PHYSICAL REVIEW A VOLUME 56, NUMBER 5 NOVEMBER 1997

Quantum copying: A network

V. Bužek,[1,2] S. L. Braunstein,[3] M. Hillery,[4] and D. Bruß[5]
[1] *Optics Section, The Blackett Laboratory, Imperial College, London SW7 2BZ, United Kingdom*
[2] *Institute of Physics, Slovak Academy of Sciences, Dubravská cesta 9, 842 28 Bratislava, Slovakia*
[3] *University of Wales, Dean Street, Bangor LL57 1UT, United Kingdom*
[4] *Department of Physics and Astronomy, Hunter College, CUNY, 695 Park Avenue, New York, New York 10021*
[5] *Clarendon Laboratory, Department of Physics, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom*
(Received 24 March 1997)

We present a network consisting of quantum gates that produces two imperfect copies of an arbitrary qubit. The quality of the copies does not depend on the input qubit. We also show that for a restricted class of inputs it is possible to use a very similar network to produce three copies instead of two. For qubits in this class, the copy quality is again independent of the input and is the same as the quality of the copies produced by the two-copy network. [S1050-2947(97)09510-3]

PACS number(s): 03.65.Bz

I. INTRODUCTION

Since the work of Wootters and Zurek it has been known that it is impossible to copy (i.e., clone) perfectly an arbitrary quantum state [1,2]. These authors considered a quantum copy machine that is supposed to copy a qubit and demonstrated that if it copies two basis vectors correctly, it cannot copy superpositions of these vectors without introducing errors. This result follows directly from the fact that quantum mechanical transformations are implemented by linear operators.

If one is only interested in producing imperfect copies, however, then it is possible to design machines (actually, find unitary transformations) that copy quantum states. A number of these were analyzed in recent papers by two of us [3,4]. The copy machine considered by Wootters and Zurek, for example, produces two identical copies at its output, but the quality of these copies depends upon the input state. They are perfect for the basis vectors that we denote as $|0\rangle$ and $|1\rangle$, but, because the copying process destroys the off-diagonal information of the input density matrix, they are poor for input states of the form $(|1\rangle + e^{i\varphi}|0\rangle)/\sqrt{2}$, where φ is arbitrary. A different copy machine, the Universal Quantum Copy Machine (UQCM), produces two identical copies whose quality is independent of the input state. In addition, its performance is, on average, better than that of the Wootters-Zurek machine, and the action of the machine simply scales the expectations values of certain operators. In particular the expectation value in one of the copies of any operator which is a linear combination of the Pauli matrixes is 2/3 that of its expectation value in the input state. Gisin has recently generalized the UQCM for the cases in which there are N identical inputs and $N+1$ outputs, that is, one copy is produced, and also in which there are N inputs and $N+2$ outputs, i.e., there are two copies produced [6]. In both cases all of the output copies are identical and their fidelity, that is, their overlap with the input state, goes to 1 as N goes to infinity.

In this paper we want to do two things. First, we present a quantum logic network that realizes the UQCM. An analysis of this network suggests that it should be possible to produce not two (imperfect) copies of the input state at the output, but three. Second, we find that a very similar quantum network can also be used as a quantum ''triplicator,'' i.e., a copying machine that produces three (imperfect) copies of the original qubit. In general, the triplicator has the undesirable feature that the quality of the copies that emerge from it is state dependent. However, if the original qubit is in a superposition state $\alpha|0\rangle + \beta|1\rangle$ with α and β *real* then the quality of the copied qubits does not depend on the particular value of α. Moreover we show that in this case the quality of the triplicated qubits is the same as those that emerge from the UQCM, which is a ''duplicator.''

In addition, we discuss the quantum entanglement of the qubits at the output of our quantum copying networks. The fact that the copies are entangled means that they are not independent; measuring one copy can have an effect on the other. This feature is something that must be kept in mind when determining how to make use of the copies.

The quantum logic networks that we propose consist of one- and two-bit quantum gates for which proposed designs already exist. They should, therefore, be useful in the experimental realization of quantum copy machines.

This paper is organized as follows. In Sec. II we briefly review the unitary transformation that specifies the UQCM. The quantum copying networks are described in Sec. III, while in Sec. IV we discuss the inseparability of the copied qubits. The quantum triplicator is described in Sec. V.

II. UNIVERSAL QUANTUM COPY MACHINE

Let us assume we want to copy an arbitrary pure state $|\Psi\rangle_{a_1}$, which in a particular basis $\{|0\rangle_{a_1},|1\rangle_{a_1}\}$ is described by the state vector $|\Psi\rangle_{a_1}$:

$$|\Psi\rangle_{a_1} = \alpha|0\rangle_{a_1} + \beta|1\rangle_{a_1}, \quad \alpha = \sin\vartheta e^{i\varphi}, \quad \beta = \cos\vartheta. \tag{2.1}$$

The two numbers that characterize the state (2.1) can be associated with the ''amplitude'' $|\alpha|$ and the ''phase'' φ of the qubit. Even though ideal copying, i.e., the transformation

$$|\Psi\rangle_{a_1} \to |\Psi\rangle_{a_1}|\Psi\rangle_{a_2}, \tag{2.2}$$

is prohibited by the laws of quantum mechanics for an *arbitrary* state (2.1), it is still possible to design quantum copiers that operate reasonably well. In particular, the UQCM [3] is specified by the following conditions.

