QUANTUM DOUBLE OF HOPF MONADS AND CATEGORICAL CENTERS

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ABSTRACT. The center $\mathcal{Z}(\mathcal{C})$ of an autonomous category $\mathcal{C}$ is monadic over $\mathcal{C}$ (if certain coends exist in $\mathcal{C}$). The notion of a Hopf monad naturally arises if one tries to reconstruct the structure of $\mathcal{Z}(\mathcal{C})$ in terms of its monad Z: we show that Z is a quasitriangular Hopf monad on $\mathcal{C}$ and $\mathcal{Z}(\mathcal{C})$ is isomorphic to the braided category $Z - \mathcal{C}$ of Z-modules. More generally, let T be a Hopf monad on an autonomous category $\mathcal{C}$. We construct a Hopf monad Z_T on $\mathcal{C}$, the *centralizer of T*, and a canonical distributive law $\Omega\colon TZ_T \to Z_T T$. By Beck's theory, this has two consequences. On one hand, $D_T = Z_T \circ_\Omega T$ is a quasitriangular Hopf monad on $\mathcal{C}$, called the *double of T*, and $\mathcal{Z}(T - \mathcal{C}) \simeq D_T - \mathcal{C}$ as braided categories. As an illustration, we define the double $D(A)$ of a Hopf algebra A in a braided autonomous category in such a way that the center of the category of A-modules is the braided category of $D(A)$-modules (generalizing the Drinfeld double). On the other hand, the canonical distributive law Ω also lifts Z_T to a Hopf monad $\tilde{Z}_T^\Omega$ on $T - \mathcal{C}$, and $\tilde{Z}_T^\Omega(\mathbb{1}, T_0)$ is the coend of $T - \mathcal{C}$. For $T = Z$, this gives an explicit description of the Hopf algebra structure of the coend of $\mathcal{Z}(\mathcal{C})$ in terms of the structural morphisms of $\mathcal{C}$. Such a description is useful in quantum topology, especially when $\mathcal{C}$ is a spherical fusion category, as $\mathcal{Z}(\mathcal{C})$ is then modular.

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Introduction

The center $\mathcal{Z}(\mathcal{C})$ of an autonomous category $\mathcal{C}$, introduced by Drinfeld, is a braided autonomous category. This construction establishes a bridge between the non-braided world and the braided world. It is useful, in particular, for comparing quantum invariants of 3 manifolds. Indeed, the center $\mathcal{Z}(\mathcal{C})$ of spherical fusion category $\mathcal{C}$ is modular (see [Mü03]), and it is conjectured that the Turaev-Viro invariant $\mathrm{TV}_{\mathcal{C}}$ (as revisited in [BW96]) is equal to the Reshetikhin-Turaev invariant $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ (see [RT91, Tur94]).

Let $\mathcal{C}$ be an autonomous category. If the coend

$$Z(X) = \int^{Y \in \mathcal{C}} {}^{\vee}Y \otimes X \otimes Y$$

exists for all object X of $\mathcal{C}$, then Day and Street [DS07] showed that Z is a monad on $\mathcal{C}$ and the center $\mathcal{Z}(\mathcal{C})$ is isomorphic to the category $Z\text{-}\mathcal{C}$ of Z-modules in $\mathcal{C}$ (also called Z-algebras). By Tannaka reconstruction, we endow the monad Z with a quasitriangular Hopf monad structure which reflects the braided autonomous structure of $\mathcal{Z}(\mathcal{C})$ in the sense that $\mathcal{Z}(\mathcal{C}) \simeq Z\text{-}\mathcal{C}$ as braided categories. The notion of a Hopf monad, which generalizes Hopf algebras to the non-braided (and non-linear) setting, was introduced in [BV07] for this very purpose.

Given a modular category $\mathcal{B}$, the original construction of the Reshetikhin-Turaev invariant $\mathrm{RT}_{\mathcal{B}}$ involves the simple objects of $\mathcal{B}$ (see [RT91, Tur94]). However, when $\mathcal{B} = \mathcal{Z}(\mathcal{C})$ is the center of a spherical fusion category $\mathcal{C}$, we cannot compute explicitly by this method the invariant $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$ in terms of $\mathcal{C}$, since there is no workable description of the simple objects of $\mathcal{Z}(\mathcal{C})$. Now Lyubashenko gave an alternate construction of the invariant $\mathrm{RT}_{\mathcal{B}}$ by means of the coend of the category $\mathcal{B}$, which is a Hopf algebra in $\mathcal{B}$ (see [Lyu95, BV05]). So, to compute $\mathrm{RT}_{\mathcal{Z}(\mathcal{C})}$, all we need is an explicit description (in terms of $\mathcal{C}$) of the coend of $\mathcal{Z}(\mathcal{C})$ and its algebraic structure. To arrive at such a description, we develop the theory of the double of a Hopf monad.

Let T be a Hopf monad on an autonomous category $\mathcal{C}$. We denote by $T\text{-}\mathcal{C}$ the category of T-modules (also called T-algebras), which is autonomous. Assume T is *centralizable*, meaning that the coend

$$Z_T(X) = \int^{Y \in \mathcal{C}} {}^{\vee}T(Y) \otimes X \otimes Y$$

exists for every object X of $\mathcal{C}$. We endow Z_T with a structure of a Hopf monad on $\mathcal{C}$ and call Z_T the *centralizer of T*. In particular, the monad Z defined above is the centralizer of the identity endofunctor $1_{\mathcal{C}}$ of the category $\mathcal{C}$.

Using adjunction and exactness properties of Hopf monads, we show that the identity endofunctor $1_{T\text{-}\mathcal{C}}$ of the category $T\text{-}\mathcal{C}$ is centralizable and

$$U_T Z_{1_{T\text{-}\mathcal{C}}} = Z_T U_T,$$

where $U_T \colon T\text{-}\mathcal{C} \to \mathcal{C}$ is the forgetful functor. This equality means that the Hopf monad $Z_{1_{T\text{-}\mathcal{C}}}$ is a lift to $T\text{-}\mathcal{C}$ of the Hopf monad Z_T. Extending Beck's theory of distributive laws to Hopf monads, we show that such a lift is encoded by an invertible comonoidal distributive law $\Omega \colon T Z_T \to Z_T T$, called the *canonical distributive law of T*. The coend of $T\text{-}\mathcal{C}$ is therefore $(Z_T(\mathbb{1}), Z_T(T_0)\Omega_{\mathbb{1}})$. When T is quasitriangular, this coend has a structure of a Hopf algebra in the braided autonomous category $T\text{-}\mathcal{C}$, which we elucidate in terms of T. In particular, for $T = Z$, we obtain

an explicit description of the coend of $\mathcal{Z}(\mathcal{C})$. The case of fusion categories is treated in detail.

The canonical distributive law Ω also endows the composition of Z_T and T with a Hopf monad structure, denoted by $D_T = Z_T \circ_\Omega T$ and called the *double of* T. We prove that D_T is quasitriangular and give a braided isomorphism:

$$D_T\text{-}\mathcal{C} \simeq \mathcal{Z}(T\text{-}\mathcal{C}).$$

This construction of the double of a centralizable Hopf monad on an autonomous category generalizes the Drinfeld double to a non-braided setting. As an illustration, we apply it to Hopf monads associated with Hopf algebras in braided categories. This leads to the double $D(A)$ of a Hopf algebra A in a braided autonomous category $\mathcal{B}$ (which is different from the double considered in [Bes97] as it involves the coend of $\mathcal{B}$). More precisely, the endofunctor $? \otimes A$ is a Hopf monad on $\mathcal{B}$. Assuming $\mathcal{B}$ admits a coend C, the Hopf monad $? \otimes A$ is centralizable, and its centralizer is of the form $? \otimes Z(A)$, where $Z(A) = {}^\vee A \otimes C$ is a Hopf algebra in $\mathcal{B}$. The canonical distributive law of $? \otimes A$ is of the form $\mathrm{id}_{1_\mathcal{B}} \otimes \Omega$, where $\Omega\colon Z(A) \otimes A \to A \otimes Z(A)$ is a distributive law of Hopf algebras. Then

