Quantum Degeneracy in Two Dimensional Systems

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ABSTRACT

Degeneracy is an important concept in physics and chemistry. Degeneracy and symmetry are closely connected. We distinguish between two types of degeneracy - symmetric or systematic and accidental one. In this paper, we illustrate the concept of degeneracy through some two dimensional quantum mechanical problems. We have also indicated the breaking of the degeneracy by suitable application of perturbing potentials.

1. Introduction

Degenerate and non-degenerate eigenstates are important concepts in quantum mechanics. If there are two or more *distinct* eigen solutions having same energy eigenvalues of Schrödinger time-independent equation, then they are termed as degenerate. The word distinct in the above sentence needs some clarification. The distinct solutions are those which are linearly independent. In other words, the solutions which are differing by only multiplicative phase factors $e^{i\phi}$, however, they are not designated as distinct. In this paper, we would like to discuss the origin of degenerate solutions in various two dimensional quantum problems.

The paper is organized as follows. With a brief discussion of an important theorem in one dimensional system in the next section, we mathematically justify the relation between symmetry and degeneracy. Then, we start with

a very simple example of two dimensional infinite box and two dimensional linear harmonic oscillators to illustrate the various degeneracy. Finally, we give our conclusions in section 3.

2. An Important Theorem

We would like to start the discussion of the degeneracy with a simple theorem. The statement of this theorem is: *In one dimension, (for normalizable wave functions), there are no degenerate bound states*.[1,2,3] In other words, it implies that degeneracy should occur in higher spatial dimensions. And in fact, we will encounter that this degeneracy is very common to any higher dimensional system. The degeneracy is a dimensionless quantity which counts the number of states with same energy eigenvalue. The degeneracy of a system could be finite or infinite. Let us now discuss the proof of this simple theorem. Let us assume that the two wave functions ψ_1 and ψ_2 give rise to the same energy eigenvalues E in a one dimensional quantum problem with a potential $V(x)$. Then, we have the following two equations:

$$-\frac{\hbar}{2m}\frac{\partial^2\psi_1}{\partial x^2}+V(x)\psi_1 = E\psi_1 \qquad (1)$$

$$-\frac{\hbar}{2m}\frac{\partial^2\psi_2}{\partial x^2}+V(x)\psi_2 = E\psi_2 \qquad (2)$$

Multiplying (1) by ψ_2 and (2) by ψ_1 and then subtracting from one another we get

$$\left(\psi_1\frac{\partial^2\psi_2}{\partial x^2}-\psi_2\frac{\partial^2\psi_1}{\partial x}\right)=0 \qquad (3)$$

On further simplification, we obtain a simple equation of the form

$$\left(\psi_1\frac{\partial\psi_2}{\partial x}-\psi_2\frac{\partial\psi_1}{\partial x}\right)=\text{Const} \qquad (4)$$

Now, we will use the basic postulates of quantum mechanics to evaluate the constant in the above equation (4). For well-behaved and normalizable wave functions ($\psi(x)\to 0$ as $x\to\pm\infty$ and $\int_{-\infty}^{+\infty}|\psi(x)|^2\,dx=1$), the constant becomes zero. Hence, from equation (4), by integrating, we find

$$\psi_1 = c\,\psi_2 \qquad (5)$$

Therefore, the two wave functions are not linearly independent and hence, they are same. Thus, there is no degeneracy in one dimensional quantum problems. This result is remarkable in the sense that it *cannot* be generalized to any higher dimension.

The conclusion is true for any wave function including the ground state. Moreover, by definition, the ground state (ψ_0) of any bound quantum mechanical system is nodeless, i.e. it does not vanish at any point in the space. It must keep its sign same throughout the region. This nodeless feature can be utilized[4] to show the non-degeneracy of the energy levels. Let us assume that the contrary is true. We assume that there are two functions ψ_0 and $\tilde{\psi}_0$ are the two ground state wave functions corresponding to the ground state energy. Then, any linear combination such as $c_0\psi_0+\tilde{c}\tilde{\psi}_0$ is also an eigenfunction of the same Hamiltonian with same ground state energy. However, we can make this wave function vanish at some point in space by choosing $c_0=-\tilde{c}=1/\sqrt{2}$. This implies that we can get a ground state wave function with node. Hence, our assumption is wrong. This argument can be easily generalized to show the ground state of

higher dimensional system (except charged particle in a magnetic field) as non-degenerate.