(i) The state of the original system and its quantum copy at the output of the quantum copier, described by density operators $\hat{\rho}_{a_1}^{(\mathrm{out})}$ and $\hat{\rho}_{a_2}^{(\mathrm{out})}$, respectively, are identical, i.e.,

$$\hat{\rho}_{a_1}^{(\mathrm{out})} = \hat{\rho}_{a_2}^{(\mathrm{out})}. \tag{2.3}$$

(ii) If no *a priori* information about the *in*-state of the original system is available, then it is reasonable to require that *all* pure states should be copied equally well. One way to implement this assumption is to design a quantum copier such that the distances between density operators of each system at the output ($\hat{\rho}_{a_j}^{(\mathrm{out})}$, where $j=1,2$) and the ideal density operator $\hat{\rho}^{(\mathrm{id})}$ which describes the *in*-state of the original mode are input state independent. Quantitatively this means that if we employ the square of the Hilbert-Schmidt norm

$$d(\hat{\rho}_1;\hat{\rho}_2) := \mathrm{Tr}[(\hat{\rho}_1 - \hat{\rho}_2)^2], \tag{2.4}$$

as a measure of distance between two operators, then the quantum copier should be such that

$$d_1(\hat{\rho}_{a_j}^{(\mathrm{out})};\hat{\rho}_{a_j}^{(\mathrm{id})}) = \mathrm{const}, \quad j=1,2. \tag{2.5}$$

Here we use the subscript 1 in the definition of the distance d_1 to signify that this is the distance between single-qubit states.

(iii) Finally, we would also like to require that the copies are as close as possible to the ideal output state, which is, of course, just the input state. This means that we want our quantum copying transformation to satisfy

$$d_1(\hat{\rho}_{a_j}^{(\mathrm{out})};\hat{\rho}_{a_j}^{(\mathrm{id})}) = \min\{d_1(\hat{\rho}_{a_j}^{(\mathrm{out})};\hat{\rho}_{a_j}^{(\mathrm{id})})\} \quad (j=1,2). \tag{2.6}$$

Originally, the UQCM was found by estimating a transformation that contained two free parameters, and then determining them by demanding that condition (ii) be satisfied, and that the distance between the two-qubit output density matrix and the ideal two-qubit output be input state independent. That the UQCM machine obeys the condition (2.6) has only been shown recently by one of us [7].

The unitary transformation that implements the UQCM [3] is given by

$$\begin{aligned}
|0\rangle_{a_1}|Q\rangle_x &\to \sqrt{\frac{2}{3}}|00\rangle_{a_1a_2}|\uparrow\rangle_x + \sqrt{\frac{1}{3}}|+\rangle_{a_1a_2}|\downarrow\rangle_x, \\
|1\rangle_{a_1}|Q\rangle_x &\to \sqrt{\frac{2}{3}}|11\rangle_{a_1a_2}|\downarrow\rangle_x + \sqrt{\frac{1}{3}}|+\rangle_{a_1a_2}|\uparrow\rangle_x,
\end{aligned} \tag{2.7}$$

where

$$|+\rangle_{a_1a_2} = \frac{1}{\sqrt{2}}(|10\rangle_{a_1a_2} + |01\rangle_{a_1a_2}), \tag{2.8}$$

and satisfies the conditions (2.3)–(2.6). The system labeled by a_1 is the original (input) qubit, while the other system a_2 represents the qubit onto which the information is copied. This qubit is supposed to be initially in a state $|0\rangle_{a_2}$ (‘‘blank paper’’ in a copier). The states of the copy machine are labeled by x. The state space of the copy machine is two dimensional, and we assume that it is always in the same state $|Q\rangle_x$ initially. If the original qubit is in the superposition state (2.1) then the reduced density operator of both copies at the output are equal [see condition (2.3)] and they can be expressed as

$$\hat{\rho}_{a_j}^{(\mathrm{out})} = \tfrac{5}{6}|\Psi\rangle_{a_j}\langle\Psi| + \tfrac{1}{6}|\Psi_\perp\rangle_{a_j}\langle\Psi_\perp|, \quad j=1,2, \tag{2.9}$$

where

$$|\Psi_\perp\rangle_{a_j} = \beta^*|0\rangle_{a_j} - \alpha^*|1\rangle_{a_j} \tag{2.10}$$

is the state orthogonal to $|\Psi\rangle_{a_j}$. This implies that the copy contains 5/6 of the state we want and 1/6 of that one we did not.

We note that the density operator $\rho_{a_j}^{(\mathrm{out})}$ given by Eq. (2.9) can be rewritten in a ‘‘scaled’’ form:

$$\hat{\rho}_{a_j}^{(\mathrm{out})} = s_j\hat{\rho}_{a_j}^{(\mathrm{id})} + \frac{1-s_j}{2}\hat{1}, \quad j=1,2, \tag{2.11}$$

which guarantees that the distance (2.4) is input-state independent, i.e., the condition (2.5 is automatically fulfilled. The scaling factor in Eq. (2.11) is $s_j = 2/3$.

III. COPYING NETWORK

In what follows we show how with simple quantum logic gates we can copy quantum information encoded in the original qubit onto other qubits. The copying procedure can be understood as a ‘‘spread’’ of information via a ‘‘controlled’’ entanglement between the original qubit and the copy qubits. This controlled entanglement is implemented by a sequence of controlled NOT operations operating on the original qubit and the copy qubits that are initially prepared in a specific state.

