3^{s-1}u_j$.

We claim $\lambda_1^{(1)}=\mu_1^{(1)}$. Indeed, for $i=1$ the cancellation of the poles with zeros is impossible, and as we saw, the cancellation of poles $z=q_1^{\lambda_1^{(1)}}u_1$ and $z=q_1^{\mu_1^{(1)}}u_1$ with other poles implies both $\mu_1^{(1)}>\lambda_1^{(1)}$ and $\lambda_1^{(1)}>\mu_1^{(1)}$, which is a contradiction.

Next, we claim that $\lambda_1^{(2)}=\mu_1^{(2)}$. Indeed, since the terms with $\lambda_1^{(1)}$ and $\mu_1^{(1)}$ cancel each other, the cancellation of the poles $z=q_1^{\lambda_1^{(2)}}u_1$ and $z=q_1^{\mu_1^{(2)}}u_1$ with zeros is again impossible, and cancellation with other poles leads to a contradiction.

Repeating, we obtain $\lambda_1^{(i)}=\mu_1^{(i)}$ for $i=1,\dots,n$.

Cancel the corresponding factors, and replace u_i with $u_iq_3^{-1}$. Then, a similar argument gives $\lambda_2^{(i)}=\mu_2^{(i)}$, where $i=1,\dots,n$. Repeating the argument, we prove that the module $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ is tame.

Now, to prove that $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ is irreducible, it is sufficient to show that if vectors $|\boldsymbol{\lambda}\rangle,|\boldsymbol{\lambda}+\mathbf{1}_s^{(i)}\rangle$ are both in $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$, then $\langle\boldsymbol{\lambda}+\mathbf{1}_s^{(i)}|e(z)|\boldsymbol{\lambda}\rangle$ and $\langle\boldsymbol{\lambda}|f(z)|\boldsymbol{\lambda}+\mathbf{1}_s^{(i)}\rangle$ are nonzero. It is similar to that of Proposition 3.3. We omit further details. □

The character of $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ is given in Theorem 4.6.

The tensor action of $\mathcal{E}$ on the space $\mathcal{F}(u_1)\otimes\cdots\otimes\mathcal{F}(u_n)$ for generic u_i does not have a limit to the case (3.1) in the basis $|\lambda^{(1)},\dots,\lambda^{(n)}\rangle$. This limit exists only on $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$. However, we think of $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ as "a submodule of $\mathcal{F}(u_1)\otimes\cdots\otimes\mathcal{F}(u_n)$".

REMARK 3.5

Note that in the case of (3.1), the action of operators $\psi^{\pm}_{\pm i}$ on $\mathcal{F}(u_1)\otimes\cdots\otimes\mathcal{F}(u_n)$ is well defined; however, the joint spectrum of $\psi^{\pm}_{\pm i}$ is not simple. For example, consider the case $n=2$, $a_1=b_1=0$. Thus we consider $\mathcal{F}(u)\otimes\mathcal{F}(uq_2)$. Then the vectors $|\emptyset\rangle_u\otimes|(2,2)\rangle_{uq_2}$ and $|(1)\rangle_u\otimes|(2,1)\rangle_{uq_2}$ have the same $\psi^{\pm}_{\pm i}$-eigenvalues.

3.3. Resonance in q_1,q_3

Consider the tensor product of n Fock modules $\mathcal{F}(u_1)\otimes\cdots\otimes\mathcal{F}(u_n)$ with $n\geq 2$.

Assume (3.1), and let p',p be some integers such that

$$a_n=p'-1-\sum_{i=1}^{n-1}(a_i+1),\qquad b_n=p-1-\sum_{i=1}^{n-1}(b_i+1)$$

belong to $\mathbb{Z}_{\geq 0}$. Assume further that

$$q_1^{p'} q_3^{p} = 1, \ p \neq p'. \tag{3.2}$$

More precisely, by equality (3.2) we mean that $q_1^x q_3^y = 1$ if and only if $x = p'\kappa$, $y = p\kappa$ for some $\kappa \in \mathbb{Z}$.

We use a cyclic modulo n convention for indices and suffixes: $u_{n+1} = u_1$, $\lambda^{(0)} = \lambda^{(n)}$, and so on. Let

$$\begin{aligned}&\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u)\\ &\quad = \mathrm{span}\{|\,\lambda^{(1)},\dots,\lambda^{(n)}\rangle \mid \lambda^{(i)}_s \geq \lambda^{(i+1)}_{s+b_i} - a_i, \quad \text{where } s \in \mathbb{Z}_{\geq 1}, \ i = 1,\dots,n\}.\end{aligned} \tag{3.3}$$

The following lemma shows that the definition of $\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u)$ is in fact a superposition of $n = 2$ conditions.

LEMMA 3.6

We have $|\lambda^{(1)},\dots,\lambda^{(n)}\rangle \in \mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u)$ *if and only if for all* i, j, $1 \leq i < j \leq n$, $|\lambda^{(i)}\rangle \otimes |\lambda^{(j)}\rangle \in \mathcal{M}^{p',p}_{a_{ij},b_{ij}}(u_i)$.

Proof

The lemma is straightforward. □

We have an obvious surjective map of linear spaces: $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u) \to \mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u)$ sending $|\lambda^{(1)},\dots,\lambda^{(n)}\rangle$ to either zero or to $|\lambda^{(1)},\dots,\lambda^{(n)}\rangle$. In particular, the space $\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u)$ inherits the $\mathbb{Z}$-grading.

We define the action of operators $\psi^{\pm}(z), e(z), f(z)$ on $\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u)$ as the factorized action of $\mathcal{E}$ on $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$. Namely, let the matrix coefficients of operators $\psi^{\pm}(z), e(z), f(z)$ in the basis $|\lambda^{(1)},\dots,\lambda^{(n)}\rangle$ be the same as the corresponding matrix coefficients in $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$.

PROPOSITION 3.7

The action of operators $\psi^{\pm}(z), e(z), f(z)$ *in* $\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u)$ *is well defined and gives a structure of a graded* $\mathcal{E}$*-module.*

Proof

Consider the case $n = 2$, and set $a_1 = a, b_1 = b$.

It is sufficient to perform the following checks.

(i) If $|\lambda,\mu\rangle \in \mathcal{M}^{p',p}_{a,b}(u)$, then the matrix coefficients $\langle\lambda,\mu + \mathbf{1}_s|e(z)|\lambda,\mu\rangle$, $\langle\lambda - \mathbf{1}_s,\mu|f(z)|\lambda,\mu\rangle$ are well defined.

(ii) If $|\lambda,\mu\rangle \in \mathcal{M}^{p',p}_{a,b}(u)$, then

$$|\lambda,\mu - \mathbf{1}_s\rangle \notin \mathcal{M}^{p',p}_{a,b}(u) \Rightarrow \langle\lambda,\mu|e(z)|\lambda,\mu - \mathbf{1}_s\rangle = 0,$$

$$|\lambda + \mathbf{1}_s,\mu\rangle \notin \mathcal{M}^{p',p}_{a,b}(u) \Rightarrow \langle\lambda,\mu|f(z)|\lambda + \mathbf{1}_s,\mu\rangle = 0.$$

All the checks are straightforward using Lemma 2.2.

For example, consider $\langle\lambda| \otimes \langle\mu + \mathbf{1}_s|e(z)|\lambda\rangle \otimes |\mu\rangle$.

By Lemma 2.2, the poles of this matrix coefficient happen if $vq_2^{1-s}/(uq_2^{1-l}) = q_1^{\lambda_l - l - \mu_s + s}$ or $uq_2^{1-s}/(vq_2^{1-l}) = q_1^{\lambda_l - l - \mu_s + s - 1}$. It means $q_1^{\lambda_l - \mu_s + a + 1} q_3^{l - s + b + 1} = 1$ or $q_1^{\lambda_l - \mu_s + a} q_3^{l-s+b+1} = 1$. Equivalently, due to (3.2),

$$l = s - b - 1 + \gamma(b_1 + b_2 + 2), \qquad \lambda_l - \mu_s + a_1 + 1 = \gamma(a_1 + a_2 + 2)$$

or

$$l = s - b - 1 + \gamma(b_1 + b_2 + 2), \qquad \lambda_l - \mu_s + a_1 = \gamma(a_1 + a_2 + 2)$$

for some $\gamma \in \mathbb{Z}$.