Let us pay attention to the equation (4) for a non-normalizable wave function. The simplest example arises in free particle in one dimension. The energy eigen functions e^{ikx} and e^{-ikx} are degenerate to each other. However, the above theorem is saved since they are not well-behaved and normalizable in the ordinary sense. Besides, a quick look into the equation (4) reveals that the constant is non-zero for such non-normalizable functions for free particles. Even for wave functions such as $e^{\pm\alpha x}$ which are well behaved for the whole region of x ($-\infty$ to ∞), the constant turns out to non-zero. However, for wave function ($\psi(x)=e^{-\alpha|x|}$ in a delta potential in one dimension ($V(x)=-V_0\delta(x)$), the constant is identically zero. Moreover, the potential $V(x)$ should go to zero at $x\to\pm\infty$. Thus, the non-degeneracy theorem works for potential which is bounded from below and piecewise continuous. If the potential consists of some isolated pieces separated by a region where the potential is finite, then this theorem will not hold good. For example, if there are two isolated infinite square well in one dimension, then there will be degenerate bound states.[5] The second example is the famous double oscillator[6] whose potential is given by

$$V(x)=\frac{1}{2}k(|x|-a)^2 \qquad (6)$$

A schematic variation of this potential is shown in Figure 1. It is evident that for a=0 is the usual harmonic oscillator with origin at x=0. In such a case, we find the non-degenerate equi-spaced energy levels of the particle of mass m given by

$$E_n=\left(n+\frac{1}{2}\right)\hbar\sqrt{\frac{k}{m}}=\left(n+\frac{1}{2}\right)\hbar\omega_0 \qquad (7)$$

However, with the increase of the value of a, the separate wells are developed having a large separation between them as seen from Figure 1. In fact, in $a\to\infty$ limit, since the system can have the equal probability to occupy in either of these harmonic oscillators right or left, then each of these bound state levels become doubly degenerate.

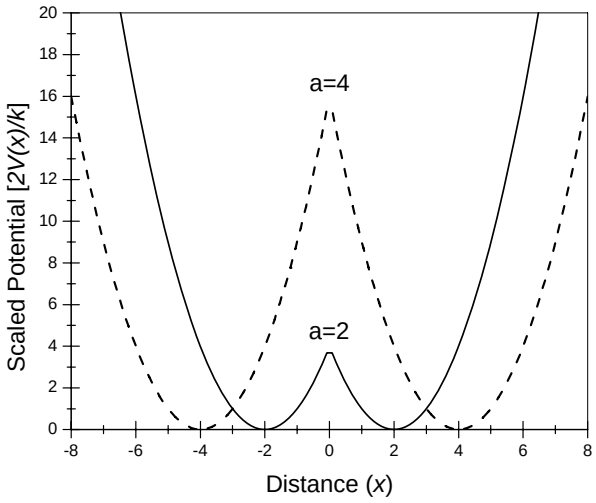
As a corollary of this theorem, it is easy to show that the eigenfunctions of the real Hamiltonian can be chosen as real in the coordinate basis in one dimension. For the problems involving a magnetic field, the Hamiltonian is no longer real in the coordinate basis. In fact, due to non-degeneracy of the bound states in one dimension, within an irrelevant scale factor, one can choose the eigenfunctions as real. In one particle picture, a non-degenerate state carries no current and is describable by a real valued wave function. This implies that wave functions for real Hamiltonian carrying current are degenerate. As a simple example, the ground state of the hydrogen atom is real and non-degenerate and does not carry any current. However, the excited states of the hydrogen atom are complex and hence, are degenerate.

2.1 Super-symmetry and Non-degeneracy

In this section we would like to discuss the close connection between the super-symmetry (SUSY)[7-9] technique and degeneracy of the quantum system. Let us first indicate how one can use the super-symmetry technique to show that the ground states as well as other states of one dimensional system are non-degenerate. Consider the one dimensional time independent Hamiltonian in equation (1) in a smooth potential $V(x)$. This Hamiltonian can be written in the factorized form[10] as

$$H=A^{\dagger}A+E \qquad (8)$$

Figure 1: Variation of potential with distance for two values of a



The explicit form of A can be chosen by assuming the real bound ground state wave function ψ_0 satisfying the above Hamiltonian with eigenvalue E_0. Thus, the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

$$= \left[\frac{\hbar}{\sqrt{2m}}\left(-\frac{d}{dx} - \frac{1}{\psi_0}\frac{d\psi_0}{dx}\right)\right]\times$$

$$\left[\frac{\hbar}{\sqrt{2m}}\left(\frac{d}{dx} - \frac{1}{\psi_0}\frac{d\psi_0}{dx}\right)\right] + E_0$$

$$= A^{\dagger}A + E_0 \qquad (9)$$

Since, the ground state wave function is nodeless, both the operators A as well as $A^{\dagger}$ are well defined. Now, for any arbitrary eigenstate ψ with eigenvalue E, we can write the equation for the operator $A^{\dagger}A$ as

$$A^{\dagger}A\,|\psi\rangle = (E - E_0)|\psi\rangle \qquad (10)$$

If the wave function happens to be the ground state one, then we get immediately $\|A\,|\psi_0\rangle\| = 0$. This indicates A just acts like an annihilation operator with $A\,|\psi_0\rangle = 0$. For any other eigenstate $\tilde{\psi}_0$ corresponding to the energy eigenvalue E_0, we must have the following relation from the operator A as

$$\frac{1}{\tilde{\psi}_0}\frac{d\tilde{\psi}_0}{dx} = \frac{1}{\psi_0}\frac{d\psi_0}{dx} \qquad (11)$$

