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To cite this article: Jean-Michel Coron *et al* 2009 *New J. Phys.* **11** 105034

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## Quantum control design by Lyapunov trajectory tracking for dipole and polarizability coupling

Jean-Michel Coron<sup>1</sup>, Andreea Grigoriu<sup>2</sup>, Cătălin Lefter<sup>3,4</sup>  
and Gabriel Turinici<sup>2,5</sup>

<sup>1</sup> Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, 175 rue du Chevaleret 75013 Paris, France et Institut Universitaire de France

<sup>2</sup> CEREMADE, Université Paris Dauphine, Place du Maréchal De Lattre De Tassigny, 75016, Paris, France

<sup>3</sup> Faculty of Mathematics, University 'Alexandru Ioan Cuza', Bd. Carol I Nr. 11, 700506, Iași, Romania

<sup>4</sup> Octav Mayer Institute of Mathematics, Romanian Academy, Bd. Carol I, Nr. 8, 700505 Iași, Romania

E-mail: [coron@ann.jussieu.fr](mailto:coron@ann.jussieu.fr), [grigoriu@ceremade.dauphine.fr](mailto:grigoriu@ceremade.dauphine.fr), [catalin.lefter@uaic.ro](mailto:catalin.lefter@uaic.ro) and [gabriel.turinici@dauphine.fr](mailto:gabriel.turinici@dauphine.fr)

*New Journal of Physics* **11** (2009) 105034 (22pp)

Received 18 May 2009

Published 30 October 2009

Online at <http://www.njp.org/>

doi:10.1088/1367-2630/11/10/105034

**Abstract.** In this paper we analyze the Lyapunov trajectory tracking of the Schrödinger equation for a coupling control operator containing both a linear (dipole) and a quadratic (polarizability) term. We show numerically that the contribution of the quadratic part cannot be exploited by standard trajectory tracking tools and propose two improvements: discontinuous feedback and periodic (time-dependent) feedback. For both cases we present theoretical results and support them by numerical illustrations.

<sup>5</sup> Author to whom any correspondence should be addressed.

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**1. Introduction**

In this work we consider the evolution of a quantum system with wavefunction  $\Psi(t)$  under the external influence of a laser field; the system satisfies the time-dependent Schrödinger equation (TDSE)

$$i\frac{d}{dt}\Psi(t) = H(t)\Psi(t), \quad (1)$$

with  $H(t)$  a Hermitian operator; the control is realized by selecting a convenient laser intensity  $u(t)$ . When the laser is shut off  $H(t)$  is the internal Hamiltonian of the system, denoted  $H_0$ ; when the laser is present  $H(t)$  is the sum of  $H_0$  and additional terms that describe the interaction of the system with the laser field. The first-order term is the dipole coupling [30] of the form  $u(t)H_1$ ; in the limit of small laser intensities this term may be enough to adequately describes the interaction.

However, situations exist where the dipole coupling does not have enough influence on the system to reach the control goal; the goal may become accessible only after adding a polarizability term  $u^2(t)H_2$  in the expansion of  $H(t)$  (see e.g. [13, 14] and related works); to make effective use of this term one needs higher laser intensities  $u(t)$ .

The focus of the paper is on practical procedures to find suitable control fields  $u(t)$  for the Hamiltonian  $H(t) = H_0 + u(t)H_1 + u^2(t)H_2$  by adapting feedback tracking control procedures to this setting. Here and in the following  $H_0$ ,  $H_1$  and  $H_2$  are  $n \times n$  Hermitian matrices with complex coefficients and the control is the laser intensity  $u(t) \in \mathbb{R}$ .

In what concerns the mere possibility to find a control, we recall that the controllability of the finite dimensional quantum system evolving with equation

$$i\frac{d}{dt}\Psi(t) = (H_0 + u(t)H_1 + u^2(t)H_2)\Psi(t), \quad (2)$$

can be studied via the general accessibility criteria [4, 32] based on Lie brackets; more specific results can be found in [34].

Let us consider for a moment the system with Hamiltonian  $H_0 + u(t)H_1 + v(t)H_2$ ,  $v(t)$  being a second control independent of  $u(t)$ . It can be shown [34] that this system is controllable

under the same circumstances as  $H_0 + u(t)H_1 + u^2(t)H_2$  i.e. all target states that can be reached with Hamiltonian  $H_0 + u(t)H_1 + v(t)H_2$  can also be reached by  $H_0 + u(t)H_1 + u^2(t)H_2$  (although obviously the second Hamiltonian is a particular case of the first for  $v(t) = u^2(t)$ ). This rather counter-intuitive result suggests that  $u^2(t)$  can be considered, for the purpose of theoretical controllability, as independent of  $u(t)$ ; however,  $u^2(t)$  having a particular functional dependence on  $u(t)$  will play a role at the level of the numerical procedure to find the control: in general, finding the control for  $H_0 + u(t)H_1 + u^2(t)H_2$  is more difficult than for  $H_0 + u(t)H_1 + v(t)H_2$ .

The characterization of the controllability does not provide in general a simple and efficient way for open-loop trajectory generation. Optimal control techniques (cf [23, 30] and the references herein) provide a first set of methods. A different approach consists of using feedback to generate trajectories and open-loop steering control [5, 19, 22]. More recent results can be found in [27] for decoupling techniques, in [3, 15, 17, 23, 31, 35, 36] for Lyapunov-based techniques and in [1, 7, 28] for factorizations techniques of the unitary group.

In order to study feedback control of systems with Hamiltonian  $H(t) = H_0 + u(t)H_1 + u^2(t)H_2$ , we adapt the analysis [20, 24], initially proposed for bilinear quantum systems  $H_0 + u(t)H_1$ . In the previous work, it has been shown that the success of the feedback control depends on whether there exists (nonzero) direct coupling, through  $H_1$ , between the target state and **all** other eigenstates. When  $H_1$  has the same property for  $H(t) = H_0 + u(t)H_1 + u^2(t)H_2$ , we show that same feedback formulae hold. However, we argued that the polarizability term  $u^2(t)H_2$  is added when dipole  $u(t)H_1$  is not enough to control the system; consequently the most interesting question is what happens when some of the (direct) coupling is realized by  $H_2$  instead of  $H_1$ . We show that the previous feedback formulae no longer hold and we propose two alternatives. Our method is valid to track any eigenstate trajectory of a Schrödinger equation (2) when the Hamiltonian includes a second-order coupling operator.

The order of the paper is as follows: in section 2, we introduce the main notations and the Lyapunov tracking feedback for a particular case. Section 3 contains the presentation of two types of feedback: discontinuous and time-dependent (periodic) forcing, that can be applied for all types of second-order coupling operators. Both sections present theoretical results on the stabilization through these feedbacks, illustrated by numerical simulations. Concluding remarks are presented in section 4.

## 2. Tracking feedback design

### 2.1. Dynamics and global phase

We consider an  $n$ -level quantum system evolving under equation (2). The wavefunction  $\Psi = (\Psi_j)_{j=1}^n$  is a vector in  $\mathbb{C}^n$ , verifying  $\sum_{j=1}^n |\Psi_j|^2 = 1$ , thus it lives on the unit sphere  $\mathbb{S}^{2n-1}$  of  $\mathbb{C}^n$ . Physically,  $\Psi$  and  $e^{i\theta(t)}\Psi$  describe the same physical state for any global phase  $\theta(t) \in \mathbb{R}$ , i.e.  $\Psi_1$  and  $\Psi_2$  are identified when there exists  $\theta(t) \in \mathbb{R}$  such that  $\Psi_1 = \exp(i\theta(t))\Psi_2$ . To take into account such non-trivial geometry we add a second control  $\omega$  corresponding to  $\dot{\theta}$  (see also [24]). Thus we consider the following control system

$$i \frac{d}{dt} \Psi(t) = (H_0 + u(t)H_1 + u^2(t)H_2 + \omega(t))\Psi(t), \quad (3)$$

where  $\omega \in \mathbb{R}$  is a new control playing the role of a gauge degree of freedom. We can choose it arbitrarily without changing the physical quantities attached to  $\Psi$ . With such additional fictitious

control  $\omega$ , we will assume in the sequel that the state space is  $\mathbb{S}^{2n-1}$  and the dynamic given by (3) admits two independent controls  $u$  and  $\omega$ .

## 2.2. Lyapunov control design

Take a reference trajectory  $t \mapsto (\Psi_r(t), u_r(t), \omega_r(t))$ , i.e. a smooth solution of (3):

$$i \frac{d}{dt} \Psi_r = (H_0 + u_r H_1 + u_r^2 H_2 + \omega_r) \Psi_r.$$

We introduce the following time varying function  $V(\Psi, t)$ :

$$V(\Psi, t) = \langle \Psi - \Psi_r(t) | \Psi - \Psi_r(t) \rangle = \|\Psi - \Psi_r(t)\|^2 \quad (4)$$

where  $\langle \cdot | \cdot \rangle$  denotes the Hermitian product. The function  $V$  is non-negative for all  $t > 0$  and all  $\Psi \in \mathbb{S}^{2n-1}$  and vanishes when  $\Psi = \Psi_r(t)$ . We search for feedback controls such that  $V$  is a Lyapunov function. To do that we compute the derivative of  $V$  along trajectories of (3)

$$\frac{dV}{dt} = 2(u - u_r) \text{Im}(\langle H_1 \Psi(t) | \Psi_r \rangle) + 2(u^2 - u_r^2) \text{Im}(\langle H_2 \Psi(t) | \Psi_r \rangle) + 2(\omega - \omega_r) \text{Im}(\langle \Psi(t) | \Psi_r \rangle), \quad (5)$$

where  $\text{Im}$  denotes the imaginary part.