$$D(A) = A \otimes_\Omega Z(A) = A \otimes {}^\vee A \otimes C$$

is a quasitriangular Hopf algebra in $\mathcal{B}$ such that

$$\mathcal{Z}(\mathcal{B}_A) \simeq \mathcal{B}_{D(A)} \simeq {}_{D(A)}\mathcal{B} \simeq \overline{\mathcal{Z}({}_A\mathcal{B})}$$

as braided categories, where ${}_A\mathcal{B}$ and $\mathcal{B}_A$ denote the categories of left and right A-modules in $\mathcal{B}$. In this context, a Hopf algebra B in $\mathcal{B}$ is quasitriangular if it is endowed with an R-matrix, that is, a morphism $r\colon C \otimes C \to B \otimes B$ which encodes a braiding on $\mathcal{B}_B$ (or equivalently ${}_B\mathcal{B}$). When $\mathcal{B}$ is the category of finite-dimensional vector spaces over a field $\Bbbk$, we recover the usual definition of an R-matrix and the Drinfeld double of a Hopf algebra H. Indeed, in that case, $C = \Bbbk$, $Z(H) = H^{*\mathrm{cop}}$, and $D(H) = H \otimes_\Omega H^{*\mathrm{cop}}$.

The canonical distributive law of a Hopf monad is in fact naturally defined in a more general setting. Let T be a Hopf monad on an autonomous category $\mathcal{C}$ and Q be a Hopf monad on $T\text{-}\mathcal{C}$. Their cross product $Q \rtimes T = U_T Q F_T$ is a Hopf monad on $\mathcal{C}$, where $F_T\colon \mathcal{C} \to T\text{-}\mathcal{C}$ is the left adjoint of the forgetful functor $U_T\colon T\text{-}\mathcal{C} \to \mathcal{C}$. If $Q \rtimes T$ is centralizable, then so is Q, and the Hopf monad Z_Q is a lift to $T\text{-}\mathcal{C}$ of the Hopf monad $Z_{Q \rtimes T}$:

$$U_T Z_Q = Z_{Q \rtimes T} U_T.$$

From this we obtain a canonical distributive law $\Omega\colon T Z_{Q \rtimes T} \to Z_{Q \rtimes T} T$ and a Hopf monad $D_{Q,T} = Z_{Q \rtimes T} \circ_\Omega T$ on $\mathcal{C}$. Moreover, we show

$$D_{Q,T}\text{-}\mathcal{C} \simeq \mathcal{Z}_Q(T\text{-}\mathcal{C}),$$

where $\mathcal{Z}_Q(T\text{-}\mathcal{C})$ is the center of $T\text{-}\mathcal{C}$ relative to Q. When $Q = \mathrm{id}_{T\text{-}\mathcal{C}}$, we obtain the previous results.

This paper is organized as follows. In Section 1, we review several facts about monoidal categories, Hopf algebras in braided categories, and graphical calculus (which flows from bottom to top in this paper). Section 2 recalls the definition and elementary properties of Hopf monads. Section 3 deals with comonoidal adjunctions, exactness properties, and cross products of Hopf monads. In Section 4, we briefly recall the basic results of Beck's theory of distributive laws and extend them to the Hopf monad setting. In Section 5, we define the centralizer Z_T of a Hopf

monad T on $\mathcal{C}$ and relate it to the center $\mathcal{Z}_T(\mathcal{C})$ of $\mathcal{C}$ relative to T. In Section 6, we define the canonical distributive law Ω of T over Z_T and the double $D_T = Z_T \circ_\Omega T$, and state their categorical properties. In Section 7, we study the centralizer Z_Q of a Hopf monad Q on T-$\mathcal{C}$ and construct the canonical distributive law of T over $Z_{Q \rtimes T}$. Section 8 is devoted to Hopf monads on a braided category $\mathcal{B}$. In particular, we define the double $D(A)$ of a Hopf algebra A in a braided autonomous category. In Section 9, we treat the case of the center of a fusion category.

1. Preliminaries and notation

In this section, in order to fix notation and conventions, we review basic notions and facts concerning monoidal categories, (co)monoidal functors and natural transformations (see [Mac98]), autonomous categories, braided categories and functors, categorical centers (see [Kas95]), Hopf algebras in braided categories and their modules (see [Maj95]), and graphical calculus.

1.1. **Categories.** Unless otherwise specified, categories are small and monoidal categories are strict.

If $\mathcal{C}$ is a category, we denote by $\mathrm{Ob}(\mathcal{C})$ the set of objects of $\mathcal{C}$ and by $\mathrm{Hom}_\mathcal{C}(X, Y)$ the set of morphisms in $\mathcal{C}$ from an object X to an object Y. The identity functor of $\mathcal{C}$ will be denoted by $1_\mathcal{C}$. We denote by $\mathcal{C}^{\mathrm{op}}$ the *opposite category* (where the arrows are reversed).

Let $\mathcal{C}$, $\mathcal{D}$ be two categories. Functors from $\mathcal{C}$ to $\mathcal{D}$ are the objects of a category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. Given two functors $F, G \colon \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\alpha \colon F \to G$ is a family $\{\alpha_X \colon F(X) \to G(X)\}_{X \in \mathrm{Ob}(\mathcal{C})}$ of morphisms in $\mathcal{D}$ satisfying the following functoriality condition: $\alpha_Y F(f) = G(f)\alpha_X$ for every morphism $f \colon X \to Y$ in $\mathcal{C}$. We denote by $\mathrm{Hom}(F, G)$ the set $\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(F, G)$ of natural transformations from F to G and by id_F the identity natural transformation of a functor F.

If $\mathcal{C}$, $\mathcal{C}'$ are two categories, we denote by $\sigma_{\mathcal{C},\mathcal{C}'}$ the flip functor $\mathcal{C} \times \mathcal{C}' \to \mathcal{C}' \times \mathcal{C}$ defined by $(X, X') \mapsto (X', X)$.

1.2. **Monoidal categories.** Let $\mathcal{C}$ be a monoidal category with monoidal product $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and unit object $\mathbb{1}$. For $n \geq 0$, we denote by $\bigotimes_n$ the functor

$$\otimes_n \colon \mathcal{C}^n = \underbrace{\mathcal{C} \times \cdots \times \mathcal{C}}_{n \text{ times}} \to \mathcal{C}, \quad (X_1, \ldots, X_n) \mapsto X_1 \otimes \cdots \otimes X_n.$$

Note that $\otimes_0$ is the constant functor equal to $\mathbb{1}$, $\otimes_1 = \mathrm{id}_{1_\mathcal{C}}$, and $\otimes_2 = \otimes$.

For a family of functors $\{F_i \colon \mathcal{A}_i \to \mathcal{C}\}_{1 \leq i \leq n}$, set

$$F_1 \otimes \cdots \otimes F_n = \otimes_n \circ (F_1, \ldots, F_n) \colon \mathcal{A}_1 \times \cdots \times \mathcal{A}_n \to \mathcal{C}.$$

For a functor $F \colon \mathcal{A} \to \mathcal{C}$, set

$$F^{\otimes n} = \underbrace{F \otimes \cdots \otimes F}_{n \text{ times}} \colon \mathcal{A}^n \to \mathcal{C}.$$

If $\mathcal{C}$ is a monoidal category, we denote by $\mathcal{C}^{\otimes \mathrm{op}}$ the monoidal category $(\mathcal{C}, \otimes^{\mathrm{op}}, \mathbb{1})$, where $X \otimes^{\mathrm{op}} = Y \otimes X$, and by $\mathcal{C}^{\mathrm{op}}$ the monoidal category $(\mathcal{C}^{\mathrm{op}}, \otimes, \mathbb{1})$.