By integrating the equation (11) we find that the wave functions are connected to each other by a constant factor as found out in equation (5). This result can be generalized to other excited states to one dimensional problem by repeated applications of appropriate super-symmetry transformation[10]. The whole argument is based on writing the Hamiltonian

H in factorized form shown in equation (8). In general, it is not possible to write in this factorized form in higher spatial dimensions.[10] That's why, this super-symmetry arguments *cannot* be translated to any eigenstates. However, for class of the Hamiltonian H in any higher spatial dimension written as

$$H = -\frac{\hbar^2}{2m}\sum_{k=1}^{d}\frac{\partial^2}{\partial x_k^2} + V(x_1, x_2, ..., x_d) \quad (12)$$

it is possible to write in factorized form as $H = \sum_{k=1}^{d} A_k^{\dagger}A_k + E_0$ with A_k defined as

$$A_k = \frac{\hbar}{\sqrt{2m}}\left(\frac{\partial}{\partial x_k} - \frac{1}{\psi_0}\frac{\partial \psi_0}{\partial x_k}\right) \quad (13)$$

Repeating the previous arguments for $k=1,2,\ldots d$, we can show that $\tilde{\psi} = C\psi_0$ signaling that the ground state of the Hamiltonian defined in equation (12) is non-degenerate. However, it is not possible to make any comment for any excited states because of the impossibility of super-symmetry partner Hamiltonian which conserves the energy spectrum[10] of the Hamiltonian. Moreover, we know that the excited states of any Hamiltonian in higher dimension $d>1$ is in general (always!) degenerate.

3. Mathematical Meaning of Degeneracy

It is a common folklore that if a system is symmetrical in some sense, its energy levels are usually degenerate. The symmetry and degeneracy are often closely linked.[11-14] However, it is not always easy to find out the symmetry which is responsible for the degeneracy in the problem. For example, classically, in a central field, the equations are invariant under rotations. As a consequence, the angular momentum of the system is conserved. This implies that corresponding to any fixed energy, there are many orbits differing in spatial orientations. In quantum mechanics, the corresponding wave functions are termed as degenerate. Let us clarify this concept mathematically. Let us consider a stationary state of a Hamiltonian H with eigenvalue E_n. In bra-ket notation, this can be stated simply as

$$H|n\rangle = E_n|n\rangle \quad (14)$$

We assume now that this Hamiltonian H has a certain symmetry denoted by the operator T. It implies that T commutes H i.e. $[T,H]=0$. Now, let us consider the following operation:

$$HT|n\rangle = TH|n\rangle = TE_n|n\rangle = E_nT|n\rangle \quad (15)$$

This indicates that $T|n\rangle$ satisfies the eigen function criteria of the Hamiltonian with the same energy eigenvalue E_n. So, $T|n\rangle$ and $|n\rangle$ are degenerate eigenkets of the Hamiltonian. In other words, the energy eigenvalue corresponds to more than one eigen function. Besides, any linear combination of the wave functions is also an eigen function of the Hamiltonian with the same energy eigenvalue. Hence, the choice of the eigen fucntions of a degenerate energy level is not unique. In general, two situations may arise for $T|n\rangle$. $T|n\rangle$ might be completely different from $|n\rangle$ or $T|n\rangle$ could be simply a multiple of $|n\rangle$. In the later situation, we may write in mathematical form as

$$T|n\rangle = g_n|n\rangle \quad (16)$$

where g_n could be termed as the degree of degeneracy of the n-th energy level.

4. Degeneracy in Two Dimensional Problems

In this section we would like to discuss the degeneracy of two dimensional quantum mechanical problems. We will consider first the particle in a rectangular box and then two dimensional harmonic oscillators.

4.1. Particle in a Rectangular Box

The potential is zero inside the rectangular box of size L_x and L_y and outside the box, the potential is infinite. Thus, the particle is free inside the box having impenetrable wall. The time-independent Schrödinger equation[1,2] of the particle of mass m in such a case is simply,

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=-\frac{2mE}{\hbar^2}\psi \qquad (17)$$

The boundary conditions for the above problem is simply $\psi(L_x,L_y) = 0$. This immediately gives the energy eigenvalues as

$$E_{n_x,n_y}=\frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2}+\frac{n_y^2}{L_y^2}\right) \qquad (18)$$

and the normalized wave function as

$$\psi_{n_x,n_y}(x,y)=$$

$$\sqrt{\frac{4}{L_xL_y}}\sin\left(\frac{n_x\pi}{L_x}\right)\sin\left(\frac{n_y\pi}{L_y}\right) \qquad (19)$$

Here, the quantum numbers can take positive integer values starting from 1. It is obvious that in this case, two quantum numbers are required to describe the energy eigenvalues and the energy is the sum of one dimensional energy eigenvalues of particle in a box problem. The lowest energy of the system is non-degenerate and is given by

$$E_{1,1}=\frac{\hbar^2\pi^2}{2m}\left(\frac{1}{L_x^2}+\frac{1}{L_y^2}\right) \qquad (20)$$