For convenience we denote:  $I_1 = \text{Im}(\langle H_1 \Psi(t) | \Psi_r \rangle)$  and  $I_2 = \text{Im}(\langle H_2 \Psi(t) | \Psi_r \rangle)$ . Note that if, for example, one takes

$$\begin{cases} u = u_r(t) - k(I_1 + 2u_r I_2)/(1 + kI_2), \\ \omega = \omega_r(t) - c \text{Im}(\langle \Psi(t) | \Psi_r \rangle), \end{cases} \quad (6)$$

with  $k$  and  $c$  strictly positive parameters, one obtains

$$\frac{dV}{dt} = -\frac{2}{k}(u - u_r)^2 - 2c(\text{Im}(\langle \Psi(t) | \Psi_r \rangle))^2 \leq 0,$$

and thus  $V$  is non-increasing.

**Remark 2.1** In order for the denominator  $1 + kI_2$  in equation (6) to be nonzero one notes that  $|I_2| \leq |\langle H_2 \Psi(t) | \Psi_r \rangle| \leq \|H_2\|$ ; therefore  $1 + kI_2 > 0$  as soon as  $k < \frac{1}{\|H_2\|}$ . From now on, unless otherwise specified, this condition will be supposed satisfied.

Let us focus on the important case when the reference trajectory corresponds to an equilibrium:  $u_r = 0$ ,  $\omega_r = -\lambda$  and  $\Psi_r = \phi$ , where  $\phi$  is an eigenvector of  $H_0$  associated with the eigenvalue  $\lambda \in \mathbb{R}$  ( $H_0 \phi = \lambda \phi$ ,  $\|\phi\| = 1$ ). We obtain

$$I_1 = \text{Im}(\langle H_1 \Psi(t) | \phi \rangle), \quad I_2 = \text{Im}(\langle H_2 \Psi(t) | \phi \rangle). \quad (7)$$

Then (6) becomes a static-state feedback

$$\begin{cases} u = -kI_1/(1 + kI_2), \\ \omega = -\lambda - c \text{Im}(\langle \Psi(t) | \phi \rangle), \end{cases} \quad (8)$$

and the Lyapunov function  $V = V(\Psi)$ .

Denote by  $\lambda_j$ , with  $j = 1, \dots, n$  the eigenvalues of the matrix  $H_0$ . Let  $\phi_1, \dots, \phi_n$  be an orthogonal system of corresponding eigenvectors.

We say that  $H_0$  has non-degenerate spectrum if  $\lambda_j \neq \lambda_l$  for all  $j \neq l$ ,  $j, l = 1, \dots, n$ .

Although  $V$  being non-increasing is a very important property, this is not enough to ensure that the target state  $\phi$  is asymptotically reached. The following theoretical result for the feedback (8) explains when convergence to target state holds:

**Theorem 2.1** Consider (3) with  $\Psi \in \mathbb{S}^{2n-1}$  and an eigenstate  $\phi \in \mathbb{S}^{2n-1}$  of  $H_0$  associated with the eigenvalue  $\lambda$ . Take the static feedback (8) with  $c > 0$ ,  $k < \frac{1}{\|H_2\|}$  and suppose that the spectrum of  $H_0$  is non-degenerate. Then the two following propositions are true:

- (i) The limit set of the closed loop system (3) is in the intersection of  $\mathbb{S}^{2n-1}$  with the vector space  $E = \mathbb{R}\phi \cup_{\alpha} \mathbb{C}\phi_{\alpha}$ , where  $\phi_{\alpha}$  is any eigenvector of  $H_0$  not co-linear to  $\phi$  such that  $\langle \phi_{\alpha} | H_1 \phi \rangle = 0$ .
- (ii) If  $E = \mathbb{R}\phi$ , the limit set is a subset of  $\{\phi, -\phi\}$ .

The proof follows the same ideas as in [24] and it can be found in [16].

**Remark 2.2** The theorem above shows that tracking to  $\phi$  works when all eigenstates of  $H_0$ ,  $\phi_2, \dots, \phi_n$ , other than  $\phi$  are coupled to  $\phi$  by  $H_1$ , i.e.  $\langle \phi_k, H_1 \phi \rangle \neq 0$ ,  $k = 2, \dots, n$ . However, we do not know what happens when some of the coupling are realized by  $H_2$  instead (the theorem does not apply but the system is still controllable cf [34]). We analyze such a case in section 3. Note that, as pointed out in the introduction, one uses the model  $H_0 + u(t)H_1 + u^2(t)H_2$  precisely in the cases when  $H_1$  coupling is not enough to control (otherwise taking low laser intensities  $u(t)$  make  $H_0 + u(t)H_1$  effective Hamiltonian instead of  $H_0 + u(t)H_1 + u^2(t)H_2$  and  $H_2$  is not longer used to model the system).

### 2.3. Examples and simulations

In order to solve (3), we use the following numerical scheme:

$$\Psi((m+1)\Delta t) = e^{-i\Delta t(H_0 + u(m\Delta t)H_1 + u^2(m\Delta t)H_2) + \omega(m\Delta t)} \Psi(m\Delta t), \quad (9)$$

where  $m$  is the index of the time step,  $\Delta t = T/M$  is the discretization time step, and  $M$  is the total number of time steps. Numerical simulations have been performed for a three-dimensional (3D) test system with  $H_0$ ,  $H_1$  and  $H_2$  given by

$$H_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (10)$$

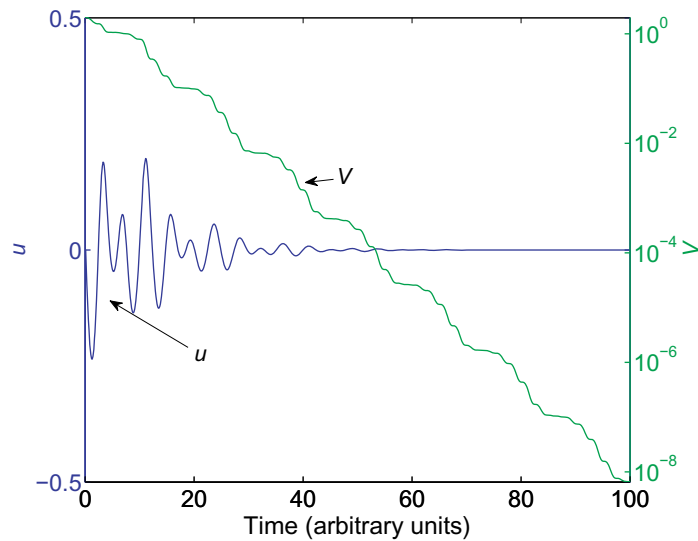
In this case the wavefunction is  $\Psi = (\Psi_1, \Psi_2, \Psi_3)^T$ . We use the Lyapunov control (8) in order to reach the first eigenstate  $\phi = (1, 0, 0)$  of energy  $\lambda = 0$ , at the final time  $T$ .

**Remark 2.3** We note that the conditions of theorem 2.1 are fulfilled since:  $\langle \phi_2 | H_1 \phi \rangle \neq 0$  and  $\langle \phi_3 | H_1 \phi \rangle \neq 0$ . In addition  $\|H_2\| = 1$ .

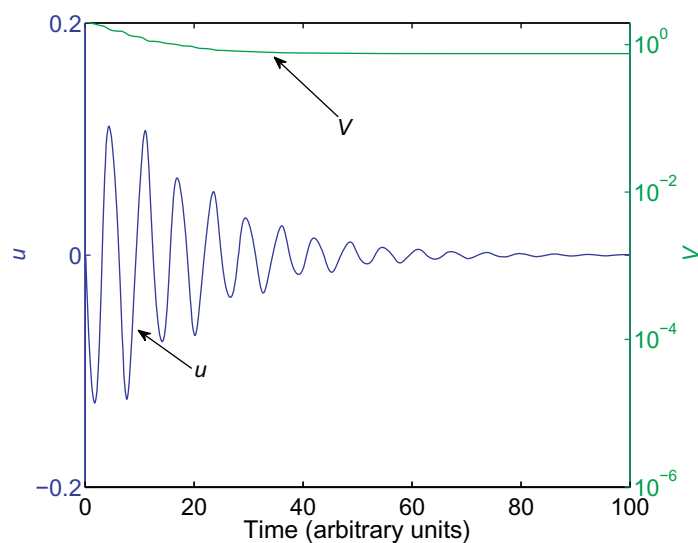
In figure 1, we plot the evolution of  $V = V(\Psi) = \langle \Psi - \phi | \Psi - \phi \rangle$  and  $u$ , corresponding to system defined by (10) with feedback (8). We can remark a fast convergence of the Lyapunov function  $V$  towards zero, that implies the convergence of  $\Psi$  towards  $\phi = (1, 0, 0)$ . The target goal is achieved with high accuracy at  $T = 100$ .

We consider another 3D test system with  $H_0$ ,  $H_1$  and  $H_2$  given by

$$H_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (11)$$



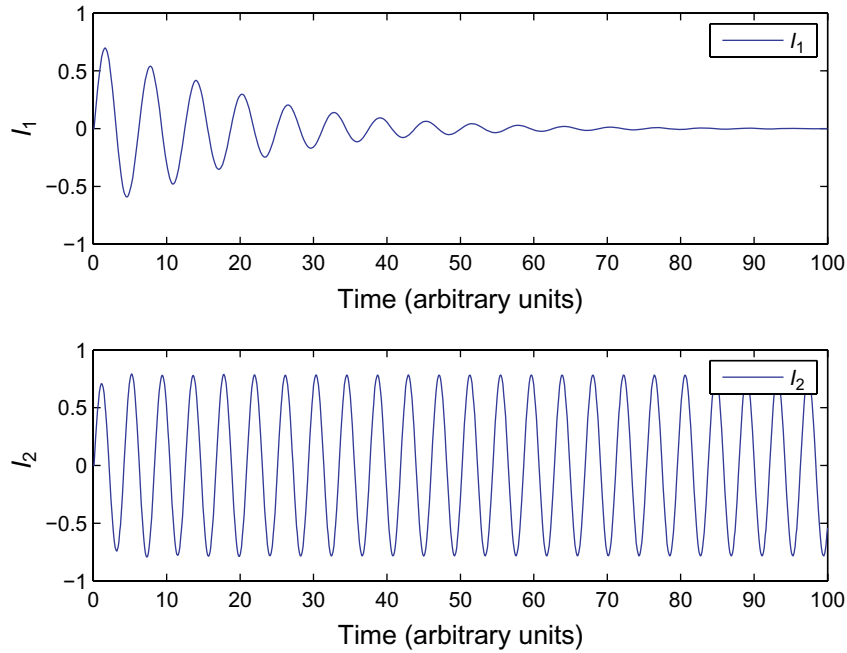
**Figure 1.** Evolution of the Lyapunov function  $V(\Psi)$  (green line) and control  $u$  (blue line); initial condition  $\Psi(t=0) = (0, 1/\sqrt{2}, 1/\sqrt{2})$ ; system defined by (10) with feedback (8). We take  $k = 0.2$ ,  $c = 0.8$ ,  $\Delta t = 0.1$ .



