

Quantum Criticality and Holographic Superconductors in M-theory

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Abstract

We present a consistent Kaluza-Klein truncation of $D = 11$ supergravity on an arbitrary seven-dimensional Sasaki-Einstein space (SE_7) to a $D = 4$ theory containing a metric, a gauge-field, a complex scalar field and a real scalar field. We use this $D = 4$ theory to construct various black hole solutions that describe the thermodynamics of the $d = 3$ CFTs dual to skew-whiffed $AdS_4 \times SE_7$ solutions. We show that these CFTs have a rich phase diagram, including holographic superconductivity with, generically, broken parity and time reversal invariance. At zero temperature the superconducting solutions are charged domain walls with a universal emergent conformal symmetry in the far infrared.

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1 Introduction

Black holes with charged hair in anti-de-Sitter space provide a holographic description of superconductivity via the AdS/CFT correspondence. This idea was first discussed in [1][2] and some of the subsequent developments have been nicely reviewed in [3][4]. In more detail, one considers a CFT with an *AdS* dual in a theory of gravity with matter fields which include a Maxwell gauge field and additional charged fields. The CFT is studied at finite temperature T and at fixed chemical potential μ (or at fixed charge density) by studying electrically charged black holes in the dual gravity theory. Charged black holes with vanishing charged matter fields describe the high temperature, normal phase of the superconductor. Below some critical temperature, one requires a new branch of charged black hole solutions to emerge that carry charged hair and are thermodynamically favoured. The charged hair spontaneously breaks the $U(1)$ gauge symmetry in the bulk which corresponds to a spontaneous breaking of a global $U(1)$ symmetry in the boundary CFT, signalling the superconductivity. More precisely, this signals superfluidity, but for certain phenomena one expects that the difference between the two is not significant [5].

Most work has focussed on $D = 4$ theories of gravity, corresponding to superconductors in $d = 3$ spacetime dimensions, as this is likely to be the most promising arena to make contact with real materials. The black holes are usually taken to have flat \mathbb{R}^2 horizons, and hence are also called black branes, corresponding to considering the CFTs in $d = 3$ Minkowski spacetime. The conformal invariance then implies that the critical temperature for the onset of superconductivity is fixed by the scale set by μ .

Most studies have been carried out within the context of “phenomenological” models of gravity without any obvious embedding into string/M-theory. This bottom up approach has the virtue of simplicity but has the drawback that one is not guaranteed that there is a well defined underlying conformal field theory. Furthermore, if the models are posited to just provide approximate supergravity solutions to string/M-theory, the approximations can obscure important physical features, such as the low-temperature behaviour of the superconductors. Many investigations have also used a “probe approximation” within these phenomenological models, in which the back reaction on

the gravitational field is ignored. This approximation again makes the analysis more tractable and while it should capture some important features it cannot be used to study, for example, low-temperature phenomena. The incorporation of back reaction for the most studied class of phenomenological models with a single charged scalar field was initiated in the foundational work [5].

There has also been recent progress in a top down approach, where one aims to find exact solutions of $D = 11$ and type IIB supergravity. The constructions have been based on new consistent KK truncations of the supergravity theories. Building on the work of [6][7], fully back reacted solutions of $D = 11$ supergravity that describe holographic superconductors in $d = 3$ spacetime dimensions were constructed in [8]. In the zero temperature limit, $T \rightarrow 0$, the event horizon of the superconducting black holes disappears implying that the entropy of the dual superconductors vanishes in this limit. More precisely, it was shown in [8] (and further studied in [9]) that as $T \rightarrow 0$ the superconducting black hole solutions approach charged domain wall solutions that interpolate between (perturbed) AdS_4 solutions in the UV that preserve the $U(1)$ gauge symmetry and different AdS_4 solutions in the IR that break the gauge symmetry. These domain wall solutions demonstrate that when the dual CFT is held at zero temperature and finite chemical potential there is an emergent conformal symmetry at low energies, exactly as in a class of phenomenological models¹ studied in [11]. The $D = 11$ superconducting black hole solutions of [8] were constructed using a $D = 4$ theory of gravity, to be discussed momentarily. This $D = 4$ theory has some similarities with the $D = 4$ phenomenological model of [5] after fixing some parameters. However, there are also important differences. In particular, it was shown in [8] that the superconducting black hole solutions in the model of [5], for these parameters², become singular as $T \rightarrow 0$.

Analogous constructions of solutions of type IIB supergravity dual to superconductors in $d = 4$ spacetime dimensions have been carried out in [13][9], but not yet in the same level of detail. It has been shown that superconducting black hole solutions should exist using a probe analysis in [13]. Furthermore, zero temperature domain wall solutions with emergent conformal symmetry have been constructed [9]; while it is expected that they arise as the zero temperature limit of the superconducting black hole solutions at finite temperature, this has not yet been shown. These superconducting black hole solutions in $D = 11$ and type IIB carry abelian charge of “ R -symmetry

¹Note that models with an interesting emergent Lifshitz scaling in the IR were also studied in [10].

²The $T \rightarrow 0$ limit for other values of the parameters in the model of [5] was discussed in [12].

type” (it is an R -symmetry for supersymmetric vacua); the construction of type IIB solutions carrying a baryonic abelian charge were recently presented in [14].

In this paper we will expand upon and extend the analysis of $d = 3$ holographic superconductivity arising in $D = 11$ supergravity that was initiated in [8]. The solutions found in [8] were obtained using the consistent KK truncation [7] of $D = 11$ supergravity on a seven-dimensional Sasaki-Einstein space down to a $D = 4$ theory of gravity that includes six real scalar fields and two gauge fields. The consistency of this KK reduction means that given an arbitrary Sasaki-Einstein metric, any solution of the $D = 4$ theory can be uplifted to obtain an exact solution of $D = 11$ supergravity. It was shown in [8] that it is possible to further truncate the $D = 4$ theory to a theory with a metric, g , a gauge field, A , with field strength F , and a charged scalar field, χ , provided that one restricts to solutions satisfying $F \wedge F = 0$ (such as electrically charged black holes). One result of this paper is that there is a consistent truncation, with no additional restrictions, that is obtained by also keeping an additional neutral scalar field h . As we now discuss this truncated $D = 4$ theory with fields (g, A, χ, h) , with action given in (4.3), has a rich structure to explore different aspects of superconductivity. A particularly interesting feature of the $D = 4$ theory is that the neutral scalar field h couples to $F \wedge F$ and this coupling implies that the solutions we discuss with $h \neq 0$ break $d = 3$ parity and time reversal invariance.

The truncated theory has three AdS_4 vacua which uplift to $AdS_4 \times SE_7$ solutions in $D = 11$. One of them uplifts to the skew-whiffed AdS_4 solutions of Freund-Rubin type [15], as shown in [7], the second uplifts to the AdS_4 solutions of Pope and Warner type [16][17], as shown in [8], and the third, which is the only one with $h \neq 0$, uplifts to the AdS_4 solutions of Englert type [18][19], as we will show here. Recall that the skew-whiffed solutions do not preserve any supersymmetry, except in the special case that the SE_7 space is the round seven-sphere in which case it preserves all supersymmetry. All skew-whiffed solutions are known to be perturbatively stable [20]. Some discussion on the possibility of $1/N$ effects destabilising the non-supersymmetric skew-whiffed solutions has been discussed in [21][22]. The Pope-Warner and the Englert solutions do not preserve any supersymmetry for any choice of SE_7 . A stability analysis for the Pope-Warner solutions has not been carried out; in light of the results presented in [8] and here, we feel that this would now be a worthwhile investigation. On the other hand the Englert solutions are known to be unstable [23]: indeed we will show that an unstable mode is already present in our truncated Lagrangian.

In the skew-whiffed AdS_4 vacuum the operators $\mathcal{O}_\chi, \mathcal{O}_h$ dual to the fields χ, h , re-

spectively, are both relevant operators with scaling dimension taken³ to be $\Delta = 2$. Before investigating superconductivity, we first construct uncharged domain wall solutions (i.e. the gauge-field is identically zero) that describe ordinary RG flows between the different *AdS* vacua. We will show that there is a one parameter family of such domain wall solutions which describe RG flows between the skew-whiffed vacuum in the UV, perturbed by \mathcal{O}_χ and \mathcal{O}_h , and the Pope-Warner vacuum in the IR. As is usual for such RG flows the operators have non-zero vevs, $\langle \mathcal{O}_\chi \rangle, \langle \mathcal{O}_h \rangle \neq 0$. We will also show that there is a single domain wall solution that flows between the skew-whiffed vacuum in the UV to the Englert vacuum in the IR and also a domain wall solution that flows from the Englert vacuum in the UV to the Pope-Warner vacuum in the IR. Of course, given the instability of the Englert vacuum, the physical relevance of these latter domain wall solutions is not clear.

We then turn our attention to superconductivity. We will be interested in the CFT dual to the skew whiffed vacuum that has been deformed by \mathcal{O}_h , generalising the analysis of [8], and the solutions we construct imply that we also have $\langle \mathcal{O}_h \rangle \neq 0$. The superconductivity is signalled by a spontaneous breaking of the $U(1)$ symmetry, so we demand that the CFT is not perturbed by \mathcal{O}_χ (in contrast to the RG flow solutions discussed in the last paragraph) and look for solutions with $\langle \mathcal{O}_\chi \rangle \neq 0$.

We first construct charged domain walls that describe the deformed skew-whiffed CFT at zero temperature and non-zero chemical potential. After observing that in the Pope-Warner vacuum the gauge-field is dual to an irrelevant operator a simple parameter count suggests that, given the one parameter family of uncharged domain wall RG flow solutions mentioned above, there should also be a one parameter family of charged domain wall solutions that interpolate between the skew-whiffed vacuum in the UV and the Pope-Warner vacuum in the IR. This is in accord with the conjecture made in [9]. In the special case that $h = 0$ such a solution was found in [8] (and further studied in [9]) and here we will construct a more general one parameter family of solutions with $h \neq 0$. It is interesting to note that these solutions only exist for a certain range of deformations by \mathcal{O}_h (at fixed μ). We then show that this one parameter family of charged domain wall solutions arises as the zero temperature limit of a more general class of superconducting black holes, once again generalising what was found in [8] for $h = 0$. We first construct Reissner-Nordström-like charged black

³This arises because of the boundary conditions we shall impose on the skew-whiffed *AdS*₄ solution; different boundary conditions would lead to $\Delta = 1$.

holes⁴ with $\chi = 0$, but with $h \neq 0$, which describe the high temperature normal phase of these superconductors. At finite temperature these solutions have a regular horizon with h going to zero at the horizon. Furthermore, in the zero temperature limit, for a certain range of deformations $\mathcal{O}_h \neq 0$, they approach $AdS_2 \times \mathbb{R}^2$, exactly as for the Reissner-Nordström black hole with $h = 0$. For larger deformations, in the zero temperature limit they have vanishing entropy and become singular. We then show that for a certain range of deformations $\mathcal{O}_h \neq 0$ a new branch of black holes carrying charged scalar hair appear and that they are thermodynamically favoured, thus demonstrating that we do indeed have holographic superconductors. Our numerics indicate that the superconducting black holes exist for exactly the same class of deformations where the Reissner-Nordström like black holes have an $AdS_2 \times \mathbb{R}^2$ limit at zero temperature and hence at zero temperature the $AdS_2 \times \mathbb{R}^2$ solutions are thermodynamically disfavoured. Furthermore, we show that the solutions with charged hair smoothly map on the zero temperature charged domain wall solutions with the Pope-Warner AdS_4 region in the IR, demonstrating the emergent $d = 3$ conformal symmetry of these holographic superconductors. It is worth highlighting that in the far IR all of these superconductors (for a given SE_7 space), when held at zero temperature and finite chemical potential, are described by exactly the same universal CFT i.e. the CFT dual to the Pope-Warner AdS_4 vacuum.

If we assume that our analysis has captured all of the relevant instabilities in M-theory, the phase diagram for our new holographic superconductors, for $\mu \neq 0$, is summarised in Figure 1. The vertical axis is the temperature, the horizontal axis is the value of h_1 which determines the isotropic deformation by the operator \mathcal{O}_h . The most striking feature is the superconducting dome that appears for $-h_1^c \leq h_1 \leq h_1^c$. Under this dome at zero temperature, and in the far IR, the superconductors are all described by the same Pope-Warner AdS_4 solution. Outside of the dome at zero temperature the system is described by singular solutions which require further investigation. We will also construct black hole solutions with $\mu = 0$ and this part of the phase diagram is incorporated in Figure 14 in the discussion section.

Finally, we calculate the electrical conductivity of our black holes using linear response theory. We find that the electrical conductivity contains both longitudinal and, when $h \neq 0$, transverse (Hall) components for both the superconducting and normal phase black holes, the latter arising from the broken parity and time reversal invariance

⁴It is interesting to compare this class of charged black holes with those that were very recently constructed in a top down model in [24] and a phenomenological model in [25].

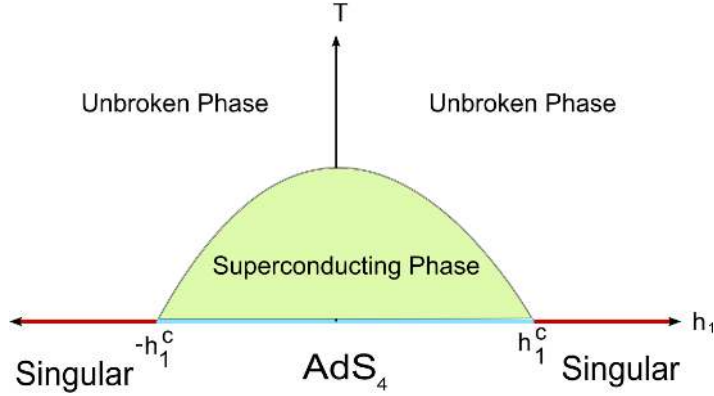


Figure 1: The phase diagram for the holographic superconductors. The vertical axis is temperature, the horizontal axis determines the deformation of the skew-whiffed CFT by the operator \mathcal{O}_h and the chemical potential μ is non-zero.

in the boundary theory.

The plan of the rest of the paper is as follows. In section 2 we briefly review the consistent KK truncation on an arbitrary Sasaki-Einstein seven-manifold that was presented in [7]. In section 3 we discuss the three AdS_4 vacua of the $D = 4$ theory and their uplifts to $D = 11$ solutions. Section 4 presents the new additional consistent KK truncation of the $D = 4$ theory. In section 5 we present the ansatz for the $D = 4$ fields that we shall use to construct black hole and domain wall solutions. We also discuss boundary counter terms in the action and some aspects of thermodynamics. As somewhat of an aside, section 6 discusses uncharged domain walls corresponding to the RG flows between the AdS_4 vacua. Section 7 discusses the zero temperature limit solutions of the charged black hole solutions that we construct in section 9. We construct the charged domain walls interpolating between the deformed skew-whiffed AdS_4 vacuum and the Pope-Warner AdS_4 vacuum and also the solutions interpolating between the skew-whiffed AdS_4 vacuum and the $AdS^2 \times \mathbb{R}^2$. Section 8 briefly discusses uncharged black hole solutions (i.e. $\mu = 0$) with $h \neq 0$. Section 9 discusses both the normal and superconducting phase charged black hole solutions for general h and presents some results on the conductivity of the black holes. We briefly conclude in section 10 and we have three appendices.

2 The consistent KK truncation of [7]

We start by summarising the consistent KK truncation of $D = 11$ supergravity on an arbitrary Sasaki-Einstein space SE_7 found in [7] (extending [26][27]). First recall that any Sasaki-Einstein metric can, locally, be written as a fibration over a six-dimensional Kähler-Einstein space

$$ds^2(SE_7) \equiv ds^2(KE_6) + \eta \otimes \eta \quad (2.1)$$

Here η is the one-form dual to the Reeb Killing vector satisfying $d\eta = 2J$ where J is the Kähler form of KE_6 . We denote the $(3, 0)$ form defined on KE_6 by Ω and $d\Omega = 4i\eta \wedge \Omega$. The volume form is taken to be $vol(SE_7) = \eta \wedge J^3/3! = (i/8)\eta \wedge \Omega \wedge \Omega^*$.

For a regular or quasi-regular Sasaki-Einstein manifold, the orbits of the Reeb vector all close, corresponding to compact $U(1)$ isometry, and the KE_6 is a globally defined manifold or orbifold, respectively. For an irregular Sasaki-Einstein manifold, the Reeb-vector generates a non-compact \mathbb{R} isometry and the KE_6 is only locally defined. For applications to holographic superconductivity one is most interested in cases with $U(1)$ isometry.