In designing a network for the UQCM we first note that since the state space of the copy machine itself is two dimensional, we can consider it to be an additional qubit. Our network, then, will take three input qubits, one for the input, one that becomes one of the copies, and one for the machine, and transform them into three output qubits, two of which will be copies of the output. In what follows we will denote the quantum copier qubit as a_3 rather than x.

The operation of this network will be slightly different from what was indicated in the previous paragraph. Rather than have the copies appear in the a_1 and the a_2 qubit, they will appear in the a_2 and a_3 qubits.

Before proceeding with the network itself let us specify the one- and two-qubit gates from which it will be constructed. Firstly we define a single-qubit rotation $\hat{R}_j(\theta)$

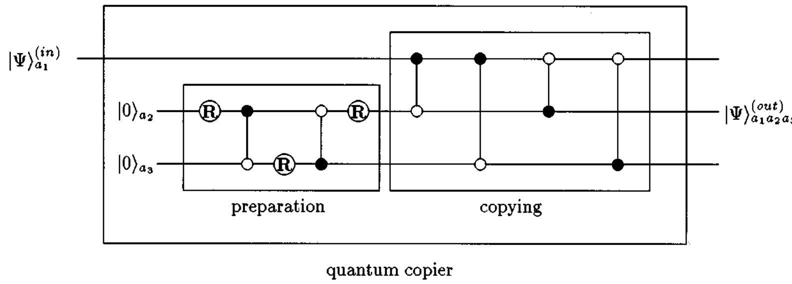

FIG. 1. Graphical representation of the UQCM network. The logical controlled NOT $\hat{P}_{kl}$ given by Eq. (3.2) has as its input a control qubit (denoted as ●) and a target qubit (denoted as ○). The action of the single-qubit operator **R** is specified by the transformation (3.1). We separate the preparation of the quantum copier from the copying process itself. The copying, i.e., the transfer of quantum information from the original qubit, is performed by a sequence of four controlled NOTs. We note that the amplitude information from the original qubit is copied in the obvious direction in an XOR or the controlled NOT operation. Simultaneously, the phase information is copied in the opposite direction making the XOR a simple model of quantum nondemolition measurement and its backaction.

$(j=1,2,3)$, which acts on the basis vectors of qubits as

$$\hat{R}_j(\theta)|0\rangle_j=\cos\,\theta|0\rangle_j+\sin\theta|1\rangle_j,$$
$$\hat{R}_j(\theta)|1\rangle_j=-\sin\,\theta|0\rangle_j+\cos\,\theta|1\rangle_j. \tag{3.1}$$

We also will utilize a two-qubit operator (a two-bit quantum gate), the so-called controlled NOT, which has as its inputs a control qubit (denoted as solid circles in Fig. 1) and a target qubit (denoted as open circles in Fig. 1). The control qubit is unaffected by the action of the gate, and if the control qubit is $|0\rangle$, the target qubit is unaffected as well. However, if the control qubit is in the $|1\rangle$ state, then a NOT operation is performed on the target qubit. The operator that implements this gate, $\hat{P}_{kl}$, acts on the basis vectors of the two qubits as follows (k denotes the control qubit and l the target):

$$\hat{P}_{kl}|0\rangle_k|0\rangle_l=|0\rangle_k|0\rangle_l,\quad \hat{P}_{kl}|0\rangle_k|1\rangle_l=|0\rangle_k|1\rangle_l,$$
$$\hat{P}_{kl}|1\rangle_k|0\rangle_l=|1\rangle_k|1\rangle_l,\quad \hat{P}_{kl}|1\rangle_k|1\rangle_l=|1\rangle_k|0\rangle_l. \tag{3.2}$$

We can decompose the quantum copier network into two parts. In the first part the replica qubits a_2 and a_3 are prepared in a specific state $|\Psi\rangle_{a_2a_3}^{(\text{prep})}$. Then in the second part of the copying network the original information from the original qubit is *redistributed* among the three qubits. That is the action of the quantum copier can be described as a sequence of two unitary transformations

$$|\Psi\rangle_{a_1}^{(\text{in})}|0\rangle_{a_2}|0\rangle_{a_3}\rightarrow|\Psi\rangle_{a_1}^{(\text{in})}|\Psi\rangle_{a_2a_3}^{(\text{prep})}\rightarrow|\Psi\rangle_{a_1a_2a_3}^{(\text{out})}. \tag{3.3}$$

The network for the quantum copying machine is displayed in Fig. 1.

A. Preparation of quantum copier

Let us first look at the preparation stage. Prior to any interaction with the input qubit we have to prepare the two quantum copier qubits (a_2 and a_3) in a very specific state $|\Psi\rangle_{a_2a_3}^{(\text{prep})}$. If we assume that initially these two qubits are in the state

$$|\Psi\rangle_{a_2a_3}^{(\text{in})}=|0\rangle_{a_2}|0\rangle_{a_3}, \tag{3.4}$$

then the arbitrary state $|\Psi\rangle_{a_1a_2}^{(\text{prep})}$,

$$|\Psi\rangle_{a_2a_3}^{(\text{prep})}=C_1|00\rangle_{a_2a_3}+C_2|01\rangle_{a_2a_3}+C_3|10\rangle_{a_2a_3}+C_4|11\rangle_{a_2a_3}, \tag{3.5}$$

with real amplitudes C_i (such that $\Sigma_{i=1}^4 C_i^2=1$) can be prepared by a simple quantum network [8] (see the ‘‘preparation’’ box in Fig. 1) with two controlled NOTs $\hat{P}_{kl}$ and three rotations $\hat{R}(\theta_j)$, i.e.,

$$|\Psi\rangle_{a_2a_3}^{(\text{prep})}=\hat{R}_2(\theta_3)\hat{P}_{32}\hat{R}_3(\theta_2)\hat{P}_{23}\hat{R}_2(\theta_1)|0\rangle_{a_2}|0\rangle_{a_3}. \tag{3.6}$$