Therefore

$$\lambda_l = \mu_{l + b_1 + 1 - \gamma(b_1 + b_2 + 2)} + \gamma(a_1 + a_2 + 2) - a_1 - 1$$

or

$$\lambda_l = \mu_{l + b_1 + 1 - \gamma(b_1 + b_2 + 2)} + \gamma(a_1 + a_2 + 2) - a_1.$$

First, let $\gamma < 0$. Then

$$\begin{aligned}\lambda_l - \mu_{l - \gamma(b_1 + b_2 + 2) + b_1 + 1} &\geq \lambda_l - \mu_{l - \gamma + b_1 + 1} + \gamma(a_1 + a_2) \geq \lambda_l - \mu_{l + b_1} + \gamma(a_1 + a_2) \\ &\geq -a_1 + \gamma(a_1 + a_2),\end{aligned}$$

and therefore the poles of the matrix coefficient do not occur.

Let $\gamma > 0$. Then

$$\begin{aligned}&\lambda_l - \mu_{l - \gamma(b_1 + b_2 + 2) + b_1 + 1)} \\ &\quad\leq \lambda_{l - (\gamma - 1)(b_1 + b_2)} - \mu_{l - \gamma(b_1 + b_2 + 2) + b_1 + 1} + (\gamma - 1)(a_1 + a_2) \\ &\quad\leq a_2 + (\gamma - 1)(a_1 + a_2),\end{aligned}$$

and again, such a pole is impossible.

Let now $\gamma = 0$. That is, $s = l + b_1 + 1$, and $\lambda_l - \mu_{l + b_1 + 1} = -a_1$ or $\lambda_l - \mu_{l + b_1 + 1} = -a_1 - 1$. Since $\lambda_l - \mu_{l + b_1 + 1} \geq \lambda_l - \mu_{l + b_1} \geq -a_1$, the second case is impossible, and in the first case we have $\mu_{l + b_1 + 1} = \mu_{l + b_1}$. Therefore our matrix coefficient was zero already for generic u, v, q_1, q_2. Note that we use $b_1 \geq 0$ here; otherwise, in the case $l = 1$ the index of $\mu_{s-1} = \mu_{l + b_1}$ is nonpositive and the coefficient does not have to be zero.

We omit further details.

The general case of Proposition 3.7 reduces to the case of $n = 2$ by Lemma 3.6.

□

THEOREM 3.8

Assume in addition that $p > n$. Then the $\mathcal{E}$-module $\mathcal{M}_{\mathbf{a},\mathbf{b}}^{p',p}(u)$ is an irreducible, tame, highest-weight $\mathcal{E}$-module with highest weight $\prod_{i=1}^{n} \psi_{\emptyset}^{\pm}(u_i/z)$.

Proof

The proof is similar to the proof of Theorem 3.4.

Assume that for some $|\lambda^{(1)},\ldots,\lambda^{(n)}\rangle, |\mu^{(1)},\ldots,\mu^{(n)}\rangle \in \mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u)$, we have

$$\prod_{i=1}^{n}\psi_{\lambda^{(i)}}(u_i/z)=\prod_{i=1}^{n}\psi_{\mu^{(i)}}(u_i/z).$$

We then show that this implies $\lambda^{(i)}=\mu^{(i)}$.

For example, let us check that the pole $z=q^{\lambda_1^{(i)}}u_i$ is not canceled by the zero $z=q_1^{\lambda_s^{(j)}}q_3^s u_j$ of $\psi_{\lambda^{(j)}}(u_j/z)$, that is, $q_1^{\lambda_1^{(i)}}u_i=q_1^{\lambda_s^{(j)}}q_3^s u_j$.

It is easy to see that $i=j$ is impossible.

Consider the case $j>i$. Then for some $\kappa\in\mathbb{Z}$ we have

$$\lambda_1^{(i)}=\lambda_s^{(j)}-\sum_{l=i}^{j-1}(a_l+1)-\kappa p',\quad s=\sum_{l=i}^{j-1}(b_l+1)+\kappa p.$$

Since $s\geq 1$, we obtain $\kappa\geq 0$. This is impossible since

$$\lambda_1^{(i)}\geq\lambda^{(i)}_{1+\kappa p}-\kappa p'\geq\lambda^{(j)}_{1+\sum_{l=i}^{j-1}b_l+\kappa p}-\sum_{l=i}^{j-1}a_l-\kappa p'>\lambda_s^{(j)}-\sum_{l=i}^{j-1}(a_l+1)-\kappa p'.$$

Here we used $a_l,b_l\geq 0$.

In the case $j<i$, we have

$$\lambda_1^{(i)}=\lambda_s^{(j)}+\sum_{l=j}^{i-1}(a_l+1)-\kappa p',\quad s=-\sum_{l=j}^{i-1}(b_l+1)+\kappa p.$$

Since $s\geq 1$, we obtain $\kappa\geq 1$. This is impossible since

$$\begin{aligned}\lambda_1^{(i)}&\geq\lambda^{(i)}_{1+(\kappa-1)p}-(\kappa-1)p'\\&>\lambda^{(j)}_{1-\sum_{l=j}^{i-1}b_l+\kappa p-n}+\sum_{l=j}^{i-1}(a_l+1)-\kappa p'\geq\lambda_s^{(j)}+\sum_{l=i}^{j-1}(a_l+1)-\kappa p'.\end{aligned}$$

The case $q_1^{\lambda_1^{(i)}}u_i=q_1^{\lambda_{s+1}^{(j)}-1}q_3^{s-1}u_j$ is again possible only if $j=i-1$, $s=0$, and $b_{i-1}=0$.

Note that since $p>n$, there exists i such that $b_{i-1}\neq 0$. In such a case the pole $z=q^{\lambda_1^{(i)}}u_i$ is not canceled with a zero.

We omit further details. □

Note that if some of b_i were negative, then the theorem would not hold.

The character of $\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u)$ is given in Theorem 4.5.

The tensor action of $\mathcal{E}$ on the space $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$ for generic q_1,q_2 does not have a limit to the case (3.2) in the basis $|\lambda^{(1)},\ldots,\lambda^{(n)}\rangle$. However, we think of $\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}(u)$ as "a quotient module of $\mathcal{M}_{\mathbf{a},\mathbf{b}}(u)$" and even as "a subquotient of $\mathcal{F}(u_1)\otimes\cdots\otimes\mathcal{F}(u_n)$".

4. Characters

All modules considered in Section 3 are graded modules with finite-dimensional graded components. Therefore we have well-defined formal characters which we study in this section.

4.1. Finitized characters and recursion

Recall that we have constructed a family of $\mathcal{E}$-modules $\mathcal{M}_{\mathbf{a},\mathbf{b}}^{p',p}$. Here p,p' are positive integers satisfying $p,p' \geq n$, $p' \neq p$, and $\mathbf{a} = (a_1,\dots,a_{n-1})$, $\mathbf{b} = (b_1,\dots,b_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$ are such that there exist $a_n, b_n \in \mathbb{Z}_{\geq 0}$ satisfying

$$\sum_{i=1}^{n}(a_i+1) = p', \qquad \sum_{i=1}^{n}(b_i+1) = p.$$

We always assume that a_n, b_n are determined from $\mathbf{a},\mathbf{b}$ as above. Throughout this section, $\mathbf{b}$ is fixed. We also assume that $p' > n$.