**Figure 2.** Evolution of the Lyapunov function  $V(\Psi)$  (green line) and control  $u$  (blue line); initial condition  $\Psi(t=0) = (0, 1/\sqrt{2}, 1/\sqrt{2})$ ; system defined by (11) with feedback (8). The feedback (8) fails to reach the target,  $V$  stalls at  $V = 10^{-0.1}$ . We take  $k = 0.2$ ,  $c = 0.8$ ,  $\Delta t = 0.1$ .

In this case the wavefunction is  $\Psi = (\Psi_1, \Psi_2, \Psi_3)^T$ . We use the previous Lyapunov control in order to reach the first eigenstate  $\phi = (1, 0, 0)$  of energy  $\lambda = 0$ , at the final time  $T$ .

Simulations in figure 2 start with  $(0, 1/\sqrt{2}, 1/\sqrt{2})$  as initial condition for  $\Psi$ . Such a feedback reduces the distance to the first state but does not ensure its convergence to  $\phi = (1, 0, 0)$  (the Lyapunov function  $V(\Psi)$  does not reach the value zero, but stalls at  $V = 10^{-0.1}$ .) This is not due to a lack of controllability. This system is controllable since the Lie algebra spanned by  $H_0/i$ ,  $H_1/i$  and  $H_2/i$  coincides with  $u(3)$  (see [24]). As explained in remark 2.2, such



**Figure 3.** Time evolution of  $I_1$  and  $I_2$ ; system defined by (11) with feedback (8). We note that  $I_1$  converges to zero. Contrary to  $I_1$ ,  $I_2$  does not converge to zero.

convergence deficiency comes from the fact that operator  $H_1$  couples  $\phi$  only with the state  $\phi_2$ . We plot the evolution of  $V(\Psi)$ ,  $u$ ,  $I_1$  and  $I_2$ , corresponding to system defined by (11) with feedback (8), in figures 2 and 3.

### 3. Discontinuous and periodic feedback

In order to stabilize the system when formula (8) are ineffective, we propose two methods. The first one is to use a special discontinuous feedback ([2, 6, 12, 26], as well as [10, section 11.4] and references therein). The second approach is through periodic time-dependent feedback ([8, 9] as well as [10, sections 11.2 and 12.4] and references therein).

#### 3.1. Discontinuous feedback

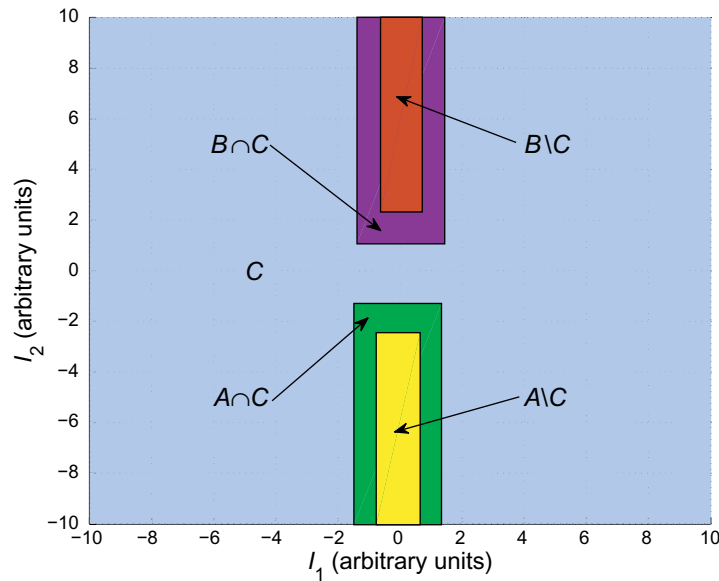
For the case of discontinuous feedback we consider the regions:  $A = \{\Psi \mid |I_1(\Psi)| < \delta \text{ and } I_2(\Psi) < -\sqrt{\delta}\}$ ,  $B = \{\Psi \mid |I_1(\Psi)| < \delta \text{ and } I_2(\Psi) > \sqrt{\delta}\}$ ,  $C = \{\Psi \mid |I_1(\Psi)| > \delta/2 \text{ or } |I_2(\Psi)| < 2\sqrt{\delta}\}$ . Note that  $A$ ,  $B$ ,  $C$  are open sets; the regions  $A$ ,  $C$ , respectively  $B$ ,  $C$  are overlapping as shown in figure 4.

For  $k_1, k_2, c, \delta > 0$  we define the control as follows:

$$u(\Psi) = u(I_1(\Psi), I_2(\Psi)) = \begin{cases} k_1 I_2, & \text{in } A \setminus C, \\ 0, & \text{in } B \setminus C, \\ -k_2 I_1 / (1 + k_2 I_2), & \text{in } C \setminus (A \cup B), \\ \text{for } A \cap C \text{ and } A \cap B \text{ see below,} \end{cases} \quad (12)$$

$$\omega = -\lambda - c \operatorname{Im}(\langle \Psi(t) | \phi \rangle).$$





**Figure 4.** The overlapping sets  $A$ ,  $B$ ,  $C$ .

The definition of  $u(I_1, I_2)$  on  $A \cap C$  is either  $u(I_1, I_2) = k_1 I_2$  (i.e. formula for region  $A$ ) or  $u(I_1, I_2) = -k_2 I_1 / (1 + k_2 I_2)$  (i.e. formula for region  $C$ ); the switching of the feedback control will take place upon reaching the boundary of  $A$  when using the feedback for  $A$  or upon reaching the boundary of  $C$  when using the feedback for  $C$ . Note that a discontinuity may appear when reaching the part of the border  $\partial C$  of  $C$  situated in the interior of  $A$  or the boundary  $\partial A$  which is a subset of the interior of  $C$ . The same considerations apply for  $A \cap B$ .

The idea is the following: in the first attempt, we tried to divide the space into disjoint regions with corresponding feedback laws assigned, in order to take into account different aspects which are not handled by choosing a feedback law defined in the whole space continuous with respect to  $I_1, I_2$ . But we encountered a problem: the system thus obtained may not have a global Carathéodory solution or the solution should be taken in a generalized sense (for example Filippov solution) and this is harder to handle. Indeed, if the regions have disjoint interiors (i.e. they may have only common boundary points) then discontinuities may appear at the boundary points which are switching points and, after switching it may happen that the solution cannot be continued in a Carathéodory sense. So, we will divide the space in overlapping regions such that the boundary of one region is contained in the interior of the neighboring regions and at most two regions overlap in a given point. Now, the switching will take place in the interior of the next region. If the system uses a feedback corresponding to a given region, it will evolve using the corresponding feedback until it touches the boundary. The boundary belongs to the interior of a neighboring region and when touching such a boundary point we switch to the next feedback law. Observe that the time spent by the system between two switching points is bounded from below by a strictly positive constant. The only fact which has to be discussed is what happens if the initial point is in the intersection of two regions. Then, the choice of the initial feedback uniquely determines the solution of the Cauchy problem, as well as the global existence.

More precisely, in our situation we may define first the propagator  $S_1(t)\Psi_0$  by solving the feedback equation such that: if the initial state  $\Psi_0 \in A \cap C$ , we initiate with the feedback corresponding to  $C$  and if the initial state  $\Psi_0 \in B \cap C$ , we initiate also with the feedback

corresponding to  $C$ . We define also the propagator  $S_2(t)\Psi_0$  by solving the feedback equation such that: if the initial state  $\Psi_0 \in A \cap C$ , we initiate with the feedback corresponding to  $A$  and if the initial state  $\Psi_0 \in B \cap C$ , we initiate with the feedback corresponding to  $B$ . We continue with the feedback for the given region until reaching the boundary of this one, then we switch to the feedback corresponding to the next overlapping region. Observe that neither  $S_1$  nor  $S_2$  defines a classical dynamical system, that is the semigroup property is lost. One has instead

$$S_1(t+s)\Psi_0 = S_1(t)S_1(s)\Psi_0 \text{ or } S_1(t+s)\Psi_0 = S_2(t)S_1(s)\Psi_0. \quad (13)$$

The propagators  $S_1(t)\Psi_0$  and  $S_2(t)\Psi_0$  are solutions in the sense of Carathéodory for the feedback controlled system and depend continuously on the initial data. These solutions are well defined locally and they are globally defined on  $[0, \infty[$ ; this follows from the fact that intervals of time between switching moments are bounded from below by a strictly positive constant.

It is now a matter of choice and we consider the feedback system corresponding to the propagator  $S_1$ . The constants  $k_1$  and  $k_2$  are to be chosen such that  $dV/dt \leq 0$  along the trajectories corresponding to  $S_1$ . If this is true the same choice is valid also in the case of  $S_2$ .