Comparing Eqs. (3.5) and (3.6) we find a set of equations

$$\cos\,\theta_1\cos\,\theta_2\cos\,\theta_3+\sin\,\theta_1\sin\,\theta_2\sin\,\theta_3=C_1,$$
$$-\cos\,\theta_1\sin\,\theta_2\sin\,\theta_3+\sin\,\theta_1\cos\,\theta_2\cos\,\theta_3=C_2,$$
$$\cos\,\theta_1\cos\,\theta_2\sin\,\theta_3-\sin\,\theta_1\sin\,\theta_2\cos\,\theta_3=C_3,$$
$$\cos\,\theta_1\sin\,\theta_2\cos\,\theta_3+\sin\,\theta_1\cos\,\theta_2\sin\,\theta_3=C_4, \tag{3.7}$$

from which the angles θ_j $(j=1,2,3)$ of rotations can be specified as functions of parameters C_i. In particular, for the purpose of the UQCM we need that

$$|\Psi\rangle_{a_2a_3}^{(\text{prep})}=\frac{1}{\sqrt{6}}(2|00\rangle_{a_2a_3}+|01\rangle_{a_2a_3}+|10\rangle_{a_2a_3}). \tag{3.8}$$

With the help of Eq. (3.7) we find that the rotation angles necessary for the preparation of the state given in Eq. (3.8) are

$$\theta_1=\theta_3=\frac{\pi}{8},\quad \theta_2=-\arcsin\left(\frac{1}{2}-\frac{\sqrt{2}}{3}\right)^{1/2}. \tag{3.9}$$

B. Quantum copying

Once the qubits of the quantum copier are properly prepared then the copying of the initial state $|\Psi\rangle_{a_1}^{(\mathrm{in})}$ of the original qubit can be performed by a sequence of four controlled NOT operations (see Fig. 1)

$$|\Psi\rangle_{a_1a_2a_2}^{(\mathrm{out})}=\hat{P}_{a_3a_1}\hat{P}_{a_2a_1}\hat{P}_{a_1a_3}\hat{P}_{a_1a_2}|\Psi\rangle_{a_1}^{(\mathrm{in})}|\Psi\rangle_{a_2a_3}^{(\mathrm{prep})}. \tag{3.10}$$

When this operation is combined with the preparation stage, we find that the basis states of the original qubit (a_1) are copied as

$$|0\rangle_{a_1}|00\rangle_{a_2a_3}\rightarrow\sqrt{\frac{2}{3}}|0\rangle_{a_1}|00\rangle_{a_2a_3}+\frac{1}{\sqrt{3}}|1\rangle_{a_1}|+\rangle_{a_2a_2}, \tag{3.11}$$

$$|1\rangle_{a_1}|00\rangle_{a_2a_3}\rightarrow\sqrt{\frac{2}{3}}|1\rangle_{a_1}|11\rangle_{a_2a_3}+\frac{1}{\sqrt{3}}|0\rangle_{a_1}|+\rangle_{a_2a_2}, \tag{3.12}$$

where $|+\rangle_{a_2a_3}=(|01\rangle_{a_2a_3}+|10\rangle_{a_2a_3})/\sqrt{2}$. When the original qubit is in the superposition state (2.1) then the state vector of the three qubits after the copying has been performed reads

$$|\Psi\rangle_{a_1a_2a_3}^{(\mathrm{out})}=|0\rangle_{a_1}|\Phi_0\rangle_{a_2a_3}+|1\rangle_{a_1}|\Phi_1\rangle_{a_2a_3}, \tag{3.13}$$

with

$$|\Phi_0\rangle_{a_2a_2}=\alpha\sqrt{\frac{2}{3}}|00\rangle_{a_2a_3}+\beta\frac{1}{\sqrt{3}}|+\rangle_{a_2a_3},$$

$$|\Phi_1\rangle_{a_2a_3}=\beta\sqrt{\frac{2}{3}}|11\rangle_{a_2a_3}+\alpha\frac{1}{\sqrt{3}}|+\rangle_{a_2a_3}. \tag{3.14}$$

From this it follows that at the output of the quantum copier we find a pair of entangled qubits in a state described by the density operator

$$\hat{\rho}_{a_2a_3}^{(\mathrm{out})}=|\Phi_0\rangle_{a_2a_3}\langle\Phi_0|+|\Phi_1\rangle_{a_2a_3}\langle\Phi_1|. \tag{3.15}$$

Each of the copy qubits at the output of the quantum copier has a reduced density operator $\hat{\rho}_{a_j}^{(\mathrm{out})}$ ($j=2,3$) given by Eq. (2.11). The distance $d_1(\hat{\rho}_{a_j}^{(\mathrm{out})};\hat{\rho}_{a_j}^{(\mathrm{id})})$ ($j=2,3$) between the output qubit and the ideal qubit is constant and can expressed as a function of the scaling parameter s in Eq. (2.11):

$$d_1(\hat{\rho}_{a_j}^{(\mathrm{out})};\hat{\rho}_{a_j}^{(\mathrm{id})})=\frac{(1-s)^2}{2}=\frac{1}{18}. \tag{3.16}$$