The module $\mathcal{M}_{\mathbf{a},\mathbf{b}}^{p',p}$ has a basis labeled by the set of n-tuples of partitions

$$(4.1)\qquad \begin{aligned} P_{\mathbf{a},\mathbf{b}}^{p',p} := \big\{(\lambda^{(1)},\dots,\lambda^{(n)}) \,\big|\, \lambda^{(j)} \in \mathcal{P}^+, \lambda_j^{(i)} \geq \lambda_{j+b_i}^{(i+1)} - a_i,\\ \text{where } i=1,\dots,n,\ j \in \mathbb{Z}_{>0}\big\}, \end{aligned}$$

where $\lambda^{(i)} = (\lambda_j^{(i)})_{j>0}$ and $\lambda^{(n+1)} = \lambda^{(1)}$. In this section, we study their characters

$$(4.2)\qquad \chi_{\mathbf{a},\mathbf{b}}^{p',p} := \sum_{(\lambda^{(1)},\dots,\lambda^{(n)}) \in P_{\mathbf{a},\mathbf{b}}^{p',p}} q^{\sum_{i=1}^{n}\sum_{j=1}^{\infty}\lambda_j^{(i)}}.$$

Our goal is to show that they coincide with the characters of modules from the $\mathcal{W}_n$-minimal series of $\mathfrak{sl}_n$-type, up to an overall factor corresponding to the presence of an extra Heisenberg algebra (see Theorem 4.5 below).

As a technical tool for studying (4.2), let us introduce a finitized version of the characters. For $\mathbf{N} \in \mathbb{Z}^n$, define the subset

$$P_{\mathbf{a},\mathbf{b}}^{p',p}[\mathbf{N}] := \big\{(\lambda^{(1)},\dots,\lambda^{(n)}) \in P_{\mathbf{a},\mathbf{b}}^{p',p} \,\big|\, \lambda_{N_i+1}^{(i)} = 0, \quad \text{where } i=1,\dots,n\big\}$$

and its character

$$\chi_{\mathbf{a},\mathbf{b}}^{p',p}[\mathbf{N}] := \sum_{(\lambda^{(1)},\dots,\lambda^{(n)}) \in P_{\mathbf{a},\mathbf{b}}^{p',p}[\mathbf{N}]} q^{\sum_{i=1}^{n}\sum_{j=1}^{\infty}\lambda_j^{(i)}}.$$

We set also

$$(4.3)\qquad \chi_{\mathbf{a},\mathbf{b}}^{p',p}[\mathbf{N}] = 0 \quad \text{if } N_i < 0 \text{ for some } i.$$

Clearly we have

$$(4.4)\qquad \chi_{\mathbf{a},\mathbf{b}}^{p',p}[\mathbf{0}] = 1.$$

In the following, we extend the suffix i for a_i by $a_{i+n} = a_i$. A similar convention is used for b_i, N_i.

PROPOSITION 4.1

The finitized characters $\chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}]$ *satisfy the following recursion relations for each* $i=1,\ldots,n$:

$$\chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}] = \chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}-\mathbf{1}_i] + q^{N_i}\chi^{p',p}_{\mathbf{a}-\mathbf{1}_{i-1}+\mathbf{1}_i,\mathbf{b}}[\mathbf{N}] \quad \text{if } N_{i+1}-N_i \le b_i \text{ and } a_{i-1} \ge 1, \tag{4.5}$$

$$\chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}] = \chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}-\mathbf{1}_i] \quad \text{if } N_i - N_{i-1} = b_{i-1}+1 \text{ and } a_{i-1}=0. \tag{4.6}$$

In the right-hand side of (4.5), $\mathbf{a}-\mathbf{1}_{i-1}+\mathbf{1}_i$ *means* $\mathbf{a}+\mathbf{1}_1$ *for* $i=1$ *and* $\mathbf{a}-\mathbf{1}_{n-1}$ *for* $i=n$.

Proof

We fix i and assume first that $N_i > 0$. Then the set $P^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}]$ is partitioned into a disjoint union of subsets $P' \sqcup P''$, where

$$P' = \{\boldsymbol{\lambda} \in P^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}] \mid \lambda^{(i)}_{N_i} = 0\}, \qquad P'' = \{\boldsymbol{\lambda} \in P^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}] \mid \lambda^{(i)}_{N_i} > 0\}.$$

By the definition, P' coincides with $P^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}-\mathbf{1}_i]$.

Suppose $N_{i+1}-N_i \le b_i$ and $a_{i-1} \ge 1$. For $\boldsymbol{\lambda} \in P''$, the conditions involving $\lambda^{(i)}$ read

$$\lambda^{(i-1)}_j \ge \lambda^{(i)}_{j+b_{i-1}} - 1 - (a_{i-1}-1),$$
$$\lambda^{(i)}_j \ge \lambda^{(i+1)}_{j+b_i} - a_i.$$

Since $N_i+1+b_i > N_{i+1}$ and $a_i \ge 0$, the second condition is void if $j > N_i$. Hence it can be replaced by

$$\lambda^{(i)}_j - \theta(j \le N_i) \ge \lambda^{(i+1)}_{j+b_i} - (a_i+1),$$

where $\theta(P)=1$ if the statement P is true and $\theta(P)=0$ otherwise. This gives rise to a bijection $P'' \to P^{p',p}_{\mathbf{a}-\mathbf{1}_{i-1}+\mathbf{1}_i,\mathbf{b}}[\mathbf{N}]$ sending $\boldsymbol{\lambda}$ to $\tilde{\boldsymbol{\lambda}}$ with $\tilde{\lambda}^{(i')}_j = \lambda^{(i')}_j - \delta_{i',i}\theta(j \le N_i)$. The recursion (4.5) follows from this. When $N_i=0$, the same consideration applies to show that $P^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}]$ is in bijective correspondence with $P^{p',p}_{\mathbf{a}-\mathbf{1}_{i-1}+\mathbf{1}_i,\mathbf{b}}[\mathbf{N}]$.

Next, suppose $N_i - N_{i-1} = b_{i-1}+1$ and $a_{i-1}=0$. (In this case necessarily $N_i > 0$.) The condition $\lambda^{(i-1)}_{N_{i-1}+1} \ge \lambda^{(i)}_{N_i} - a_{i-1}$ implies $\lambda^{(i)}_{N_i}=0$, so that $P''=\emptyset$. Hence (4.6) holds true. □

In general, the recursions (4.5) and (4.6) are not enough to determine the characters $\chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}]$ completely. Nevertheless, they are in certain region of the parameters $\mathbf{a},\mathbf{b},\mathbf{N}$, as the following proposition shows.

PROPOSITION 4.2

The set

$$\{\chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}] \mid N_i, a_i \in \mathbb{Z}_{\ge 0}, N_{i+1}-N_i \le b_i+1, \quad \text{where } i=1,\ldots,n\}$$

is uniquely determined by the recursion relations (4.5) and (4.6), along with the initial condition (4.4) and the boundary condition (4.3).

Proof
The proof is by induction on $d=\sum_{i=1}^n N_i$. When $d=0$, there is nothing to show. Suppose $d>0$, and assume that the assertion is true for $\sum_{i=1}^n N_i<d$. We divide into two cases:

(a) $N_{i+1}-N_i\le b_i$ for all $i=1,\dots,n$,
(b) $N_{i+1}-N_i=b_i+1$ for some i.