In the KK ansatz the $D = 11$ metric is written as

$$\frac{1}{(2L)^2} ds^2 = e^{-6U-V} ds_4^2 + e^{2U} ds^2(KE_6) + e^{2V} (\eta + A_1) \otimes (\eta + A_1) , \quad (2.2)$$

while the four-form is written

$$\begin{aligned} \frac{1}{(2L)^3} G_4 = & 6e^{-18U-3V} (\epsilon + h^2 + |\chi|^2) vol_4 + H_3 \wedge (\eta + A_1) + H_2 \wedge J \\ & + dh \wedge J \wedge (\eta + A_1) + 2hJ \wedge J \\ & + \sqrt{3} [\chi(\eta + A_1) \wedge \Omega - \frac{i}{4} D\chi \wedge \Omega + \text{c.c.}] . \end{aligned} \quad (2.3)$$

where ds_4^2 is a four-dimensional metric⁵, U, V, h are real scalars, χ is a complex scalar defined on the four-dimensional space. Furthermore, also defined on this four-dimensional space are A_1 a one-form potential, with field strength $F_2 \equiv dA_1$, two-form and three-form field strengths H_2 and H_3 , related to one-form and two-form potentials via

$$\begin{aligned} H_3 &= dB_2 \\ H_2 &= dB_1 + 2B_2 + hF_2 \end{aligned} \quad (2.4)$$

Note that the scalar χ is charged with respect to A_1 and in particular we have $D\chi \equiv d\chi - 4iA_1\chi$. Also note that ϵ appearing in the four-form flux is a constant:

$$\epsilon = \pm 1 \quad (2.5)$$

⁵Note that we are using the four-dimensional Einstein frame metric.

whose significance will be explained below. Our conventions for $D = 11$ supergravity are as in [28]; in particular we note that the $D = 11$ volume form is given by $vol_4 \wedge vol(SE_7)$. Finally, L is an arbitrary length scale which we will later set to $1/2$.

This provides a consistent KK truncation of $D = 11$ supergravity in the sense that if one finds a solution to the equations of motion for the $D = 4$ metric g and matter fields $U, V, A_1, B_1, B_2, h, \chi$, as given in [7], then one has found a solution to the $D = 11$ supergravity equations of motion. The $D = 4$ equations of motion can be derived from an action given in [7]. In this paper we will find it convenient to work with an action that is obtained after dualising the one-form B_1 to another one form \tilde{B}_1 and the two-form B_2 to a scalar a as explained in section 2.3 of [7]. The dual action is given by

$$\begin{aligned}
S = & \frac{(2L)^2}{16\pi G} \int d^4x \sqrt{-g} \left[R - 24(\nabla U)^2 - \frac{3}{2}(\nabla V)^2 - 6\nabla U \cdot \nabla V - \frac{3}{2}e^{-4U-2V}(\nabla h)^2 \right. \\
& - \frac{3}{2}e^{-6U}|D\chi|^2 - \frac{1}{4}e^{6U+3V}F_{\mu\nu}F^{\mu\nu} - \frac{3}{4}\frac{e^{2U+V}}{4h^2 + e^{4U+2V}}(\tilde{H}_2 + h^2F_2)_{\mu\nu}(\tilde{H}_2 + h^2F_2)^{\mu\nu} \\
& - \frac{1}{2}e^{-12U} \left[\nabla a + 6(\tilde{B}_1 - \epsilon A_1) - \frac{3}{4}i(\chi^* D\chi - \chi D\chi^*) \right]^2 + 48e^{-8U-V} - 6e^{-10U+V} \\
& \left. - 24h^2e^{-14U-V} - 18(\epsilon + h^2 + |\chi|^2)^2 e^{-18U-3V} - 24e^{-12U-3V}|\chi|^2 \right] \\
& + \frac{(2L)^2}{16\pi G} \int \left[-h^3F_2 \wedge F_2 - 3h\tilde{H}_2 \wedge F_2 + \frac{3h}{(4h^2 + e^{4U+2V})}(\tilde{H}_2 + h^2F_2) \wedge (\tilde{H}_2 + h^2F_2) \right].
\end{aligned} \tag{2.6}$$

where $\tilde{H}_2 \equiv d\tilde{B}_1$. The dual fields are related to the original fields via

$$\begin{aligned}
H_2 &= \frac{1}{4h^2 + e^{4U+2V}} \left[2h(\tilde{H}_2 + h^2F_2) - e^{2U+V} * (\tilde{H}_2 + h^2F_2) \right] \\
H_3 &= -e^{-12U} * \left[da + 6(\tilde{B}_1 - \epsilon A_1) + \frac{3}{4}i(\chi^* D\chi - \chi D\chi^*) \right]
\end{aligned} \tag{2.7}$$

Note that the factors of L appear in a more conventional way if one uses the rescaled metric $\tilde{g} = (2L)^2g$.

3 AdS_4 vacuum solutions

The $D = 4$ equations of motion arising from (2.6) admit various AdS_4 solutions with $F_2 = \tilde{H}_2 = a = 0$ and for various values of the scalar fields U, V, h and χ . In the following $ds^2(AdS_4)$ will always denote the standard unit radius AdS_4 metric and $Vol(AdS_4)$ the corresponding volume-form as given in appendix A of [7]. We will determine the masses of the other fields considered as perturbations around each AdS_4 solution to obtain the scaling dimensions of the dual operators in the boundary CFT. For scalar

fields with mass m the scaling dimensions are given by

$$\Delta = \frac{3}{2} \pm \frac{1}{2}[9 + 4m^2 R_{AdS}^2]^{1/2} \quad (3.1)$$

while those for vector fields are given by

$$\Delta = \frac{3}{2} \pm \frac{1}{2}[1 + 4m^2 R_{AdS}^2]^{1/2} \quad (3.2)$$

where R_{AdS}^2 is the radius squared of the Einstein frame AdS metric g . Note that if we used the metric $\tilde{g} = (2L)^2 g$ then $R_{AdS}^2 \rightarrow (2L)^2 R_{AdS}^2$ and $m^2 \rightarrow m^2/(2L)^2$.

3.1 $AdS_4 \times SE_7$ solutions: supersymmetric and skew-whiffed

The simplest AdS_4 vacua have $\epsilon = \pm 1$,

$$U = 0, \quad V = 0, \quad \chi = 0, \quad h = 0 \quad (3.3)$$

and the radius squared of the Einstein AdS_4 metric is given by

$$R_{AdS}^2 = \frac{1}{4}. \quad (3.4)$$

These uplift to the $D = 11$ solutions:

$$\begin{aligned} \frac{1}{(2L)^2} ds^2 &= \frac{1}{4} ds^2(AdS_4) + ds^2(SE_7) \\ \frac{1}{(2L)^3} G_4 &= \epsilon \frac{3}{8} Vol(AdS_4) \end{aligned} \quad (3.5)$$

When $\epsilon = +1$, these $AdS_4 \times SE_7$ solutions are supersymmetric and are dual to $d = 3$ SCFTs with, generically, $\mathcal{N} = 2$ supersymmetry. For these solutions the Killing vector dual to the one-form η in the SE_7 metric (2.1) is dual to an R -symmetry. On the other hand when $\epsilon = -1$ the solutions are skew-whiffed $AdS_4 \times SE_7$ solutions and generically do not preserve any supersymmetry at all. An important exception is when SE_7 is the round seven-sphere in which case both AdS_4 solutions preserve maximal supersymmetry. The skew-whiffed solutions have been shown to be perturbatively stable [20] and thus should be dual to well defined CFTs at least in the supergravity approximation. Some discussion on the possibility of $1/N$ effects destabilising the non-supersymmetric skew-whiffed solutions has been discussed in [21][22] and it would be interesting to explore this further. Note that for the skew-whiffed $AdS_4 \times SE_7$ solutions the Killing vector dual to the one-form η in the SE_7 metric (2.1) is dual to a global symmetry in the dual CFT and is an R -symmetry just for the case of $SE_7 = S^7$.

The spectrum of the $D = 4$ theory in these backgrounds was discussed in [7]. For $\epsilon = +1$, $m_h^2 = 40$, $m_\chi^2 = 40$ and U, V mix to give $m^2 = 16, 72$. These give scaling dimensions $\Delta = 5, 5$ and $\Delta = 4, 6$ respectively. There is also a massless gauge field and a massive gauge field with $m^2 = 48$ corresponding to $\Delta = 2$ and $\Delta = 5$, respectively.

When $\epsilon = -1$, U, V mix to again give $m^2 = 16, 72$, corresponding to scaling dimensions $\Delta = 4, 6$ respectively. On the other hand now $m_h^2 = m_\chi^2 = -8$ with each corresponding to scaling dimension $\Delta_\pm = 1, 2$. The masses of the gauge fields are unchanged. Note that in both cases the field a becomes the longitudinal mode of the massive gauge field.

3.2 Pope-Warner solutions

Another AdS_4 vacuum is obtained when $\epsilon = -1$,

$$e^U = 2^{-1/6}, \quad e^V = 2^{1/3}, \quad \chi^2 = 2/3, \quad h = 0 \quad (3.6)$$

and the radius squared of the Einstein AdS_4 metric is given by

$$R_{AdS}^2 = \frac{3}{16} \quad (3.7)$$

Choosing $\chi = +(2/3)^{1/2}$ for definiteness we find that this uplifts to the $D = 11$ class solutions

$$\begin{aligned} \frac{1}{(2L)^2} ds^2 &= 2^{2/3} \left[\frac{3}{16} ds^2(AdS_4) + \frac{1}{2} ds^2(KE_6) + \eta \otimes \eta \right] \\ \frac{1}{(2L)^3} G_4 &= 2 \left[-\frac{9}{64} Vol(AdS_4) + \frac{1}{\sqrt{2}} (\eta \wedge \Omega + c.c) \right] \end{aligned} \quad (3.8)$$

which were first constructed in [16][17]. It has been shown that they do not preserve any supersymmetry but a stability analysis has not yet been performed. Note that these solutions are topologically $AdS_4 \times SE_7$: the fibre of the SE_7 metric being stretched by a factor of $\sqrt{2}$ compared to the SE_7 metric in (2.1).

In this background, the $D = 4$ theory gives two massive scalars with $m^2 = 32$ and two with $m^2 = 96$ corresponding to scaling dimensions $\Delta = 3/2 + \sqrt{33}/2$ and $\Delta = 6$, respectively. There are also two massive gauge fields with $m^2 = 32$ and $m^2 = 96$ corresponding to scaling dimensions $\Delta = 4$ and $\Delta = 3/2 + \sqrt{73}/2$, respectively. Note that the phase of χ and the field a become longitudinal modes of the massive gauge-fields.

3.3 Englert solutions

Another AdS_4 vacuum is obtained when $\epsilon = -1$,

$$e^U = (4/5)^{1/6}, \quad e^V = (4/5)^{1/6}, \quad \chi^2 = 4/15, \quad h^2 = 1/5 \quad (3.9)$$

and the radius squared of the Einstein AdS_4 metric is given by

$$R_{AdS}^2 = \frac{12}{25\sqrt{5}} \quad (3.10)$$

Choosing $\chi = +(2/15)^{1/2}$, $h = +1/\sqrt{5}$ for definiteness we find that this uplifts to the $D = 11$ class of solutions

$$\begin{aligned} \frac{1}{(2L)^2} ds^2 &= \left(\frac{4}{5}\right)^{1/3} \left[\frac{3}{10} ds^2(AdS_4) + ds^2(KE_6) + \eta \otimes \eta \right] \\ \frac{1}{(2L)^3} G_4 &= \left(\frac{4}{5}\right)^{1/2} \left[-\frac{9}{25} Vol(AdS_4) + J \wedge J + (\eta \wedge \Omega + c.c) \right] \end{aligned} \quad (3.11)$$

This an Englert-type solution and note that the metric on SE_7 is exactly the same as in (2.1). For the special case when $SE_7 = S^7$ it was first constructed in [18] and the generalisation was suggested in [19]. This solution is known not to preserve any supersymmetry (for the S^7 case this was shown in [29] and the results of [23] show this more generally) and to be unstable [23].

In this background, the $D = 4$ theory gives four massive scalars with $m^2 = -5\sqrt{5}$, $25\sqrt{5}/2$, $20\sqrt{5}$, $75\sqrt{5}/2$. Note that the first mode has complex scaling dimension and thus violates the BF bound. This unstable mode, as well as the other three modes, are precisely the same modes considered in [23]. The scaling dimensions of the three other scalars are $\Delta = 3/2 + \sqrt{33}/2$, $3/2 + (237/20)^{1/2}$, 6 , respectively. There are also two massive gauge fields with $m^2 = 5\sqrt{5}$ and $m^2 = 30\sqrt{5}$ corresponding to scaling dimensions $\Delta = 3/2 + (53/20)^{1/2}$ and $\Delta = 3/2 + (293/20)^{1/2}$, respectively.

3.4 Flux quantisation and central charges

The $D = 11$ equation of motion for the four-form is $d * G_4 + \frac{1}{2} G_4 \wedge G_4 = 0$ and the quantised membrane charge is given by

$$N = \frac{1}{(2\pi l)^6} \int (*G_4 + \frac{1}{2} C_3 \wedge G_4) \quad (3.12)$$

where $dC_3 = G_4$ and l is the $D = 11$ Planck length. For each of the above solutions with $\epsilon = -1$, we find

$$N = 6 \left(\frac{(2L)}{2\pi l} \right)^6 vol(SE_7) \quad (3.13)$$

Note that in our conventions this is counting the number of anti-membranes. We can also calculate the central charge using the formula [30]

$$c \equiv 384 \frac{(2L)^2 R_{AdS}^2}{16\pi G} \quad (3.14)$$

In our conventions the D=11 action is

$$S = \frac{1}{(2\pi)^8 l^9} \int d^{11}x \sqrt{-g_{11}} [R_{11} + \dots] \quad (3.15)$$

and hence

$$\frac{1}{16\pi G} = \frac{(2L)^7 \text{vol}(SE_7)}{(2\pi)^8 l^9} \quad (3.16)$$

We thus deduce

$$c = \frac{128\pi N^{3/2}}{6^{1/2} \text{vol}(SE_7)^{1/2}} R_{AdS}^2 \quad (3.17)$$

By comparing R_{AdS}^2 of the different AdS_4 vacua, one concludes that it might be possible to find a domain wall solution that interpolates between a perturbed skew-whiffed vacuum in the UV and the Pope-Warner vacuum in the IR, which would be dual to an RG flow between the corresponding dual CFTs. We shall see later that this is indeed the case. In fact we will see that there is a one parameter family of domain walls that interpolates between these vacua. We will also find a domain wall solution that interpolates from the skew-whiffed vacuum in the UV to the Englert vacuum in the IR and another domain wall solution that interpolates between the Englert vacuum in the UV and the Pope-Warner in the IR. The instability of the Englert vacuum makes these latter solutions less interesting as far as RG flows are concerned.

4 Further consistent KK truncation

When $\epsilon = -1$ there is a further consistent truncation of the $D = 4$ theory described by (2.6) that is obtained by setting

$$\begin{aligned} a &= 0 \\ \tilde{B}_1 &= -A_1 \\ e^{6U} &= 1 - \frac{3}{4}|\chi|^2 \\ e^{6V} &= \frac{(1 - h^2)^3}{(1 - \frac{3}{4}|\chi|^2)^2} \end{aligned} \quad (4.1)$$

Note that this implies

$$\begin{aligned} H_3 &= \frac{3i}{4(1 - \frac{3}{4}|\chi|^2)^2} * [\chi^* D\chi - \chi D\chi^*] \\ H_2 &= \frac{(1 - h^2)}{1 + 3h^2} [-2hF_2 + (1 - h^2)^{1/2} * F_2] \end{aligned} \quad (4.2)$$

After substituting this into the equations of motion derived from (2.6) (see appendix B of [7]) we obtain equations that can be derived from the following action

$$\begin{aligned} S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} & \left[R - \frac{(1 - h^2)^{3/2}}{1 + 3h^2} F_{\mu\nu} F^{\mu\nu} - \frac{3}{2(1 - \frac{3}{4}|\chi|^2)^2} |D\chi|^2 \right. \\ & \left. - \frac{3}{2(1 - h^2)^2} (\nabla h)^2 - \frac{24(-1 + h^2 + |\chi|^2)}{(1 - \frac{3}{4}|\chi|^2)^2 (1 - h^2)^{3/2}} \right] + \frac{1}{16\pi G} \int \frac{2h(3 + h^2)}{(1 + 3h^2)} F \wedge F \end{aligned} \quad (4.3)$$

This action is also obtained by substituting the ansatz (4.1) directly into (2.6). From now on we have set $L = 1/2$. Observe that we must have $|\chi| < \frac{2}{\sqrt{3}}$ and $|h| < 1$. We also observe that all terms in the Lagrangian except for the coupling of h to $F \wedge F$ are invariant under $h \rightarrow -h$. As we shall see later this coupling leads to breaking of parity and time reversal invariance in the dual CFT.

It is worth pointing out that there are further consistent truncations that one can consider. For example, one can consistently set $\chi = 0$ to obtain a theory of gravity coupled to a gauge field A_1 and a neutral scalar h , and we will find black hole solutions of this theory (it is interesting to compare and contrast with the phenomenological models studied in [24][25]). It is also possible to then further consistently set $A_1 = 0$. Alternatively, we can set $A_1 = 0$ provided that we restrict χ to be real: $\chi = \chi_R$. This theory with two real scalars can then be further consistently truncated⁶ by either setting $\chi_R = (2/\sqrt{3})h$ or by setting $h = 0$. Finally, observe that we can set $h = 0$ provided that we impose $F \wedge F = 0$ by hand [8] (and hence setting $h = 0$ is not a consistent KK truncation).