Analogously we find the distance $d_2(\hat{\rho}_{a_2a_3}^{(\mathrm{out})};\hat{\rho}_{a_2a_3}^{(\mathrm{id})})$ between the two-qubit output of the quantum copying and the ideal output to be constant, i.e.,

$$d_2(\hat{\rho}_{a_2a_3}^{(\mathrm{out})};\hat{\rho}_{a_2a_3}^{(\mathrm{id})})=\frac{s^2}{2}=\frac{2}{9}. \tag{3.17}$$

The original qubit after the copying is performed is in a state

$$\hat{\rho}_{a_1}^{(\mathrm{out})}=\frac{1}{3}(\hat{\rho}_{a_1}^{(\mathrm{in})})^T+\frac{1}{3}\hat{1}, \tag{3.18}$$

where the superscript T denotes the transpose. We note that in spite of the fact that the distance between this density operator and the ideal qubit depends on the initial state of the original qubit, i.e.,

$$d_1(\hat{\rho}_{a_1}^{(\mathrm{out})};\hat{\rho}_{a_1}^{(\mathrm{id})})=\frac{2}{9}(1+12|\alpha|^2|\beta|^2\sin^2\varphi), \tag{3.19}$$

the output state of the original qubit still contains information about the input state, though less than either of the copies. In order to extract this information we note that for a Hermitian operator $\hat{A}$

$$\mathrm{Tr}(\hat{\rho}_{a_1}^{(\mathrm{in})}\hat{A})=\mathrm{Tr}[(\hat{\rho}_{a_1}^{(\mathrm{in})})^T\hat{A}^T]. \tag{3.20}$$

This means that to obtain information about $\hat{A}$ at the input, we measure $\hat{A}^{\mathrm{T}}$ for the original qubit at the output.

We note that the flow of information in our quantum network can be controlled by the choice of the preparation state $|\Psi\rangle_{a_2a_3}^{(\mathrm{prep})}$ [5]. In particular, if we chose the rotation angles θ_j in our network (4.6) such that

$$\cos 2\theta_1=\frac{1}{\sqrt{5}},\quad \cos 2\theta_2=\frac{\sqrt{5}}{3},\quad \cos 2\theta_3=\frac{2}{\sqrt{5}}, \tag{3.21}$$

then the state$|\Psi\rangle_{a_2a_3}^{(\mathrm{prep})}$ reads

$$|\Psi\rangle_{a_2a_3}^{(\mathrm{prep})}=\frac{1}{\sqrt{6}}(2|00\rangle_{a_2a_3}+|01\rangle_{a_2a_3}+|11\rangle_{a_2a_3}). \tag{3.22}$$

In this case the copies appear in the a_1 and a_2 qubits, while the qubit a_3 plays the role of the copying machine (ancila). That is, with the preparation (3.22) the transformation (2.7) is realized.

IV. INSEPARABILITY OF COPIED QUBITS

An ideal copy machine would produce two copies that are completely independent of each other; i.e., the reduced density matrix for the two copies, $\hat{\rho}_{a_2a_3}$, would be a product of $\hat{\rho}_{a_2}$ and $\hat{\rho}_{a_3}$. For the UQCM, however, this is not the case and there are correlations between the copies. These correlations can be either quantum mechanical or classical, and we would like to determine whether the two copies are quantum-mechanically entangled. To do so, we first recall that a density operator of two subsystems is inseparable if it *cannot* be written as the convex sum

$$\hat{\rho}_{a_2a_3}=\sum_m w^{(m)}\hat{\rho}_{a_2}^{(m)}\otimes\hat{\rho}_{a_3}^{(m)}. \tag{4.1}$$

Inseparability is one of the most fundamental quantum phenomenon, which, in particular, may result in the violation of Bell's inequality (to be specific, a separable system always satisfies Bell's inequality, but the contrary is not necessarily true). Note that distant parties cannot prepare an inseparable state from a separable state if they only use local operations and classical communication channels [9].

In the case of two qubits (i.e., spin-1/2) we can utilize the Peres-Horodecki theorem [9,10], which states that the positivity of the partial transposition of a state is both a *necessary* and *sufficient* condition for its separability. Before we proceed further we briefly describe how to ''use'' this theorem: the density matrix associated with the density operator of two spin-1/2 can be written as

$$\rho_{m\mu,n\nu}=\langle e_m|\langle f_\mu|\hat{\rho}|e_n\rangle|f_\nu\rangle, \tag{4.2}$$

where $\{|e_m\rangle\}$ ($\{|f_\mu\rangle\}$) denotes an orthonormal basis in the Hilbert space of the first (second) spin-1/2 (for instance, $|e_0\rangle=|0\rangle_{a_2}$; $|e_1\rangle=|1\rangle_{a_2}$, and $|f_0\rangle=|0\rangle_{a_3}$; $|f_1\rangle=|1\rangle_{a_3}$). The partial transposition $\hat{\rho}^{T_2}$ of $\hat{\rho}$ is defined as

$$\rho^{T_2}_{m\mu,n\nu}=\rho_{m\nu,n\mu}. \tag{4.3}$$

Then the necessary and sufficient condition for the state $\hat{\rho}_{a_2a_3}$ of two spin-1/2 to be inseparable is that at least one of the eigenvalues of the partially transposed operator $\rho^{T_2}_{m\mu,n\nu}$ is negative.