Observe first that  $dV/dt = u(I_1 + uI_2)$ . We fix now  $k_1 > 1$  and  $0 < k_2 < \frac{1}{\|H_2\|}$ . We have the following possibilities:

1. If  $\Psi(t) \in A \setminus C$  then  $u = k_1 I_2$  and  $dV/dt = k_1 I_2 (I_1 + k_1 I_2^2) \leq -k_1^2 \delta |I_2| + k_1 \delta |I_2| < 0$ .
2. If  $\Psi(t) \in B \setminus C$  then  $u = 0$  and  $dV/dt = 0$ .
3. If  $\Psi(t) \in C \setminus (A \cup B)$  then  $u = -k_2 I_1 / (1 + k_2 I_2)$  (the denominator does not vanish by the bound imposed to  $k_2$ ) so  $dV/dt = -k_2 I_1^2 / (1 + k_2 I_2)^2 \leq 0$ .
4. If  $\Psi(t) \in A \cap C$  or  $\Psi(t) \in B \cap C$  then we are in one of the previous situations depending on which feedback happens to be used at the given moment  $t$ .

Moreover, one has the following convergence result for the feedback (12).

**Theorem 3.1.** Consider (3) with  $\Psi \in \mathbb{S}^{2n-1}$  and an eigenstate  $\phi \in \mathbb{S}^{2n-1}$  of  $H_0$  associated with the eigenvalue  $\lambda$ . Take the feedback (12) with  $k_1 > 1$ ,  $k_2 < \frac{1}{\|H_2\|}$  and  $c, \delta > 0$ . If  $H_0$  is not degenerate and for every  $k$  with  $\phi_k \neq \phi$  either  $\langle \phi_k | H_1 \phi \rangle \neq 0$  or  $\langle \phi_k | H_2 \phi \rangle \neq 0$  then the limit set of  $\Psi(t)$  reduces to a solution of the uncontrolled system, with  $|I_1| < \delta$ ,  $-2\sqrt{\delta} \leq I_2 \leq C\sqrt{\delta}$  with a constant  $C$  depending only on  $H_0$ .

**Proof of theorem 3.1** Up to a shift on  $\omega$  and  $H_0$ , we can assume that  $\lambda = 0$ .

Trajectories corresponding to the propagator  $S_1$  are relatively compact so the limit points at infinity form a limit set  $\Omega_\delta$  which is compact and connected.

On the limit set  $\Omega_\delta$ ,  $V$  is constant and from the relation (13),  $\Omega_\delta$  is invariant either to  $S_1$  or to  $S_2$ , that is if  $\Psi_1 \in \Omega_\delta$  then  $S_1(t)\Psi_1 \in \Omega_\delta, t > 0$  or  $S_2(t)\Psi_1 \in \Omega_\delta, t > 0$ .

The limit set  $\Omega_\delta$  is a union of trajectories of equation (3) corresponding either to the propagator  $S_1$  or to the propagator  $S_2$ . Along these trajectories  $V$  is constant, so  $dV/dt = 0$  where  $V$  is defined by (4). The equation  $dV/dt = 0$  means that

$$u(I_1 + uI_2) = 0, \quad (14)$$

$$\text{Im}\langle \Psi, \phi \rangle = 0. \quad (15)$$

Since  $u$  is defined by (12) it follows that the limit set  $\Omega_\delta$  consists in fact of trajectories of the uncontrolled system

$$i \frac{d}{dt} \Psi = H_0 \Psi. \quad (16)$$

Indeed, a trajectory in the  $\Omega_\delta$  limit set, corresponding either to the propagator  $S_1$  or to the propagator  $S_2$ , cannot have points in  $A$  since there  $dV/dt < 0$ . So,  $\Omega_\delta \subset C \cup B$ . In  $C \setminus B$ ,  $dV/dt = 0$  implies  $I_1 = 0$  and thus  $u = 0$  while when using the feedback corresponding to  $B$ , this is always 0 and the above assertion is proved. It is clear that the solutions of (16) are of the form:

$$\Psi = \sum_{j=1}^n b_j e^{-i\lambda_j t} \phi_j. \quad (17)$$

For the same reasons as above we obtain that the limit set  $\Omega_\delta$  is characterized by

$$\Omega_\delta \subset \{I_1 = 0 \text{ and } |I_2| < 2\sqrt{\delta}\} \cup \{|I_1| < \delta \text{ and } I_2 > \sqrt{\delta}\}. \quad (18)$$

From relation (18), we have that on the limit set  $|I_1| < \delta$ . Without loss of generality we take  $\phi = \phi_1$ . We substitute (17) in (15) and we have

$$\text{Im} \langle \Psi, \phi \rangle = \text{Im}(b_1) \langle \phi, \phi \rangle + \sum_{j=2}^n \text{Im}(b_j \langle \phi_j, \phi \rangle e^{-i\lambda_j t}) = 0. \quad (19)$$

Since  $\langle \phi_j, \phi \rangle = 0$  for all  $j = 2, \dots, n$ , we obtain  $\text{Im}(b_1) = 0$ . We denote by  $J_1 = \{j | j \neq 1, \langle H_1 \phi_j | \phi \rangle \neq 0\}$  and  $J_2 = \{j | j \neq 1, \langle H_2 \phi_j | \phi \rangle \neq 0\}$ . We have the hypothesis that  $J_1 \cup J_2 = \{2, 3, \dots, n\}$ .

We substitute (17) in (7), and we obtain

$$I_1 = \text{Im}(b_1) \langle H_1 \phi, \phi \rangle + \sum_{j \in J_1} \text{Im}(b_j \langle H_1 \phi_j, \phi \rangle e^{-i\lambda_j t}), \quad (20)$$

$$I_2 = \text{Im}(b_1) \langle H_2 \phi, \phi \rangle + \sum_{j \in J_2} \text{Im}(b_j \langle H_2 \phi_j, \phi \rangle e^{-i\lambda_j t}). \quad (21)$$

Since  $\text{Im}(b_1) = 0$  we have

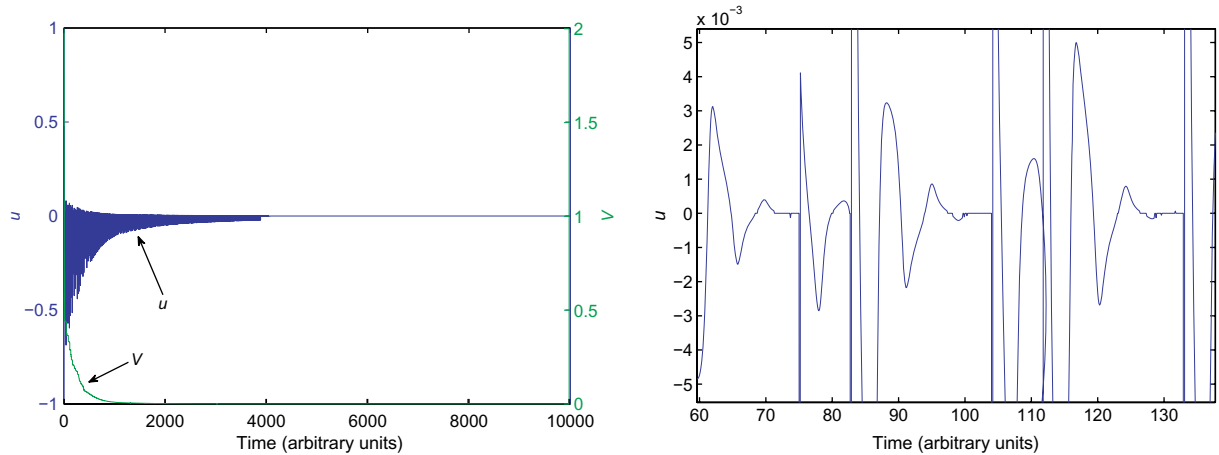
$$I_2 = \sum_{j \in J_2} \text{Im}(b_j \langle H_2 \phi_j, \phi \rangle e^{-i\lambda_j t}) = \sum_{j \in J_2} B_j \sin(\lambda_j t + \theta_j). \quad (22)$$

where the coefficients  $B_j = 0$  if and only if  $b_j = 0$ ,  $j \in J_2$ .

We define  $M = \sup(I_2)$  and  $m = \inf(I_2)$ . We claim that there exists  $\mathcal{C} > 0$  independent of  $B_j$  and  $\theta_j$  such that  $M \leq -\mathcal{C}m$ . Indeed, let  $\kappa = \text{card} J_2$ , the number of elements of  $J_2$  and  $T^\kappa = S^1 \times \dots \times S^1$ ,  $\kappa$  times, be the  $\kappa$  dimensional real torus (we denoted by  $S^k$  the  $k$  dimensional real unit sphere). Then the function  $t \rightarrow (\lambda_j t)_{j \in J_2}$  defines a conditionally periodic trajectory in  $T^\kappa$  (it is either dense in the torus or the closure of its image is a lower dimensional torus in  $T^\kappa$ ) and we denote its image by  $\mathcal{M} \subset T^\kappa$ . Choose  $B_j \in \mathbb{R}$ ,  $j \in J_2$  with  $\sum_{j \in J_2} B_j^2 = 1$ . The function

$$(B_j, \mu_j, \theta_j)_{j \in J_2} \rightarrow h(B_j, \mu_j, \theta_j) = \sum_{j \in J_2} B_j \sin(\mu_j + \theta_j)$$

is continuous on the compact set  $S^{2\kappa-1} \times \overline{\mathcal{M}} \times T^\kappa \rightarrow \mathbb{C}$  and it cannot have constant sign for any fixed  $B_j, \theta_j$ . Indeed, otherwise, denoting by  $\mu_j = \lambda_j t$ , the function  $t \rightarrow \sum_{j \in J_2} B_j \cos(\lambda_j t + \theta_j) / \lambda_j$  would be strictly monotonic and this is not possible because this is a function defined



**Figure 5.** Left panel: evolution of the Lyapunov function  $V(\Psi)$  (green line) and control  $u$  (blue line); initial condition:  $\Psi(t=0) = (0, 1/\sqrt{2}, 1/\sqrt{2})$ ; system defined by (11) with feedback (12). Right panel: zoom of the evolution of  $u$  from  $T = 60$  to 135. We take:  $k_1 = 1.1$ ,  $k_2 = c = 0.8$ ,  $\delta = 10^{-4}$ ,  $\Delta t = 0.1$ . For this example, the constant  $C$  defined by (23) is equal to 2.