Observe that all of the AdS_4 vacua discussed in the last subsection are solutions of the consistent truncation (4.3). For the skew-whiffed AdS_4 solution the perturbed fields $\delta\chi$ and δh have masses given by $m_\chi^2 = m_h^2 = -8$, as before, and the gauge field is massless. For the Pope-Warner AdS_4 solution we have $m_\chi^2 = m_h^2 = 32$ and the

⁶Observe that this latter theory of a single real scalar field can be obtained from the truncation considered in equation (3.3) of [7]. In particular, we obtain the same theory after setting $e^{2v} = 1 - h^2$ in (3.3) of [7], after flipping the sign of 3 in the last term in the potential which is needed for the skew-whiffed case with $\epsilon = -1$.

gauge field is massive with $m^2 = 32$. For the Englert solution we find that the scalar perturbations mix and that $\delta h - (\sqrt{3}/2)\delta\chi$ and $\delta\chi + (\sqrt{3}/2)\delta h$ have mass squared $-5\sqrt{5}$ and $25\sqrt{5}/2$, respectively. In particular, the unstable, BF violating mode about the Englert solution is contained within the truncation (4.3). The gauge field is massive in the Englert solution with $m^2 = 5\sqrt{5}$.

In the skew-whiffed vacuum h, χ are dual to operators $\mathcal{O}_h, \mathcal{O}_\chi$ each with conformal dimension $\Delta_\pm = 1, 2$. It is worth emphasising that in the solutions of the truncated theory (4.3) with $h, \chi \neq 0$, in addition to activating these operators in the dual CFT we are also activating the operators with $\Delta = 4, 6$ that are dual to linear combinations of the fields U, V , via (4.1). More precisely, in the solutions we shall consider that asymptotically approach the skew-whiffed AdS_4 vacuum, the asymptotic falloffs of h, χ will include cases where the skew-whiffed CFT is deformed by \mathcal{O}_h and, for the RG flows in section 6 only, by \mathcal{O}_χ , and also where these operators acquire vevs. This means that the operators dual to U, V are not deforming the skew-whiffed CFT but they are acquiring vevs which can easily be worked out for our solutions. However, we will not include the details.

Finally, we observe that if we set $h = 0$ in the truncated action (4.3) (which is only possible for configurations with $F \wedge F = 0$), and then linearize in $|\chi|$, we make contact with the model with a simple mass term considered in [5] (after rescaling their gauge field by a factor of 2 and setting their $q = 2$). Also when $h = 0$ the action (4.3) is in the class considered in [31].

5 Ansatz for black hole and domain wall solutions

For the remainder of the paper we will consider the following ansatz for the $D = 4$ fields in the truncated theory described by the action (4.3). For the metric we take

$$ds^2 = -ge^{-\beta}dt^2 + g^{-1}dr^2 + r^2(dx^2 + dy^2) \quad (5.1)$$

where g, β are functions of r only. The gauge-field A_1 is taken to be purely electric

$$A_1 = \phi(r)dt, \quad (5.2)$$

and we will also impose

$$\chi \equiv \xi(r) \in \mathbb{R}, \quad h = h(r) \quad (5.3)$$

We now substitute into the equations of motion arising from the action (4.3). After some calculation we are led to five ordinary differential equations, which we have

presented in (A.1), (A.2), (A.3), (A.5) and (A.6), for five real functions ϕ, ξ, h, g and β . These equations can also be obtained from an action obtained by substituting the above ansatz directly in to the action (4.3):

$$\begin{aligned}
S = c_0 \int dr r^2 e^{-\beta/2} & \left[-g'' + g' \left(\frac{3}{2} \beta' - \frac{4}{r} \right) + g \left(\beta'' - \frac{1}{2} (\beta')^2 + 2 \frac{\beta'}{r} - \frac{2}{r^2} \right) \right. \\
& - \frac{3(g\xi'^2 - 16g^{-1}e^\beta \phi^2 \xi^2)}{2(1 - \frac{3}{4}\xi^2)^2} - \frac{3g(h')^2}{2(1 - h^2)^2} + \frac{2(1 - h^2)^{3/2} e^\beta \phi'^2}{1 + 3h^2} \\
& \left. - \frac{24(-1 + h^2 + \xi^2)}{(1 - \frac{3}{4}\xi^2)^2 (1 - h^2)^{3/2}} \right]
\end{aligned} \tag{5.4}$$

where

$$c_0 \equiv \frac{1}{16\pi G} \int dt dx dy \tag{5.5}$$

It will be helpful to note the following scaling symmetries:

$$r \rightarrow ar, \quad (t, x, y) \rightarrow a^{-1}(t, x, y), \quad g \rightarrow a^2 g, \quad \phi \rightarrow a\phi, \tag{5.6}$$

and

$$e^\beta \rightarrow a^2 e^\beta, \quad t \rightarrow at, \quad \phi \rightarrow a^{-1} \phi \tag{5.7}$$

which leave the metric, A_1 , and all equations of motion invariant. Notice that this ansatz has the symmetry $h \rightarrow -h$. We also have the \mathbb{Z}_2 symmetries $\xi \rightarrow -\xi$ and $\phi \rightarrow -\phi$. Also notice that it is consistent to separately set $\xi = 0$, $\phi = 0$ or $h = 0$.

We will be almost exclusively interested in solutions that asymptote to a perturbation of the skew-whiffed AdS_4 vacuum. We recall that a scalar field dual to an operator in the CFT with scaling dimension Δ has the two asymptotic behaviours

$$r^{\Delta-3} \quad \text{and} \quad r^{-\Delta} \tag{5.8}$$

whereas a vector field behaves as

$$r^{\Delta-2} \quad \text{and} \quad r^{1-\Delta} \tag{5.9}$$

We thus focus on the asymptotic expansion ⁷

$$\begin{aligned}
g &= 4r^2 + 16\pi G (h_1^2 + \xi_1^2) - 8\pi G[\varepsilon - 4\xi_1\xi_2 - 4h_1h_2]\frac{1}{r} + \dots \\
\beta &= \beta_a + 4\pi G (h_1^2 + \xi_1^2) \frac{1}{r^2} + \frac{32\pi G}{3}(h_1h_2 + \xi_1\xi_2)\frac{1}{r^3} + \dots \\
\xi &= \sqrt{\frac{16\pi G}{3}} \left[\frac{\xi_1}{r} + \frac{\xi_2}{r^2} + \dots \right] \\
h &= \sqrt{\frac{16\pi G}{3}} \left[\frac{h_1}{r} + \frac{h_2}{r^2} + \dots \right] \\
\phi &= \sqrt{4\pi G} e^{-\beta_a/2} \left[\mu - \frac{q}{r} + \dots \right]
\end{aligned} \tag{5.10}$$

It is worthwhile noting that

$$e^{-\beta}g = e^{-\beta_a} \left[4r^2 - \frac{8\pi G[\varepsilon + \frac{4}{3}\xi_1\xi_2 + \frac{4}{3}h_1h_2]}{r} + \dots \right] \tag{5.11}$$

5.1 Skew-whiffed to Pope-Warner uncharged and charged domain walls

As we will discuss in more detail later, we will be interested in both charged and uncharged domain wall solutions that asymptote in the IR to the Pope-Warner vacuum as $r \rightarrow 0$. Hence we will consider the following expansion

$$\begin{aligned}
\xi &= \sqrt{\frac{2}{3}} + a_\xi r^{\Delta_{\text{PW}}^\xi - 3} + \dots \\
h &= a_h r^{\Delta_{\text{PW}}^h - 2} + \dots \\
\beta &= a_\beta + \dots \\
g &= \frac{16r^2}{3} + \dots \\
\phi &= a_\phi r^{\Delta_{\text{PW}}^\phi - 2} + \dots
\end{aligned} \tag{5.12}$$

where $\Delta_{\text{PW}}^\xi = \Delta_{\text{PW}}^h = (3 + \sqrt{33})/2$ and $\Delta_{\text{PW}}^\phi = 4$. Here $\{a_\xi, a_h, a_\beta, a_\phi\}$ parametrises all possible marginal and irrelevant operators of the Pope-Warner vacuum within our truncation.

⁷To compare with [8] we should identify $\varepsilon, \mu, q, \xi_i$ with $m, \hat{\mu}, \hat{q}, \sigma_i$, respectively.

5.2 Skew-whiffed to $AdS_2 \times \mathbb{R}^2$ solutions

The Reissner-Nordström electrically charged black hole solution (with planar horizon) is given by,

$$g = 4r^2 - \frac{1}{r}(4r_+^3 + \frac{\alpha^2}{r_+}) + \frac{\alpha^2}{r^2}, \quad \phi = \alpha(\frac{1}{r_+} - \frac{1}{r}) \quad (5.13)$$

with $h = \xi = \beta = 0$, where $\alpha = \sqrt{4\pi G}q$, $r_+ = q/\mu$. At zero temperature, when $\alpha^2 = 12r_+^4$ and $g = 4(r - r_+)^2(r^2 + 2r_+r + 3r_+^2)/r^2$, the near horizon geometry of this solution is given by $AdS_2 \times \mathbb{R}^2$ with the AdS_2 having radius squared $1/24$. In other words, at zero temperature this charged solution interpolates between skew-whiffed AdS_4 in the UV and $AdS_2 \times \mathbb{R}^2$ in the IR.

Later we will consider deformations of this interpolating solution by marginal and irrelevant operators with respect to the AdS_2 factor in the IR which allow us to find zero temperature charged solutions with $h \neq 0$ and $\chi = 0$ which again interpolate from a deformed skew-whiffed AdS_4 vacuum in the UV to an $AdS_2 \times \mathbb{R}^2$ geometry in the IR. Specifically we find the $AdS_2 \times \mathbb{R}^2$ IR geometry is asymptotically given by

$$\begin{aligned} h &= h_+(r - r_+)^{(\sqrt{105}-3)/6} + \dots \\ \phi &= 2\sqrt{3}e^{-\beta_+/2} \left(\frac{1}{r_+} - \frac{1}{r} \right) + \dots \\ g &= 4r^2 - \frac{1}{r} \left(4r_+^3 + \frac{12}{r_+} \right) + \frac{12}{r^2} + \dots \\ \beta &= \beta_+ + \dots \end{aligned} \quad (5.14)$$

where β_+ parameterizes the only marginal deformation, and h_+ the only irrelevant deformation in our ansatz.

5.3 Black hole solutions

For the general black hole solutions we demand that there is a regular finite temperature horizon located at $r = r_+$. Specifically we impose that

$$g(r_+) = \phi(r_+) = 0 \quad (5.15)$$

and so the solution is specified by five parameters at the horizon

$$r_+, \quad \beta_+ = \beta(r_+), \quad \phi_+ = \phi'(r_+), \quad \xi_+ = \xi(r_+), \quad h_+ = h(r_+) \quad (5.16)$$

(we will return to this counting later). We have the expansion as $r \rightarrow r_+$

$$\begin{aligned}
\xi &= \xi_+ + \xi_+^{(1)}(r - r_+) + \dots \\
h &= h_+ + h_+^{(1)}(r - r_+) + \dots \\
\phi &= \phi_+(r - r_+) + \phi_+^{(2)}(r - r_+)^2 \dots \\
g &= g_+^{(1)}(r - r_+) + g_+^{(2)}(r - r_+)^2 \dots \\
\beta &= \beta_+ + \beta_+^{(1)}(r - r_+) + \dots
\end{aligned} \tag{5.17}$$

where for example,

$$g_+^{(1)} = \frac{r_+ \left[12(1 + 3h_+^2)(-1 + h_+^2 + \xi_+^2) + e^{\beta_+} \phi_+^2 (1 - h_+^2)^3 \left(1 - \frac{3}{4}\xi_+^2\right)^2 \right]}{\sqrt{1 - h_+^2} \left(1 - \frac{3}{4}\xi_+^2\right)^2 (-1 - 2h_+^2 + 3h_+^4)} \tag{5.18}$$

and analogous expressions can be obtained for $g_+^{(2)}, \xi_+^{(1)}, h_+^{(1)}, \beta_+^{(1)}, \phi_+^{(2)}, \dots$ in terms of the data given in (5.16).

5.4 Counter terms and black hole thermodynamics

To calculate thermodynamic quantities for the black hole solutions we would like to calculate the on-shell Euclidean action I . As usual this will require adding counter terms which are also relevant for the domain wall solutions. Interesting early work analysing the thermodynamics of R -charged Reissner-Nordström black holes using AdS/CFT techniques was carried out in [32][33].

We analytically continue by writing

$$I = -iS, \quad t = -i\tau \tag{5.19}$$

The temperature of the black hole is $T = e^{\beta_a/2}/\Delta\tau$ where $\Delta\tau$ is fixed by demanding regularity of the Euclidean metric at $r = r_+$. We find using (A.5) and (5.17) that

$$\begin{aligned}
T &= \frac{e^{\beta_a/2}}{4\pi} [g'e^{-\beta/2}]_{r=r_+} \\
&= \frac{r_+ e^{\beta_a/2}}{4\pi} \left[\frac{12e^{-\beta/2}(1 - h^2 - \xi^2)}{(1 - \frac{3}{4}\xi^2)^2(1 - h^2)^{3/2}} - \frac{e^{\beta/2}(1 - h^2)^{3/2}}{1 + 3h^2} (\phi')^2 \right]_{r=r_+}
\end{aligned} \tag{5.20}$$

As explained in appendix B, we find that the on-shell action can be expressed as

$$I_{OS} = \frac{\Delta\tau vol_2}{16\pi G} \int_{r_+}^{\infty} dr [2rge^{-\beta/2}]' \tag{5.21}$$

where $\text{vol}_2 \equiv \int dx dy$. Since $g(r_+) = 0$, this expression only gets contributions from the on-shell functions at $r = \infty$. An alternative expression for the on-shell action is given by

$$I_{OS} = \frac{\Delta\tau \text{vol}_2}{16\pi G} \int_{r_+}^{\infty} dr \left[r^2 e^{-\beta/2} (g' - g\beta' - 4e^{\beta} \frac{(1-h^2)^{3/2}}{1+3h^2} \phi\phi') \right]' \quad (5.22)$$

which gets contributions from both $r = r_+$ and $r = \infty$.

The on-shell action diverges and we need to regulate by adding appropriate counter terms. By examining the asymptotic expansion of the fields given in (5.10), we find that the following counter-term action renders the total action finite:

$$I_{ct} = \frac{1}{16\pi G} \int d\tau d^2x \sqrt{g_{\infty}} [-2K + 8 + 3\xi^2 + 3h^2] \quad (5.23)$$

where $\sqrt{g_{\infty}} = \lim_{r \rightarrow \infty} g^{1/2} r^2 e^{-\beta/2}$ and $K = \lim_{r \rightarrow \infty} g^{\mu\nu} \nabla_{\mu} n_{\nu}$ is the trace of the extrinsic curvature. For the class of solutions under consideration we find

$$I_{ct} = \frac{\Delta\tau \text{vol}_2}{16\pi G} \lim_{r \rightarrow \infty} e^{-\beta/2} [-r^2 e^{\beta} (ge^{-\beta})' - 4rg + r^2 g^{1/2} (8 + 3\xi^2 + 3h^2)] \quad (5.24)$$

Defining

$$I_{Tot} = I + I_{ct} \quad (5.25)$$

we find that corresponding to the two expressions (5.21), (5.22), the on-shell total action can be written as

$$\begin{aligned} [I_{Tot}]_{OS} &= \frac{\text{vol}_2}{T} \left(-\frac{1}{2}\varepsilon - 2\xi_1\xi_2 - 2h_1h_2 \right) \\ &= \frac{\text{vol}_2}{T} (\varepsilon - \mu q - Ts), \end{aligned} \quad (5.26)$$

respectively, where we have defined the entropy density s to be

$$s = \frac{(r_+^2)}{4G} \quad (5.27)$$

(the total entropy is $s \text{vol}_2$). The equality of these two expressions imply the Smarr-type relation

$$\frac{3}{2}\varepsilon = \mu q + Ts - 2\xi_1\xi_2 - 2h_1h_2 \quad (5.28)$$

A variation of the action I yields the equations of motion together with surface

terms. For an on-shell variation the only terms remaining are the surface terms

$$\begin{aligned}
\delta I_{OS} = & \frac{\Delta \tau vol_2}{16\pi G} \int dr \partial_r \left\{ \delta g [e^{-\beta/2} r (2 - r\beta')] + \delta g' [e^{-\beta/2} r^2] \right. \\
& + \delta \beta \left[\frac{1}{2} e^{-\beta/2} r^2 (g\beta' - g') \right] - \delta \beta' [e^{-\beta/2} r^2 g] \\
& + \delta \xi \left[\frac{3r^2 e^{-\beta/2} g}{(1 - \frac{3}{4}\xi^2)^2} \xi' \right] + \delta h \left[3r^2 e^{-\beta/2} g \frac{h'}{(1 - h^2)^2} \right] \\
& \left. - \delta \phi \left[4r^2 e^{\beta/2} \frac{(1 - h^2)^{3/2}}{1 + 3h^2} \phi' \right] \right\} \quad (5.29)
\end{aligned}$$

In the Euclidean black hole solution the only boundary is the conformal boundary $r \rightarrow \infty$ and hence this integral only gets contributions there. In addition one must also add the variation of the counter terms,

$$\begin{aligned}
\delta I_{ct} = & \frac{\Delta \tau vol_2}{16\pi G} \lim_{r \rightarrow \infty} \left\{ \delta g [e^{-\beta/2} r (r\beta' - 4 + \frac{1}{2} r (8 + 3\xi^2 + 3h^2) g^{-1/2})] - \delta g' [e^{-\beta/2} r^2] \right. \\
& + \delta \beta \left[\frac{1}{2} r e^{-\beta/2} (r g' - r g \beta' + 4g - r g^{1/2} (8 + 3\xi^2 + 3h^2)) \right] + \delta \beta' [e^{-\beta/2} r^2 g] \\
& \left. + \delta \xi [6e^{-\beta/2} r^2 g^{1/2} \xi] + \delta h [6e^{-\beta/2} r^2 g^{1/2} h] \right\} \quad (5.30)
\end{aligned}$$

Combining these expressions we deduce that

$$[\delta I_{Tot}]_{OS} = \Delta \tau vol_2 e^{-\beta_a/2} \left[(-\frac{1}{2}\varepsilon + \frac{1}{2}\mu q) \delta \beta_a - q \delta \mu - 4\xi_2 \delta \xi_1 - 4h_2 \delta h_1 \right] \quad (5.31)$$

(which corrects equation (17) of [8] by a factor of 16π). Note that we are keeping $\Delta \tau$ fixed in this variation, and hence $\delta \beta_a = 2\delta T/T$. Hence we see that I_{Tot} is stationary for fixed temperature and chemical potential (i.e. $\delta \beta_a = \delta \mu = 0$) and for either $\xi_2 = 0$ or fixed ξ_1 and similarly either $h_2 = 0$ or fixed h_1 . In our applications we will always fix ξ_1 and h_1 .