1.3. Monoidal functors. Let $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{D}, \otimes, \mathbb{1})$ be two monoidal categories. A *monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, F_2, F_0), where $F\colon \mathcal{C} \to \mathcal{D}$ is a functor,

$$F_2 = \{F_2(X,Y)\colon F(X) \otimes F(Y) \to F(X \otimes Y)\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$$

is a natural transformation (from $F \otimes F$ to $F\otimes$), and $F_0\colon \mathbb{1} \to F(\mathbb{1})$ is a morphism in $\mathcal{D}$, such that

$$F_2(X, Y \otimes Z)(\mathrm{id}_{F(X)} \otimes F_2(Y,Z)) = F_2(X \otimes Y, Z)(F_2(X,Y) \otimes \mathrm{id}_{F(Z)}),$$
$$F_2(X, \mathbb{1})(\mathrm{id}_{F(X)} \otimes F_0) = \mathrm{id}_{F(X)} = F_2(\mathbb{1}, X)(F_0 \otimes \mathrm{id}_{F(X)}),$$

for all objects X, Y, Z of $\mathcal{C}$.

A monoidal functor (F, F_2, F_0) is said to be *strong* (resp. *strict*) if F_2 and F_0 are isomorphisms (resp. identities).

By a *monoidal isomorphism*, we mean a strong monoidal functor which is an isomorphism.

1.4. Monoidal natural transformations. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ be two monoidal functors. A natural transformation $\varphi\colon F \to G$ is *monoidal* if it satisfies

$$\varphi_{X \otimes Y} F_2(X,Y) = G_2(X,Y)(\varphi_X \otimes \varphi_Y) \quad \text{and} \quad G_0 = \varphi_{\mathbb{1}} F_0.$$

1.5. Comonoidal functors. Let $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{D}, \otimes, \mathbb{1})$ be two monoidal categories. A *comonoidal functor*[1] from $\mathcal{C}$ to $\mathcal{D}$ is a triple (F, F_2, F_0), where $F\colon \mathcal{C} \to \mathcal{D}$ is a functor,

$$F_2 = \{F_2(X,Y)\colon F(X \otimes Y) \to F(X) \otimes F(Y)\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$$

is a natural transformation (from $F\otimes$ to $F \otimes F$), and $F_0\colon F(\mathbb{1}) \to \mathbb{1}$ is a morphism in $\mathcal{D}$ such that

$$\big(\mathrm{id}_{F(X)} \otimes F_2(Y,Z)\big)F_2(X, Y \otimes Z) = \big(F_2(X,Y) \otimes \mathrm{id}_{F(Z)}\big)F_2(X \otimes Y, Z),$$
$$(\mathrm{id}_{F(X)} \otimes F_0)F_2(X, \mathbb{1}) = \mathrm{id}_{F(X)} = (F_0 \otimes \mathrm{id}_{F(X)})F_2(\mathbb{1}, X),$$

for all objects X, Y, Z of $\mathcal{C}$.

We denote by $F_3\colon F\otimes_3 \to F^{\otimes 3}$ the natural transformation defined by

$$F_3(X,Y,Z) = (\mathrm{id}_{F(X)} \otimes F_2(Y,Z))F_2(X, Y \otimes Z)$$
$$= (F_2(X,Y) \otimes \mathrm{id}_{F(Z)})F_2(X \otimes Y, Z).$$

A comonoidal functor (F, F_2, F_0) is said to be *strong* (resp. *strict*) if F_2 and F_0 are isomorphisms (resp. identities). In that case, (F, F_2^{-1}, F_0^{-1}) is a strong (resp. strict) monoidal functor.

1.6. Comonoidal natural transformations. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{D}$ be two comonoidal functors. A natural transformation $\varphi\colon F \to G$ is *comonoidal* if it satisfies

$$G_2(X,Y)\varphi_{X \otimes Y} = (\varphi_X \otimes \varphi_Y)F_2(X,Y) \quad \text{and} \quad G_0\varphi_{\mathbb{1}} = F_0.$$

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[1]Comonoidal functors are also called *opmonoidal functors*.

1.7. **Autonomous categories.** Recall that a *duality* in a monoidal category $\mathcal{C}$ is a quadruple (X, Y, e, d), where X, Y are objects of $\mathcal{C}$, $e\colon X \otimes Y \to \mathbb{1}$ (the *evaluation*) and $d\colon \mathbb{1} \to Y \otimes X$ (the *coevaluation*) are morphisms in $\mathcal{C}$ such that

$$(e \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes d) = \mathrm{id}_X \quad \text{and} \quad (\mathrm{id}_Y \otimes e)(d \otimes \mathrm{id}_Y) = \mathrm{id}_Y.$$

Then (X, e, d) is a *left dual* of Y and (Y, e, d) is a *right dual* of X.

If $D = (X, Y, e, d)$ and $D' = (X', Y', e', d')$ are two dualities, two morphisms $f\colon X \to X'$ and $g\colon Y' \to Y$ are *in duality with respect to D and D'* if

$$e'(f \otimes \mathrm{id}_{Y'}) = e(\mathrm{id}_X \otimes g) \quad \left(\text{or, equivalently, } (\mathrm{id}_{Y'} \otimes f)d = (g \otimes \mathrm{id}_X)d'\right).$$

In that case we write $f = {}^{\vee}g_{D,D'}$ and $g = f^{\vee}_{D,D'}$, or simply $f = {}^{\vee}g$ and $g = f^{\vee}$. Note that this defines a bijection between $\mathrm{Hom}_{\mathcal{C}}(X, X')$ and $\mathrm{Hom}_{\mathcal{C}}(Y', Y)$.

Left and right duals, if they exist, are essentially unique: if (Y, e, d) and (Y', e', d') are right duals of some object X, then there exists a unique isomorphism $u\colon Y \to Y'$ such that $e' = e(\mathrm{id}_X \otimes u^{-1})$ and $d' = (u \otimes \mathrm{id}_X)d$.

A *left autonomous* (resp. *right autonomous*, resp. *autonomous*) category is a monoidal category for which every object admits a left dual (resp. a right dual, resp. both a left and a right dual).

Assume $\mathcal{C}$ is a left autonomous category and, for each object X, pick a left dual $({}^{\vee}X, \mathrm{ev}_X, \mathrm{coev}_X)$. This data defines a strong monoidal functor ${}^{\vee}?\colon \mathcal{C}^{\mathrm{op},\otimes\mathrm{op}} \to \mathcal{C}$.

Likewise, if $\mathcal{C}$ is a right autonomous category, then picking a right dual $(X^{\vee}, \widetilde{\mathrm{ev}}_X, \widetilde{\mathrm{coev}}_X)$ for each object X defines a strong monoidal functor $?^{\vee}\colon \mathcal{C}^{\mathrm{op},\otimes\mathrm{op}} \to \mathcal{C}$.

Subsequently, when dealing with left or right autonomous categories, we shall always assume tacitly that left duals or right duals have been chosen. Moreover, in formulae, we abstain from writing the following canonical isomorphisms:

$$\begin{aligned}
{}^{\vee}?_2(X, Y)\colon {}^{\vee}Y \otimes {}^{\vee}X \to {}^{\vee}(X \otimes Y), & \qquad {}^{\vee}?_0\colon \mathbb{1} \to {}^{\vee}\mathbb{1}, \\
?^{\vee}_2(X, Y)\colon Y^{\vee} \otimes X^{\vee} \to (X \otimes Y)^{\vee}, & \qquad ?^{\vee}_0\colon \mathbb{1} \to \mathbb{1}^{\vee},
\end{aligned}$$

and

$$\begin{aligned}
(\widetilde{\mathrm{ev}}_X \otimes \mathrm{id}_{{}^{\vee}(X^{\vee})})(\mathrm{id}_X \otimes \mathrm{coev}_{X^{\vee}})\colon X \to {}^{\vee}(X^{\vee}), \\
(\mathrm{id}_{({}^{\vee}X)^{\vee}} \otimes \mathrm{ev}_X)(\widetilde{\mathrm{coev}}_{{}^{\vee}X} \otimes \mathrm{id}_X)\colon X \to ({}^{\vee}X)^{\vee}.
\end{aligned}$$

1.8. **Braided categories.** Recall that a *braiding* on a monoidal category $\mathcal{C}$ is a natural isomorphism

$$\tau = \{\tau_{X,Y}\colon X \otimes Y \to Y \otimes X\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$$

(from $\otimes$ to $\otimes\sigma_{\mathcal{C},\mathcal{C}}$) such that

$$\tau_{X,Y \otimes Z} = (\mathrm{id}_Y \otimes \tau_{X,Z})(\tau_{X,Y} \otimes \mathrm{id}_Z) \quad \text{and} \quad \tau_{X \otimes Y, Z} = (\tau_{X,Z} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes \tau_{Y,Z}).$$

A *braided category* is a monoidal category endowed with a braiding.