along a conditionally periodic orbit in the torus  $T^\kappa$ . Now we take

$$C = -\frac{2 \sup_{(B_j, \mu_j, \theta_j) \in S^{2\kappa-1} \times \overline{M} \times T^\kappa} h(B_j, \mu_j, \theta_j)}{\sup_{(B_j, \theta_j) \in S^{2\kappa-1} \times T^\kappa} \inf_{\mu_j \in \overline{M}} h(B_j, \mu_j, \theta_j)}. \quad (23)$$

Since on the limit set  $\Omega_\delta$ ,  $I_2 \geq -2\sqrt{\delta}$  it is easy to verify that  $I_2 \leq C\sqrt{\delta}$ .  $\square$

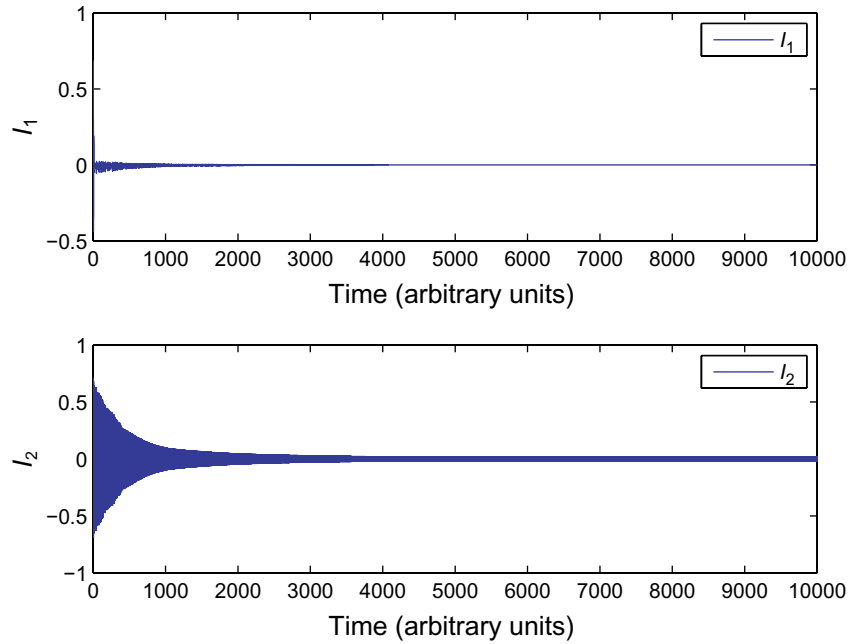
**Remark 3.1** In order to make the conclusion of the theorem more precise note that if  $\Psi_\delta$  is a trajectory of (16) belonging to  $\Omega_\delta$ , then when  $\delta$  converges to zero,  $\Psi_\delta \rightarrow \phi$ , if the initial state is different from  $-\phi$ . Accordingly, when  $I_1, I_2$  are small  $V(\Psi)$  will also be small and the system is close to the target state.

In examples (11) and (24),  $\kappa = 1$  hence  $C = 2$ .

**Remark 3.2** An important ingredient of the proof is finding the limit sets of the evolution, which itself depends very much on the choice of the sets  $A, B, C$  and of the controls  $u$ . The general rationale behind these choices is to modify formula (8) minimally in order to have good properties of  $\Omega_\delta$ .

**3.1.1. Examples for non-degenerate cases.** We take the system (11) and apply the discontinuous feedback (12). Simulations in figure 5, (left panel) describe the evolution of the Lyapunov function  $V(\Psi)$  and control  $u$ , for the initial state  $\Psi(t=0) = (0, 1/\sqrt{2}, 1/\sqrt{2})$ , the right image shows a zoom of the evolution of the control  $u$  and one can observe from the form of  $u$  the jumps between the regimes corresponding to regions  $A, C$  and  $B, C$ . The constant  $C$  defined by (23) is equal to 2. In this case:  $k_1 = 1.1$ ,  $k_2 = c = 0.8$  and  $\delta = 10^{-4}$ .

It appears that this feedback is quite efficient for system (11). We present the evolution of  $I_1$  and  $I_2$  corresponding to system defined by (11), with feedback (12), in figure 6.



**Figure 6.** Time evolution of  $I_1$  and  $I_2$ ; system defined by (11) with feedback (12).

We consider next the 5D system (see [33]) defined by

$$H_0 = \begin{pmatrix} 1.0 & 0 & 0 & 0 & 0 \\ 0 & 1.2 & 0 & 0 & 0 \\ 0 & 0 & 1.3 & 0 & 0 \\ 0 & 0 & 0 & 1.4 & 0 \\ 0 & 0 & 0 & 0 & 2.15 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (24)$$

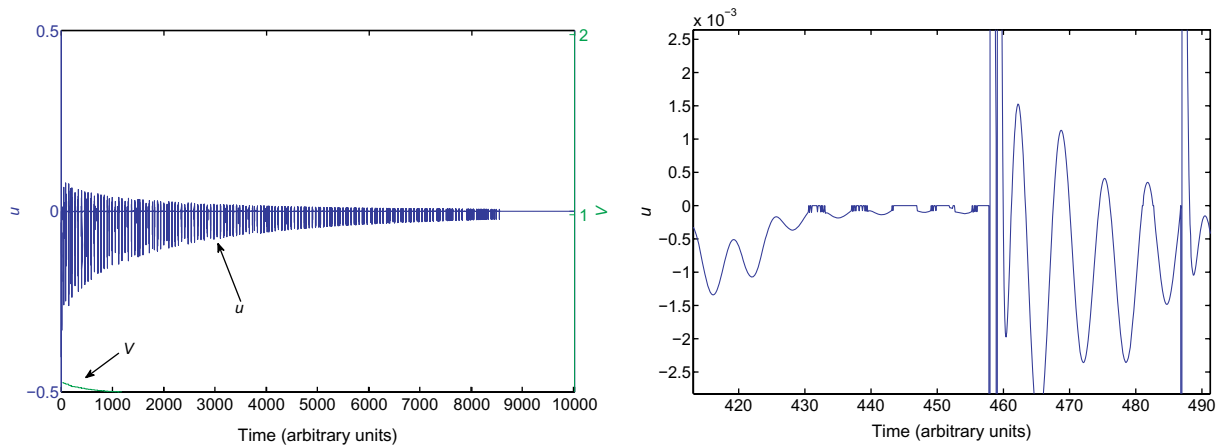
We use the previous Lyapunov control in order to reach the first eigenstate  $\phi = (1, 0, 0, 0, 0, 0)$  of energy  $\lambda = 1$ , at the final time  $T$ . Note that here  $\|H_2\| = 1$ .

Simulations in figure 7, (left panel), describe the evolution of the Lyapunov function  $V(\Psi)$  and control  $u$ , for the initial state  $\Psi(t=0) = (0, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4})$ , the right panel shows a zoom of the evolution of control  $u$ . The constant  $\mathcal{C}$  defined by (23) is equal to 2. We take  $k_1 = 1.1$ ,  $k_2 = c = 0.8$  and  $\delta = 10^{-4}$ .

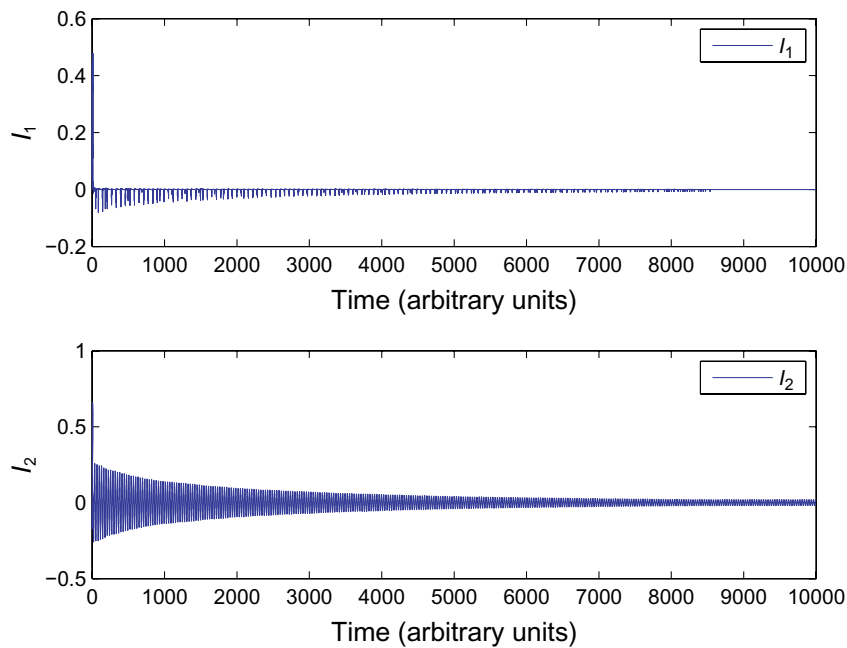
We present the evolution of  $I_1$  and  $I_2$  corresponding to system defined by (24), with feedback (12), in figure 8.