We now define the thermodynamic potential for a grand canonical ensemble via $W \equiv T[I_{Tot}]_{OS} \equiv w vol_2$. From the above variations we see that $w = w(T, \mu, \xi_1, h_1)$ and using the second expression in (5.26) we deduce the first law

$$\delta w = -s \delta T - q \delta \mu - 4\xi_2 \delta \xi_1 - 4h_2 \delta h_1 \quad (5.32)$$

Note that from the second expression in (5.26) we can write $w = \varepsilon - Ts - \mu q$. We therefore have $\varepsilon = \varepsilon(s, q, \xi_1, h_1)$ with

$$\delta \varepsilon = T \delta s + \mu \delta q - 4\xi_2 \delta \xi_1 - 4h_2 \delta h_1 \quad (5.33)$$

and we can identify ε as the energy density of the thermal system.

As a consistency check we can also calculate the energy by calculating the holographic energy-momentum tensor. From [34] we have

$$(8\pi G)T_{ij} = K_{ij} + \gamma_{ij}(-K + 4 + \frac{3}{2}\xi^2 + \frac{3}{2}h^2) \quad (5.34)$$

where γ_{ij} is the spatial metric at a fixed radius r , K_{ij} is the extrinsic curvature tensor and we note that the ξ and h terms have arisen from the corresponding terms in I_{ct} given in (5.24). We find that the T_{ij} is diagonal with

$$\begin{aligned} T_t^t &= \frac{1}{r^3} \frac{1}{2}(\varepsilon) \\ T_x^x = T_y^y &= -\frac{1}{r^3} \frac{1}{2}(\frac{1}{2}\varepsilon + 2\xi_1\xi_2 + 2h_1h_2) \end{aligned} \quad (5.35)$$

Using equation (45) of [34] to calculate the total energy we obtain

$$E = e^{\beta_a/2} \varepsilon \text{vol}_2 \quad (5.36)$$

which agrees upon setting $\beta_a = 0$ which we will do. It is also worth noting that from the spatial part of the stress tensor we deduce that the pressure is $p = \frac{1}{2}\varepsilon + \frac{3}{2}\xi_1\xi_2 + \frac{3}{2}h_1h_2$ and we thus see that the Smarr-type formula (5.28) can be written in the familiar form $\varepsilon + p = \mu q + Ts$.

6 Uncharged domain wall solutions: holographic RG flows

Recall that the scalar potential for the truncated action (4.3) with $\chi = \xi \in \mathbb{R}$ is given by

$$V = 24 \frac{-1 + h^2 + \xi^2}{(1 - h^2)^{3/2} (1 - \frac{3}{4}\xi^2)^2} \quad (6.1)$$

and has extrema for $(h, \xi) = (0, 0), (0, \pm\sqrt{2/3}), (\pm 1/\sqrt{5}, \pm 2/\sqrt{15})$. These correspond to the skew-whiffed, Pope-Warner and Englert AdS_4 vacua, respectively, that were discussed in section 3. As before, we will restrict our considerations to vacua with positive values of the fields. Before discussing new solutions related to holographic superconductivity in subsequent sections, in this section we pause to numerically construct interpolating uncharged domain-wall solutions, with vanishing gauge field, $\phi = 0$, which have the interpretation as ordinary holographic RG flows.

For these solutions, as $r \rightarrow \infty$ we require the asymptotic expansion given in (5.10) (with $\mu = q = 0$), corresponding to a perturbed skew-whiffed vacuum in the UV. For

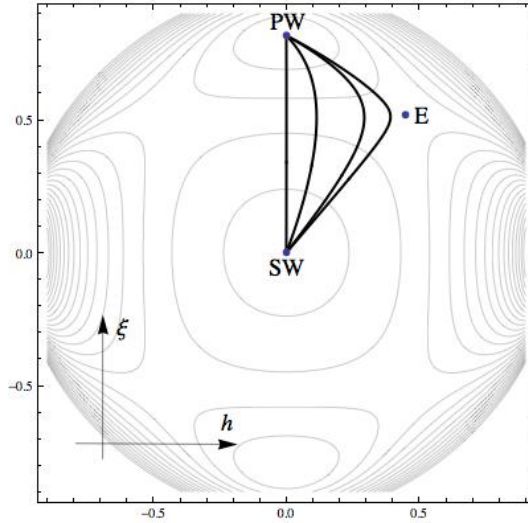


Figure 2: The plot shows the scalar potential of our model in the (h, ξ) plane. The extrema indicated by dots correspond to the skew-whiffed AdS_4 vacuum (SW), the Pope-Warner vacuum (PW) and the Englert vacuum (E). The interpolating trajectories are domain wall solutions that describe holographic flows interpolating between a deformed skew-whiffed vacuum in the UV and a Pope-Warner vacuum in the IR.

uncharged domain wall solutions that flow to the Pope-Warner vacuum in the IR, we demand that as $r \rightarrow 0$ we have the expansion (5.12) (with $\phi_{IR} = 0$). Note that in the UV ξ, h are both dual to relevant operators and non-zero values of ξ_1, h_1 correspond to deforming the skew-whiffed CFT by the corresponding operators, \mathcal{O}_h and \mathcal{O}_ξ , while ξ_2, h_2 correspond to giving vevs for these operators in the deformed CFT. By contrast, in the IR ξ, h are both dual to irrelevant operators in the Pope-Warner CFT.

Do such interpolating domain wall solutions exist? A simple counting suggests the following picture. We have four fields, g, β, ξ, h , two of which satisfy⁸ first-order equations and the rest second order equations (given in appendix A). Thus we must specify six constants to obtain a unique solution. We can use the scaling symmetries of the theory given in (3.1), (3.2) to set $\beta_a = 0$ as well as $(16\pi G/3)^{1/2}\xi_1 = 1$. We therefore have seven parameters, $\varepsilon, \xi_2, h_1, h_2, a_\beta, a_\xi, a_h$ left to specify and thus we expect to ob-

⁸Note that the domain walls we are interested in do not satisfy first order RG flow equations. For example when $h = 0$, after redefining $\xi = (2/\sqrt{3}) \tanh(s/2)$, the $D = 4$ Lagrangian can be written in the form

$$16\pi G\mathcal{L} = \sqrt{-g}[R - \frac{1}{2}(\nabla s)^2 - V] \quad (6.2)$$

with the potential V given in terms of a superpotential W via $V = 8(4(W')^2 - 3W^2)$ and $W = 1/2(1 + \cosh(s))$. This is of the form considered in e.g. [35] and we note that while for the skew-whiffed vacuum $W'(0) = 0$, by contrast for the Pope-Warner vacuum $W'(1) \neq 0$.

tain a one-parameter family of solutions. Let us take this parameter to be h_1 . We will take $h_1 \geq 0$ and solutions with h_1 negative can be obtained using the $h \rightarrow -h$ symmetry of the equations of motion.

Using a shooting technique (see Appendix C), we have constructed this one-parameter family of solutions numerically, finding solutions with h_1 in the range $(16\pi G/3)^{1/2}h_1 \in [0, \sim .86)$. Figure 2 shows a contour plot of the scalar potential with domain-wall trajectories superposed. The solution with $h_1 = 0$ has $h = 0$ identically. As h_1 approaches the maximum value $(16\pi G/3)^{1/2}h_1 \sim 0.86$ the solution gets closer and closer to the unstable Englert vacuum. In Figure 3 we show the values of h_2 and ξ_2 , which are fixing the vevs $\langle \mathcal{O}_h \rangle$ and $\langle \mathcal{O}_\xi \rangle$, as a function of h_1 .

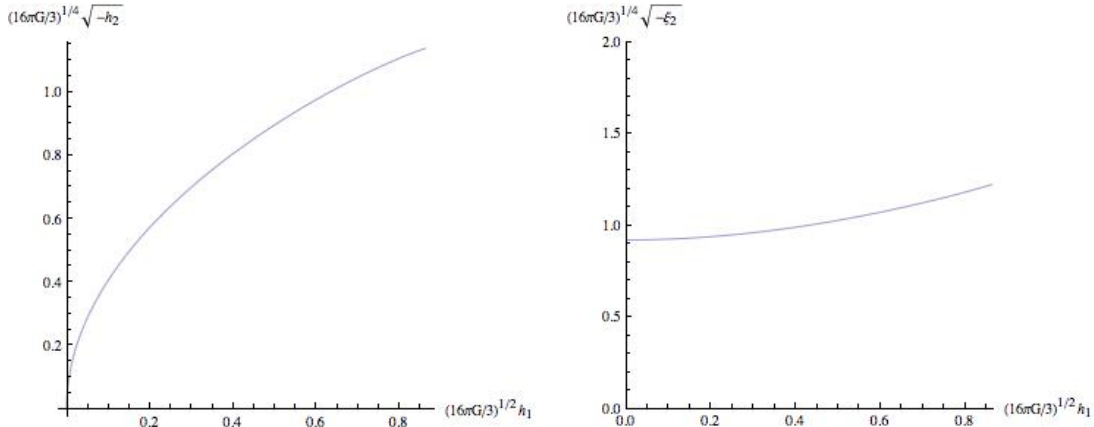


Figure 3: Plot showing the behaviour of h_2 and ξ_2 as a function of h_1 in the uncharged domain wall solutions.

Following similar considerations we have also constructed two further domain wall solutions, one that interpolates between the skew-whiffed vacuum in the UV and the Englert vacuum in the IR and another between the Englert vacuum in the UV and the Pope-Warner vacuum in the IR. Since the Englert solution is unstable, the physical significance of such solutions, if any, is not clear.

For the one parameter family of domain wall solutions flowing between the skew-whiffed vacuum to the Pope Warner vacuum to describe sensible RG flows we require that the corresponding AdS_4 solutions of $D = 11$ supergravity are stable, at least perturbatively. While perturbative stability has been demonstrated for the skew-whiffed solutions it has not yet been shown for the Pope-Warner solutions. Assuming that they are in fact stable, one might then be concerned that the instability of the Englert vacuum implies a concomitant pathology of the RG flows, especially for values of h_1

near the maximum value $(16\pi G/3)^{1/2}h_1 \sim 0.86$ for which the domain wall solutions are getting close to the Englert solution. We think that this is unlikely to be a problem. While the solutions do have a region that is approximated by the Englert solution, the unstable mode of the Englert solution will not be localised in that region. Furthermore, if there was a critical value of h_1 for the solutions in which they become unstable, one would expect a marginal static mode to appear which we are able to explicitly test for numerically, as described in detail in appendix C, and do not find.

7 Interpolating solutions with $T = 0$, $\mu \neq 0$

In this section we will study two classes of regular interpolating solutions with non-zero gauge field, $\phi \neq 0$, that arise as the zero temperature limit of black hole solutions which will be constructed in section 9. The first class is a one parameter family of charged domain walls that interpolate between deformed skew-whiffed AdS_4 in the UV and the Pope-Warner AdS_4 solution in the IR. These solutions have scalar ξ hair and, as we will show in section 9, are the zero temperature limit of superconducting black holes. In particular, the AdS_4 region in the IR corresponds to an emergent $d = 3$ conformal symmetry in the IR. For the special case when $h = 0$ these solutions were found in [8][9]. The second class of solutions is a one parameter family of charged solutions that interpolate between deformed skew-whiffed AdS_4 in the UV and $AdS_2 \times \mathbb{R}^2$ in the IR. These solutions have no scalar hair and, as we show later, will give the zero temperature limit of some of the normal phase black holes. For the special case when $h = 0$ these solutions are simply the zero temperature limit of the Reissner-Nordström black hole solution given in (5.13). As we discuss in section 9, our numerical results indicate that the class of interpolating solutions with AdS_2 factors in the IR are never thermodynamically favoured while those with AdS_4 factors are.

7.1 Pope-Warner IR: zero temperature superconductors

As $r \rightarrow \infty$ we again impose the asymptotic expansion given in (5.10), corresponding to a perturbed skew-whiffed vacuum in the UV. Similarly as $r \rightarrow 0$ we impose the expansion (5.12) in order that we approach the Pope-Warner vacuum in the IR (observe that ϕ is dual to an irrelevant operator in the CFT dual to the Pope-Warner vacuum). We now have five fields, g, β, ξ, h, ϕ , two of which satisfy first-order equations and the rest second order equations (given in appendix A). Thus we must specify eight constants

to obtain a unique solution. We next use the scaling symmetries of the theory given in (3.1), (3.2) to set $(16\pi G)^{1/2}\mu = 1$ as well as $\beta_a = 0$. This leaves the ten parameters $\varepsilon, q, \xi_1, \xi_2, h_1, h_2, a_\beta, a_\xi, a_h, a_\phi$ and so we expect a two parameter family of solutions. We will fix one of these parameters by choosing to set $\xi_1 = 0$ (as we discuss further below). This then leaves us with a single parameter which we choose to be h_1 . Once again we take $h_1 \geq 0$ and we can recover negative h_1 by using the symmetry $h \rightarrow -h$. Using a shooting techniques described in appendix C, we do indeed find a one parameter family of such charged domain wall solutions for $h_1 \leq h_1^c$ with $(16\pi G/3)^{1/2}h_1^c \sim 0.35$ (for $(16\pi G)^{1/2}\mu = 1$). In Figure 4 we have plotted the trajectories of the scalar fields and Figure 5 displays the dependence of h_2 , ξ_2 and q on h_1 . Notice that as $h \rightarrow h_1^c$, the charge carried by the black hole is going to zero since $q \rightarrow 0$.

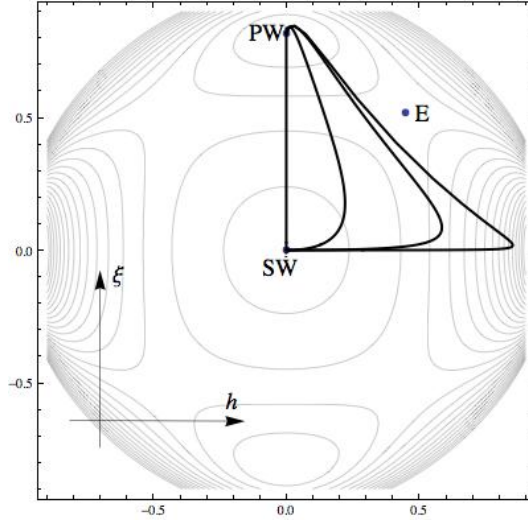


Figure 4: Plot in the (h, ξ) plane showing the interpolating trajectories of the charged domain wall solutions interpolating between the skew-whiffed vacuum in the UV and the Pope-Warner vacuum in the IR. Along the trajectories $\phi \neq 0$.