The *mirror of a braiding* τ is the braiding $\overline{\tau}$ defined by $\overline{\tau}_{X,Y} = \tau^{-1}_{Y,X}$. The *mirror of a braided category* $\mathcal{B}$ is the braided category $\overline{\mathcal{B}}$ which coincides with $\mathcal{B}$ as a monoidal category but is endowed with the mirror braiding.

If $\mathcal{B}$ is braided with braiding τ, then $\mathcal{B}^{\otimes\mathrm{op}}$ is braided with braiding τ^{o} defined by $\tau^{\mathrm{o}}_{X,Y} = \tau_{Y,X}$. Note that $\tau \mapsto \tau^{\mathrm{o}}$ is a bijection between braidings on $\mathcal{B}$ and braidings on $\mathcal{B}^{\otimes\mathrm{op}}$.

1.9. Braided functors. A *braided functor* between two braided categories $\mathcal{B}$ and $\mathcal{B}'$ is a strong monoidal functor $F \colon \mathcal{B} \to \mathcal{B}'$ such that

$$F(\tau_{X,Y})F_2(X,Y) = F_2(Y,X)\tau'_{F(X),F(Y)}$$

for all objects X, Y of $\mathcal{B}$, where τ and τ' are the braidings of $\mathcal{B}$ and $\mathcal{B}'$.

Example 1.1. If $\mathcal{B}$ is a braided category with braiding τ, the monoidal functor $(1_{\mathcal{B}}, \tau, \mathrm{id}_{\mathbb{1}}) \colon \mathcal{B}^{\otimes\mathrm{op}} \to \mathcal{B}$ is a braided isomorphism.

1.10. The center of a monoidal category. Let $\mathcal{C}$ be a monoidal category. A *left half braiding* of $\mathcal{C}$ is a pair (M, σ), where M is an object of $\mathcal{C}$ and

$$\sigma = \{\sigma_X \colon M \otimes X \to X \otimes M\}_{X \in \mathrm{Ob}(\mathcal{C})}$$

is a natural transformation (from $M \otimes 1_{\mathcal{C}}$ to $1_{\mathcal{C}} \otimes M$) such that:

   (i) $\sigma_{Y \otimes Z} = (\mathrm{id}_Y \otimes \sigma_Z)(\sigma_Y \otimes \mathrm{id}_Z)$ for all $Y, Z \in \mathrm{Ob}(\mathcal{C})$;
   (ii) $\sigma_{\mathbb{1}} = \mathrm{id}_M$;
   (iii) σ is an isomorphism.

Note that if $\mathcal{C}$ is autonomous, (iii) is a consequence of (i) and (ii).

The *center of* $\mathcal{C}$ is the braided category $\mathcal{Z}(\mathcal{C})$ defined as follows. Its objects are left half braidings of $\mathcal{C}$. A morphism in $\mathcal{Z}(\mathcal{C})$ from (M, σ) to (M', σ') is a morphism $f \colon M \to M'$ in $\mathcal{C}$ such that $(\mathrm{id}_{1_{\mathcal{C}}} \otimes f)\sigma = \sigma'(f \otimes \mathrm{id}_{1_{\mathcal{C}}})$. The monoidal product and braiding τ are

$$(M, \sigma) \otimes (N, \gamma) = \bigl(M \otimes N, (\sigma \otimes \mathrm{id}_N)(\mathrm{id}_M \otimes \gamma)\bigr) \quad \text{and} \quad \tau_{(M,\sigma),(N,\gamma)} = \sigma_N.$$

Note that if $\mathcal{C}$ is autonomous, so is $\mathcal{Z}(\mathcal{C})$.

Remark 1.2. Likewise, define a *right half braiding* of a monoidal category $\mathcal{C}$ to be a pair (M, σ), where M is an object of $\mathcal{C}$ and $\sigma \colon 1_{\mathcal{C}} \otimes M \to M \otimes 1_{\mathcal{C}}$ is a natural transformation satisfying analogous axioms. Right half braidings form a braided category $\mathcal{Z}'(\mathcal{C})$ with braiding: $\tau'_{(M,\sigma),(N,\gamma)} = \gamma_M$. We have

$$\mathcal{Z}'(\mathcal{C}) = \mathcal{Z}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{Z}(\mathcal{C}^{\otimes\,\mathrm{op}})^{\otimes\,\mathrm{op}}.$$

The braided category $\mathcal{Z}'(\mathcal{C})$ is isomorphic to the mirror of $\mathcal{Z}(\mathcal{C})$ via the braided isomorphism given by $(M, \sigma) \mapsto (M, \sigma^{-1})$.

1.11. Algebras, bialgebras, and Hopf algebras in categories. Let $\mathcal{C}$ be a monoidal category. An *algebra in* $\mathcal{C}$ is an object A of $\mathcal{C}$ endowed with morphisms $m \colon A \otimes A \to A$ (the product) and $u \colon \mathbb{1} \to A$ (the unit) such that

$$m(m \otimes \mathrm{id}_A) = m(\mathrm{id}_A \otimes m) \quad \text{and} \quad m(\mathrm{id}_A \otimes u) = \mathrm{id}_A = m(u \otimes \mathrm{id}_A).$$

A *coalgebra in* $\mathcal{C}$ is an object C of $\mathcal{C}$ endowed with morphisms $\Delta \colon C \to C \otimes C$ (the coproduct) and $\varepsilon \colon C \to \mathbb{1}$ (the counit) such that

$$(\Delta \otimes \mathrm{id}_C)\Delta = (\mathrm{id}_C \otimes \Delta)\Delta \quad \text{and} \quad (\mathrm{id}_C \otimes \varepsilon)\Delta = \mathrm{id}_C = (\varepsilon \otimes \mathrm{id}_C)\Delta.$$

Let $\mathcal{B}$ be a braided category with braiding τ. A *bialgebra in* $\mathcal{B}$ is an object A of $\mathcal{B}$ endowed with an algebra structure (m, u) and a coalgebra structure (Δ, ε) in $\mathcal{B}$ satisfying

$$\Delta m = (m \otimes m)(\mathrm{id}_A \otimes \tau_{A,A} \otimes \mathrm{id}_A)(\Delta \otimes \Delta), \qquad \Delta u = u \otimes u,$$

$$\varepsilon m = \varepsilon \otimes \varepsilon, \qquad\qquad\qquad\qquad\qquad\qquad \varepsilon u = \mathrm{id}_{\mathbb{1}}.$$

Let A be a bialgebra in $\mathcal{B}$. Set

$$m^{\mathrm{op}} = m \tau_{A,A}^{-1} \quad \text{and} \quad \Delta^{\mathrm{cop}} = \tau_{A,A}^{-1} \Delta.$$

Then $(A, m^{\mathrm{op}}, u, \Delta, \varepsilon)$ is a bialgebra in the mirror $\overline{\mathcal{B}}$ of $\mathcal{B}$, called the *opposite of* A, and denoted by A^{op}. Similarly $(A, m, u, \Delta^{\mathrm{cop}}, \varepsilon)$ is a bialgebra in $\overline{\mathcal{B}}$, called the *co-opposite of* A, and denoted by A^{cop}. Consequently $A^{\mathrm{cop,op}} = (A^{\mathrm{cop}})^{\mathrm{op}}$ is a bialgebra in $\mathcal{B}$ (with product $m\tau_{A,A}$ and coproduct $\tau_{A,A}^{-1}\Delta$).