Consider case (a). Since $p'-n>0$, there is an i such that $a_{i-1}>0$. Applying (4.5) successively for i, $i+1$, …, we obtain

$$\begin{aligned}\chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}] &= \chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}-\mathbf{1}_i]+q^{N_i}\chi^{p',p}_{\mathbf{a}-\mathbf{1}_{i-1}+\mathbf{1}_i,\mathbf{b}}[\mathbf{N}]\\ &= \chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}-\mathbf{1}_i]+q^{N_i}\chi^{p',p}_{\mathbf{a}-\mathbf{1}_{i-1}+\mathbf{1}_i,\mathbf{b}}[\mathbf{N}-\mathbf{1}_{i+1}]\\ &\quad + q^{N_i+N_{i+1}}\chi^{p',p}_{\mathbf{a}-\mathbf{1}_{i-1}+\mathbf{1}_{i+1},\mathbf{b}}[\mathbf{N}]\\ &= \cdots\\ &= \sum_{j=i}^{i+n-1} q^{N_i+\cdots+N_{j-1}}\chi^{p',p}_{\mathbf{a}-\mathbf{1}_{i-1}+\mathbf{1}_{j-1},\mathbf{b}}[\mathbf{N}-\mathbf{1}_j]\\ &\quad + q^{N_1+\cdots+N_n}\chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}].\end{aligned}$$

Since $N_1+\cdots+N_n=d>0$, $\chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}]$ is determined in terms of those with $\sum_{i=1}^n N_i<d$.

Next, consider case (b). Since $\sum_{i=1}^n(b_i+1)>0$, we cannot have the equality $N_{i+1}-N_i=b_i+1$ for all i. Choose an i such that $N_i-N_{i-1}=b_{i-1}+1$ and $N_{i+1}-N_i\le b_i$. If $a_{i-1}=0$, then (4.6) implies $\chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}]=\chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}-\mathbf{1}_i]$ and we are done. Otherwise, (4.5) is applicable. Repeating it a_{i-1} times we obtain

$$\begin{aligned}\chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}] &= \sum_{j=1}^{a_{i-1}} q^{(j-1)N_i}\chi^{p',p}_{\mathbf{a}-(j-1)(\mathbf{1}_{i-1}-\mathbf{1}_i),\mathbf{b}}[\mathbf{N}-j\mathbf{1}_i]\\ &\quad + q^{a_{i-1}N_i}\chi^{p',p}_{\mathbf{a}-a_{i-1}(\mathbf{1}_{i-1}-\mathbf{1}_i),\mathbf{b}}[\mathbf{N}].\end{aligned}$$

The last term reduces to the case $a_{i-1}=0$ already discussed above. □

4.2. Bosonic formulas and comparison to $\mathcal{W}_n$ characters

Our next task is to relate $\chi^{p',p}_{\mathbf{a},\mathbf{b}}$ to the characters from the $\mathcal{W}_n$-minimal series. Let us prepare some notation concerning the affine Lie algebra $\widehat{\mathfrak{sl}}_n$. Denote the simple roots by $\alpha_0,\dots,\alpha_{n-1}$ and the fundamental weights by $\omega_0,\dots,\omega_{n-1}$. We set $\rho=\sum_{i=0}^{n-1}\omega_i$. Let $W=S_n\ltimes Q$ be the affine Weyl group of type $A^{(1)}_{n-1}$, where $Q=\bigoplus_{i=1}^{n-1}\mathbb{Z}\alpha_i$ denotes the classical root lattice. Further, let $L=\bigoplus_{i=0}^{n-1}\mathbb{Z}\omega_i$ be the weight lattice, and let $L^+_l=\{\sum_{i=0}^{n-1}c_i\omega_i\mid c_0,\dots,c_{n-1}\in\mathbb{Z}_{\ge0},\sum_{i=0}^{n-1}c_i=l\}$ be the set of dominant integral weights of level l.

The characters of the irreducible modules from the $\mathcal{W}_n$-minimal series of $\mathfrak{sl}_n$-type are parameterized by a pair of dominant integral weights $(\boldsymbol{\eta},\boldsymbol{\xi}) \in L^+_{p'-n} \times L^+_{p-n}$. Explicitly, they are given by the alternating series (see [FKW]),

$$
\begin{aligned}
\overline{\chi}^{p',p}_{\boldsymbol{\eta},\boldsymbol{\xi}} &= \sum_{w\in W} (-1)^{\ell(w)} q^{p'p/2|(w*\boldsymbol{\xi}-\boldsymbol{\xi})/p|^2+((w*\boldsymbol{\xi}-\boldsymbol{\xi})/p,p'(\boldsymbol{\xi}+\rho)-p(\boldsymbol{\eta}+\rho))} \\
&= \sum_{\sigma\in S_n} (-1)^{\ell(\sigma)} \sum_{\alpha\in Q} q^{p'p/2(\alpha,\alpha)+(p'\sigma(\boldsymbol{\xi}+\rho)-p(\boldsymbol{\eta}+\rho),\alpha)+(\boldsymbol{\xi}+\rho-\sigma(\boldsymbol{\xi}+\rho),\boldsymbol{\eta}+\rho)}.
\end{aligned}
\tag{4.7}
$$

Here $w*\boldsymbol{\xi} = w(\boldsymbol{\xi}+\rho)-\rho = \sigma(\boldsymbol{\xi}+\rho)-\rho+p\alpha$, where $w=(\sigma,\alpha)$, and $\ell(w)$ denotes the length function.

We need also their finitization. For $\mathbf{N}\in\mathbb{Z}^n_{\geq 0}$ and $\boldsymbol{\eta},\boldsymbol{\xi}\in L$, define

$$
\begin{aligned}
\overline{\chi}^{p',p}_{\boldsymbol{\eta},\boldsymbol{\xi}}[\mathbf{N}] = \sum_{w\in W} &(-1)^{\ell(w)} q^{p'p/2|(w*\boldsymbol{\xi}-\boldsymbol{\xi})/p|^2+((w*\boldsymbol{\xi}-\boldsymbol{\xi})/p,p'(\boldsymbol{\xi}+\rho)-p(\boldsymbol{\eta}+\rho))} \\
&\times (q)_{|\mathbf{N}|} \prod_{i=1}^{n} \frac{1}{(q)_{N_i-(w*\boldsymbol{\xi}-\boldsymbol{\xi},\omega_i-\omega_{i-1})}}.
\end{aligned}
$$

Here $(q)_m = \prod_{i=1}^m (1-q^i)$ for $m\in\mathbb{Z}_{\geq 0}$, $|\mathbf{N}| = \sum_{i=1}^n N_i$. We set also

$$
\frac{1}{(q)_m} = 0 \quad \text{if } m<0.
$$

We retain the modulo n convention for the indices, such as $\omega_n = \omega_0$.

PROPOSITION 4.3

(i) *For all $\boldsymbol{\xi},\boldsymbol{\eta}\in L$ and $i=1,\dots,n$, we have*

$$
\overline{\chi}^{p',p}_{\boldsymbol{\eta},\boldsymbol{\xi}}[\mathbf{N}] = q^{N_i}\overline{\chi}^{p',p}_{\boldsymbol{\eta}-\omega_{i-1}+\omega_i,\boldsymbol{\xi}}[\mathbf{N}] + (1-q^{|\mathbf{N}|})\overline{\chi}^{p',p}_{\boldsymbol{\eta},\boldsymbol{\xi}}[\mathbf{N}-\mathbf{1}_i].
$$

(ii) *If $N_{i+1} = N_i + (\boldsymbol{\xi}+\rho,\alpha_i)$ and $(\boldsymbol{\eta}+\rho,\alpha_i) = 0$ for $i=1,\dots,n$, then*

$$
\overline{\chi}^{p',p}_{\boldsymbol{\eta},\boldsymbol{\xi}}[\mathbf{N}] = 0.
$$

(iii) *If $\boldsymbol{\xi}\in L^+_{p-n}$, then*

$$
\overline{\chi}^{p',p}_{\boldsymbol{\eta},\boldsymbol{\xi}}[\mathbf{0}] = 1.
$$

Proof

The relation (i) can be verified directly, term by term. To see (ii), let σ_i be the simple reflection with respect to the root α_i. The assumption can be written as

$$
\begin{aligned}
&N_i - (w*\boldsymbol{\xi}-\boldsymbol{\xi},\omega_i-\omega_{i-1}) = N_{i+1} - \big((\sigma_i w)*\boldsymbol{\xi}-\boldsymbol{\xi},\omega_{i+1}-\omega_i\big), \\
&\sigma_i * \boldsymbol{\eta} = \boldsymbol{\eta}.
\end{aligned}
$$

Under these circumstances, the terms with w and $\sigma_i w$ cancel out in the sum pairwise.