The special solution with $h_1 = 0$, which has $h = 0$ identically, has been shown to arise as the zero temperature limit of holographic superconducting black holes with non-zero chemical potential in [8]. We will see in the next section that all of the new charged domain walls arise in a similar way. As in [8] we have imposed $\xi_1 = 0$ because it corresponds to allowing the operator \mathcal{O}_χ , dual to ξ , to obtain a vev, determined by ξ_2 , without being sourced i.e. without adding the operator to the CFT dual to the skew-whiffed vacuum. Equivalently, the abelian symmetry in the dual CFT is then

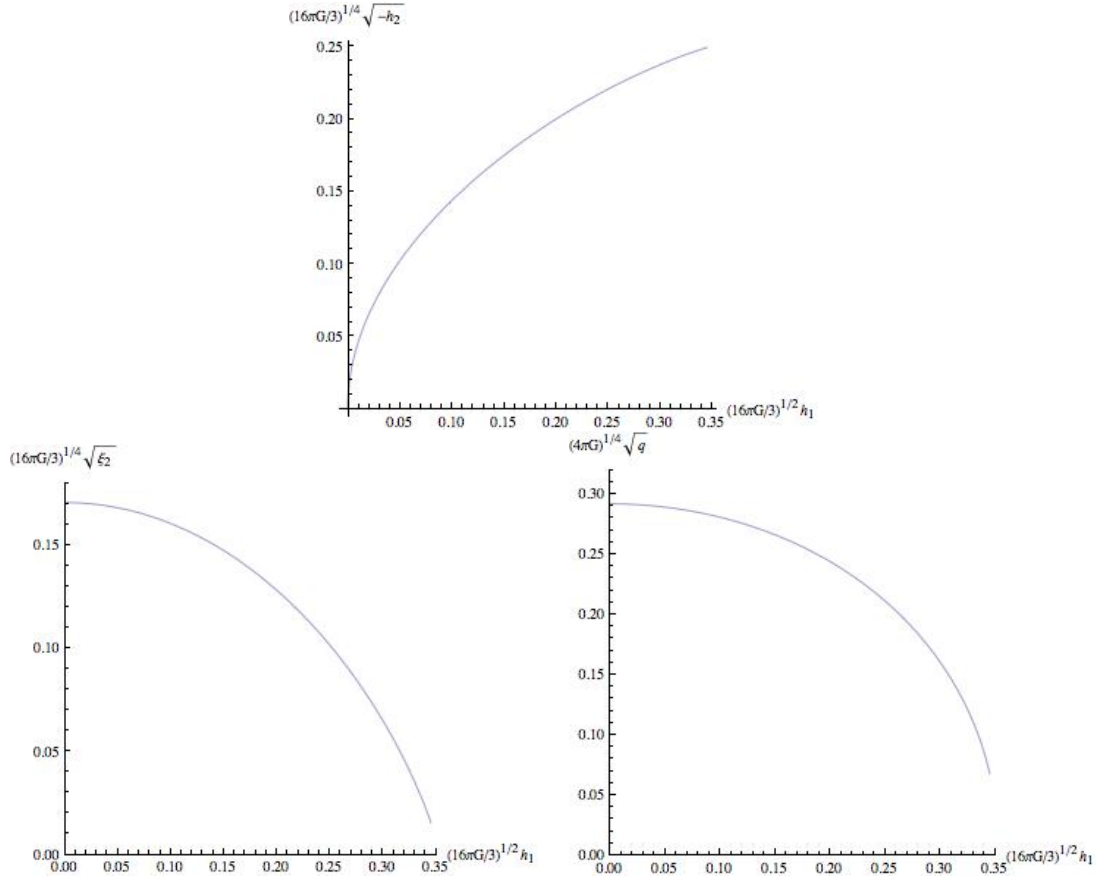


Figure 5: Plots showing h_2 , ξ_2 and q for the one-parameter family of charged domain wall solutions labelled by h_1 with $(16\pi G)^{1/2}\mu = 1$.

broken spontaneously and not explicitly⁹. In our new charged domain walls with $h \neq 0$ we necessarily have $h_1 \neq 0$, $h_2 \neq 0$ corresponding to the dual $d = 3$ CFT having been perturbed by the operator \mathcal{O}_h , dual to h , as well as this operator acquiring a vev. As we will discuss later the coupling of h to $F \wedge F$ in (4.3) implies that when $h \neq 0$ the dual theory breaks parity and time reversal invariance.

In the far IR, all of the new charged domain wall solutions approach the Pope-Warner CFT. Thus, in this limit, all of the h -deformed skew-whiffed CFTs with $T = 0$ and $\mu \neq 0$ are described by the same universal CFT.

⁹Note that we expect entirely analogous results if we set $\xi_2 = 0$ and $\xi_1 \neq 0$.

7.2 $AdS_2 \times \mathbb{R}^2$ IR: zero temperature normal phase

To construct the solutions interpolating between deformed skew-whiffed AdS_4 in the UV and $AdS_2 \times \mathbb{R}^2$ in the IR, we proceed in a similar fashion. However, since the scalar ξ has no irrelevant behaviour about $AdS_2 \times \mathbb{R}^2$ (see (5.14)) these solutions have $\xi = 0$ identically (i.e. no scalar hair). The counting of parameters is exactly the same as that for the Pope-Warner domain walls after excluding ξ , and so we again expect a one parameter family of solutions, which we label by $h_1 \geq 0$.

We do indeed find a one parameter family of interpolating solutions which for $h_1 = 0$ (in fact $h = 0$ identically) includes the $T = 0$ Reissner-Nordström solution (5.13). As we will see later all of these solutions arise as the zero temperature limit of charged black holes with $\xi = 0$ corresponding to the normal phase of the system. In Figure 6 we have plotted the dependence of h_2 and q on h_1 . Note that as for the Pope-Warner charged domain walls, as $h \rightarrow h_1^c$ it appears that the charge carried by the black hole is vanishing, $q \rightarrow 0$.

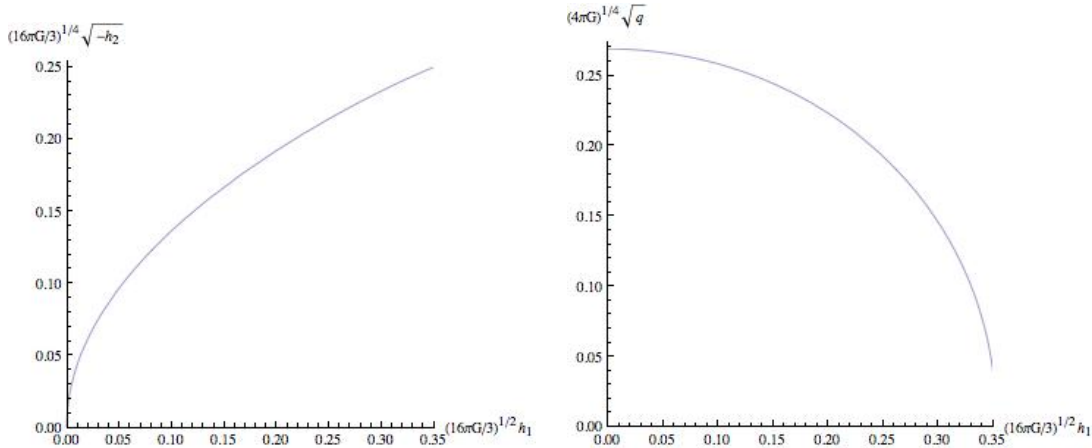


Figure 6: Plots showing h_2 and q for the one-parameter family of charged domain wall solutions labelled by h_1 with $(16\pi G)^{1/2} \mu = 1$.

Interestingly we find that these solutions exist for a very similar, if not identical, range of h_1 as for the Pope-Warner charged domain wall solutions of the last section, i.e. $h_1 \leq h_1^c$ with $(16\pi G/3)^{1/2} h_1^c \sim 0.35$ (for $(16\pi G)^{1/2} \mu = 1$). We will see later that the broken phase superconducting phase solutions are thermodynamically preferred when $h_1 < h_1^c$. Thus, if the ranges are indeed identical (as we expect), for $h_1 < h_1^c$ the system at zero temperature is described by the interpolating solutions with AdS_4 factors in the IR and not those with AdS_2 factors. More detailed investigations near $h_1 = h_1^c$ would

certainly be worthwhile.

8 Uncharged black hole solutions

In this section we construct uncharged black hole solutions that are asymptotic to the perturbed skew-whiffed AdS_4 solution. These describe the skew-whiffed CFT deformed by the relevant operator \mathcal{O}_h at finite temperature T and $\mu = 0$. These solutions have unbroken gauge symmetry and we will see later how they interface with the superconducting solutions.

The asymptotic behaviour of the uncharged black holes is given in (5.10) and the behaviour at the black hole horizon is given by (5.17). The black hole solutions that we construct have $\xi = \phi = 0$ identically, but can have $h_1 \neq 0$, which corresponds to a deformation of the skew-whiffed CFT by the operator dual to h , \mathcal{O}_h , as well as having $h_2 \neq 0$ corresponding to giving \mathcal{O}_h a vev. For solutions with $h_1 = 0$ (which have $h = 0$ identically) we have the usual neutral AdS-Schwarzschild black hole,

$$g = 4r^2 - \frac{8\pi G\varepsilon}{r} \quad (8.1)$$

As the temperature is taken to zero, one recovers the skew-whiffed AdS_4 vacuum in the usual manner. For $h_1 \neq 0$ we have found new solutions for all temperatures. To solve the differential equations we use the scaling symmetries (3.1), (3.2) to set $\beta_a = 0$ and $(16\pi G/3)^{1/2}h_1 = 1$. Since $\phi = \xi = 0$ we set $\mu = q = \xi_1 = \xi_2 = 0$. This leaves us with five parameters $\varepsilon, h_2, r_+, \beta(r_+), h(r_+)$ and since a solution to the differential equations for g, β, h is specified by four parameters we expect a one parameter family of black hole solutions. We take this parameter to be the temperature of the black hole.

In Figure 7 we have plotted the dependence of h_2 , the thermodynamic potential $w = -\varepsilon/2 - 2h_1h_2$ (since $\xi_1 = 0$) and also $r_+^2 = 4Gs$, where s is the entropy density, against temperature. Observe that as the temperature goes to zero the entropy goes to zero. In Figure 8 we have plotted the value of h at the horizon, $h(r_+)$ and we see that as the temperature goes to zero it is approaching the singular value of 1. We have also plotted the Ricci scalar at the horizon as a function of temperature in Figure 8 which confirms that, unlike the solutions with $h_1 = 0$, as the temperature is decreased to zero, the solutions become singular.

We have verified that as the temperature goes to zero, the solution appears to approach Poincaré invariant behaviour with $ge^{-\beta} = r^2$. First observe that it is consistent with the equations of motion to set $ge^{-\beta} = r^2$ when $\phi = \xi = 0$ and that the equations

then boil down to solving a second order ODE for h . We find that as $T \rightarrow 0$ our solutions approach the behaviour $1 - h \sim r^{4/3}$ and hence $g \sim r^{4/3}$ near the singularity at $r = 0$. Note that this behaviour implies that the distance to the singularity from any fiducial point in the spacetime is finite. In fact we have found a full analytic solution of the second order ODE for h given by

$$\begin{aligned} h &= \frac{\sqrt{3}\sqrt{\frac{6\sqrt{3}(2r^8+4r^4+1)}{\sqrt{Z}}} - Z + 9 - \sqrt{3}\sqrt{Z} + 3}{6(r^4 + 1)} \\ Z &= \frac{2 \cdot 6^{2/3} (r^4 + 1) r^{8/3}}{(\sqrt{48r^4 + 81} - 9)^{1/3}} - 6^{1/3} (r^4 + 1) \left(\sqrt{48r^4 + 81} - 9\right)^{1/3} r^{4/3} + 3 \quad (8.2) \end{aligned}$$

which appears to describe the $T \rightarrow 0$ solution. As $r \rightarrow 0$ it has the behaviour $h = 1 - \frac{1}{2^{1/3}}r^{4/3} = \dots$, while as $r \rightarrow \infty$, $h = 1/r - 1/(2r^2) + \dots$, i.e. $(16\pi G/3)^{1/2}h_1 = 1$, $(16\pi G/3)^{1/2}h_2 = -1/2$ and $g = 4r^2 + 3 - 4/r + \dots$ i.e. $4\pi G\varepsilon = 4\pi Gw = 1/2$. We have indicated this behaviour on Figure 7 with dots for comparison.

9 Charged black hole solutions and superconductivity

In this section we construct charged black hole solutions both without and with charged scalar hair that describe unbroken and superconducting phases of holographic superconductors, respectively. We also connect the zero temperature limit of these black holes to the interpolating solutions discussed in section 7. Finally, we describe our calculations on the electrical conductivity of the black holes.

The asymptotic behaviour of the charged black holes is given in (5.10), corresponding to a perturbed skew-whiffed vacuum, and the behaviour at the horizon is given by (5.17). The high temperature unbroken phase black hole solutions that we construct have $\xi = 0$ identically but can have $h_1 \neq 0$, which corresponds to a deformation of the skew-whiffed CFT by the operator dual to h , \mathcal{O}_h , as well as having $h_2 \neq 0$ corresponding to giving \mathcal{O}_h a vev. On the other hand the charged black hole solutions with charged scalar hair, corresponding to the low temperature superconducting phase, will have $\xi_1 = 0$ and $\xi_2 \neq 0$, corresponding to allowing the operator dual to ξ in the skew-whiffed CFT to acquire a vev without being sourced¹⁰. These black hole solutions will also have, generically, $h_1, h_2 \neq 0$.

¹⁰Note that as for the charged domain walls discussed in section 7.1, we again expect analogous results if we set $\xi_2 = 0$ and $\xi_1 \neq 0$.

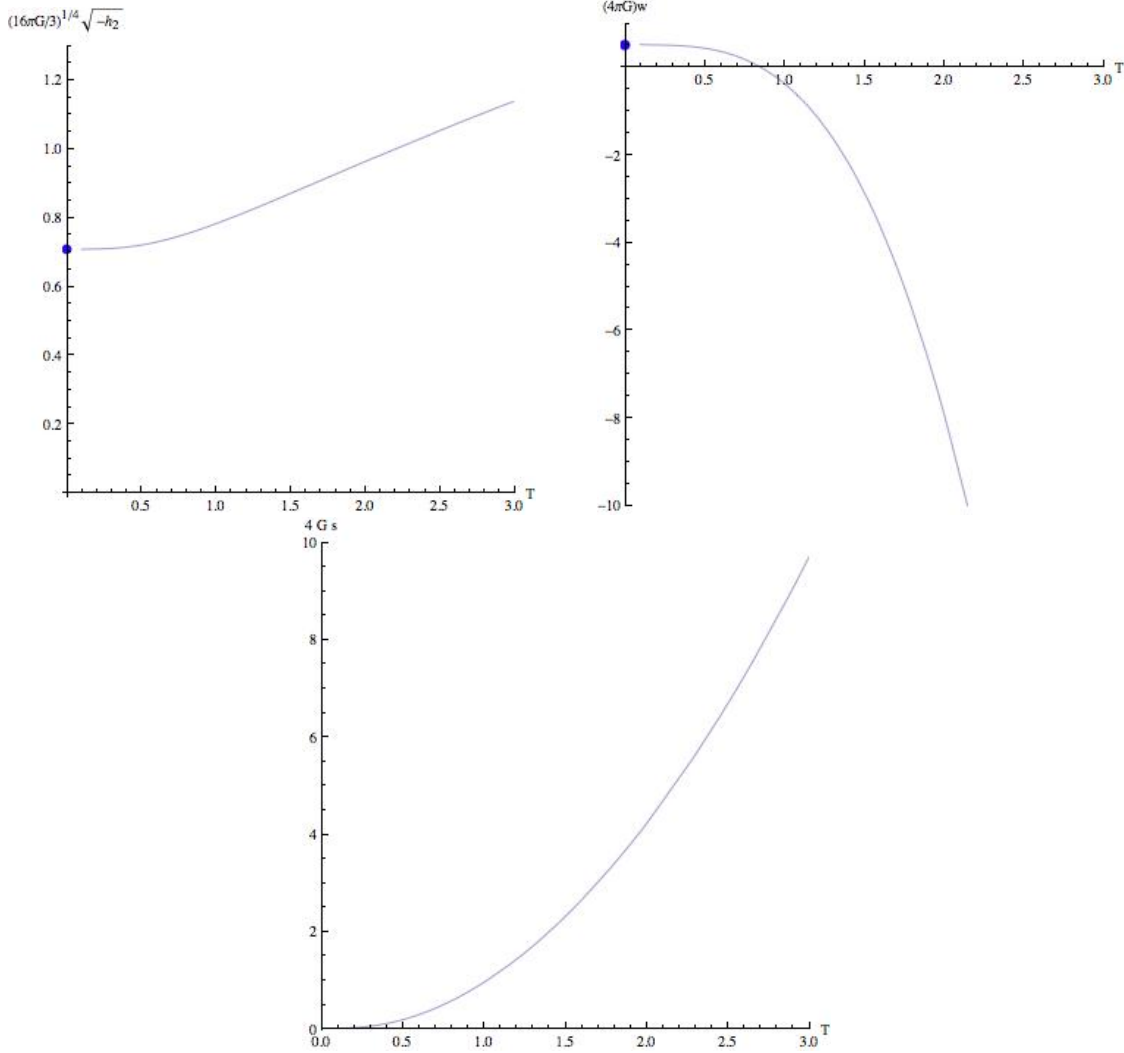


Figure 7: Plot showing the dependence of h_2 , the thermodynamic potential w and the entropy density s against temperature for the uncharged black hole solutions with $(16\pi G/3)^{1/2}h_1 = 1$. The dots are explained in the text.