Although it is stated in many text books that the energy eigenvalues are non-degenerate ($E_{n_x,n_y}\neq E_{n_x,n_y}$), however, there are peculiar type of degeneracy present in the system for some specific commensurate ratio of the two lengths L_x and L_y. It is known as *accidental or non-geometric*[13,15] in contrast to more familiar one. Suppose, the ratio of two lengths L_x and L_y is a ratio of two prime integers $\left(\frac{L_x}{L_y}=p/q\right)$, then it is easy to notice that the states (n_x, n_y) and $\left(\frac{pn_y}{q},\frac{qn_x}{p}\right)$ are degenerate to each other. To show this explicitly, we obtain the energy eigenvalues from equation (12) for $pL_y=qL_x=L_1$((say)) as

$$E_{n_x,n_y}=\frac{\hbar^2\pi^2}{2mL_1^2}(q^2n_x^2+p^2n_y^2) \qquad (21)$$

Therefore, the energy levels remain the same for distinct wave functions under the transformation ($qn_x\Leftrightarrow pn_y$). As a specific example, we take $L_x=2L_y$, then the energy levels corresponding to eigenstates (n_x, n_y) and $\left(2n_y,\frac{1}{2}n_x\right)$ are degenerate. Thus, (4,1) and (2,2) are degenerate. This type of degeneracy is accidental because they are not related to any geometrical symmetry but to some hidden symmetry[16,17] in the problem.

4.2. Particle in a Square Box

For a square box ($L_x=L_y=L$), the usual type degeneracy occurs known as systematic or symmetric one. However, it is also better known as geometrical one because of the obvious reason. The energy levels are invariant under the transformation ($n_x \Leftrightarrow n_y$). This is due to the fact that the box is square, one can interchange the sides x and y without changing the magnitude of the energy levels. Therefore, each energy level is at least doubly degenerate when n_x and n_y are different. Hence, (1,2) and (2,1) are degenerate. The eigenfunctions corresponding to these sates are

$$\psi_{1,2}(x,y)=\left(\frac{2}{L}\right)\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi}{L}y\right)$$

$$\psi_{2,1}(x,y)=\left(\frac{2}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{\pi}{L}y\right) \qquad (22)$$

Apart from this type of geometric degeneracy, there is also accidental one. In this case, two or different pairs of integers give the same sum of the squares. For example,

$$5^2+5^2 = 7^2+1^2 \qquad (23)$$

This implies that $E_{5,5}=E_{7,1}$ although $\psi_{5,5} \neq \psi_{7,1}$. As stated earlier, this degeneracy is inherent in the structure and related to a hidden dynamical symmetry. It is also clear that the symmetry degeneracy disappears for rectangular box while the accidental persists in rectangular as well as square box. There are other few examples of non-geometric degeneracy ($7^2+4^2=8^2+1^2$ and $4^2+13^2=8^2+11^2$). This problem being a mathematical one, have been addressed to find out the degree of degeneracy as well as the set/sets of quantum number required for a given energy level[18].

The connection between accidental degeneracy and the symmetric degeneracy can be illustrated through a simple diagram in Figure 2.

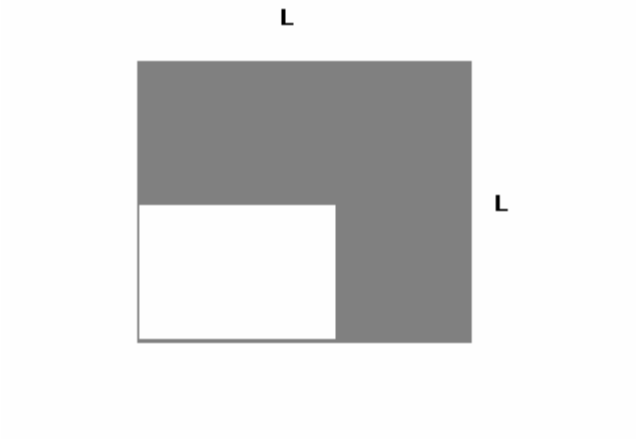


Figure 2. Illustration of accidental degeneracy in box

We have incorporated a small rectangular box (white in color)) having sizes (pL_x=qL_y=L) with p and q prime positive integers. It can be shown that the degenerate wave functions for the smaller box when extended to the bigger one preserve their degeneracy property.