An *antipode* for a bialgebra A is a morphism $S\colon A \to A$ in $\mathcal{B}$ such that

$$m(S \otimes \mathrm{id}_A)\Delta = u\varepsilon = m(\mathrm{id}_A \otimes S)\Delta.$$

If it exists, an antipode is unique, and it is a morphism of bialgebras $A \to A^{\mathrm{cop,op}}$. A *Hopf algebra in* $\mathcal{B}$ is a bialgebra in $\mathcal{B}$ which admits an invertible antipode.

If A is a Hopf algebra in $\mathcal{B}$, with antipode S, then A^{op} and A^{cop} are Hopf algebras in the mirror $\overline{\mathcal{B}}$ of $\mathcal{B}$, with antipode S^{-1}.

1.12. **Modules in categories.** Let (A, m, u) be an algebra in a monoidal category $\mathcal{C}$. A *left A-module* (in $\mathcal{C}$) is a pair (M, r), where M is an object of $\mathcal{C}$ and $r\colon A \otimes M \to M$ is a morphism in $\mathcal{C}$, such that

$$r(m \otimes \mathrm{id}_M) = r(\mathrm{id}_A \otimes r) \quad \text{and} \quad r(u \otimes \mathrm{id}_M) = \mathrm{id}_M.$$

An *A-linear morphism* between two left A-modules (M, r) and (N, s) is a morphism $f\colon M \to N$ such that $fr = s(\mathrm{id}_A \otimes f)$. Hence the category $_A\mathcal{C}$ of left A-modules. Likewise, one defines the category $\mathcal{C}_A$ of right A-modules.

Let A be a bialgebra in a braided category $\mathcal{B}$. Then the category $_A\mathcal{B}$ is monoidal, with unit object $(\mathbb{1}, \varepsilon)$ and monoidal product

$$(M, r) \otimes (N, s) = (r \otimes s)(\mathrm{id}_A \otimes \tau_{A,M} \otimes \mathrm{id}_N)(\Delta \otimes \mathrm{id}_{M \otimes N}),$$

where Δ and ϵ are the coproduct and counit of A, and τ is the braiding of $\mathcal{B}$. Likewise the category $\mathcal{B}_A$ is monoidal, with unit object $(\mathbb{1}, \varepsilon)$ and monoidal product

$$(M, r) \otimes (N, s) = (r \otimes s)(\mathrm{id}_M \otimes \tau_{N,A} \otimes \mathrm{id}_A)(\Delta \otimes \mathrm{id}_{M \otimes N}).$$

Assume $\mathcal{B}$ is autonomous. Then $_A\mathcal{B}$ is autonomous if and only if $\mathcal{B}_A$ is autonomous, if and only if A is a Hopf algebra. If A is a Hopf algebra, with antipode S, then the duals of a left A-module (M, r) are

$$^{\vee}(M, r) = \left({}^{\vee}M, (\mathrm{ev}_M \otimes \mathrm{id}_{^{\vee}M})(\mathrm{id}_{^{\vee}M} \otimes r(S \otimes \mathrm{id}_M) \otimes \mathrm{id}_{^{\vee}M})(\tau_{A,^{\vee}M} \otimes \mathrm{coev}_M) \right),$$

$$(M, r)^{\vee} = \left(M^{\vee}, (\mathrm{id}_{M^{\vee}} \otimes \widetilde{\mathrm{ev}}_M)(\mathrm{id}_{M^{\vee}} \otimes r\tau_{A,M}^{-1} \otimes \mathrm{id}_{M^{\vee}})(\widetilde{\mathrm{coev}}_M \otimes S^{-1} \otimes \mathrm{id}_{M^{\vee}}) \right),$$

and the duals of a right A-module (M, r) are

$$^{\vee}(M, r) = \left({}^{\vee}M, (\mathrm{ev}_M \otimes \mathrm{id}_{^{\vee}M})(\mathrm{id}_{^{\vee}M} \otimes r\tau_{M,A}^{-1} \otimes \mathrm{id}_{^{\vee}M})(\mathrm{id}_{^{\vee}M} \otimes S^{-1} \otimes \mathrm{coev}_M) \right),$$

$$(M, r)^{\vee} = \left(M^{\vee}, (\mathrm{id}_{M^{\vee}} \otimes \widetilde{\mathrm{ev}}_M)(\mathrm{id}_{M^{\vee}} \otimes r(\mathrm{id}_M \otimes S) \otimes \mathrm{id}_{M^{\vee}})(\widetilde{\mathrm{coev}}_M \otimes \tau_{M^{\vee},A}) \right).$$

Remark 1.3. Let A be a Hopf algebra in a braided category $\mathcal{B}$, with braiding τ. The functor $F_A \colon {}_A\mathcal{B} \to \mathcal{B}_A$, defined by $F_A(M, r) = \big(M, r\tau_{M,A}(\mathrm{id}_M \otimes S)\big)$ and $F_A(f) = f$, gives rise to a monoidal isomorphism of categories:

$$F_A = (F_A, \tau, \mathbb{1}) \colon ({}_A\mathcal{B})^{\otimes\mathrm{op}} \to \mathcal{B}_A.$$

Therefore braidings on ${}_A\mathcal{B}$ are in bijection with braidings on $\mathcal{B}_A$. More precisely, if c is a braiding on $\mathcal{B}_A$, then

$$c'_{(M,r),(N,s)} = \tau_{M,N}\, c_{F_A(N,s),F_A(M,r)}\, \tau_{N,M}^{-1}$$

is a braiding on ${}_A\mathcal{B}$, making F_A braided, and the correspondence $c \mapsto c'$ is bijective.

1.13. **Penrose graphical calculus.** We represent morphisms in a category by diagrams to be read from bottom to top. Thus we draw the identity id_X of an object X, a morphism $f \colon X \to Y$, and its composition with a morphism $g \colon Y \to Z$ as follows:

$$\mathrm{id}_X = \Big|\ , \quad f = \boxed{f}\ , \quad \text{and} \quad gf = \frac{\boxed{g}}{\boxed{f}}\ .$$

In a monoidal category, we represent the monoidal product of two morphisms $f \colon X \to Y$ and $g \colon U \to V$ by juxtaposition:

$$f \otimes g = \boxed{f}\ \boxed{g}\ .$$

The duality morphisms of an autonomous category are depicted as

$$\mathrm{ev}_X = \bigcap\ , \quad \mathrm{coev}_X = \bigcup\ , \quad \widetilde{\mathrm{ev}}_X = \bigcap\ , \quad \text{and} \quad \widetilde{\mathrm{coev}}_X = \bigcup\ .$$

The braiding τ of a braided category, and its inverse, are depicted as

$$\tau_{X,Y} = \ \diagdown\!\!\!\diagup\ \quad \text{and} \quad \tau_{Y,X}^{-1} = \ \diagup\!\!\!\diagdown\ .$$

Given a Hopf algebra A in a braided category, we depict its product m, unit u, coproduct Δ, counit ε, antipode S, and S^{-1} as follows:

$$m = \ \wedge\ , \quad u = \ |\ , \quad \Delta = \ \vee\ , \quad \varepsilon = \ |\ , \quad S = \ \oplus\ , \quad S^{-1} = \ \ominus\ .$$

2. Hopf monads and their modules

In this section, we review the notion of a Hopf monad. For a general treatment, we refer to [BV07].