**3.1.2. Examples for degenerate cases.** There are various situations where the condition of non-degeneracy of the Hamiltonian  $H_0$ , present in theorems 2.1 and 3.1 is non-fulfilled. One such example is given below (see [18, 25]):

$$H_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.04556 & 0 & 0 \\ 0 & 0 & 0.095683 & 0 \\ 0 & 0 & 0 & 0.095683 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (25)$$



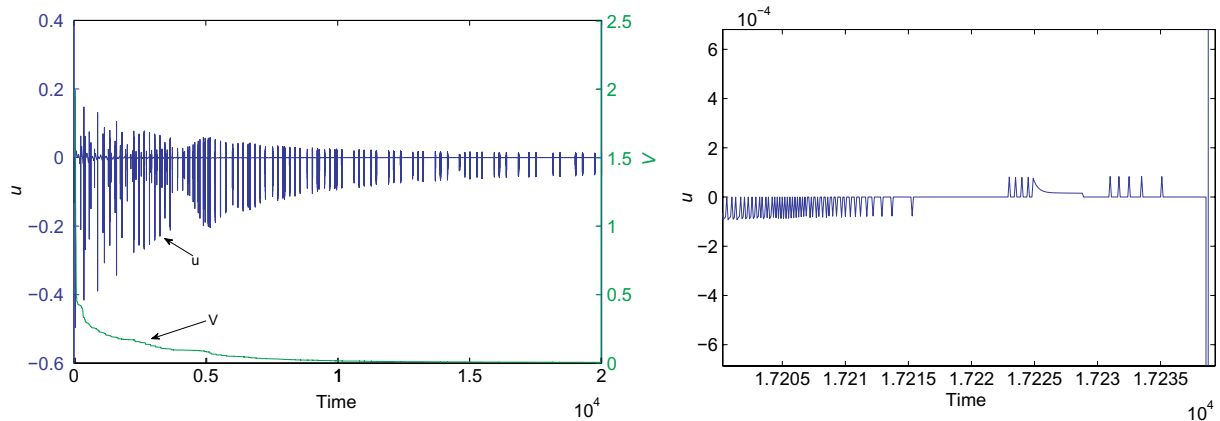
**Figure 7.** Left panel: evolution of the Lyapunov function  $V(\Psi)$  (green line) and control  $u$  (blue line); initial condition:  $\Psi(t=0) = (0, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4})$ ; system defined by (24) with feedback (12). Right panel: zoom of the evolution of  $u$  from  $T = 410$  to 490. We take  $k_1 = 1.1, k_2 = c = 0.8, \delta = 10^{-4}, \Delta t = 0.1$ . For this example, the constant  $\mathcal{C}$  defined by (23) is equal to 2.



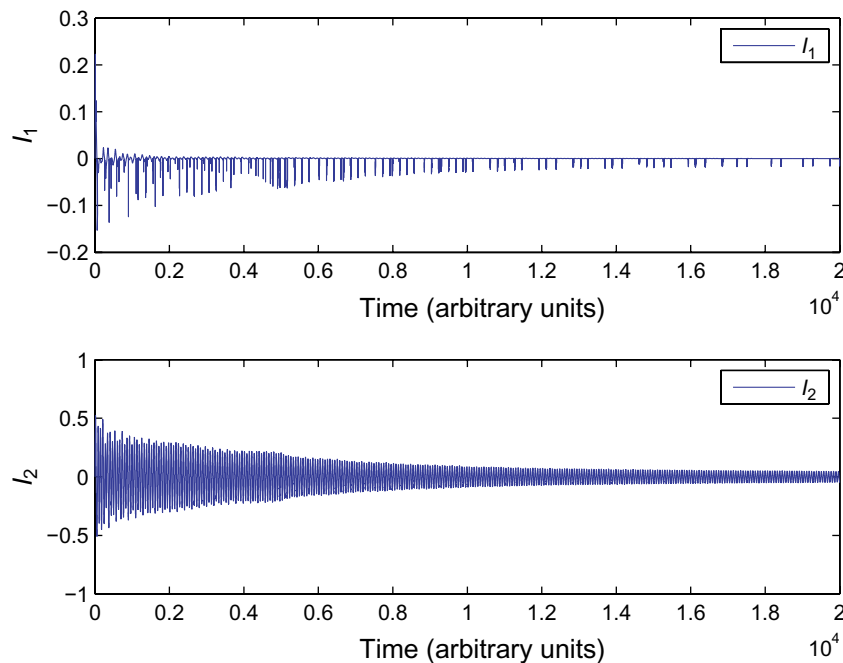
**Figure 8.** Time evolution of  $I_1$  and  $I_2$ ; system defined by (24) with feedback (12).

The internal Hamiltonian  $H_0$  is degenerate since  $\lambda_3 = \lambda_4 = 0.095683$ , but it can be stabilized using the discontinuous feedback defined by (12). Here  $\|H_2\| = 1$ .

Simulations in figure 9 (left panel) describe the evolution of the Lyapunov function  $V(\Psi)$  and control  $u$ , system defined by (25) starting from the initial state  $\Psi(t=0) = (0, 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ ; the right panel shows a zoom of the evolution of the control. We



**Figure 9.** Left panel: evolution of the Lyapunov function  $V(\Psi)$  (green line) and control  $u$  (blue line); initial condition  $\Psi(t=0) = (0, 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ ; system defined by (25) with feedback (12). Right panel: zoom of the evolution of  $u$  from  $T = 17205 \times 10^4$  to  $17235 \times 10^4$ . We take:  $k_1 = 1.1$ ,  $k_2 = c = 0.8$ ,  $\delta = 10^{-4}$  and  $\Delta t = 0.1$ .



**Figure 10.** Time evolution of  $I_1$  and  $I_2$ ; system defined by (25) with feedback (12).

present the evolution of  $I_1$  and  $I_2$  corresponding to system defined by (25), with feedback (12), in figure 10.

This positive result for degenerate system shows that the theoretical results are sufficient but not necessary; however, the approach may fail in some particular degenerate cases. This is consistent with the literature on quantum control that shows that degenerate cases have special structure (starting even with controllability criteria).

### 3.2. Periodic feedback

Although the discontinuous feedback (12) gives satisfactory results in terms of the control quality, the fact that it is discontinuous motivates trying to find additional procedures. To this end we introduce in this section a periodic, time-dependent feedback  $u = u(t, \Psi)$  stabilizing (3) to the ground state  $\phi$ . The idea is to use a highly oscillatory field component whose linear contribution averages to zero while the quadratic part averages to a constant; then we compare the asymptotic behavior of the system with the behavior of the averaged system. We recall that we are in the case when the reference trajectory corresponds to an equilibrium.

We consider the following time-dependent feedback

$$u(t, \Psi) = \alpha(\Psi) + \beta(\Psi) \sin(t/\varepsilon). \quad (26)$$

We substitute (26) in (3) and we obtain the system

$$i \frac{d}{dt} \Psi(t) = (H_0 + \alpha(\Psi)H_1 + \beta(\Psi) \sin(t/\varepsilon)H_1 + \alpha^2(\Psi)H_2 + 2\alpha(\Psi)\beta(\Psi) \sin(t/\varepsilon)H_2 + \beta^2(\Psi) \sin^2(t/\varepsilon)H_2 + \omega(\Psi))\Psi(t). \quad (27)$$

**Remark 3.3** For a differential system  $\dot{x} = f(t, x)$ , with  $f$  a  $T$ -periodic function,  $f(t+T, x) = f(t, x)$ , the averaged system is defined by  $\dot{x}_{\text{av}} = f_{\text{av}}(x)$  where  $f_{\text{av}}(x) = \frac{1}{T} \int_0^T f(t, x) dt$  (see [21] pp 402–10).

In our case the averaged system corresponding to (27) is given by

$$i \frac{d}{dt} \Psi_{\text{av}} = (H_0 + \alpha H_1 + (\alpha^2 + \frac{1}{2}\beta^2)H_2 + \omega)\Psi_{\text{av}}. \quad (28)$$

We identify the coefficients  $\alpha$  and  $\beta$  such that the averaged system is asymptotically stable. Since the trajectories of the system (27) are close to the trajectories of the averaged system (28) (see lemma 3.1 below), one can use the stability of the averaged system to obtain an approximate stability result for the system (27).

We use a Lyapunov technique to stabilize the averaged system (28) around the ground state  $\phi$ . We take again the function  $V$  defined by (4), which is non-negative for all  $\Psi \in \mathbb{S}^{2n-1}$  and vanishes when  $\Psi = \phi$ .

The derivative of  $V$  along a trajectory of the averaged system (27) is given by

$$\begin{aligned} \frac{d}{dt} V(\Psi_{\text{av}}(t)) &= 2\alpha \operatorname{Im}(\langle H_1 \Psi_{\text{av}}(t) | \phi \rangle) + 2\alpha^2 \operatorname{Im}(\langle H_2 \Psi_{\text{av}}(t) | \phi \rangle) + \beta^2 \operatorname{Im}(\langle H_2 \Psi_{\text{av}}(t) | \phi \rangle) \\ &\quad + 2(\omega + \lambda) \operatorname{Im}(\langle \Psi_{\text{av}}(t) | \phi \rangle). \end{aligned} \quad (29)$$

We denote  $I_1^{\text{av}} = \operatorname{Im}(\langle H_1 \Psi_{\text{av}}(t) | \phi \rangle)$  and  $I_2^{\text{av}} = \operatorname{Im}(\langle H_2 \Psi_{\text{av}}(t) | \phi \rangle)$ . For instance, we take

$$\alpha = -kI_1^{\text{av}}, \quad \beta = (I_2^{\text{av}})^-, \quad \omega = -\lambda - c \operatorname{Im}(\langle \Psi_{\text{av}}(t) | \phi \rangle), \quad (30)$$

where we have denoted by  $(I_2^{\text{av}})^- = -\min(I_2^{\text{av}}, 0)$ , the negative part of  $I_2^{\text{av}}$ . We obtain

$$\frac{d}{dt} V(\Psi_{\text{av}}(t)) = -2 \left( k(I_1^{\text{av}})^2 (1 - kI_2^{\text{av}}) + \frac{((I_2^{\text{av}})^-)^3}{2} + c \operatorname{Im}^2(\langle \Psi_{\text{av}}(t) | \phi \rangle) \right) \quad (31)$$



and thus  $dV/dt \leq 0$ , for  $c > 0$  and  $k < \frac{1}{\|H_2\|}$  (see also remark 2.1), i.e.  $V$  is non-increasing along the trajectories of the averaged system. In particular  $\phi$  is a stable point for the averaged system, i.e. such that

$$\forall \delta > 0, \exists \delta' > 0 \text{ such that } (|\Psi_{\text{av}}(0) - \phi| < \delta') \Rightarrow (|\Psi_{\text{av}}(t) - \phi| < \delta, \forall t \in [0, +\infty)). \quad (32)$$

We have the following asymptotic stability result:

**Theorem 3.2** *Under the hypotheses*

- (i)  $\lambda_j \neq \lambda_l$  for  $j \neq l$ ,
- (ii) for any  $j = 2, \dots, n: \langle H_1 \phi_j | \phi \rangle \neq 0$  or  $\langle H_2 \phi_j | \phi \rangle \neq 0$ ,

the averaged system (28) is globally asymptotically stable on  $\mathbb{S}^{2n-1} \setminus \{-\phi\}$  in the sense (recall (32)) that every solution  $\Psi_{\text{av}}$  of (28) with an initial state other than  $-\phi$  tends to  $\phi$  as  $t$  tends to  $+\infty$ .