4.3 Breaking the Degeneracy in a Square Box

To break the degeneracy in this system, we have to add a perturbation to it. Let us apply a very simple perturbation given in the exercise of Sakurai's book[19] and consider its effect on the first three eigenstates of the problem. The perturbation $V(x,y)=\lambda\, xy$ within the square box of length L. The first order correction to the system for any n-th eigenstate is given as

$$\Delta E_n = \langle n|\lambda xy|n) \propto \lambda \qquad (24)$$

The first excited state eigenfunction is doubly degenerate and its energy is given by $E_{fe}=\frac{5\pi^2\hbar^2}{2mL^2}$. Using the two eigenfunctions given in equation (22), we construct the (2×2) perturbing matrix whose matrix elements are shown below.

$$P_{11}=\frac{4\pi}{L^2}\int_0^L x\sin^2\left(\frac{\pi x}{L}\right)dx\int_0^L y\sin^2\left(\frac{2\pi y}{L}\right)dy$$

$$=\frac{1}{4}\lambda L^2$$

$$P_{12}=\frac{4\lambda}{L^2}\int_0^L x\sin^2\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)dx$$

$$\int_0^L y\sin\left(\frac{\pi y}{L}\right)\sin\left(\frac{2\pi y}{L}\right)dy$$

$$=\frac{256}{81\pi^4}\lambda L^2 \qquad (25)$$

By symmetry, we have P_{22}=P_{11} and P_{12}=P_{21}. Diagonalizing this P matrix, we find the eigenvalues as $\frac{(\pi^4+\frac{4\times256}{81})}{4\pi^4}$ and $\frac{(\pi^4-\frac{4\times256}{81})}{4\pi^4}\lambda L^2$. Hence, the degenerate energy eigenvalue after the perturbation becomes

$$E_{ne1}=E_{fe}+\frac{(\pi^4+\frac{4\times256}{81})}{4\pi^4}\lambda L^2;$$

$$E_{ne2}=E_{fe}+\frac{(\pi^4-\frac{4\times256}{81})}{4\pi^4}\lambda L^2 \qquad (26)$$

Thus, the systematic degeneracy in the first excited state for this square box has been removed by the perturbation potential.

Note that in this problem, all the excited energy eigenstates are not degenerate. For example, the second excited state is non-degenerate with energy eigenvalue $E_{se}=\frac{4\pi^2\hbar^2}{mL^2}$. One may also ask what happens to energy eigenvalues corresponding to the states in equation (23) giving rise to accidental degeneracy after the application of above perturbation. The energy eigenfunctions are

$$\psi_{5,5}(x,y)=\left(\frac{2}{L}\right)\sin\left(\frac{5\pi x}{L}\right)\sin\left(\frac{5\pi}{L}y\right)$$

$$\psi_{7,1}(x,y)=\left(\frac{2}{L}\right)\sin\left(\frac{7\pi x}{L}\right)\sin\left(\frac{\pi}{L}y\right) \qquad (27)$$

In this case, a straightforward algebra reveals that $P_{11}=P_{22}=\frac{1}{4}\lambda L^2$ while $P_{12}=P_{21}=0$. Thus, the new energy values are $\frac{25\pi^2\hbar^2}{mL^2}+\frac{1}{4}\lambda L^2$. Therefore, although the magnitude of the eigenvalue in the first order perturbation calculation is changed but the accidental degeneracy for the states (*5,5*) and (*7,1*) is not removed by the perturbation.

5 Degeneracy in 2*d* Harmonic Oscillator Problem

The *2d* harmonic oscillator is a system where the symmetry and degeneracy can be beautifully demonstrated without invoking too much mathematical tools.[13,15,20] The accidental degeneracy and related symmetry group of the harmonic oscillator have been extensively discussed by Quesne.[21]

5.1 Anisotropic Harmonic Oscillator

The Hamiltonian of the two dimensional harmonic oscillators is given by

$$E_{n_x,n_y}(\omega_x,\omega_y)=(n_x\hbar\omega_x+n_y\hbar\omega_y)+\frac{1}{2}\hbar(\omega_x+\omega_y) \quad (29)$$

The last term is the zero point of energy of the system and can be understood from the Uncertainty principle. Because of the positivity of the Hamiltonian, all the energy levels are positive definite and this is ensured by the quantum numbers which can take the positive integer values like *0,1, 2* . . ., etc. Although it might appear at first sight from the equation (29) that the energy levels are non-degenerate due to anisotropic nature of the system, however, there does appear a special kind of degeneracy known as accidental degeneracy or non-geometric one as discussed in *2d* particle in a box problem. If we can choose the ratio of the frequencies as the ratio of the positive integers in the following way

$$H=\frac{1}{2m}(p_x^2+m\omega_x^2x^2)+\frac{1}{2m}(p_y^2+m\omega_y^2y^2) \quad (28)$$

Since this is sum of two harmonic motions in *x* and *y* directions, we can easily write down the energy levels of this anisotropic system as

The degeneracy and the first few energy levels are shown in tabular form.

n_x	n_y	$n=n_x+n_y$	Comments
0	0	0	Ground state *Non-degenerate*
1 0	0 1	1 1	First Excited State *Doubly Degenerate*
0 2 **1**	2 0 **1**	2 2 **2**	Second Excited State *Triply Degenerate*
0 3 1 2	3 0 2 1	3 3 3 3	Third Excited State *Four Fold* *Degenerate*