2.1. **Monads.** Let $\mathcal{C}$ be a category. Recall that the category $\mathrm{End}(\mathcal{C})$ of endofunctors of $\mathcal{C}$ is strict monoidal with composition for monoidal product and identity functor $1_\mathcal{C}$ for a unit object. A *monad* on $\mathcal{C}$ is an algebra in $\mathrm{End}(\mathcal{C})$, that is, a triple (T, μ, η), where $T \colon \mathcal{C} \to \mathcal{C}$ is a functor, $\mu \colon T^2 \to T$ and $\eta \colon 1_\mathcal{C} \to T$ are natural transformations, such that

$$\mu_X T(\mu_X) = \mu_X \mu_{T(X)} \quad \text{and} \quad \mu_X \eta_{T(X)} = \mathrm{id}_{T(X)} = \mu_X T(\eta_X)$$

for any object X of $\mathcal{C}$.

Example 2.1. Let A be an algebra in a monoidal category $\mathcal{C}$, with product m and unit u. Then the endofunctor $? \otimes A$ of $\mathcal{C}$, defined by $X \mapsto X \otimes A$, has a structure of a monad on $\mathcal{C}$ with product $\mu = \mathrm{id}_{1_{\mathcal{C}}} \otimes m$ and unit $\eta = \mathrm{id}_{1_{\mathcal{C}}} \otimes u$. Similarly, the endofunctor $A \otimes ?$ is a monad on $\mathcal{C}$ with product $m \otimes \mathrm{id}_{1_{\mathcal{C}}}$ and unit $u \otimes \mathrm{id}_{1_{\mathcal{C}}}$.

2.2. Bimonads. A *bimonad*[2] on a monoidal category $\mathcal{C}$ is a monad (T, μ, η) on $\mathcal{C}$ such that the functor $T \colon \mathcal{C} \to \mathcal{C}$ is comonoidal and the natural transformations $\mu \colon T^2 \to T$ and $\eta \colon 1_{\mathcal{C}} \to T$ are comonoidal. In other words, T is endowed with a natural transformation $T_2 \colon T \otimes \to T \otimes T$ and a morphism $T_0 \colon T(\mathbb{1}) \to \mathbb{1}$ in $\mathcal{C}$ such that

$$\big(\mathrm{id}_{T(X)} \otimes T_2(Y, Z)\big) T_2(X, Y \otimes Z) = \big(T_2(X, Y) \otimes \mathrm{id}_{T(Z)}\big) T_2(X \otimes Y, Z),$$
$$(\mathrm{id}_{T(X)} \otimes T_0) T_2(X, \mathbb{1}) = \mathrm{id}_{T(X)} = (T_0 \otimes \mathrm{id}_{T(X)}) T_2(\mathbb{1}, X)$$

and

$$T_2(X, Y) \mu_{X \otimes Y} = (\mu_X \otimes \mu_Y) T_2(T(X), T(Y)) T(T_2(X, Y)),$$
$$T_0 \mu_{\mathbb{1}} = T_0 T(T_0), \qquad T_2(X, Y) \eta_{X \otimes Y} = (\eta_X \otimes \eta_Y), \qquad T_0 \eta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}.$$

Remark 2.2. A bimonad on a monoidal category $\mathcal{C}$ is nothing but an algebra in the strict monoidal category of comonoidal endofunctors of $\mathcal{C}$ (with a monoidal product $\circ$ and a unit object $1_{\mathcal{C}}$).

Remark 2.3. A bimonad T on a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ may be viewed as a bimonad T^{cop} on the monoidal category $\mathcal{C}^{\otimes \mathrm{op}} = (\mathcal{C}, \otimes^{\mathrm{op}}, \mathbb{1})$, with comonoidal structure $T_2^{\mathrm{cop}}(X, Y) = T_2(Y, X)$ and $T_0^{\mathrm{cop}} = T_0$. The bimonad T^{cop} is called the *coopposite* of the bimonad T. We have $T^{\mathrm{cop}}\text{-}\mathcal{C}^{\otimes \mathrm{op}} = (T\text{-}\mathcal{C})^{\otimes \mathrm{op}}$.

2.3. Antipodes. Right and left antipodes of a Hopf monad generalize the antipode of a Hopf algebra and its inverse. Let (T, μ, η) be a bimonad on a monoidal category $\mathcal{C}$.

Assume $\mathcal{C}$ is left autonomous. A *left antipode for T* is a natural transformation $s^l = \{s_X^l \colon T({}^{\vee}T(X)) \to {}^{\vee}X\}_{X \in \mathrm{Ob}(\mathcal{C})}$ satisfying

$$T_0 T(\mathrm{ev}_X) T({}^{\vee}\eta_X \otimes \mathrm{id}_X) = \mathrm{ev}_{T(X)}(s_{T(X)}^l T({}^{\vee}\mu_X) \otimes \mathrm{id}_{T(X)}) T_2({}^{\vee}T(X), X),$$
$$(\eta_X \otimes \mathrm{id}_{{}^{\vee}X}) \mathrm{coev}_X T_0 = (\mu_X \otimes s_X^l) T_2(T(X), {}^{\vee}T(X)) T(\mathrm{coev}_{T(X)}),$$

for every object X of $\mathcal{C}$. By [BV07, Theorem 3.7], a left antipode s^l is 'anti-(co)multiplicative': for all objects X, Y of $\mathcal{C}$,

$$s_X^l \mu_{{}^{\vee}T(X)} = s_X^l T(s_{T(X)}^l) T^2({}^{\vee}\mu_X), \qquad\qquad s_X^l \eta_{{}^{\vee}T(X)} = {}^{\vee}\eta_X,$$
$$s_{X \otimes Y}^l T({}^{\vee}T_2(X, Y)) = (s_Y^l \otimes s_X^l) T_2({}^{\vee}T(Y), {}^{\vee}T(X)), \qquad s_{\mathbb{1}}^l T({}^{\vee}T_0) = T_0.$$

Assume $\mathcal{C}$ is right autonomous. A *right antipode for T* is a natural transformation $s^r = \{s_X^r \colon T(T(X)^{\vee}) \to X^{\vee}\}_{X \in \mathrm{Ob}(\mathcal{C})}$ satisfying

$$T_0 T(\widetilde{\mathrm{ev}}_X) T(\mathrm{id}_X \otimes \eta_X^{\vee}) = \widetilde{\mathrm{ev}}_{T(X)}(\mathrm{id}_{T(X)} \otimes s_{T(X)}^r T(\mu_X^{\vee})) T_2(X, T(X)^{\vee}),$$
$$(\mathrm{id}_{X^{\vee}} \otimes \eta_X) \widetilde{\mathrm{coev}}_X T_0 = (s_X^r \otimes \mu_X) T_2(T(X)^{\vee}, T(X)) T(\widetilde{\mathrm{coev}}_{T(X)}),$$

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[2]Bimonads were introduced in [Moe02] under the name 'Hopf monads', which we prefer to reserve for bimonads with antipodes by analogy with Hopf algebras.

for every object X of $\mathcal{C}$. By [BV07, Theorem 3.7], a right antipode s^r is also 'anti-(co)multiplicative': for all objects X, Y of $\mathcal{C}$,

$$s_X^r \mu_{T(X)^\vee} = s_X^r T(s_{T(X)}^r) T^2(\mu_X{}^\vee), \qquad\qquad s_X^r \eta_{T(X)^\vee} = \eta_X{}^\vee,$$
$$s_{X \otimes Y}^r T(T_2(X, Y)^\vee) = (s_Y^r \otimes s_X^r) T_2(T(Y)^\vee, T(X)^\vee), \qquad s_{\mathbb{1}}^r T(T_0{}^\vee) = T_0.$$

Note that if a left (resp. right) antipode exists, then it is unique. Furthermore, when both exist, the left antipode s^l and the right antipode s^r are 'inverse' to each other in the sense that $\mathrm{id}_{T(X)} = s_{{}^\vee T(X)}^r T((s_X^l)^\vee) = s_{T(X)^\vee}^l T({}^\vee(s_X^r))$ for any object X of $\mathcal{C}$.

2.4. Hopf monads.

A *Hopf monad* is a bimonad on an autonomous category which has a left antipode and a right antipode.