Now we will check whether the density operator $\hat{\rho}^{(\mathrm{out})}_{a_2a_3}$ given by Eq. (3.15) is separable. In the basis $\{|11\rangle_{a_2a_3},|10\rangle_{a_2a_3},|01\rangle_{a_2a_3},|00\rangle_{a_2a_3}\}$ this density operator is described by a matrix

$$\hat{\rho}^{(\mathrm{out})}_{a_2a_3}=\frac{1}{6}\begin{pmatrix}4|\beta|^2 & 2\alpha^*\beta & 2\alpha^*\beta & 0\\ 2\alpha\beta^* & 1 & 1 & 2\alpha^*\beta\\ 2\alpha\beta^* & 1 & 1 & 2\alpha^*\beta\\ 0 & 2\alpha\beta^* & 2\alpha\beta^* & 4|\alpha|^2\end{pmatrix}, \tag{4.4}$$

while the corresponding partially transposed operator in the matrix representation reads

$$\hat{\rho}^{T_2}_{a_2a_3}=\frac{1}{6}\begin{pmatrix}4|\beta|^2 & 2\alpha\beta^* & 2\alpha^*\beta & 1\\ 2\alpha^*\beta & 1 & 0 & 2\alpha^*\beta\\ 2\alpha\beta^* & 0 & 1 & 2\alpha\beta^*\\ 1 & 2\alpha\beta^* & 2\alpha^*\beta & 4|\alpha|^2\end{pmatrix}. \tag{4.5}$$

From the fact that one of the four eigenvalues

$$\left\{\frac{1}{6},\frac{1}{6},\frac{2-\sqrt{5}}{6},\frac{2+\sqrt{5}}{6}\right\}, \tag{4.6}$$

of this partially transposed operator is negative for all values of α (i.e., for arbitrary state of the original qubit), it follows that the two qubits at the output of the quantum copier are nonclassically entangled. The fact that the eigenvalues of the transposed density operator are input-state independent (and combined with the fact that distance d_2 between $\hat{\rho}^{(\mathrm{out})}_{a_2a_3}$ and $\hat{\rho}^{(\mathrm{id})}_{a_2a_3}$ is also input-state independent) suggests that the degree of entanglement between the copied qubits is also input-state independent.

V. QUANTUM TRIPLICATOR

When it is *a priori* known that the original qubit is initially in a superposition state (2.1) with the mean value of the observable $\hat{\sigma}_y$ equal to zero (i.e., α and β are real) then the quantum copying network presented in Fig. 1 can serve also as a quantum triplicator. That is, out of a single original qubit this device can create three identical qubits with equal density operator $\hat{\rho}^{(\mathrm{out})}_{a_j}$, i.e.,

$$\hat{\rho}^{(\mathrm{out})}_{a_1}=\hat{\rho}^{(\mathrm{out})}_{a_2}=\hat{\rho}^{(\mathrm{out})}_{a_3}, \tag{5.1}$$

such that the distances $d_1(\hat{\rho}^{(\mathrm{out})}_{a_j};\hat{\rho}^{(\mathrm{id})}_{a_j})$ given by Eq. (2.4) are constant (i.e., they do not depend on α). This quantum triplicator is input-state independent, but we have to remember that the class of original qubits for which this is true is restricted.

The triplicator network is exactly the same as the one considered in the previous section except we have to perform the rotation $\hat{R}_3(\theta_2)$ in the opposite direction. That is, the angles θ_1 and θ_2 are the same as specified by Eq. (3.9), but $\theta_2=\arcsin(1/2-\sqrt{2}/3)^{1/2}$. In this case the state $|\Psi\rangle^{(\mathrm{prep})}_{a_2a_3}$ reads

$$|\Psi\rangle^{(\mathrm{prep})}_{a_2a_3}=\frac{1}{\sqrt{12}}(3|00\rangle+|01\rangle+|10\rangle+|11\rangle). \tag{5.2}$$

With the help of Eq. (3.10) we now find the output state of the quantum triplicator:

$$|\Psi\rangle^{(\mathrm{out})}_{a_1a_2a_3}=\frac{1}{\sqrt{12}}(3\alpha|000\rangle+\alpha|101\rangle+\alpha|110\rangle+\alpha|011\rangle +3\beta|111\rangle+\beta|010\rangle+\beta|001\rangle+\beta|100\rangle). \tag{5.3}$$

When α and β are real then we find that the three qubits at the output of the triplicator have *identical* density operators given by Eq. (2.11) with the scaling factor $s=2/3$.