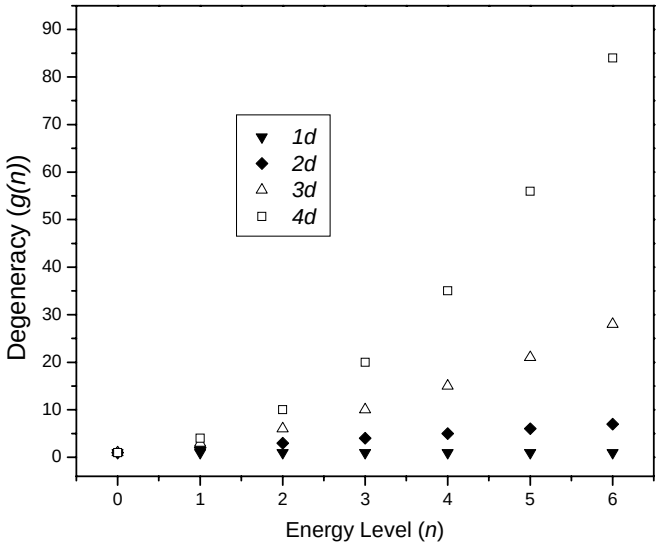
$$\frac{\omega_x}{\omega_y} = \frac{n_y}{n_x} \tag{30}$$

then, it is easy to notice from equation (29) that the energy remains the same. Thus, for such set of pairs of frequencies and quantum numbers, the energy levels are degenerate.

5.2 Isotropic Harmonic Oscillator

For isotropic case ($\omega_x=\omega_y=\omega$), the energy levels are geometric or systematic degeneracy. Using the property of confluent hypergeometric function, the degeneracy has been computed[22] for *d*-dimensional confined harmonic oscillator. In fact, the degeneracy of any *d*-dimensional harmonic oscillator[23,24] can be computed in the following way. In such a situation, the total quantum numbers (n_i, 1<i<*d*) required are *d*; however, their sum is restricted to *n*. In addition, there is no degeneracy in one dimensional system and hence

Figure 3: Variation of Degeneracy with energy level *n*



$$E_{n_x,n_y}(\omega) = (n_x + n_y + 1)\hbar\omega \tag{31}$$

For a given *n*, the energy is determined from $n_x+n_y=n$ equation. However, each n_x and n_y can take (*n*+1) values starting from 0 to *n*. Hence, the degeneracy of *n*-th energy level is simply (*n*+1) fold. The degeneracy in second excited state marked with bold font is known as the accidental one while the others are related with the total number independent arrangements would be simply

$$g(n) = \frac{(n+d-1)!}{n!(d-1)!} \tag{32}$$

We show in Figure 3, the variation of degeneracy with quantum number n for spatial dimensions. It is obvious from Figure 3 that with increase of spatial dimension d, the degeneracy increases non-linearly with discrete integer n. It is to be noted the degeneracy in one dimension is one and hence, all the energy levels are non-degenerate.

Like the energy eigenvalues, the wave functions are also labeled by n_x and n_y. Thus, $\psi_{0,1}(x,y)$ and $\psi_{1,0}(x,y)$ are degenerate eigen functions corresponding to first excited state having the same energy eigenvalue $2\hbar\omega$. As discussed earlier, the second excited state has one peculiar so called non-geometric degeneracy aside from two usual degenerate systematic or geometric states. The root of this accidental degeneracy lies in some extra hidden symmetry[13,15] in the Hamiltonian. Let us define three operator

$$F_1 = \left[\frac{p_x p_y}{2m\omega} + m\omega xy\right]$$

$$F_2 = \left[\frac{p_y^2 - p_x^2}{4m\omega} + \frac{m\omega}{4}(y^2 - x^2)\right] \qquad (33)$$

$$F_3 = \frac{1}{2}[xp_y - yp_x]$$

to note that these three are constants of motion, i.e. $[H, F_i]=0$ for $i=1,2,3$. Third component F_3 is simply related to third component of angular momentum. The second one can be viewed as the difference between two Hamiltonians in y and x direction. In fact, for two degrees of freedom, there cannot be more than 3 ($2\times2-1=3$) constants of motion. In this case, we have exactly three independent constants of motion. More importantly, these operators satisfy the commutations as followed by the usual angular momentum operators $[F_1,F_2]=i\hbar F_3$. They are the generators of O(3) group or more generally SU(2) group. The commutation relation enforces one to rewrite the equation of eigenvalues of the Hamitonian as

$$\left(\frac{E_l}{2}\right)^2 - \left(\frac{\hbar\omega}{2}\right)^2 = \omega^2(F_1^2 + F_2^2 + F_3^2)$$

$$= \hbar^2\omega^2 l(l+1) \qquad (34)$$

giving the energy eigenvalues

$$E_l = \hbar\omega(2l+1) \qquad (35)$$

with $l=0, ½, 1, 3/2, 2, 5/2, \ldots$. A comparison with equation (21) indicates that $l=2n$ with equal spacing and characteristic zero point energy $\hbar\omega$. Moreover, the degeneracy of n-th level is $(2l+1)=n+1$ as discussed earlier.