Hopf monads generalize Hopf algebras in a non-braided setting. In particular, finite-dimensional Hopf algebras and several generalizations (Hopf algebras in braided autonomous categories, bialgebroids, etc.) provide examples of Hopf monads. In fact, any comonoidal adjunction between autonomous categories gives rise to a Hopf monad (see Theorem 3.3). It turns out that much of the theory of finite-dimensional Hopf algebras (such as the decomposition of Hopf modules, the existence of integrals, Maschke's criterium of semisimplicity, etc.) extends to Hopf monads; see [BV07].

Example 2.4 (Hopf monads associated with Hopf algebras)**.** Let A be a Hopf algebra in a braided autonomous category $\mathcal{B}$, with braiding τ. According to [BV07], the endofunctor $? \otimes A$ of $\mathcal{B}$ has a structure of a Hopf monad on $\mathcal{B}$, with product $\mu = \mathrm{id}_{1_\mathcal{B}} \otimes m$, unit $\eta = \mathrm{id}_{1_\mathcal{B}} \otimes u$, comonoidal structure given by

$$(? \otimes A)_2(X, Y) = (\mathrm{id}_X \otimes \tau_{Y,A} \otimes \mathrm{id}_A)(\mathrm{id}_{X \otimes Y} \otimes \Delta) \quad \text{and} \quad (? \otimes A)_0 = \varepsilon,$$

and left and right antipodes

$$s_X^l = (\mathrm{ev}_A \otimes \mathrm{id}_{{}^\vee X})(\mathrm{id}_{{}^\vee A} \otimes \tau_{{}^\vee X, A})(\mathrm{id}_{{}^\vee A \otimes {}^\vee X} \otimes S^{-1}),$$
$$s_X^r = (\widetilde{\mathrm{ev}}_A \otimes \mathrm{id}_{X^\vee}) \tau_{A^\vee \otimes X^\vee, A}(\mathrm{id}_{A^\vee \otimes X^\vee} \otimes S).$$

Pictorially, the structural morphisms of $? \otimes A$ are

Similarly, the endofunctor $A \otimes ?$ of $\mathcal{B}$ has a structure of a Hopf monad on $\mathcal{B}$, with product $\mu = m \otimes \mathrm{id}_{1_\mathcal{B}}$, unit $\eta = u \otimes \mathrm{id}_{1_\mathcal{B}}$, comonoidal structure

$$(A \otimes ?)_2(X, Y) = (\mathrm{id}_A \otimes \tau_{A,X} \otimes \mathrm{id}_Y)(\Delta \otimes \mathrm{id}_{X \otimes Y}) \quad \text{and} \quad (A \otimes ?)_0 = \varepsilon,$$

and left and right antipodes

$$s_X^l = (\mathrm{id}_{{}^\vee X} \otimes \mathrm{ev}_A) \tau_{A, {}^\vee X \otimes {}^\vee A}(S \otimes \mathrm{id}_{{}^\vee X \otimes {}^\vee A}),$$
$$s_X^r = (\mathrm{id}_{X^\vee} \otimes \widetilde{\mathrm{ev}}_A)(\tau_{A, X^\vee} \otimes \mathrm{id}_{{}^\vee A})(S^{-1} \otimes \mathrm{id}_{X^\vee \otimes A^\vee}).$$

Pictorially, the structural morphisms of $A\otimes?$ are

$$\mu_X = \begin{array}{c} {}^{A\ \ X} \\ \wedge \ \ | \\ {}_{A\ A\ X} \end{array}, \qquad \eta_X = \begin{array}{c} {}^{A\ \ X} \\ \text{\Large \circ} \ | \\ {}_{X} \end{array}, \qquad (A\otimes?)_2(X,Y) = \begin{array}{c} {}^{A\ X\ A\ Y} \\ \text{\Large $\diagdown\!\!\diagup\!\!\diagdown$} \ | \\ {}_{A\ \ X\ Y} \end{array},$$

$$(A\otimes?)_0 = \begin{array}{c} \text{\Large \circ} \\ | \\ {}_{A} \end{array}, \qquad s_X^l = \begin{array}{c} {}^{\vee X} \\ \oplus\!\diagdown\!| \diagdown \\ {}_{A\ \ {}^\vee X\ \ {}^\vee A} \end{array}, \qquad s_X^r = \begin{array}{c} {}^{X^\vee} \\ \ominus \!| \diagdown \\ {}_{A\ \ X^\vee\ A^\vee} \end{array}.$$

Example 2.5. The previous example can be extended to the non-braided setting as follows. Let $\mathcal{C}$ be an autonomous category and (A,σ) be a Hopf algebra in the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ (see Section 1.10). Denote by m, u, Δ, ε, S the product, unit, coproduct, counit, and antipode of (A,σ). Observe that (A,m,u) is an algebra in $\mathcal{C}$. Then the endofunctor $A\otimes?$ of $\mathcal{C}$ has a structure of a Hopf monad on $\mathcal{C}$, denoted by $A\otimes_\sigma?$, with product $\mu = m \otimes \mathrm{id}_{1_\mathcal{C}}$, unit $\eta = u \otimes \mathrm{id}_{1_\mathcal{C}}$, comonoidal structure

$$(A\otimes_\sigma?)_2(X,Y) = (\mathrm{id}_A \otimes \sigma_X \otimes \mathrm{id}_Y)(\Delta \otimes \mathrm{id}_{X\otimes Y}) \quad \text{and} \quad (A\otimes_\sigma?)_0 = \varepsilon,$$

and left and right antipodes

$$s_X^l = (\mathrm{id}_{^\vee X} \otimes \mathrm{ev}_A)\sigma_{^\vee X \otimes ^\vee A}(S \otimes \mathrm{id}_{^\vee X \otimes ^\vee A}),$$
$$s_X^r = (\mathrm{id}_{X^\vee} \otimes \widetilde{\mathrm{ev}}_A)(\sigma_{X^\vee} \otimes \mathrm{id}_{^\vee A})(S^{-1} \otimes \mathrm{id}_{X^\vee \otimes A^\vee}).$$

Likewise, if (A,σ) is a Hopf algebra in $\mathcal{Z}'(\mathcal{C})$ (see Remark 1.2), then the endofunctor $?\otimes A$ of $\mathcal{C}$ has a structure of a Hopf monad on $\mathcal{C}$, denoted by $?\otimes_\sigma A$, with product $\mu = \mathrm{id}_{1_\mathcal{C}} \otimes m$, unit $\eta = \mathrm{id}_{1_\mathcal{C}} \otimes u$, comonoidal structure given by

$$(?\otimes_\sigma A)_2(X,Y) = (\mathrm{id}_X \otimes \sigma_Y \otimes \mathrm{id}_A)(\mathrm{id}_{X\otimes Y} \otimes \Delta) \quad \text{and} \quad (?\otimes_\sigma A)_0 = \varepsilon,$$

and left and right antipodes

$$s_X^l = (\mathrm{ev}_A \otimes \mathrm{id}_{^\vee X})(\mathrm{id}_{^\vee A} \otimes \sigma_{^\vee X})(\mathrm{id}_{^\vee A \otimes ^\vee X} \otimes S^{-1}),$$
$$s_X^r = (\widetilde{\mathrm{ev}}_A \otimes \mathrm{id}_{X^\vee})\sigma_{A^\vee \otimes X^\vee}(\mathrm{id}_{A^\vee \otimes X^\vee} \otimes S).$$

Note that if A is a Hopf algebra in an autonomous braided category $\mathcal{B}$ with braiding τ, then $(A,\tau_{A,-})$ is a Hopf algebra in $\mathcal{Z}(\mathcal{B})$, $(A,\tau_{-,A})$ is a Hopf algebra in $\mathcal{Z}'(\mathcal{B})$, and we have $A\otimes? = A\otimes_{\tau_{A,-}}?$ and $?\otimes A =?\otimes_{\tau_{-,A}} A$ as Hopf monads on $\mathcal{B}$.