To solve the differential equations we first use the scaling symmetries (3.1), (3.2) to set $\beta_a = 0$ and $(16\pi G)^{1/2}\mu = 1$. We also impose $\xi_1 = 0$. This leaves us with ten parameters $\varepsilon, q, \xi_2, h_1, h_2, \beta(r_+), \xi(r_+), h(r_+), \phi'(r_+)$ and r_+ , and since a solution to the differential equations for g, β, ξ, h, ϕ is specified by eight parameters we expect a two-parameter family of black hole solutions. We can choose these parameters to be the temperature T and $h_1 \geq 0$ (with negative h_1 obtained via $h \rightarrow -h$). For further discussion see appendix C. It is worth noting that when $T \rightarrow 0$ we have the same parameter counting that arose in the charged domain-wall solutions discussed in

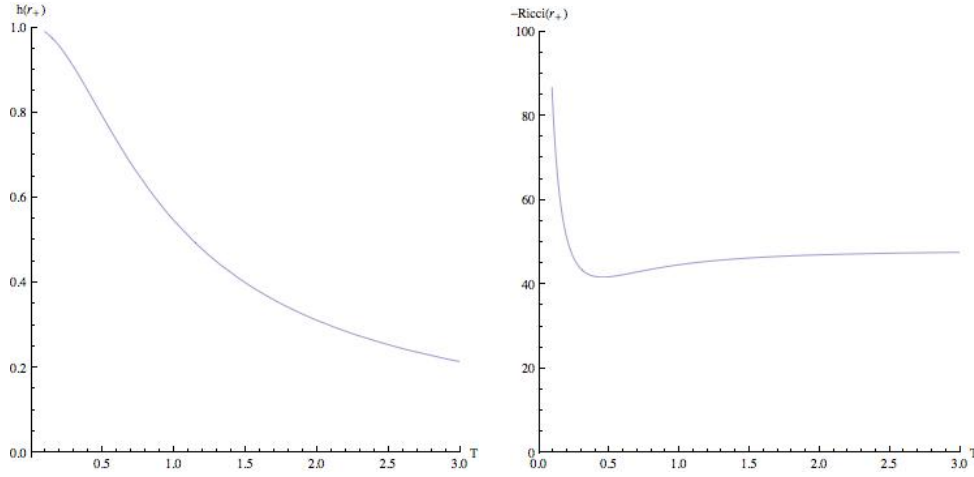


Figure 8: Plot showing the value of the h and the Ricci scalar at the horizon against temperature for the uncharged black hole solutions with $(16\pi G/3)^{1/2}h_1 = 1$.

section 7.1. This is because the operators dual to ξ , h and ϕ are all irrelevant in the Pope-Warner CFT. When $h_1 = 0$ the solutions have $h = 0$ identically and were all constructed in [8].

9.1 Black hole solutions with $\mu \neq 0$ and no scalar hair: the unbroken phase

The unbroken phase black hole solutions have $\xi = 0$ identically. When $h_1 = 0$ (in fact $h = 0$ identically) we have the usual Reissner-Nordström kind of electrically charged black hole solutions given in (5.13). Recall that at zero temperature the Reissner-Nordström solutions have finite entropy and near the event horizon approach $AdS_2 \times \mathbb{R}^2$ with the AdS_2 having radius squared $1/24$ and hence the Ricci scalar is -48 . For non-zero values of h_1 with $h_1 < h_1^c$, with $(16\pi G/3)^{1/2}h_1^c \sim 0.35$ (for $(16\pi G)^{1/2}\mu = 1$), we have found solutions with very similar properties. In particular, at zero temperature they approach the interpolating solutions presented in section 7.2 that approach $AdS_2 \times \mathbb{R}^2$ in the IR. As we will see below, this is the same range of h_1 for which superconducting black hole solutions also exist. We have also constructed charged unbroken phase solutions for $h_1 > h_1^c$ and for this range the solutions become singular in the zero temperature limit.

In Figure 9 we have plotted the dependence of h_2 , $r_+^2 = 4Gs$, where s is the entropy density, and also the thermodynamic potential $w = -\varepsilon/2 - 2h_1h_2$ (since $\xi_1 = 0$) against

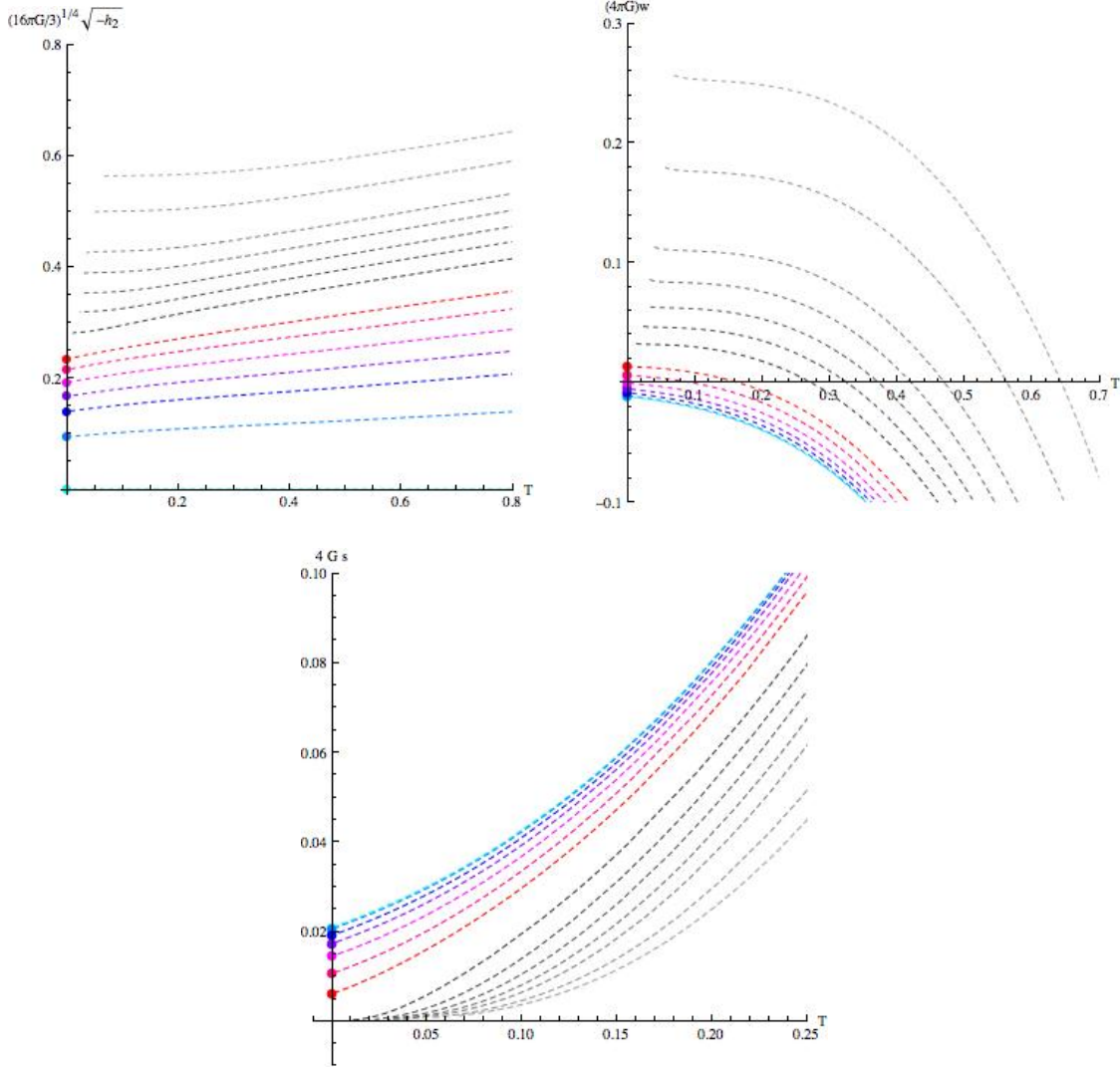


Figure 9: Plot showing the dependence of h_2 , the thermodynamic potential w and entropy density s against temperature for the unbroken phase black hole solutions for various values of h_1 ranging from $(16\pi G/3)^{1/2}h_1 = 0$ (light blue) to $(16\pi G/3)^{1/2}h_1 = 0.3$ (red). Increasing values of $(16\pi G/3)^{1/2}h_1 = 0.4, \dots, 0.8$ are shown in decreasing shades of grey. The zero charged domain wall solutions are added at $T = 0$ as coloured dots. All plots have $(16\pi G)^{1/2}\mu = 1$.

temperature for various values of h_1 . In Figure 10 we have plotted the value of h and the value of the Ricci scalar at the horizon $r = r_+$ against temperature, which clearly demonstrates the change from non-singular to singular behaviour at zero temperature as h_1 becomes bigger than h_1^c .

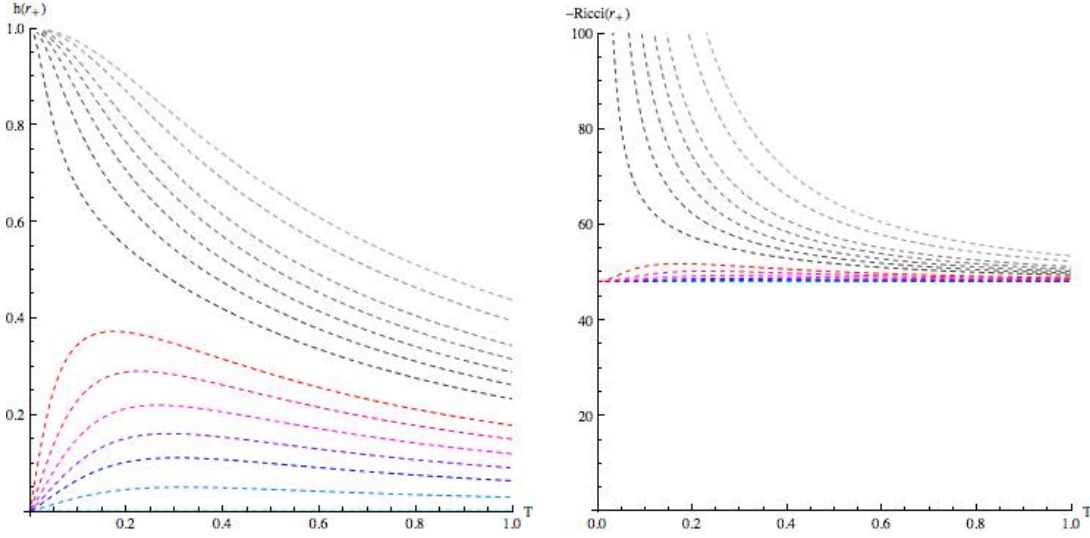


Figure 10: Plots showing the value of h and of the Ricci scalar at the horizon against temperature for the unbroken black hole solutions for various values for various values of h_1 ranging from $(16\pi G/3)^{1/2}h_1 = 0$ (light blue) to $(16\pi G/3)^{1/2}h_1 = 0.3$ (red). Increasing values of $(16\pi G/3)^{1/2}h_1 = 0.4, \dots, 0.8$ are shown in decreasing shades of grey. The zero charged domain wall solutions are added at $T = 0$ as coloured dots. Both plots have $(16\pi G)^{1/2}\mu = 1$.

9.2 Black hole solutions with $\mu \neq 0$ and scalar hair: the superconducting phase

We have constructed charged black holes with charged scalar hair, for various values of h_1 , with $h_1 < h_1^c$ where $(16\pi G/3)^{1/2}h_1^c \sim 0.35$ (for $(16\pi G)^{1/2}\mu = 1$). As we have already noted this is the same range of h_1 that is required for the existence of a zero temperature charged domain wall solution that we constructed in section 7.1. In Figure 11 we have plotted h_2 , ξ_2 (recall $\xi_1 = 0$ for these solutions) and w against temperature for various values of h_1 for these new solutions¹¹. Importantly Figure 11 shows that the superconducting phase solutions, with charged scalar hair, are thermodynamically favoured over the unbroken phase solutions. Our numerical results also clearly indicate that as the temperature goes to zero, the broken phase black hole solutions do indeed approach the charged domain wall solutions that we constructed in section 7.1. We have made various checks of this including comparing scalar curvature invariants as was

¹¹To compare with Figures 1 and 2 in [8] which plots the solutions for $h = 0$ one should note that the normalisation in [8] was $(16\pi G)^{1/2}\hat{\mu} = 1$, as we are using here. In addition the vertical axis in Figure 1 of [8] was missing a factor of 4π .

done for $h = 0$ in [8]. In particular, we find that as the temperature goes to zero the entropy of the superconductors goes to zero and that the solutions have an emergent conformally invariant behaviour in the far IR that is captured by the Pope-Warner AdS_4 vacuum for all values of $h < h_1^c$.

These results are consistent with the simple picture that the CFT deformed by the operator \mathcal{O}_h and held at finite chemical potential $\mu \neq 0$ has a phase diagram as summarised in Figure 1. For $h_1 < h_1^c$ there is high temperature phase in which the abelian symmetry is unbroken and then below a critical temperature there is a superconducting phase in which it is broken. At low temperatures, and in the far IR, the superconductors are dual to the Pope-Warner AdS_4 solution. For $h_1 > h_1^c$ there is only an unbroken phase and at low temperatures and in the far IR the system is dual to a singular solution.

To demonstrate that this is indeed the correct picture would require showing that there are no other black hole solutions which branch off at temperatures higher than the critical temperatures that we have found and that also dominate the free energy. Within our radial truncation, as we discuss in appendix C, we find no evidence for any such additional solutions. It is possible that there are such black hole solutions of $D = 11$ supergravity outside our consistent KK truncation (4.3); perhaps it can be shown, at least for a class of SE_7 spaces, that additional black hole solutions do not exist, but we leave that for future work.

9.3 Conductivity

Linear response theory can be used to determine the various conductivities of the dual field theories for both the unbroken and broken phase black hole solutions. We will consider small perturbations of the form

$$\begin{aligned} g_{tx} &= e^{-i\omega t} g_{tx}(r), & g_{ty} &= e^{-i\omega t} g_{ty}(r), \\ \delta A &= e^{-i\omega t} [a_x(r)dx + a_y(r)dy] \\ &= e^{-i\omega t} [a_-(r)(dx + idy) + a_+(r)(dx - idy)] \end{aligned} \tag{9.1}$$

where we have introduced the circularly polarised perturbations

$$a_{\pm}(r) \equiv \frac{1}{2}[a_x(r) \pm ia_y(r)] \tag{9.2}$$

All other fields take their background values.

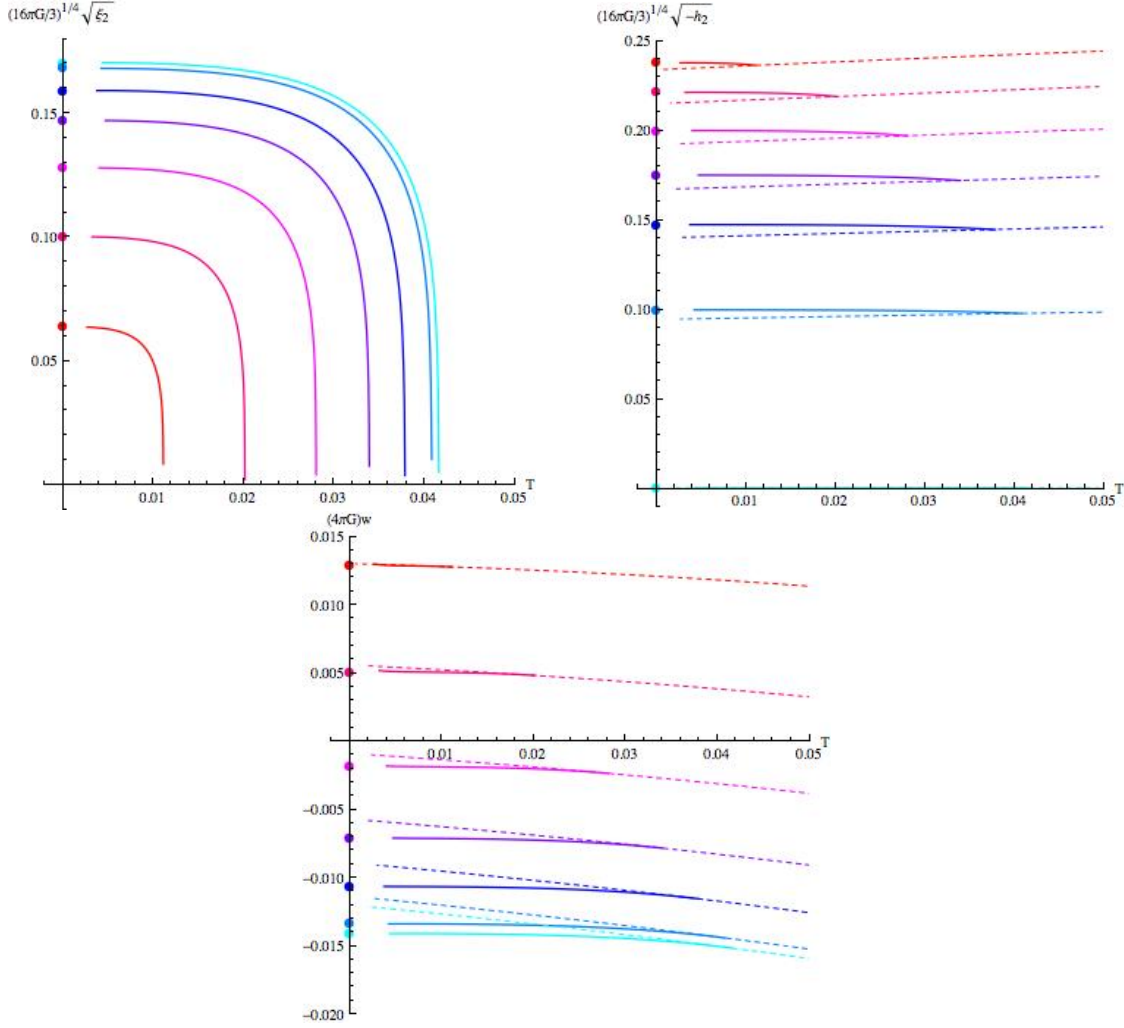


Figure 11: Plot showing dependence of ξ_2 , h_2 and the thermodynamic potential w with temperature for the broken phase black hole solutions for various values of h_1 ranging from $(16\pi G/3)^{1/2}h_1 = 0$ (light blue) to $(16\pi G/3)^{1/2}h_1 = 0.3$ (red) with $(16\pi G)^{1/2}\mu = 1$. These broken phase solutions are plotted as solid lines and the zero temperature charged domain wall solutions are added at $T = 0$ as coloured dots. The unbroken phase black hole solutions for the same values of h_1 are plotted as dashed lines for comparison.