Finally, under the assumption of (iii) and $\mathbf{N}=\mathbf{0}$, only the term with $w=\mathrm{id}$ survives. ☐

PROPOSITION 4.4

For all $\mathbf{N}, \mathbf{a}, \mathbf{b}$ *such that* $N_i, a_i, b_i \geq 0$ *and* $N_{i+1} - N_i \leq b_i + 1$ *for* $i = 1, \dots, n$, *we have the equality*

$$\chi^{p',p}_{\mathbf{a},\mathbf{b}}[\mathbf{N}] = \frac{1}{(q)_{|\mathbf{N}|}} \overline{\chi}^{p',p}_{\boldsymbol{\eta},\boldsymbol{\xi}}[\mathbf{N}],$$

$$\boldsymbol{\eta} = \sum_{i=1}^{n} a_i \omega_i, \quad \textit{where } \boldsymbol{\xi} = \sum_{i=1}^{n} b_i \omega_i.$$

We recall that $a_n = p' - n - \sum_{i=1}^{n-1} a_i$, $b_n = p - n - \sum_{i=1}^{n-1} b_i$.

Proof

This follows from Propositions 4.3 and 4.2. □

Letting $N_i \to \infty$, we arrive at the following result.

THEOREM 4.5

The character of the module $\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}$ *is given by*

$$\chi^{p',p}_{\mathbf{a},\mathbf{b}} = \frac{1}{(q)_\infty} \overline{\chi}^{p',p}_{\boldsymbol{\eta},\boldsymbol{\xi}},$$

$$\boldsymbol{\eta} = \sum_{i=1}^{n} a_i \omega_i, \quad \textit{where } \boldsymbol{\xi} = \sum_{i=1}^{n} b_i \omega_i.$$

4.3. Characters of $\mathcal{M}_{\mathbf{a},\mathbf{b}}$

The module $\mathcal{M}_{\mathbf{a},\mathbf{b}}$ has a basis labeled by the set of n-tuples of partitions

$$P_{\mathbf{a},\mathbf{b}} := \big\{ (\lambda^{(1)}, \dots, \lambda^{(n)}) \mid \lambda^{(i)} \in \mathcal{P}, \lambda^{(i)}_j \geq \lambda^{(i+1)}_{j+b_i} - a_i, \\ \text{where } i = 1, \dots, n-1,\ j \in \mathbb{Z}_{>0} \big\}.$$

Define their characters

$$\chi_{\mathbf{a},\mathbf{b}} := \sum_{(\lambda^{(1)}, \dots, \lambda^{(n)}) \in P_{\mathbf{a},\mathbf{b}}} q^{\sum_{i=1}^{n} \sum_{j=1}^{\infty} \lambda^{(i)}_j}.$$

THEOREM 4.6

We have

$$\chi_{\mathbf{a},\mathbf{b}} = \frac{1}{(q)^n_\infty} \sum_{w \in S_n} (-1)^{\ell(w)} q^{(\boldsymbol{\xi} + \rho - w(\boldsymbol{\xi} + \rho), \boldsymbol{\eta} + \rho)}.$$

Proof

Clearly, the set $P_{\mathbf{a},\mathbf{b}}$ is the limit of the set $P^{p',p}_{\mathbf{a},\mathbf{b}}$ as $p', p \to \infty$. The theorem then follows from Theorem 4.5. □

5. Isomorphisms of representations

In this section we establish several isomorphisms between representations of $\mathcal{E}$ discussed in this paper. All these isomorphisms preserve the basis described in terms of partitions. This is no wonder since the modules are tame. In general, we expect that any two highest-weight $\mathcal{E}$-modules with the same highest weight are isomorphic. We check this statement here in several cases. At the moment our proofs are strictly computational.

5.1. Permutations of factors in the tensor products of Fock spaces

In this section we assume that $q_1, q_2, u_1, \dots, u_n$ are generic.

THEOREM 5.1

Let $\sigma \in S_n$, and let $q_1, q_2, u_1, \dots, u_n$ be generic. There exist nonzero constants $b_{\boldsymbol\lambda}$, where $\boldsymbol\lambda = (\lambda^{(1)}, \dots, \lambda^{(n)}) \in \mathcal{P}^n$, such that the map

$$\iota : \mathcal{F}(u_1) \otimes \cdots \otimes \mathcal{F}(u_n) \to \mathcal{F}(u_{\sigma(1)}) \otimes \cdots \otimes \mathcal{F}(u_{\sigma(n)}),$$
$$|\lambda^{(1)}\rangle \otimes \cdots \otimes |\lambda^{(n)}\rangle \mapsto b_{\boldsymbol\lambda} |\lambda^{(\sigma(1))}\rangle \otimes \cdots \otimes |\lambda^{(\sigma(n))}\rangle$$

is an isomorphism of $\mathcal{E}$-modules.

Proof

It is sufficient to prove the theorem in the case $n = 2$.

Let $n = 2$ and $\sigma = (12)$. It is necessary and sufficient to show that there exist coefficients $b_{\boldsymbol\lambda}$ satisfying the conditions

$$b_{\boldsymbol\lambda + \mathbf{1}_s^{(i)}} \langle \boldsymbol\lambda + \mathbf{1}_s^{(i)} | e(z) | \boldsymbol\lambda \rangle = b_{\boldsymbol\lambda} \langle \boldsymbol\lambda' + \mathbf{1}_s^{(3-i)} | e(z) | \boldsymbol\lambda' \rangle,$$
$$b_{\boldsymbol\lambda} \langle \boldsymbol\lambda | f(z) | \boldsymbol\lambda + \mathbf{1}_s^{(i)} \rangle = b_{\boldsymbol\lambda + \mathbf{1}_s^{(i)}} \langle \boldsymbol\lambda' | f(z) | \boldsymbol\lambda' + \mathbf{1}_s^{(3-i)} \rangle.$$

Here $i = 1, 2$, and if $\boldsymbol\lambda = (\lambda^{(1)}, \lambda^{(2)})$, then $\boldsymbol\lambda' = (\lambda^{(2)}, \lambda^{(1)})$.

In order that these equations for $b_{\boldsymbol\lambda}$ be consistent, the following conditions are necessary and sufficient:

$$\frac{\langle \boldsymbol\lambda | f(w) | \boldsymbol\lambda + \mathbf{1}_s^{(i)} \rangle}{\langle \boldsymbol\lambda' | f(w) | \boldsymbol\lambda' + \mathbf{1}_s^{(3-i)} \rangle} = \frac{\langle \boldsymbol\lambda' + \mathbf{1}_s^{(3-i)} | e(z) | \boldsymbol\lambda' \rangle}{\langle \boldsymbol\lambda + \mathbf{1}_s^{(i)} | e(z) | \boldsymbol\lambda \rangle}, \tag{5.1}$$

$$\frac{\langle \boldsymbol\lambda + \mathbf{1}_s^{(i)} + \mathbf{1}_t^{(j)} | e(z) | \boldsymbol\lambda + \mathbf{1}_s^{(i)} \rangle}{\langle \boldsymbol\lambda' + \mathbf{1}_s^{(3-i)} + \mathbf{1}_t^{(3-j)} | e(z) | \boldsymbol\lambda' + \mathbf{1}_s^{(3-i)} \rangle} \cdot \frac{\langle \boldsymbol\lambda + \mathbf{1}_s^{(i)} | e(w) | \boldsymbol\lambda \rangle}{\langle \boldsymbol\lambda' + \mathbf{1}_s^{(3-i)} | e(w) | \boldsymbol\lambda' \rangle} \tag{5.2}$$
$$= \frac{\langle \boldsymbol\lambda + \mathbf{1}_s^{(i)} + \mathbf{1}_t^{(j)} | e(z) | \boldsymbol\lambda + \mathbf{1}_t^{(j)} \rangle}{\langle \boldsymbol\lambda' + \mathbf{1}_s^{(3-i)} + \mathbf{1}_t^{(3-j)} | e(z) | \boldsymbol\lambda' + \mathbf{1}_t^{(3-j)} \rangle} \cdot \frac{\langle \boldsymbol\lambda + \mathbf{1}_t^{(j)} | e(w) | \boldsymbol\lambda \rangle}{\langle \boldsymbol\lambda' + \mathbf{1}_t^{(3-j)} | e(w) | \boldsymbol\lambda' \rangle}.$$