Moreover, we find that the three two-qubit density operators at the output of the triplicator are mutually equal. In the matrix form they read

$$\hat{\rho}^{(\mathrm{out})}_{a_2a_3}=\hat{\rho}^{(\mathrm{out})}_{a_1a_2}=\hat{\rho}^{(\mathrm{out})}_{a_1a_3} =\frac{1}{12}\begin{pmatrix}8\beta^2+1 & 4\alpha\beta & 4\alpha\beta & 3\\ 4\alpha\beta & 1 & 1 & 4\alpha\beta\\ 4\alpha\beta & 1 & 1 & 4\alpha\beta\\ 3 & 4\alpha\beta & 4\alpha\beta & 8\alpha^2+1\end{pmatrix}. \tag{5.4}$$

This quantum triplicator operates in such way that all distances between output qubits and ideal copies, i.e.,

$$d_1(\hat{\rho}_{a_j}^{(\mathrm{out})};\hat{\rho}_{a_j}^{(\mathrm{id})})=\frac{1}{18},\quad d_2(\hat{\rho}_{a_j a_l}^{(\mathrm{out})};\hat{\rho}_{a_j a_l}^{(\mathrm{id})})=\frac{2}{9},$$

$$d_3(\hat{\rho}_{a_1 a_2 a_3}^{(\mathrm{out})};\hat{\rho}_{a_1 a_2 a_3}^{(\mathrm{id})})=\frac{1}{2}, \tag{5.5}$$

are constant for arbitrary real values of α.

We note that the two-qubit density matrices (5.4) are inseparable, because one of the eigenvalues

$$\left\{-\frac{1}{6},\frac{1}{3},\frac{5+\sqrt{17}}{12},\frac{5-\sqrt{17}}{12}\right\} \tag{5.6}$$

of the corresponding partially transposed matrix is negative. Because this negative eigenvalue does not depend on α, and the fact that d_2 is input-state independent, we can conclude that the quantum triplicator creates a specific class of two-qubit states characterized by the same degree of entanglement.

Next we turn our attention to the fact that the scaling factor $s=2/3$, which relates the output qubits to the original qubit, is in our case larger than that found in Gisin's triplication procedure [6], where it is $s=5/9$. While our scaling factor is larger, there is a price to pay. Namely, our triplication network requires *a priori* knowledge; the original qubit must be described by the state vector (2.1) with real α and β. Gisin's scheme is more general, because it triplicates all qubits (2.1) and the quality of the copies is independent of the input state. However, the quality of his copies is not as good, which can be seen directly from the fact that for his procedure the distance between the copied and original qubits $d_1(\hat{\rho}_{a_j}^{(\mathrm{out})};\hat{\rho}_{a_j}^{(\mathrm{id})})$ is almost two times (to be precise, 16/9 times) larger than with ours. In fact, there exists a general tradeoff between the *a priori* knowledge of the state of the original qubit and the quality of the copying: the better we know the initial state of the original qubit the better copying transformation can be. For example, if we know exactly the state of the original qubit, we can produce as many perfect copies as we want.

Finally we analyze the output state of the triplicator network described in Fig. 1 when the original qubit is in an arbitrary superposition state (2.1) (with α and β complex). Using the general expression (5.3) for the output of the triplicator we find that the individual qubits at the output are equal, i.e., $\hat{\rho}_{a_1}^{(\mathrm{out})}=\hat{\rho}_{a_2}^{(\mathrm{out})}=\hat{\rho}_{a_3}^{(\mathrm{out})}$, with the density matrices given by the expression

$$\hat{\rho}_{a_j}^{(\mathrm{out})}=\frac{1}{6}\begin{pmatrix} 4|\beta|^2+1 & 3\alpha^*\beta+\alpha\beta^* \\ 3\alpha\beta^*+\alpha^*\beta & 4|\alpha|^2+1\end{pmatrix},\qquad j=1,2,3. \tag{5.7}$$

In general, these density operators cannot be written in the scaled form (2.11) and consequently, the distance between the output and input qubits depends on the initial state of the original qubit, i.e.,

$$d_1(\hat{\rho}_{a_j}^{(\mathrm{out})};\hat{\rho}_{a_j}^{(\mathrm{id})})=\frac{1}{18}(1+12|\alpha|^2|\beta|^2\ \sin^2\varphi). \tag{5.8}$$

The two-qubit density operators at the output of the triplicator are also equal, and they can be described by the density matrix

$$\hat{\rho}_{a_2 a_3}^{(\mathrm{out})}=\hat{\rho}_{a_1 a_2}^{(\mathrm{out})}=\hat{\rho}_{a_1 a_3}^{(\mathrm{out})}=\frac{1}{12}\begin{pmatrix} 8|\beta|^2+1 & 3\alpha^*\beta+\alpha\beta^* & 3\alpha^*\beta+\alpha\beta^* & 3 \\ 3\alpha\beta^*+\alpha^*\beta & 1 & 1 & 3\alpha^*\beta+\alpha\beta^* \\ 3\alpha\beta^*+\alpha^*\beta & 1 & 1 & 3\alpha^*\beta+\alpha\beta^* \\ 3 & 3\alpha\beta^*+\alpha^*\beta & 3\alpha\beta^*+\alpha^*\beta & 8|\alpha|^2+1\end{pmatrix}. \tag{5.9}$$

From this expression we can easily find that the two-qubit distances $d_2(\hat{\rho}_{a_k a_l}^{(\mathrm{out})};\hat{\rho}_{a_k a_l}^{(\mathrm{id})})$ between the actual output of the triplicator and the ideal case are input-state dependent, i.e.,

$$d_2(\hat{\rho}_{a_k a_l}^{(\mathrm{out})};\hat{\rho}_{a_k a_l}^{(\mathrm{id})})=\frac{2}{9}(1+12|\alpha|^2|\beta|^2\sin^2\varphi),$$

$$k,l=1,2,3,\quad k\neq l. \tag{5.10}$$

Analogously for the three-qubit distance $d_3(\hat{\rho}_{a_1 a_2 a_3}^{(\mathrm{out})};\hat{\rho}_{a_1 a_2 a_3}^{(\mathrm{id})})$ we find

$$d_3(\hat{\rho}_{a_1 a_2 a_3}^{(\mathrm{out})};\hat{\rho}_{a_1 a_2 a_3}^{(\mathrm{id})})=\tfrac{1}{2}(1+12|\alpha|^2|\beta|^2\sin^2\varphi). \tag{5.11}$$

Here the minimum values of the distances d_j ($j=1,2,3$) are obtained when $\varphi=0,\pi$ and in this case they do not depend on the particular value of $|\alpha|$.