The perturbations behave in the UV as

$$a_i(r) \sim \sqrt{4\pi G} \left[a_i^{(0)} + \frac{a_i^{(1)}}{r} \right], \quad g_{ti} \sim g_{ti}^{(0)} r^2 + \frac{g_{ti}^{(1)}}{r} \quad (9.3)$$

where we have used the notation $(x, y) \equiv (x^1, x^2)$. At finite temperature, where there is a black-hole horizon, we impose purely ingoing boundary conditions, with the perturbations behaving as $(r - r_+)^{-i\omega/4\pi T}$. For the superconducting black holes in the

zero temperature limit, we impose similar ingoing boundary conditions at the IR end of the domain-wall geometry. As explained in [3] the above perturbation produces an electric field given by $E_i = i\omega(a_i^{(0)} + \mu g_{ti}^{(0)})$ while the induced current J_i is obtained by calculating the on-shell variation of the action and we find $\delta S/\delta a_i^{(0)} = 4a_i^{(1)}$. Thus the electric conductivity matrix is given by

$$\sigma_{ij} = \frac{-i4a_i^{(1)}}{\omega a_j^{(0)}} \quad (9.4)$$

The rotational invariance of our setup implies that $\sigma_{ij} = s_1\delta_{ij} + s_2\epsilon_{ij}$ and that s_2 is non-zero only if parity and time reversal invariance is violated. As we will see when $h \neq 0$ the coupling to $F \wedge F$ in (4.3) gives rise to such parity violation. Note that the thermal $\bar{\kappa}$ and thermoelectric α conductivities can be easily determined once σ is known [36][3].

The xr and yr Einstein equations imply the first order equations

$$r^2 \left(\frac{g_{ti}}{r^2} \right)' = -\frac{4(1-h^2)^{3/2}}{1+3h^2} \phi' a_i \quad i = x, y \quad (9.5)$$

We also find that the Maxwell equations for a_{\pm} decouple. After using (9.5) and also (A.1), we deduce that

$$a_{\pm}'' + \left[\frac{g'}{g} - \frac{\beta'}{2} + \frac{L_1'}{L_1} \right] a_{\pm}' + \left[\left(\frac{\omega^2}{g^2} - \frac{4\phi'^2 L_1}{g} \right) e^{\beta} - \frac{12e^{-12U}\xi^2}{L_1 g} \pm \frac{3L_3 \omega h' e^{\beta/2}}{L_1 g} \right] a_{\pm} = 0 \quad (9.6)$$

where

$$L_1 = \frac{(1-h^2)^{3/2}}{1+3h^2}, \quad L_3 = \frac{(1-h^2)^2}{(1+3h^2)^2} \quad (9.7)$$

The only other non-zero components of the Einstein equations are in the xt and yt components and they lead to second order equations which are implied by the above.

Thus the problem boils down to solving the linear equations (9.6) subject to the boundary conditions mentioned above. The parity operation in the three-dimensional boundary field theory can be defined as the reversal of one of the spatial directions, x say, and this transforms $a_{\pm} \rightarrow -a_{\pm}$. On the other hand time reversal changes the sign of ω . Thus, the last term in (9.6) implies that configurations with non-constant h break parity and time-reversal invariance of the boundary field theory.

Since the equations for a^{\pm} decouple, the conductivity matrix is diagonal in this basis. We first calculate

$$\sigma^{\pm} \equiv \frac{-i4a_{\pm}^{(1)}}{\omega a_{\pm}^{(0)}} \quad (9.8)$$

which can be used to obtain

$$\begin{aligned}\sigma_{xx} &= \sigma_{yy} = \frac{1}{2}(\sigma^+ + \sigma^-) \\ \sigma_{xy} &= -\sigma_{yx} = \frac{i}{2}(\sigma^+ - \sigma^-)\end{aligned}\tag{9.9}$$

In particular, the special solutions with $h = 0$ found in [8] have vanishing Hall conductivity $\sigma_{xy} = 0$ but the new solutions found here with $h \neq 0$ have $\sigma_{xy} \neq 0$.

We have calculated the electrical conductivity for various values of h_1 and for various temperatures for both broken and unbroken phase black holes. Here we will just plot the real and imaginary parts of σ_{xx} and σ_{xy} for the zero temperature superconducting charged domain wall solutions. The longitudinal conductivity is shown in Figure 12 and the Hall conductivity in Figure 13. At finite temperature we find similar results.

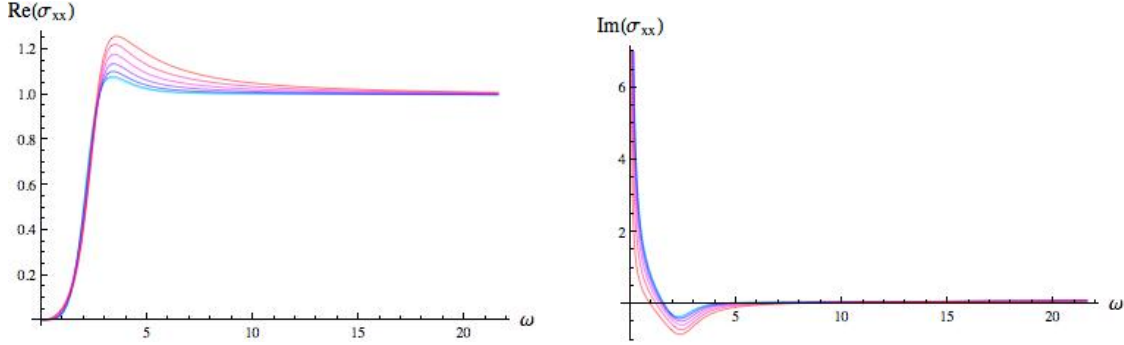


Figure 12: The left panel shows the real part of the direct conductivity of the superconducting solutions at zero temperature for $(16\pi G/3)^{1/2}h_1 = 0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3$ (from bottom to top) and $(16\pi G)^{1/2}\mu = 1$. The right panel shows the corresponding imaginary part (now from top to bottom). We see that there is a pole in the imaginary part, at $\omega = 0$, from which we deduce by analyticity that the real part has a delta function there, which the numerics don't show.

10 Discussion

Building on the results of [7], we have presented a simple consistent KK truncation of $D = 11$ supergravity on an arbitrary seven-dimensional Sasaki-Einstein space, SE_7 , to a $D = 4$ theory that contains a metric, a gauge-field, a charged scalar field χ and a neutral scalar field h . Given any SE_7 , each solution of the $D = 4$ theory gives rise to an exact solution of $D = 11$ supergravity.

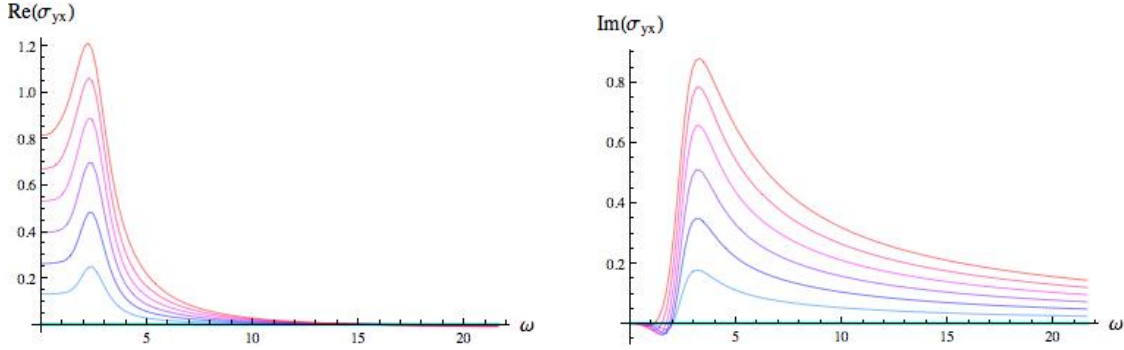


Figure 13: The left panel shows the real part of the Hall conductivity of the superconductors at zero temperature for $\sqrt{16\pi G/3}h_1 = 0, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3$ (from bottom to top) and $(16\pi G)^{1/2}\mu = 1$. The right panel shows the corresponding imaginary part (again from bottom to top). Note that the undeformed ($h_1 = 0$) Hall conductivity, shown in cyan, vanishes identically because P and T remain unbroken at $h_1 = 0$.

We have used this $D = 4$ theory to construct a range of solutions that describe the thermodynamics of the CFT dual to the skew-whiffed AdS_4 vacuum, allowing for finite temperature, T , finite chemical potential μ with respect to the current dual to the gauge-field, and also deformations by the relevant operator \mathcal{O}_h dual to the field h . This latter deformation is encoded in the asymptotic behaviour of the h -field: $h = (16\pi G/3)^{1/2}(h_1/r + \dots)$. The phase diagram is summarised in Figure 14 and extends the results of [8] which constructed the solutions with $h = 0$.

When $\mu = 0$, we constructed new uncharged black hole solutions in section 8 for $T \neq 0$ and $h_1 \neq 0$. As $T \rightarrow 0$ the entropy of the solutions goes to zero, in accordance with the third law, and they become nakedly singular. Interestingly, in this limit they develop a Poincaré symmetry and furthermore the singularity is at finite proper distance from any fiducial point in the bulk of the spacetime; it would be interesting to see if this leads to a gapped spectrum. For the solutions with $h_1 = 0$ (which have $h = 0$), the black hole solutions are the usual Schwarzschild AdS solutions and smoothly map onto the skew-whiffed AdS_4 vacuum at $T = 0$ as indicated in Figure 14.

Richer structures appear when $\mu \neq 0$. We first point out that because of the underlying conformal symmetry the physics is captured by any fixed slice $mu \neq 0$ as presented in Figure 1. In section 9 we constructed electrically charged black holes with vanishing scalar hair $\chi = 0$ which describe the unbroken phase of the system. For $|h_1| < h_1^c$ we showed that below a critical temperature there is a transition

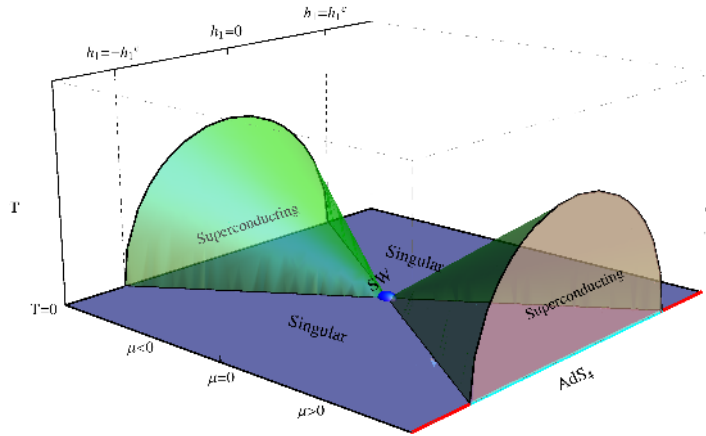


Figure 14: The phase diagram of the holographic superconductors in the (T, μ, h_1) space. Note any $\mu \neq 0$ slice gives the $2d$ projection of Figure 1. At the origin we have the skew-whiffed vacuum. The $T = 0$ plane is null singular except under the two half cones.

to a superconducting phase described by electrically charged black holes with scalar hair $\chi \neq 0$. The region of superconductivity corresponds to the region below the two green half cones in Figure 14. At zero temperature the superconducting solutions are the non-singular charged domain wall solutions interpolating between the skew-whiffed AdS_4 vacuum and the Pope-Warner AdS_4 vacuum constructed in section 7. It is interesting that in the far IR these solutions have emergent conformal symmetry in three spacetime dimensions, described by the same CFT.

At $|h_1| = h_1^c$ the critical temperature goes to zero, and for $|h_1| > h_1^c$ the unbroken phase black hole solutions extend all the way down to zero temperature. In the zero temperature limit these solutions again have vanishing entropy and become nakedly singular; it would be interesting to explore them in more detail. Interestingly, in all of these solutions with $h \neq 0$, parity and time reversal invariance are broken in the boundary theory.

A striking aspect of our numerical investigations is that $|h_1| = h_1^c$ also seems to precisely correspond to a change of behaviour of the zero temperature limit of the unbroken phase black holes. In particular, for $|h_1| < h_1^c$ at zero temperature these solutions become smooth interpolating solutions that approach $AdS_2 \times \mathbb{R}^2$ in the IR and are thermodynamically disfavoured. It would be worthwhile further exploring both the zero temperature superconducting black holes and the unbroken phase black holes near their critical values of h_1 to confirm that h_1^c is indeed governing both behaviours

and also to elucidate what is physically happening precisely at $|h_1| = h_1^c$.

For simplicity, in all of our solutions that carry scalar charge we only considered asymptotic behaviours of $(16\pi G/3)^{1/2}\xi = \xi_1/r + \xi_2/r^2 + \dots$ with $\xi_1 = 0$ and $\xi_2 \neq 0$ corresponding to a spontaneous breaking of the abelian gauge symmetry by the operator dual to ξ acquiring a vev. All of our solutions should have analogues with $\xi_1 \neq 0$ and $\xi_2 = 0$, describing the same phenomenon in the CFT dual to skew-whiffed AdS_4 but with different boundary conditions. We do not anticipate any significant qualitative differences in behaviour but we think it would be worthwhile to investigate this in detail.

Given the rich phase structure that we have uncovered in this work, one may ask how this might map onto the phase diagram of a real material exhibiting quantum critical behaviour. One may define a quantum critical point (QCP) to be a zero-temperature second order phase transition with emergent scaling symmetry, and we will restrict our discussion here to cases where this is enlarged to a conformal symmetry. In the vicinity of such a QCP a condensed-matter system, which typically has a discrete underlying lattice structure, is well-described by a relativistic continuum CFT. The coupling that one has to dial in order to reach the quantum critical point shows up as a relevant deformation in the CFT. Real-world examples include the applied pressure in heavy-fermion systems and also, conjecturally, the hole-doping ratio in the cuprate superconductors.

The simplest possibility is that the skew-whiffed CFT, dual to the skew-whiffed AdS_4 vacuum (for a given choice of SE_7 space), provides an AdS/CFT realization of a material with such a QCP. Specifically, when $\mu = 0$ we observe a phase transition at zero temperature when $h_1 = 0$. In particular, tuning h_1 corresponds to tuning the operator \mathcal{O}_h , which is a relevant operator in the skew-whiffed CFT, and so one might try and identify \mathcal{O}_h with some macroscopic quantity in the material. When we switch on $\mu \neq 0$, at zero temperature the phase transition opens up into the two phase transitions at either end of the superconducting dome (see Figures 1 and 14) at $h_1 = \pm h_{1,c}$. Interestingly underneath the superconducting dome at zero temperature there is an emergent conformal scaling behaviour in the far IR (with the scale set by μ), which is a universal strongly coupled CFT dual to the Pope-Warner AdS_4 vacuum. It would be very interesting if there are real world materials with QCP's with such a phase structure.

It might be too optimistic to be able to find a material with a QCP that is exactly described by the skew-whiffed CFT for some choice of SE_7 . One might then look

for some approximate features. Alternatively, one might also wonder if it is instead possible to interpret the CFT dual to the Pope-Warner AdS_4 vacuum as *the* QCP. In this point of view, the IR description of the material near its QCP would coincide with the Pope-Warner CFT that the skew-whiffed CFT flows to in the far IR. Thus, in this interpretation the skew-whiffed CFT would be a fictitious UV theory, whose usefulness for the real material lies solely in the fact that at low energies it flows to an IR fixed point that describes the quantum critical region of that material. One problem with this point of view is that, within our truncation, the PW fixed point has no relevant operators and so one cannot deform away from it, as one could with the skew-whiffed CFT. This could potentially be solved by going outside the truncation used in this paper, where one might find the required relevant operator. We feel this warrants further investigation.