The precise meaning of such equations is as follows. Suppose that $\delta_i(z) = c_i \sum_{n \in \mathbb{Z}} (u/z)^n$, where $i = 1, 2$, are delta functions with the same support multiplied by nonzero constants c_i. Then by the ratio $\delta_1(z)/\delta_2(z)$ we mean the ratio

c_1/c_2. For example, we have

$$\frac{\langle \lambda + \mathbf{1}_i | e(z)_i | \lambda \rangle}{\langle \lambda | f(z)_i | \lambda + \mathbf{1}_i \rangle} = q_1^{-1}.$$

Equations (5.1) and (5.2) are checked by a straightforward computation. □

5.2. The Z_n-symmetry of $\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}$

In this section we assume that the parameters $q_1, q_2, u_1, \ldots, u_n$ satisfy (3.1) and (3.2).

THEOREM 5.2

Let parameters $q_1, q_2, u_1, \ldots, u_n$ satisfy (3.1) and (3.2). There exist nonzero constants $c_{\boldsymbol{\lambda}}$, where $\boldsymbol{\lambda} \in P^{p',p}_{\mathbf{a},\mathbf{b}}$, such that the map

$$\iota : \mathcal{M}^{p',p}_{(a_1,\ldots,a_{n-1}),(b_1,\ldots,b_{n-1})} \to \mathcal{M}^{p',p}_{(a_2,\ldots,a_n),(b_2,\ldots,b_n)},$$
$$|\lambda^{(1)}\rangle \otimes \cdots \otimes |\lambda^{(n-1)}\rangle \otimes |\lambda^{(n)}\rangle \mapsto c_{\boldsymbol{\lambda}} |\lambda^{(2)}\rangle \otimes \cdots \otimes |\lambda^{(n)}\rangle \otimes |\lambda^{(1)})\rangle$$

is an isomorphism of $\mathcal{E}$-modules.

Proof

Clearly the set-theoretic map

$$\iota : P^{p',p}_{(a_1,\ldots,a_{n-1}),(b_1,\ldots,b_{n-1})} \to P^{p',p}_{(a_2,\ldots,a_n),(b_2,\ldots,b_n)},$$
$$(\lambda^{(1)}, \ldots, \lambda^{(n-1)}, \lambda^{(n)}) \mapsto (\lambda^{(2)}, \ldots, \lambda^{(n)}, \lambda^{(1)})$$

is a bijection.

Then the equations on the constants $c_{\boldsymbol{\lambda}}$ are the same as in Theorem 5.1 with $\sigma = (1, 2, \ldots, n)$. Therefore, the theorem follows. □

5.3. The modules $\mathcal{M}^{n+r,n+1}_{\mathbf{a},\mathbf{0}}$ and $G^{n,r}_{\mathbf{a}}$

In this section we assume the resonance condition

$$q_1^{n+r} q_3^{n+1} = 1. \tag{5.3}$$

We consider the special case of $\mathcal{M}^{p',p}_{\mathbf{a},\mathbf{b}}$, where

$$p' = n + r, \qquad p = n + 1, \qquad r = \sum_{i=1}^{n} a_i,$$
$$\mathbf{a} = (a_1, \ldots, a_{n-1}), \mathbf{b} = \mathbf{0} = (\underbrace{0, \ldots, 0}_{n-1}).$$

We abbreviate the representation $\mathcal{M}^{n+r,n+1}_{\mathbf{a},0}$ to $\mathcal{M}_{\bar{\mathbf{a}}}$, where $\bar{\mathbf{a}} = (a_1, \ldots, a_n)$, and the set of n-tuple partitions $P^{n+r,n+1}_{\mathbf{a},\mathbf{0}}$ to $P_{\bar{\mathbf{a}}}$:

$$P_{\bar{\mathbf{a}}} = \big\{ \boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(n)}) \mid \lambda^{(i)} \in \mathcal{P}^+, \lambda^{(i)}_j \geq \lambda^{(i+1)}_j - a_i,$$
$$\text{where } i = 1, \ldots, n,\ j \in \mathbb{Z}_{\geq 0} \big\}.$$

Similarly, we abbreviate the representation $G^{n,r}_{\mathbf{a}}$ to $G_{\bar{\mathbf{a}}}$ and the set $\mathcal{P}^{n,r}_{\mathbf{a}}$ to $\mathcal{P}_{\bar{\mathbf{a}}}$.

Our aim is to prove that two representations $\mathcal{M}_{\bar{\mathbf{a}}}$ and $G_{\bar{\mathbf{a}}}$ are isomorphic:

$$\iota : \mathcal{M}_{\bar{\mathbf{a}}} \xrightarrow{\simeq} G_{\bar{\mathbf{a}}}. \tag{5.4}$$

Recall that the vector space $G_{\bar{\mathbf{a}}}$ has a basis labeled by the set $\mathcal{P}_{\bar{\mathbf{a}}}$ of (n,r)-admissible partitions. Let ι be a bijection given by

$$\iota : P_{\bar{\mathbf{a}}} \to \mathcal{P}_{\bar{\mathbf{a}}}, \qquad \boldsymbol{\lambda} \mapsto \Lambda,$$

$$\Lambda_{ns+i} = \lambda^{(i)}_{s+1} + \Lambda^0_{ns+i} \quad \text{for } i = 1, \ldots, n,\ s \in \mathbb{Z}_{\geq 0}.$$

THEOREM 5.3

There exist nonzero constants c_Λ, $\Lambda \in \mathcal{P}_{\bar{\mathbf{a}}}$, *such that the linear map*

$$\iota : \mathcal{M}_{\bar{\mathbf{a}}} \to G_{\bar{\mathbf{a}}}, \tag{5.5}$$

$$|\boldsymbol{\lambda}\rangle \mapsto c_\Lambda |\Lambda\rangle, \qquad \Lambda = \iota(\boldsymbol{\lambda}), \tag{5.6}$$

is an isomorphism of graded $\mathcal{E}$*-modules.*

Proof

The proof is similar to the proof of Theorem 5.1. However, the checks are slightly more involved, and we give some details.

Note that the vectors $e(z)|\Lambda\rangle$ are a finite linear combination of the vectors $|\Lambda + \mathbf{1}_j\rangle$, where $\Lambda + \mathbf{1}_j = (\Lambda_1, \Lambda_2, \ldots, \Lambda_{j-1}, \Lambda_j + 1, \Lambda_{j+1}, \ldots)$, and $f(z)|\Lambda\rangle$ is a finite linear combination of the vectors $|\Lambda - \mathbf{1}_j\rangle$.

Therefore, it is necessary and sufficient to show that there exist coefficients c_Λ satisfying the conditions

$$c_{\Lambda+\mathbf{1}_{ns+i}} \langle \boldsymbol{\lambda} + \mathbf{1}^{(i)}_{s+1} | e(z) | \boldsymbol{\lambda}\rangle = c_\Lambda \langle \Lambda + \mathbf{1}_{ns+i} | e(z) | \Lambda\rangle,$$

$$c_\Lambda \langle \boldsymbol{\lambda} | f(z) | \boldsymbol{\lambda} + \mathbf{1}^{(i)}_{s+1}\rangle = c_{\Lambda+\mathbf{1}_{ns+i}} \langle \Lambda | f(z) | \Lambda + \mathbf{1}_{ns+i}\rangle.$$

Here $\langle \boldsymbol{\lambda} + \mathbf{1}^{(i)}_{s+1} | e(z) | \boldsymbol{\lambda}\rangle$ and $\langle \boldsymbol{\lambda} | f(z) | \boldsymbol{\lambda} + \mathbf{1}^{(i)}_{s+1}\rangle$ denote the matrix coefficients of $e(z)$ and $f(z)$, respectively, in the module $\mathcal{M}_{\bar{\mathbf{a}}}$, and $\langle \Lambda + \mathbf{1}_{ns+i} | e(z) | \Lambda\rangle$ and $\langle \Lambda | f(z) | \Lambda + \mathbf{1}_{ns+i}\rangle$ are those in the module $G_{\bar{\mathbf{a}}}$.