To have a clear picture of the symmetry, it is better to rewrite the above Hamiltonian in polar coordinates. The isotropic Hamiltonian in the polar coordinates reads as

$$H = -\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)$$

$$-\frac{\hbar^2}{2mr^2}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2}m\omega^2 r^2 \qquad (36)$$

It is not surprising to verify that the z-component of the angular momentum $L_z = -i\hbar\frac{\partial}{\partial\phi}$ which is perpendicular to the x-y plane is a constant of motion of the system. In other words, $[H,L_z]=0$.

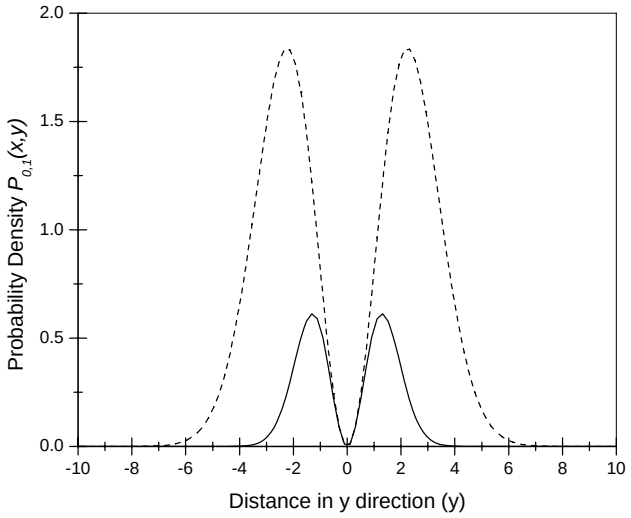


Figure 4: Variation of the probability density $P_{0,1}(x,y)$ as a function of y for two values of α. The dashed curve is drawn for α=0.1 while for continuous curve, the value of for α=0.3.

This is due to the fact that the Hamiltonian is invariant under rotation by any angle ϕ. After rotation, the rotated Hamiltonian remains the same as the original Hamiltonian. Note that for three dimensional systems, when the Hamiltonian is spherically symmetric, it is invariant under rotations about any axis. In that situation, the Hamiltonian commutes with all three components (L_x, L_y and L_z) of the angular momentum. The vanishing commutator relation also points out that the L_z and H must have simultaneously same eigenstate provided the states are non-degenerate. The degeneracy can be viewed from the eigenstates $\psi_{0,1}(x,y)$ and $\psi_{1,0}(x,y)$. The wave function $\psi_{0,1}(x,y)$ can be written within a normalization constant N_1 as

$$\psi_{0,1}(x,y) = N_1 y e^{-\alpha(x^2+y^2)} \qquad (37)$$

In Figure 4, we show schematically the probability density $P_{0,1}(x,y)$ corresponding to $\psi_{0,1}(x,y)$ as a function of y keeping x constant for two values of α. It is seen from the figure that the probability density $P_{0,1}(x,y)$ is peaked at two points along y directions. Moreover, the magnitudes as well as the positions of the peaks are functions of α. It is amazing to note that although the oscillator does not possess any preferred direction; however, one of the first excited states namely $\psi_{0,1}(x,y)$ has acquired a particular direction. By symmetry principle, since x and y are equivalent for this *2d* harmonic oscillator, there must be another wave functions whose probability density must peak up along x directions. A quick look reveals that the probability density of $\psi_{1,0}(x,y)$ has the required property. This implies that these two wave functions must have the same

energy. One should also note that there could be other wave functions – linear combinations of these two wave functions - which also have the same energy. Now, let us pay attention to the ground state wave function $\psi_{0,0}(x,y)$

$$\psi_{0,0}(x,y) = N_0 e^{-\alpha(x^2+y^2)} = N_0 e^{-\alpha r^2} \quad (38)$$

This state is independent of ϕ, so rotation does not produce any new state and hence, this is the only state which is non-degenerate. A close inspection points out that this is also an eigen state of L_z with eigen value zero. Thus, the commutator between H and L_z is satisfied by this wave function. However, the situation is slightly complicated for the two degenerate eigen functions. In polar coordinates, the wave functions are given by

$$\psi_{1,0}(r,\phi) = N_1 r\cos(\phi)e^{-\alpha r^2}$$

$$\psi_{0,1}(r,\phi) = N_1 r\sin(\phi)e^{-\alpha r^2} \quad (39)$$

A straightforward calculation shows that they are not the eigen functions of L_z. In fact, $L_z\psi_{1,0}=\lambda\psi_{0,1}$ and vice versa. Then, one might worry about the commutator relation between the Hamiltonian and L_z. These wave functions are the eigen functions of H; but surprisingly, *not* the eigen functions of L_z. It is due to the fact that degenerate eigen functions are not necessarily the eigen functions of L_z. However, a linear combination of the degenerate eigen functions can be constructed to form the eigen functions of L_z. For example, the wave functions

$$\tilde{\psi} = \psi_{1,0} + i\psi_{0,1} = N_1 e^{i\phi}e^{-\alpha r^2}$$

$$\bar{\psi} = \psi_{1,0} - i\psi_{0,1} = N_1 e^{-i\phi}e^{-\alpha r^2} \quad (40)$$

correspond to $\hbar$ and – $\hbar$ eigen values of L_z respectively.