We conclude by briefly mentioning some additional avenues for further exploration. The first issue concerns the stability of the skew-whiffed AdS_4 solutions. For the special case that $SE_7 = S^7$ the stability is guaranteed by supersymmetry. For the generic class, it is known that they are perturbatively stable [20] and hence that they define CFTs in the strict $N \rightarrow \infty$ limit. It would be interesting to further study whether or not they are destabilised by $1/N$ effects of the type discussed in [21][22]. It would also be interesting to have an explicit construction of the CFTs dual to the skew-whiffed $AdS_4 \times SE_7$ solutions. Since the Pope-Warner solutions play an important role in our solutions, it would also be worthwhile analysing their stability, starting with a perturbative analysis.

Another outstanding issue is to determine whether or not there are additional unstable modes in the skew-whiffed $AdS_4 \times SE_7$ background which condense at higher temperatures. If they do exist, and dominate the free energy then the corresponding black hole solutions would be the relevant ones for describing the thermodynamics. It would be nice to show, perhaps for a subclass of Sasaki-Einstein manifolds, that this does not occur. In addition to further explorations of superconductivity we envisage that our $D = 4$ action will also have other applications in studying condensed matter systems in the context of M-theory.

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A The differential equations

Starting with the equations of motion arising from (4.3) and substituting in the ansatz (5.1), (5.2), (5.3) we obtain differential equations for g, β, ϕ, ξ, h . The following differential equations arise from the matter fields:

$$\left[e^{\beta/2} r^2 \frac{(1-h^2)^{3/2}}{1+3h^2} \phi' \right]' - \frac{12g^{-1} e^{\beta/2} r^2 \phi \xi^2}{(1 - \frac{3}{4}\xi^2)^2} = 0 \quad (\text{A.1})$$

$$\begin{aligned} & r^{-2} e^{\beta/2} (1 - \frac{3}{4}\xi^2) \left[\frac{g e^{-\beta/2} r^2 \xi'}{(1 - \frac{3}{4}\xi^2)^2} \right]' \\ & + \left[\frac{9}{2} g (\xi')^2 - \frac{12}{(1-h^2)^{3/2}} (-2 + 6h^2 + 3\xi^2) + 12g^{-1} e^{\beta} \phi^2 (4 + 3\xi^2) \right] \xi = 0 \end{aligned} \quad (\text{A.2})$$

$$r^{-2} e^{\beta/2} \left[\frac{r^2 e^{-\beta/2} g}{(1-h^2)} h' \right]' - \frac{2h(1-h^2)^{3/2}(3+h^2)}{(1+3h^2)^2} e^{\beta} (\phi')^2 - \frac{8h(-1+h^2+3\xi^2)}{(1 - \frac{3}{4}\xi^2)^2 (1-h^2)^{3/2}} = 0 \quad (\text{A.3})$$

For the components of the Einstein tensor we have

$$\begin{aligned} E_{tt} &= -g^2 e^{-\beta} \left(\frac{g'}{gr} + \frac{1}{r^2} \right) \\ E_{rr} &= \frac{g'}{gr} - \frac{\beta'}{r} + \frac{1}{r^2} \\ E_{xx} &= \frac{r}{2} [2g' - g\beta'] + \frac{r^2}{2} \left[g'' - \frac{3}{2}g'\beta' - g\beta'' + \frac{1}{2}g(\beta')^2 \right] \end{aligned} \quad (\text{A.4})$$

The tt and rr components of the Einstein equations are equivalent to

$$\begin{aligned} & \left[\frac{g'}{gr} + \frac{1}{r^2} \right] + g^{-1} e^{\beta} \frac{(1-h^2)^{3/2}}{1+3h^2} (\phi')^2 + \frac{3(h')^2}{4(1-h^2)^2} + \frac{3}{4(1 - \frac{3}{4}\xi^2)^2} (\xi')^2 \\ & + \frac{12e^{\beta} \phi^2 \xi^2}{g^2 (1 - \frac{3}{4}\xi^2)^2} + \frac{12(-1+h^2+\xi^2)}{g(1 - \frac{3}{4}\xi^2)^2 (1-h^2)^{3/2}} = 0 \end{aligned} \quad (\text{A.5})$$

and

$$\frac{\beta'}{r} + \frac{3}{2} \frac{(h')^2}{(1-h^2)^2} + \frac{3(\xi')^2}{2(1-\frac{3}{4}\xi^2)^2} + \frac{24e^\beta \phi^2 \xi^2}{g^2(1-\frac{3}{4}\xi^2)^2} = 0 \quad (\text{A.6})$$

One can check that the remaining xx component of the Einstein equations, $E_{xx} = T_{xx}$ with

$$\begin{aligned} r^{-2}g^{-1}T_{xx} &= g^{-1}e^\beta \frac{(1-h^2)^{3/2}}{1+3h^2}(\phi')^2 - \frac{3}{4} \frac{(h')^2}{(1-h^2)^2} - \frac{3(\xi')^2}{4(1-\frac{3}{4}\xi^2)^2} \\ &\quad + \frac{12e^\beta \phi^2 \xi^2}{g^2(1-\frac{3}{4}\xi^2)^2} - \frac{12(-1+h^2+\xi^2)}{g(1-\frac{3}{4}\xi^2)^2(1-h^2)^{3/2}} \end{aligned} \quad (\text{A.7})$$

is implied by (A.5), (A.6).

Observe that the equations (A.1), (A.2), (A.3), (A.5) and (A.6), can be obtained from the action given in (5.4).

B Calculating the on-shell action

We start by writing (2.6) as $S = S_0 + S_{CS}$ where S_{CS} is the metric independent Chern-Simons like contributions to the action. Since we are only considering purely electric gauge fields S_{CS} vanishes on-shell. We further write $S_0 = (16\pi G)^{-1} \int d^4x \sqrt{-g} \mathcal{L}_0$ and observe that

$$T_{xx} \equiv \frac{1}{2}r^2(\mathcal{L}_0 - R) \quad (\text{B.1})$$

Since $E_{xx} = T_{xx}$ and $E^\mu{}_\mu = -R$ we deduce that

$$\mathcal{L}_0 = -[E^t{}_t + E^r{}_r] \quad (\text{B.2})$$

and hence we can write the on-shell action as

$$S_{OS} = -\frac{1}{16\pi G} \int dt d^2x dr \sqrt{-g} (E^t{}_t + E^r{}_r) \quad (\text{B.3})$$

As in [5] we next observe that

$$\sqrt{-g} (E^t{}_t + E^r{}_r) = (2rge^{-\beta/2})' \quad (\text{B.4})$$

and hence

$$S_{OS} = -\frac{1}{16\pi G} \int dt d^2x \int_{r_+}^{\infty} dr [2rge^{-\beta/2}]' \quad (\text{B.5})$$

An alternative expression for the on-shell action can be obtained by using the fact that

$$\sqrt{-g}R = -[r^2 e^{-\beta/2}(g' - g\beta')] - 2e^{-\beta/2}gr^2\left(\frac{g'}{gr} + \frac{1}{r^2}\right) \quad (\text{B.6})$$

Then using (A.5) and (B.1) we obtain

$$S_{OS} = -\frac{1}{16\pi G} \int dt d^2x \int_{r_+}^{\infty} dr \left[r^2 e^{-\beta/2}(g' - g\beta' - 4e^{\beta} \frac{(1-h^2)^{3/2}}{1+3h^2} \phi\phi') \right]' \quad (\text{B.7})$$

Finally, we analytically continue by setting $t = -i\tau$ and $iS = -I$, we correspondingly obtain the following two expressions for the on-shell euclidean action

$$\begin{aligned} I_{OS} &= \frac{\Delta\tau \text{vol}_2}{16\pi G} \int_{r_+}^{\infty} dr [2rge^{-\beta/2}]' \\ &= \frac{\Delta\tau \text{vol}_2}{16\pi G} \int_{r_+}^{\infty} dr \left[r^2 e^{-\beta/2}(g' - g\beta' - 4e^{\beta} \frac{(1-h^2)^{3/2}}{1+3h^2} \phi\phi') \right]' \end{aligned} \quad (\text{B.8})$$

where $\text{vol}_2 \equiv \int dx dy$.

C Numerical shooting problem

In this paper we have used ODE shooting methods to find the various domain wall and black hole solutions of interest. We solve the shooting problem by providing data in the IR of the geometry and then integrating into the UV, and also providing data in the UV of the geometry and using this to integrate into the IR. We then require that these solutions match correctly at some point in between. In the UV the data is given by the boundary expansion of perturbations about the skew-whiffed AdS_4 solution. The data in the IR is given by the data at the horizon $r = r_+$ for a black hole solution, or the marginal and irrelevant perturbations to the AdS_4 Pope-Warner and $AdS_2 \times \mathbb{R}^2$ solutions for the zero temperature interpolating solutions that we construct. Let us denote the IR data as $\{x_i\}$ and the UV data as $\{y_a\}$. We have first order equations for g, β and second order equations for ξ, h, ϕ and hence if we match the solutions integrated from the IR up to some intermediate position $r = r_0$ and from the UV down to $r = r_0$, we must match 8 integration constants associated to these equations. To this end, let us define a function $\Phi^{IR}(x)$ via

$$\Phi^{IR}(x) = \left(g_x(r_0), \beta_x(r_0), \xi_x(r_0), \xi'_x(r_0), h_x(r_0), h'_x(r_0), \phi_x(r_0), \phi'_x(r_0) \right), \quad (\text{C.1})$$

which is obtained by integrating the differential equations out to a fixed radius r_0 by evolving given IR data $\{x_i\}$. Here $g_x(r_0)$ denotes the result of integrating the metric function $g(r)$ out to $r = r_0$ subject to IR initial conditions $\{x_i\}$ and we use analogous notation for the other fields. Let us also define the function

$$\Phi^{UV}(y) = \left(g_y(r_0), \beta_y(r_0), \xi_y(r_0), \xi'_y(r_0), h_y(r_0), h'_y(r_0), \phi_y(r_0), \phi'_y(r_0) \right), \quad (\text{C.2})$$

which is obtained by evolving given UV initial data $\{y_a\}$ inward to the same radius r_0 . We next define a function $V(x, y)$, which depends on the entire set of initial data

$$V(x, y) = \Phi^{IR}(x) - \Psi^{UV}(y) \quad (\text{C.3})$$

The condition that we have a solution to the differential equations is now simply

$$V(x, y) = 0, \quad (\text{C.4})$$

which gives rise to 8 conditions. Thus we require that 8 elements of the data $\{x, y\}$ must be tuned in order to satisfy this. Let us denote the 8 tuning data variables as α , and denote the remaining data that fixes the solution of interest as λ . Thus for some λ we need to tune the α in order that $V(\alpha; \lambda) = 0$.

Let us illustrate this using the concrete example of finite temperature black holes. We parameterize the putative solution by imposing $\xi_1 = 0$, using the scaling symmetries to fix $(16\pi G)^{1/2}\mu = 1, \beta_a = 0$, and then choosing some fixed h_1 in the UV and some fixed r_+ . Thus $\lambda = \{r_+, \mu, \xi_1, \beta_a, h_1\}$. We then tune the remaining data, the α , which in the IR is $\{\beta_+, \xi_+, \phi_+, h_+\}$ and in the UV is $\{\varepsilon, \xi_2, h_2, q\}$. These 8 variables are exactly sufficient degrees of freedom to solve the 8 constraints $V(\alpha; \lambda) = 0$, and thus one expects that if a solution exists, it is locally unique except at isolated points.

Denote the solution for some λ as $\alpha = \alpha_0(\lambda)$. Then we may consider perturbing about that solution $\alpha = \alpha_0 + \delta\alpha$, and the corresponding variation in V is given by $\delta V_i = M_{ij}\delta\alpha_j$ where,

$$M_{ij}(\lambda) = \partial V_i / \partial \alpha_j |_{\alpha_0(\lambda)} \quad (\text{C.5})$$

The local uniqueness of the solution for some λ can be expressed as $\det M_{ij}(\lambda) \neq 0$ ie. there is no perturbation to the tuning data α about the solution α_0 such that one can maintain $V = 0$.

Whilst generically tuning 8 variables α to fix 8 conditions V to vanish will yield a locally unique solution, it may happen that at specific λ the determinant, $\det M_{ij}(\lambda)$

can vanish. In particular, if the original family of solutions has a different family that branches off at some value of λ , then this will occur. The zero eigenvector $\delta\alpha$ of M then gives the linear deformation of the solution at λ that generates this new branch of solutions. Considering our example, the unbroken finite temperature black holes have precisely such behaviour at their critical temperature, where the broken phase solutions emerge as a new branch.

Hence given our solutions, simply by checking $\det M$ we may numerically check for the existence of marginal deformations that generate new branches of solutions. We have done so for all the solutions presented in the main text. In Figure 15 and 16 we plot the value of $\det M$ as a function of the various solution parameters for the unbroken and broken finite temperature black holes, and also for the RG domain walls and the charged skew-whiffed to Pope-Warner and domain wall solutions, respectively. As expected we see the determinant vanishes for the unbroken black holes at some critical temperature, for $h_1 < h_1^c$, which gives us the superconducting phase transition temperature. This indeed corresponds to the linear deformation that generates the broken phase branch of solutions. However, apart from this expected branching, we see no evidence of any other new solution branches emerging from any of the other solutions.

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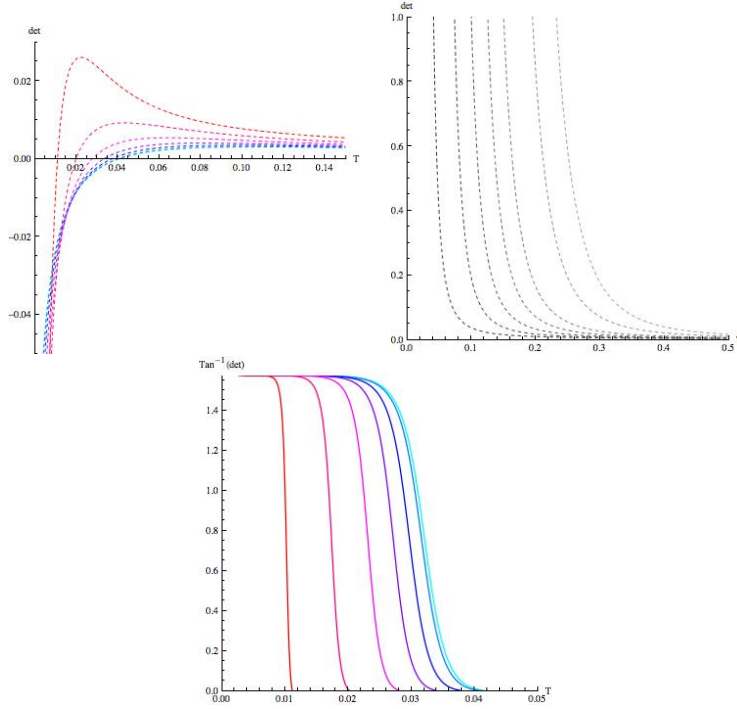


Figure 15: Plot showing a quantity $\det M$ (see text) for the unbroken phase solutions with $h_1 < h_1^c$ (top left), the unbroken phase solutions with $h_1 > h_1^c$ (top right) and the broken phase solutions which only exist for $h_1 < h_1^c$ (bottom). The vanishing of $\det M$ signals a joining point of two solution branches. We see that indeed this quantity vanishes for the critical temperature of the unbroken solutions with $h_1 < h_1^c$, indicating the broken phase emanates from there. For the broken phase, we see that it is correspondingly vanishing at the beginning of the branches where they join the broken phase. However, for no other values of temperature do we see evidence, within our ansatz, of new branches of solutions emerging.

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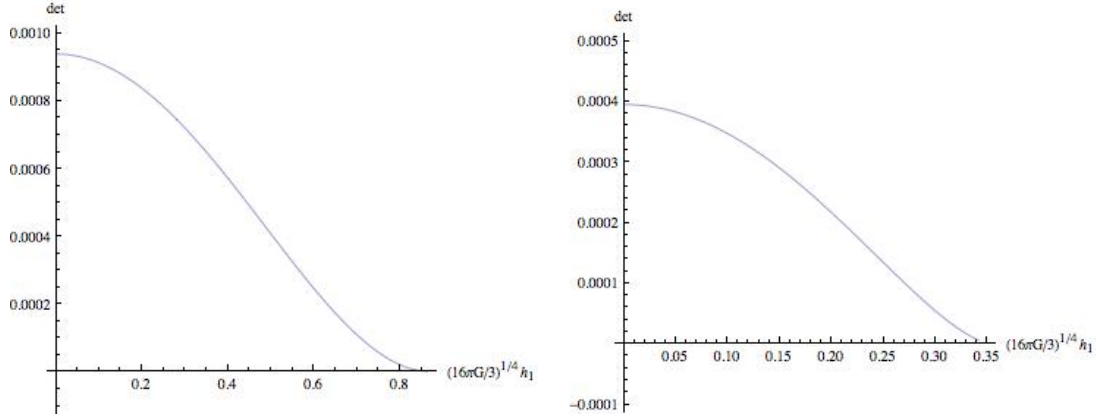


Figure 16: Plot showing a quantity $detM$ for the uncharged RG domain wall solutions (left) and charged domain wall solutions (right). Since $detM$ appears to be non-vanishing we expect that no other branches of solutions joining these solutions.

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