In order that these equations for c_Λ be consistent, the following conditions are necessary and sufficient:

$$\frac{\langle \boldsymbol{\lambda} | f(w) | \boldsymbol{\lambda} + \mathbf{1}^{(i)}_{s+1}\rangle}{\langle \Lambda | f(w) | \Lambda + \mathbf{1}_{ns+i}\rangle} = \frac{\langle \Lambda + \mathbf{1}_{ns+i} | e(z) | \Lambda\rangle}{\langle \boldsymbol{\lambda} + \mathbf{1}^{(i)}_{s+1} | e(z) | \boldsymbol{\lambda}\rangle}, \tag{5.7}$$

$$\begin{aligned} &\frac{\langle \boldsymbol{\lambda} + \mathbf{1}^{(i)}_{s+1} + \mathbf{1}^{(j)}_{t+1} | e(z) | \boldsymbol{\lambda} + \mathbf{1}^{(i)}_{s+1}\rangle}{\langle \Lambda + \mathbf{1}_{ns+i} + \mathbf{1}_{nt+j} | e(z) | \Lambda + 1_{ns+i}\rangle} \cdot \frac{\langle \boldsymbol{\lambda} + \mathbf{1}^{(i)}_{s+1} | e(w) | \boldsymbol{\lambda}\rangle}{\langle \Lambda + \mathbf{1}_{ns+i} | e(w) | \Lambda\rangle} \\ &\quad = \frac{\langle \boldsymbol{\lambda} + \mathbf{1}^{(i)}_{s+1} + \mathbf{1}^{(j)}_{t+1} | e(z) | \boldsymbol{\lambda} + \mathbf{1}^{(j)}_{t+1}\rangle}{\langle \Lambda + \mathbf{1}_{ns+i} + \mathbf{1}_{nt+j} | e(z) | \Lambda + \mathbf{1}_{nt+j}\rangle} \cdot \frac{\langle \boldsymbol{\lambda} + \mathbf{1}^{(j)}_{t+1} | e(w) | \boldsymbol{\lambda}\rangle}{\langle \Lambda + \mathbf{1}_{nt+j} | e(w) | \Lambda\rangle}, \end{aligned} \tag{5.8}$$

and the condition similar to (5.8) for $f(z)$. Here we follow the conventions for the ratios of delta functions with the same support as in (5.1) and (5.2).

We rewrite the action of $\mathcal{E}$ on $G_{\bar{\mathbf{a}}}$ in such a way that the matrix coefficients of $e(z)$ and $f(z)$ acting on $G_{\bar{\mathbf{a}}}$ are given in terms of certain quantities which are used in the expressions for the matrix coefficients of $e(z)$ and $f(z)$ acting on $\mathcal{M}_{\bar{\mathbf{a}}}$. This makes the proof of these equations shorter.

The vector in $G_{\bar{\mathbf{a}}}$ which corresponds to $\Lambda \in \mathcal{P}_{\bar{\mathbf{a}}}$ is given by a semiinfinite tensor product:

$$|\Lambda\rangle = |\Lambda_1\rangle_u \otimes |\Lambda_2 - 1\rangle_{uq_3^{-1}} \otimes \cdots \otimes |\Lambda_j - j + 1\rangle_{q_3^{-j+1}u} \otimes \cdots .$$

Let $\langle\Lambda|\psi^{\pm}(z)_{ns+i}|\Lambda\rangle$ be the matrix coefficient of $\psi^{\pm}(z)$ in the module $V(q_3^{-ns-i+1}u)$:

$$\psi^{\pm}(z)|\Lambda_{ns+i} - ns - i + 1\rangle_{q_3^{-ns-i+1}u} = \langle\Lambda|\psi^{\pm}(z)_{ns+i}|\Lambda\rangle|\Lambda_{ns+i} - ns - i + 1\rangle_{q_3^{-ns-i+1}u}.$$

Recall that we also write $\langle\lambda^{(i)}|\psi^{\pm}(z)_{s+1}|\lambda^{(i)}\rangle$ for the matrix coefficient of $\psi^{\pm}(z)$ in the module $V(q_2^{-s}u_i)$:

$$\psi^{\pm}(z)|\lambda^{(i)}_{s+1} - s\rangle_{q_2^{-s}u_i} = \langle\lambda^{(i)}|\psi^{\pm}(z)_{s+1}|\lambda^{(i)}\rangle|\lambda^{(i)}_{s+1} - s\rangle_{q_2^{-s}u_i},$$

where

$$u_i = q_1^{-\sum_{j=1}^{i-1}(a_j+1)} q_3^{-i+1}u = q_1^{-c_{i-1}} q_2^{i-1}u. \tag{5.9}$$

Note that we consider these quantities as rational functions in z, not as series in $z^{\pm 1}$; therefore, there is no distinction between $\psi^{\pm}(z)$.

The following is the basic equality for the proof of the isomorphism:

$$q_1^{\lambda^{(i)}_{s+1}} q_3^s u_i = q_1^{\Lambda_{ns+i}} q_2^{ns+i-1}u.$$

From this follows

$$\langle\lambda^{(i)}|\psi^{\pm}(z)_{s+1}|\lambda^{(i)}\rangle = \langle\Lambda|\psi^{\pm}(z)_{ns+i}|\Lambda\rangle.$$

Using these equalities one can write the formula for the actions of $e(z), f(z)$ in $G_{\bar{\mathbf{a}}}$ in the form

$$\begin{aligned}
&\langle\Lambda + \mathbf{1}_{ns+i}|e(z)|\Lambda\rangle \\
&\quad= \frac{\delta(q_1^{\lambda^{(i)}_{s+1}} q_3^s u_i/z)}{1-q_1} \prod_{j=1}^{i-1}\Big(\frac{1-q_2u_j/z}{1-u_j/z}\,\frac{1-q_3^{s+1}u_j/z}{1-q_2q_3^{s+1}u_j/z} \prod_{m=1}^{s+1}\frac{\langle\lambda^{(j)}|\psi^-(z)_m|\lambda^{(j)}\rangle}{\langle\emptyset^{(j)}|\psi^-(z)_m|\emptyset^{(j)}\rangle}\Big) \\
&\qquad\times \prod_{j=i}^{n}\Big(\frac{1-q_2u_j/z}{1-u_j/z}\,\frac{1-q_3^{s}u_j/z}{1-q_2q_3^{s}u_j/z} \prod_{m=1}^{s}\frac{\langle\lambda^{(j)}|\psi^-(z)_m|\lambda^{(j)}\rangle}{\langle\emptyset^{(j)}|\psi^-(z)_m|\emptyset^{(j)}\rangle}\Big), \\
&\langle\Lambda|f(z)|\Lambda + \mathbf{1}_{ns+i}\rangle \\
&\quad= \frac{q_1\delta(q_1^{\lambda^{(i)}_{s+1}} q_3^s u_i/z)}{1-q_1} \prod_{j=1}^{i}\Big(\frac{1-q_2q_3^{s+1}u_j/z}{1-q_3^{s+1}u_j/z} \prod_{m=s+2}^{\infty}\frac{\langle\lambda^{(j)}|\psi^+(z)_m|\lambda^{(j)}\rangle}{\langle\emptyset^{(j)}|\psi^+(z)_m|\emptyset^{(j)}\rangle}\Big) \\
&\qquad\times \prod_{j=i+1}^{n}\Big(\frac{1-q_2q_3^{s}u_j/z}{1-q_3^{s}u_j/z} \prod_{m=s+1}^{\infty}\frac{\langle\lambda^{(j)}|\psi^+(z)_m|\lambda^{(j)}\rangle}{\langle\emptyset^{(j)}|\psi^+(z)_m|\emptyset^{(j)}\rangle}\Big).
\end{aligned}$$