**Proof of theorem 3.2** Up to a shift on  $\omega$  and  $H_0$ , we may assume that  $\lambda = 0$ . LaSalle's principle (see, e.g. [21, theorem 3.4, pp 115]) says that the trajectories of the system (28) converge to the largest invariant set contained in  $dV/dt = 0$ . The equation  $dV/dt = 0$  means that

$$I_1^{\text{av}} = 0, \quad (I_2^{\text{av}})^- = 0, \quad \text{Im}(\langle \Psi_{\text{av}}(t) | \phi \rangle) = 0, \quad (33)$$

and therefore  $\alpha = \beta = 0$ .

On the  $\Omega$ -limit set of a trajectory,  $V$  is constant. Since the  $\Omega$ -limit set is also invariant under the flow generated by (28) it follows, taking into account (33), that it consists in fact of trajectories of the uncontrolled system:

$$i \frac{d}{dt} \Psi_{\text{av}} = H_0 \Psi_{\text{av}}. \quad (34)$$

The solutions of (34) have the form

$$\Psi_{\text{av}} = \sum_{j=1}^n b_j e^{-i\lambda_j t} \phi_j. \quad (35)$$

We substitute (35) in (33) and we obtain

$$\text{Im}(\langle \Psi_{\text{av}}(t) | \phi \rangle) = \text{Im}(b_1) \langle \phi, \phi \rangle + \sum_{j=2}^n \text{Im}(b_j \langle \phi_j, \phi \rangle e^{-i\lambda_j t}), \quad (36)$$

$$I_1^{\text{av}}(\Psi_{\text{av}}) = \text{Im}(b_1) \langle H_1 \phi, \phi \rangle + \sum_{j \in J_1} \text{Im}(b_j \langle H_1 \phi_j, \phi \rangle e^{-i\lambda_j t}), \quad (37)$$

$$I_2^{\text{av}}(\Psi_{\text{av}}) = \text{Im}(b_1) \langle H_2 \phi, \phi \rangle + \sum_{k \in J_2} \text{Im}(b_k \langle H_2 \phi_k, \phi \rangle e^{-i\lambda_k t}). \quad (38)$$

Without loss of generality we take  $\phi = \phi_1$ . From equation (33) and (36), together with  $\langle \phi_j, \phi \rangle = 0$  for all  $j = 2, \dots, n$  we obtain  $\text{Im}(b_1) = 0$ . Since along the trajectories  $\Psi_{\text{av}}$  in  $\Omega$ ,  $I_1^{\text{av}}(\Psi_{\text{av}}) \equiv 0$ , we have  $\sum_{j \in J_1} \text{Im}(b_j \langle H_1 \phi_j, \phi \rangle e^{-i\lambda_j t}) = \sum_{j \in J_1} B_j' \sin(\lambda_j t + \theta_j) = 0$ . The functions  $\sin(\lambda_j t + \theta_j)$  are linearly independent as the  $\lambda_j$  are all different, hence the sum

can only vanish if all coefficients  $B'_j$  vanish. Observe now that  $B'_j = 0$ ,  $j \in J_1$  if and only if  $b_j = 0$ ,  $j \in J_1$ . Using  $\text{Im}(b_1) = 0$  we have:

$$I_2(\Psi_{\text{av}}) = \sum_{j \in J_2} \text{Im}(b_j \langle H_2 \phi_j, \phi \rangle e^{-i\lambda_j t}) = \sum_{j \in J_2} B_j \sin(\lambda_j t + \theta_j). \quad (39)$$

Since  $I_2^{\text{av}}(\Psi_{\text{av}}) \geq 0 \forall t$ , it follows that  $I_2^{\text{av}}(\Psi_{\text{av}}) \equiv 0$  (see the argument at the end of the proof of theorem 3.1). We have thus  $b_j = 0$  for  $j = 2, \dots, n$ . Now considering the form of the limit trajectories (35) this leaves only  $\Psi_{\text{av}} = b_1 e^{-i\lambda t} \phi = b_1 \phi$  (we assumed  $\lambda = 0$ ). Since  $\text{Im}(b_1) = 0$  the only case remained is  $\Psi_{\text{av}} = \pm \phi$  that is  $\Omega \subset \{\phi, -\phi\}$ . This concludes the proof of theorem 3.2.  $\square$

Our next theorem shows that our time-varying feedback laws lead to some kind of ‘practical global asymptotic stability on  $\mathbb{S}^{2n-1} \setminus \{-\phi\}$ ’ if  $\varepsilon > 0$  is small enough and if the assumptions of theorem 3.2 hold (see also [11]).

**Theorem 3.3** *Assume the hypothesis (i) and (ii) of Theorem 3.2 hold and let  $\mathcal{V}$  be a neighborhood of  $-\phi$  and  $\delta$  be a positive number. Then there exist a time  $T > 0$  and  $\varepsilon_0 > 0$  (depending both on  $\delta$  and  $\mathcal{V}$ ) such that every solution  $\Psi(t)$  of (27) with  $\varepsilon \in (0, \varepsilon_0)$  that satisfies  $\Psi(\tau) \in \mathbb{S}^{2n-1} \setminus \mathcal{V}$  for some  $\tau > 0$  also satisfies  $|\Psi(t) - \phi| < \delta$  for every  $t \geq \tau + T$ .*

**Proof of theorem 3.3** The idea of the proof is as follows. One knows that every trajectory of the averaged system (28) which is not identically equal to  $-\phi$  converges to the target state  $\phi$ . Hence, since the trajectories of the system (27) are close to the trajectory of the averaged system (28), one may expect that a trajectory which at some time is not in a given neighborhood  $\mathcal{V}$  of  $-\phi$  will be in a given neighborhood of  $\phi$  for large time if  $\varepsilon > 0$  is small enough.

The key ingredient that shows the relation between the trajectories of the oscillating system (27) and the trajectories of the averaged system (28) is the following classical lemma (see, e.g. [21, pp 415–7] or [29, section 3.2]).

**Lemma 3.1** *Let  $T > 0$ . There exists  $C$  and  $\varepsilon_0 > 0$  such that, for every  $\tau \in \mathbb{R}$  and for every  $\varepsilon \in (0, \varepsilon_0)$ , if  $\Psi : [\tau, \tau + T] \rightarrow \mathbb{S}^{2n-1}$  is a solution of (27) and  $\Psi_{\text{av}}$  is the solution of the averaged system (28) such that  $\Psi_{\text{av}}(\tau) = \Psi(\tau)$ , then*

$$|\Psi(t) - \Psi_{\text{av}}(t)| < C\varepsilon, \quad \forall t \in [\tau, \tau + T].$$

Let  $\delta_1 > 0$  be such that

$$(|\xi - \phi| < \delta_1) \Rightarrow (\xi \notin \mathcal{V}). \quad (40)$$

By (32), there exists  $\delta_2 > 0$  such that, for every solution  $\Psi_{\text{av}}$  of the averaged system (28),

$$(|\Psi_{\text{av}}(0) - \phi| < 2\delta_2) \Rightarrow \left( |\Psi_{\text{av}}(t) - \phi| < \frac{\min\{\delta, \delta_1\}}{2} \quad \forall t \in [0, +\infty) \right). \quad (41)$$

By theorem 3.2, there exists  $T > 0$  such that, for every solution  $\Psi_{\text{av}}$  of the averaged system (28),

$$(\Psi_{\text{av}}(0) \in \mathbb{S}^{2n-1} \setminus \mathcal{V}) \Rightarrow (|\Psi_{\text{av}}(t) - \phi| < \delta_2 \quad \forall t \in [T, +\infty)). \quad (42)$$

By lemma 3.1 and (42), there exists  $\epsilon_1 > 0$  such that, for every  $\epsilon \in (0, \epsilon_1)$ , for every  $\tau \in \mathbb{R}$  and for every solution  $\Psi$  of (27),

$$(\Psi(\tau) \in \mathbb{S}^{2n-1} \setminus \mathcal{V}) \Rightarrow (|\Psi(\tau + T) - \phi| < 2\delta_2). \quad (43)$$

By lemma 3.1 and (41), there exists  $\epsilon_2 > 0$  such that, for every  $\epsilon \in (0, \epsilon_2)$ , for every  $\tau' \in \mathbb{R}$  and for every solution  $\Psi$  of (27),

$$(|\Psi(\tau') - \phi| < 2\delta_2) \Rightarrow (|\Psi(\tau' + t) - \phi| < \min\{\delta, \delta_1\} \quad \forall t \in [0, T]). \quad (44)$$

Let us check that the conclusion of theorem 3.3 holds with  $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$ . Let  $\epsilon \in (0, \min\{\epsilon_1, \epsilon_2\})$ , let  $\tau > 0$  and let  $\Psi$  be a solution of (27) such that  $\Psi(\tau) \in \mathbb{S}^{2n-1} \setminus \mathcal{V}$ . By (43),

$$|\Psi(\tau + T) - \phi| < 2\delta_2. \quad (45)$$

From (44) with  $\tau' = \tau + T$  and (45), one obtains that

$$|\Psi(\tau + t) - \phi| < \min\{\delta, \delta_1\} \leq \delta \quad \forall t \in [T, 2T]. \quad (46)$$