From the explicit expression (5.9) we find that the two-qubit density matrices are inseparable for an arbitrary state of the input qubit. This means that quantum triplication ''creates'' very specific quantum correlations between the output qubits. Namely, one of the eigenvalues of the partially transposed matrix (5.9) is negative for *arbitrary* values $|\alpha|$ and φ. Moreover, there exists correspondence between the behavior of the distance $d_1(\hat{\rho}_{a_j}^{(\mathrm{out})};\hat{\rho}_{a_j}^{(\mathrm{id})})$ and the value of the negative eigenvalue E of the partially transposed matrix. In particular, when $\varphi=0,\pi$ this eigenvalue does not depend on $|\alpha|$ and is equal to $-1/6$. The corresponding distance d_1 in this case is minimal and equal to 1/18 (irrespective of $|\alpha|$). As the distance d_1 increases, this eigenvalue decreases. Specifically, for a given value of $|\alpha|$ the distance d_1 is maximal when $\varphi=\pi/2$. Correspondingly, the negative eigenvalue E of the partially transposed matrix for a given $|\alpha|$ takes its minimal value when $\varphi=\pi/2$. In this case E can be approximated by its upper bound $\overline{E}$:

$$E \lesssim \overline{E} = -\left(\frac{1+4(\sqrt{5}-2)|\alpha|^2|\beta|^2}{6}\right), \qquad (5.12)$$

which clearly reveals a dependence between the distance d_1 [given by Eq. (5.8) with $\varphi = \pi/2$] and the negative eigenvalue E. This observation suggests that the copying schemes analogous to the triplication network discussed above can serve as specific ''quantum entanglers'' and that the measure of entanglement can be operationally related to a specific distance d_1.

VI. CONCLUSION

It is possible to construct devices that copy the information in a quantum state as long as one does not demand perfect copies. One can build either a duplicator, which produces two copies, or a triplicator, which produces three. Both of these devices can be realized by simple networks of quantum gates, which should make it possible to construct them in the laboratory.

There are a number of unanswered questions about quantum copiers. Perhaps the most obvious is which quantum copier is the best. Recently it has been shown [7] that the UQCM described in this paper is the best quantum copier able to produce two copies of the original qubit. It is not known, however, how to construct the best quantum triplicator (or, in general, a device that will produce multiple copies, the so-called multiplicator). There exist bounds on how well one can do, which follow from unitarity, but they are not realized by existing copiers [11]. This is at least partially the fault of the bounds, which are probably lower than they have to be.

A quantum copier takes quantum information in one system and spreads it among several. It would be nice to be able to see how this happens qualitatively, but, at the moment, it is not clear how to do this. The problem is that we are interested in how only a part of the information flows through the machine. It is only the information in the input state, and not that in the two input qubits, which enter the machine in standard states, the so-called ''blank pieces of paper,'' which matters, but it seems to be difficult to separate the effect of the two in the action of the machine.

This issue is connected to another, which is how to best use the copies to gain information about the input state. In a previous paper we showed how nonselective measurements of a single quantity on one of the copies can be used to gain information about the original and leave the one-particle reduced density matrix of the other copy unchanged. An interesting extension of this would be to ask, for a given number of copies, how much information we can gain about the original state by performing different kinds of measurements on the copies.

It is clear that quantum copying still presents both theoretical and experimental challenges. We hope to be able to address some of issues raised by the questions in the preceding paragraphs in future publications.

ACKNOWLEDGMENTS

We thank Nicolas Gisin for communication of his recent results to us. We also thank Peter Knight and Artur Ekert for useful discussions. This work was supported by the United Kingdom Engineering and Physical Sciences Research Council, by the grant agency VEGA of the Slovak Academy of Sciences (under the project 2/1152/96), and by the National Science Foundation under Grant No. INT 9221716.

[1] W. K. Wootters and W. H. Zurek, Nature (London) **299**, 802 (1982).
[2] H. Barnum, C. M. Caves, C. A. Fuchs, R. Jozsa, and B. Schumacher, Phys. Rev. Lett. **76**, 2818 (1996).
[3] V. Bužek and M. Hillery, Phys. Rev. A **54**, 1844 (1996).
[4] V. Bužek, V. Vedral, M. Plenio, P. L. Knight, and M. Hillery, Phys. Rev. A **55**, 3327 (1997).
[5] V. Bužek and M. Hillery, Acta Phys. Slov. **47**, 193 (1997).
[6] N. Gisin and S. Massar, Phys. Rev. Lett. **79**, 2153 (1997).
[7] D. Bruß, D. P. DiVincenzo, A. Ekert, C. Macchiavello, and J. A. Smolin (unpublished).
[8] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, Phys. Rev. A **52**, 3457 (1995).
[9] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
[10] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A **223**, 1 (1996).
[11] M. Hillery and V. Bužek, Phys. Rev. A **56**, 1212 (1997).