We also have the following formulas for the matrix coefficients in $\mathcal{M}_{\bar{\mathbf{a}}}$:

$$\begin{aligned}
&\langle \boldsymbol{\lambda} + \mathbf{1}^{(i)}_{s+1} | e(z) | \boldsymbol{\lambda} \rangle \\
&\quad = \frac{\delta(q_1^{\lambda^{(i)}_{s+1}} q_3^s u_i/z)}{1-q_1} \frac{1-q_2 u_i/z}{1-u_i/z} \frac{1-q_3^s u_i/z}{1-q_2 q_3^s u_i/z} \prod_{m=1}^{s} \frac{\langle \lambda^{(i)} | \psi^-(z)_m | \lambda^{(i)} \rangle}{\langle \emptyset^{(i)} | \psi^-(z)_m | \emptyset^{(i)} \rangle} \\
&\qquad \times \prod_{j=1}^{i-1} \Bigl(\frac{1-q_2 u_j/z}{1-u_j/z} \prod_{m=1}^{\infty} \frac{\langle \lambda^{(j)} | \psi^-(z)_m | \lambda^{(j)} \rangle}{\langle \emptyset^{(j)} | \psi^-(z)_m | \emptyset^{(j)} \rangle} \Bigr), \\
&\langle \boldsymbol{\lambda} | f(z) | \boldsymbol{\lambda} + \mathbf{1}^{(i)}_{s+1} \rangle \\
&\quad = \frac{q_1 \delta(q_1^{\lambda^{(i)}_{s+1}} q_3^s u_i/z)}{1-q_1} \frac{1-q_2 q_3^{s+1} u_i/z}{1-q_3^{s+1} u_i/z} \prod_{m=s+2}^{\infty} \frac{\langle \lambda^{(i)} | \psi^+(z)_m | \lambda^{(i)} \rangle}{\langle \emptyset^{(i)} | \psi^+(z)_m | \emptyset^{(i)} \rangle} \\
&\qquad \times \prod_{j=i+1}^{n} \Bigl(\frac{1-q_2 u_j/z}{1-u_j/z} \prod_{m=1}^{\infty} \frac{\langle \lambda^{(j)} | \psi^+(z)_m | \lambda^{(j)} \rangle}{\langle \emptyset^{(j)} | \psi^+(z)_m | \emptyset^{(j)} \rangle} \Bigr).
\end{aligned}$$

Here $\emptyset^{(i)}$ is used for the trivial partition $\lambda^{(i)} = (0, 0, \dots)$.

It is straightforward to check (5.7) and (5.8) by using these formulas. □

As a corollary we have a bosonic formula for the character of set of the (k,r)-admissible partitions. Let

$$\chi^{k,r}_{\mathbf{a}} = \sum_{\Lambda \in \mathcal{P}^{k,r}_{\mathbf{a}}} q^{|\Lambda - \Lambda^0|} = \sum_{\Lambda \in \mathcal{P}^{k,r}_{\mathbf{a}}} q^{\sum_{j=1}^{\infty} (\Lambda_j - \Lambda^0_j)}$$

be the character of set of the (k,r)-admissible partitions.

COROLLARY 5.4

The character of the set of the (k,r)-admissible partitions $\mathcal{P}^{k,r}_{\mathbf{a}}$ coincides with the character of the set $P^{p',p}_{\mathbf{a},\mathbf{0}}$ with $p' = k+r$, $p = k+1$, and, in particular, we have the bosonic formula

$$\chi^{k,r}_{\mathbf{a}} = \frac{1}{(q)_\infty} \bar{\chi}^{k+r,k+1}_{\boldsymbol{\eta},\mathbf{0}},$$

where $\bar{\chi}^{k+r,k+1}_{\boldsymbol{\eta},\mathbf{0}}$ is given by (4.7) and $\boldsymbol{\eta} = \sum_{i=1}^{n} a_i \omega_i (\omega_n = \omega_0)$.

There are other known formulas of the sets of (k,r)-admissible partitions for bosonic formulas (see [FJL+]; for fermionic formulas in the case $(p=3)$, see [FJ+1], [FJ+2]).

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References

[ABB+] G. E. Andrews, R. J. Baxter, D. M. Bressoud, W. H. Burge, P. J. Forrester, and G. X. Viennot, *Partitions with prescribed hook differences*, European J. Combin. **8** (1987), 341–350.

[BS] I. Burban and O. Schiffmann, *On the Hall algebra of an elliptic curve, I*, preprint, arXiv:math/0505148v2 [math.AG]

[DI] J. Ding and K. Iohara, *Generalization of Drinfeld quantum affine algebras*, Lett. Math. Phys. **41** (1997), 181–193.

[FFJ+] B. Feigin, E. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, *Quantum continuous $\mathfrak{gl}_\infty$: Semi-infinite construction of representations*, Kyoto J. Math. **51** (2011), 337–364.

[FHH+] B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi, and S. Yanagida, *A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials*, J. Math. Phys. **50** (2009), no. 095215.

[FJL+] B. Feigin, M. Jimbo, S. Loktev, T. Miwa, and E. Mukhin, *Bosonic formulas for (k,l)-admissible partitions*, Ramanujan J. **7** (2003), 485–517.

[FJMM] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, *Symmetric polynomials vanishing on the shifted diagonals and Macdonald polynomials*, Int. Math. Res. Not. **2003**, no. 18, 1015–1034.

[FJ+1] B. Feigin, M. Jimbo, T. Miwa, E. Mukhin, and Y. Takeyama, *Fermionic formulas for $(k,3)$-admissible configurations*, Publ. Res. Inst. Math. Sci. **40** (2004), 125–162.

[FJ+2] ———, *Particle content of the $(k,3)$-configurations*, Publ. Res. Inst. Math. Sci. **40** (2004), 163–220.

[FT] B. Feigin and A. Tsymbaliuk, *Heisenberg action in the equivariant K-theory of Hilbert schemes via shuffle algebra*, preprint, arXiv:0904.1679v1 [math.RT]

[FKW] E. Frenkel, V. Kac, and M. Wakimoto, *Characters and fusion rules for W-algebras via quantized Drinfel'd-Sokolov reduction*, Comm. Math. Phys. **147** (1992), 295–328.

[K] M. Kasatani, *Subrepresentations in the polynomial representation of the double affine Hecke algebra of type* GL_n *at* $t^{k+1}q^{r-1}=1$, Int. Math. Res. Not. **2005**, no. 28, 1717–1742.

[M] I. Macdonald, *Symmetric Functions and Hall Polynomials, with contributions by A. Zelevinsky*, 2nd ed., Oxford Math. Monogr., Oxford Univ. Press, New York, 1995.

[S] O. Schiffmann, *On the Hall algebra of an elliptic curve, II*, preprint, arXiv:0508553v2 [math.RT]

[ScV1] O. Schiffmann and E. Vasserot, *The elliptic Hall algebra, Cherednik Hecke algebras and Macdonald polynomials*, preprint, arXiv:0802.4001v1 [math.QA]

[ScV2] ———, *The elliptic Hall algebra and the equivariant K-theory of the Hilbert scheme of* $\mathbb{A}^2$, preprint, arXiv:0905.2555v2 [math. QA]

[SV] A. N. Sergeev and A. P. Veselov, *Generalised discriminants, deformed Calogero-Moser-Sutherland operators and super-Jack polynomials*, Adv. Math. **192** (2005), 341–375.

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