From (40) and (46) for  $t = T$ , one obtains that

$$\Psi(\tau + T) \notin \mathcal{V}. \quad (47)$$

Using (47) and applying (46) with  $\tau + T$  for the new value of  $\tau$ , one obtains that

$$|\Psi(T + \tau + t) - \phi| < \min\{\delta, \delta_1\} \leq \delta \quad \forall t \in [T, 2T].$$

Continuing, an easy induction argument on the integer  $m$  shows that, more generally, for every non-negative integer  $m$ ,

$$|\Psi(mT + \tau + t) - \phi| < \min\{\delta, \delta_1\} \leq \delta \quad \forall t \in [T, 2T].$$

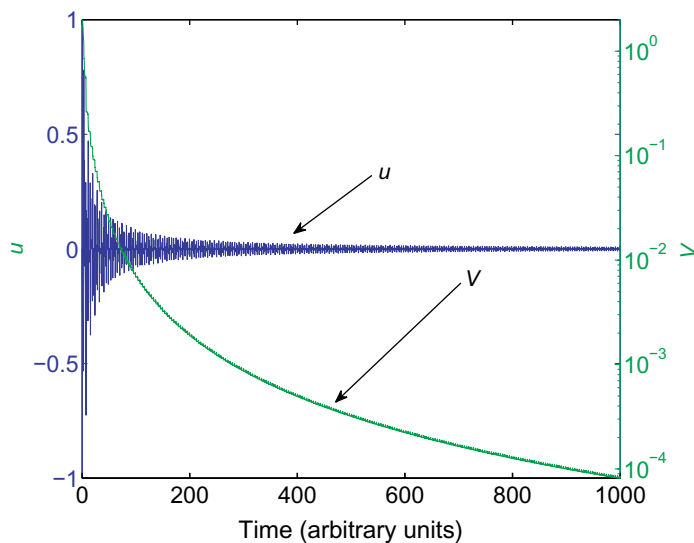
This ends the proof of theorem 3.3. □

*3.2.1. Examples for non-degenerate cases.* We take the system (11) and apply the periodic feedback (26) with  $\alpha$  and  $\beta$  defined by (30). Simulations of figure 11 describe the evolution of the Lyapunov function  $V(\Psi)$  for the initial state  $\Psi(t = 0) = (0, 1/\sqrt{2}, 1/\sqrt{2})$ .

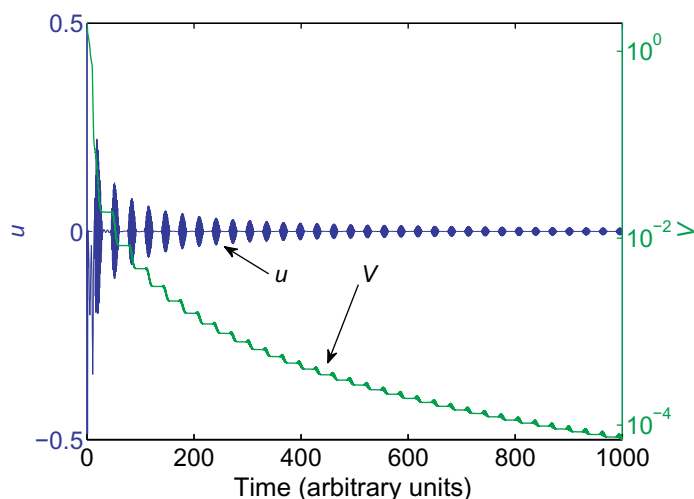
It appears that the periodic feedback is quite efficient for system (11) (see figure 11).

We take the system (24) and apply the periodic feedback (26) with  $\alpha$  and  $\beta$  defined by (30). Simulations of figure 12 describe the evolution of the Lyapunov function  $V(\Psi)$  and control  $u$  for the initial state  $\Psi(t = 0) = (0, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4})$ . Agreement with the theoretical results presented above is obtained (see figure 12).

*3.2.2. Examples for degenerate cases.* We take the system defined by (25) and we apply the periodic feedback (26) with  $\alpha$  and  $\beta$  defined by (30). Simulations of figure 13 describe the evolution of the Lyapunov function  $V(\Psi)$  and control  $u$ , system defined by (25) starting from the initial state  $\Psi(t = 0) = (0, 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ . We present the evolution of  $I_1$  and  $I_2$  corresponding to system defined by (25), with feedback (26), in figure 14.



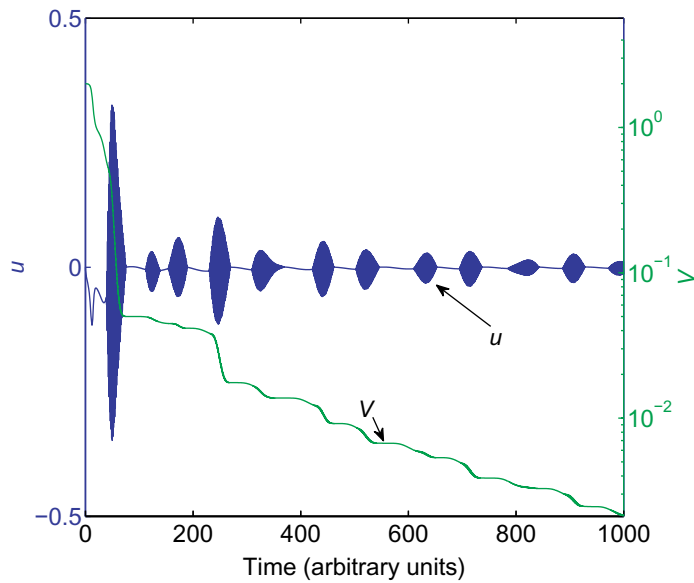
**Figure 11.** Evolution of the Lyapunov function  $V(\Psi)$  (green line) and control  $u$  (blue line); initial condition:  $\Psi(t=0) = (0, 1/\sqrt{2}, 1/\sqrt{2})$ ; system defined by (11) with feedback (26). We take  $\varepsilon = 10^{-3}$ ,  $k = 0.8$ ,  $c = 0.5$  and  $\Delta t = 0.1$ .



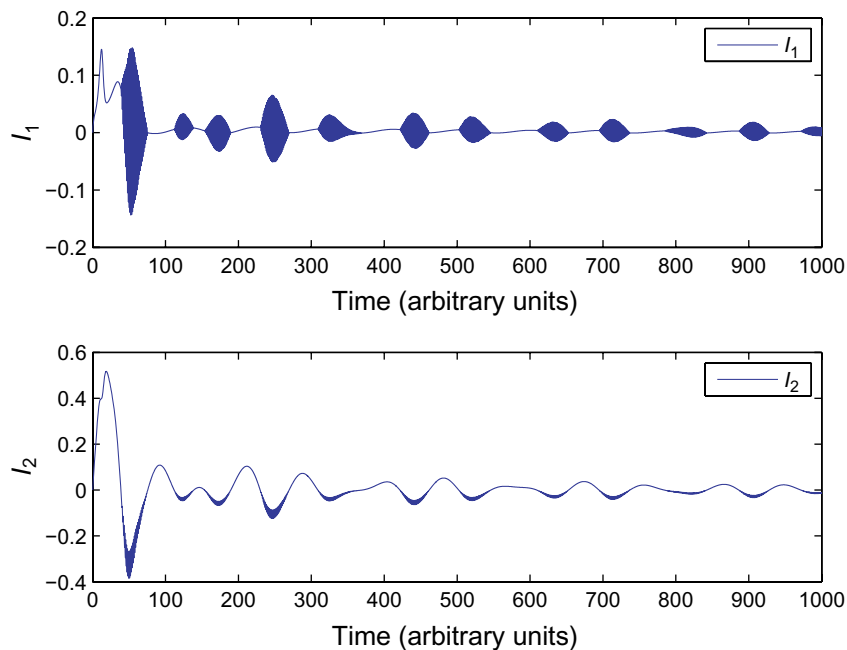
**Figure 12.** Evolution of the Lyapunov function  $V(\Psi)$  (green line) and control  $u$  (blue line); initial condition:  $\Psi(t=0) = (0, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4}, 1/\sqrt{4})$ ; system defined by (24) with feedback (26). We take  $\varepsilon = 10^{-3}$ ,  $k = 0.8$ ,  $c = 0.5$ ,  $\Delta t = 0.1$ .

#### 4. Conclusions

We focus in this paper on designing trajectory tracking (feedback) procedures for a control system with polarizability terms  $u^2(t)H_2$  present. We find that a straightforward application of the previous results only work for systems that are controllable without the polarizability term. To be able to find a control field that exploits the polarizability coupling we propose two different solutions: the first one is to use a discontinuous feedback with memory terms, the



**Figure 13.** Evolution of the Lyapunov function  $V(\Psi)$  (green line) and control  $u$  (blue line); initial condition  $\Psi(t=0) = (0, 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$ ; system defined by (25) with feedback (26). We take  $\varepsilon = 10^{-3}$ ,  $k = 0.8$ ,  $c = 0.5$  and  $\Delta t = 0.1$ .



**Figure 14.** Time evolution of  $I_1$  and  $I_2$ ; system defined by (25) with feedback (26).

other is to use time-dependent (periodic) forcing. In both cases, we present related theoretical results and numerically implement these techniques on prototypical examples. The time-dependent feedback is seen to generally produce smoother controls. The stabilizing procedures

for feedback design we have proposed in this paper may be adapted to other general situations where stabilization fails under the usual continuous feedbacks.

## Acknowledgments

JMC received financial support from ‘Agence Nationale de la Recherche’ (ANR), Projet Blanc C-QUID number BLAN-3-139579. AG and GT received financial support from INRIA Rocquencourt projet MicMac and OMQP, from ‘Agence Nationale de la Recherche’ (ANR), Projet Blanc C-QUID number BLAN-3-139579, and from PICS NSF-CNRS. CL received financial support from Université Paris Dauphine and from CNCSIS grant PNII ID-404/2007-2010.

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