5.3 Breaking the Degeneracy of Isotropic Harmonic Oscillator

To break the degeneracy of the isotropic oscillator, one can add some potential such as $H' = \lambda\, xy$ or any other functional form. We would like to illustrate here simple perturbation in terms of spring constants to remove the degeneracy in the system. To do this, we have to introduce a very simple perturbation in terms of different spring constants in two directions x and y. Let us assume that the spring constants k_x and k_y are different by an infinitesimal amount η in the following way: $\tilde{k}_x \to k+\eta$ and $\tilde{k}_y \to k-\eta$ with $\eta << k$. Thus, the Hamiltonian in this new variable reads as

$$H = \frac{1}{2m}(p_x^2 + m\omega^2 x^2) + \frac{1}{2m}(p_y^2 + m\omega^2 y^2)$$

$$+\frac{1}{2}\eta(x^2 - y^2)$$

$$= -\frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}kr^2 + \frac{1}{2}\eta r^2\cos(2\phi)$$

$$= H_0 + H' \quad (41)$$

where H_0 is the unperturbed part of 2d harmonic oscillator and H' is the perturbed part of it. Now, because of this angle dependence of the perturbed part, it is easy to visualize that $[H_0,L_z]=0$ but $[H',L_z]\neq 0$. In other words, the perturbation breaks the rotational invariance already present in the unperturbed part. We will notice immediately that this deficiency of this symmetry is responsible for the breaking of the degeneracy. It is easy to notice that from equation (41) even after the introduction of the perturbation, the Hamiltonian can be separated into two directions with the modifications of frequencies. In the limit of $\eta << k$, the frequencies along x direction and y direction

are respectively, $\omega_0\left(1+\frac{\eta}{2k}\right)$ and $\omega_0\left(1-\frac{\eta}{2k}\right)$ with $\omega_0 = \sqrt{\frac{k}{m}}$. Therefore, the energy in this limit is

$$E_{n_x,n_y}(\omega_0,\eta) = (n_x + n_y + 1)\hbar\omega_0 + \frac{1}{2}\hbar\omega_0(n_x - n_y)\left(\frac{\eta}{k}\right) \quad (42)$$

Thus, we notice that the energy depends not only on sum of the quantum numbers n_x and n_y but also their difference. For example, the first excited states are now non-degenerate

$$E_{0,1} = 2\hbar\omega_0 + \frac{1}{2}\hbar\omega_0\left(\frac{\eta}{k}\right)$$

$$E_{0,1} = 2\hbar\omega_0 - \frac{1}{2}\hbar\omega_0\left(\frac{\eta}{k}\right) \quad (43)$$

with the difference

$$E_{1,0} - E_{0,1} = \hbar\omega_0\left(\frac{\eta}{k}\right) \quad (44)$$

The higher energy in (*1,0*) is expected in compared to (*0,1*) states because of the higher value of the spring constant however small may be. This eventually points out the role played by η. In fact, in the limit η→0, the energy levels $E_{0,1}$ and $E_{1,0}$ become degenerate. Thus, not only the first excited states, but also the other excited energy levels are non-degenerate by the application of this simple perturbation. Therefore, without the presence of rotational invariance, the symmetry degeneracy disappears. One might wonder about the fate of accidental degeneracy in such a situation. Suppose $\tilde{k}_x = 9\tilde{k}_y$ and then, it immediately implies $\tilde{\omega}_x = 3\tilde{\omega}_y$. Thus, the ratio of two frequencies is 1/3, i.e. $n_y = 3n_x$. The energy levels in such a situation from equation (29) can be written as

$$E_{n_x,n_y}(\tilde{\omega}_y) = \hbar\tilde{\omega}_y(3n_x + n_y) + 2\hbar\tilde{\omega}_y \quad (45)$$

It is easy to notice that $E_{0,3} = E_{1,0}$ in such a situation. Although, the eigenfunctions $\psi_{0,3}(x,y)$ and $\psi_{1,0}(x,y)$ are not the same but they have the same energy and degenerate to each other in spite of having any continuous symmetry. Hence, remarkably, accidental degeneracy is not removed by the introduction of the above perturbation.

6. Conclusions

Thus, we have shown how the symmetry in the problem can lead to degeneracy in the problem. We have been able to distinguish between the two types of symmetry namely the systematic degeneracy and accidental one. We have also considered a simple example in two dimensional harmonic oscillators where the systematic degeneracy is broken while the accidental degeneracy persists upon the application of a perturbation into the system.

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