2.5. **Modules over a monad.** Let (T,μ,η) be a monad on a category $\mathcal{C}$. An *action* of T on an object M of $\mathcal{C}$ is a morphism $r\colon T(M) \to M$ in $\mathcal{C}$ such that

$$rT(r) = r\mu_M \quad \text{and} \quad r\eta_M = \mathrm{id}_M.$$

The pair (M,r) is then called a *T-module in $\mathcal{C}$*, or just a *T-module*.[3]

Given two T-modules (M,r) and (N,s) in $\mathcal{C}$, a *morphism of T-modules* from (M,r) to (N,r) is a morphism $f \in \mathrm{Hom}_\mathcal{C}(M,N)$ which is *T-linear*, that is, such that $fr = sT(f)$. This gives rise to the *category T-$\mathcal{C}$ of T-modules (in $\mathcal{C}$)*, with composition inherited from $\mathcal{C}$. We denote by $U_T\colon T\text{-}\mathcal{C} \to \mathcal{C}$ the *forgetful functor of T* defined by $U_T(M,r) = M$ for any T-module (M,r) and $U_T(f) = f$ for any T-linear morphism f.

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[3]Pairs (M,r) are usually called T-algebras in the literature (see [Mac98]). However, throughout this paper, pairs (M,r) are considered as the analogues of modules over an algebra, and so the term 'algebra' would be awkward in this context.

Example 2.6. Let A be an algebra in a monoidal category $\mathcal{C}$ and consider the monads $? \otimes A$ and $A \otimes ?$ of Example 2.1. Then the category of $(? \otimes A)$- modules (resp. of $(A \otimes ?)$- modules) coincides with the category $\mathcal{C}_A$ of right A-modules in $\mathcal{C}$ (resp. with the category $_A\mathcal{C}$ of left A-modules in $\mathcal{C}$):

$$(? \otimes A)\text{-}\mathcal{C} = \mathcal{C}_A \quad \text{and} \quad (A \otimes ?)\text{-}\mathcal{C} = {}_A\mathcal{C}.$$

2.6. **Tannaka dictionary.** Structures of bimonad and Hopf monad on a monad T have natural interpretations in terms of the category of T- modules:

Theorem 2.7 ([BV07]). *Let T be a monad on a monoidal category $\mathcal{C}$ and T-$\mathcal{C}$ be the category of T-modules. Then:*

(a) *If T is a bimonad, then the category T-$\mathcal{C}$ is monoidal by setting*

$$(M, r) \otimes (N, s) = \big(M \otimes N, (r \otimes s)T_2(M, N) \big) \quad \text{and} \quad \mathbb{1}_{T\text{-}\mathcal{C}} = (\mathbb{1}, T_0).$$

Moreover, this gives a bijective correspondence between bimonad structures on the monad T and monoidal structures on T-$\mathcal{C}$ such that the forgetful functor $U_T \colon T$-$\mathcal{C} \to \mathcal{C}$ is strict monoidal.

(b) *Assume T is a bimonad and $\mathcal{C}$ is left (resp. right) autonomous. Then T has a left (resp. right) antipode if and only if T-$\mathcal{C}$ is left (resp. right) autonomous. If s^l is a left antipode for T, left duals in T-$\mathcal{C}$ are given by*

$$^\vee(M, r) = (^\vee M, s^l_M T(^\vee r)), \quad \mathrm{ev}_{(M,r)} = \mathrm{ev}_M, \quad \mathrm{coev}_{(M,r)} = \mathrm{coev}_M,$$

and if s^r is a right antipode for T, right duals in T-$\mathcal{C}$ are given by

$$(M, r)^\vee = (M^\vee, s^r_M T(r^\vee)), \quad \widetilde{\mathrm{ev}}_{(M,r)} = \widetilde{\mathrm{ev}}_M, \quad \widetilde{\mathrm{coev}}_{(M,r)} = \widetilde{\mathrm{coev}}_M.$$

(c) *Assume T is a bimonad and $\mathcal{C}$ is autonomous. Then T is a Hopf monad if and only if T-$\mathcal{C}$ is autonomous.*

Example 2.8. Let A be a Hopf algebra in a braided autonomous category $\mathcal{B}$ and consider the Hopf monads $? \otimes A$ and $A \otimes ?$ of Example 2.4. Then

$$(? \otimes A)\text{-}\mathcal{B} = \mathcal{B}_A \quad \text{and} \quad (A \otimes ?)\text{-}\mathcal{B} = {}_A\mathcal{B}$$

as monoidal categories.

Example 2.9. More generally, let $\mathcal{C}$ be a monoidal category and (A, σ) be a Hopf algebra in the braided category $\mathcal{Z}(\mathcal{C})$. Then $_A\mathcal{C}$ coincides with the category of modules over the Hopf monad $A \otimes_\sigma ?$ on $\mathcal{C}$ defined in Example 2.5. Hence $_A\mathcal{C}$ is autonomous, with unit object $(\mathbb{1}, \varepsilon)$ and monoidal product

$$(M, r) \otimes (N, s) = (r \otimes s)(\mathrm{id}_A \otimes \sigma_M \otimes \mathrm{id}_N)(\Delta \otimes \mathrm{id}_{M \otimes N}).$$

Likewise, if (A, σ) is a Hopf algebra in the braided category $\mathcal{Z}'(\mathcal{C})$ (see Remark 1.2), then $\mathcal{C}_A$ coincides with the category of modules over the Hopf monad $? \otimes_\sigma A$, and so is autonomous, with a unit object $(\mathbb{1}, \varepsilon)$ and a monoidal product

$$(M, r) \otimes (N, s) = (r \otimes s)(\mathrm{id}_M \otimes \sigma_N \otimes \mathrm{id}_A)(\mathrm{id}_{M \otimes N} \otimes \Delta).$$

2.7. **Quasitriangular Hopf monads.** An R-*matrix* for a Hopf monad (T, μ, η) on an autonomous category $\mathcal{C}$ is a natural transformation

$$R = \{R_{X,Y} \colon X \otimes Y \to T(Y) \otimes T(X)\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$$

such that, for all objects X, Y, Z of $\mathcal{C}$,

$$(\mu_Y \otimes \mu_X)R_{T(X),T(Y)}T_2(X,Y) = (\mu_Y \otimes \mu_X)T_2(T(Y),T(X))T(R_{X,Y}),$$

$$(\mathrm{id}_{T(Z)} \otimes T_2(X,Y))R_{X \otimes Y, Z}$$
$$= (\mu_Z \otimes \mathrm{id}_{T(X) \otimes T(Y)})(R_{X,T(Z)} \otimes \mathrm{id}_{T(Y)})(\mathrm{id}_X \otimes R_{Y,Z}),$$

$$(T_2(Y,Z) \otimes \mathrm{id}_{T(X)})R_{X,Y \otimes Z}$$
$$= (\mathrm{id}_{T(Y) \otimes T(Z)} \otimes \mu_X)(\mathrm{id}_{T(Y)} \otimes R_{T(X),Z})(R_{X,Y} \otimes \mathrm{id}_Z),$$

$$(\mathrm{id}_{T(X)} \otimes T_0)R_{\mathbb{1},X} = \eta_X = (T_0 \otimes \mathrm{id}_{T(X)})R_{X,\mathbb{1}}.$$

A *quasitriangular Hopf monad* is a Hopf monad equipped with an R-matrix.

Remark 2.10. For a bimonad, an R-matrix is also required to be $*$-invertible (see [BV07, Section 8.2]). For a Hopf monad T, this condition is automatic and we have

$$R_{X,Y}^{*-1} = \left(\mathrm{id}_{T(X) \otimes T(Y)} \otimes \mathrm{ev}_X(s_X^l \otimes \mathrm{id}_X)\right)$$
$$(\mathrm{id}_{T(X)} \otimes R_{{}^\vee T(X),Y} \otimes \mathrm{id}_X)(\mathrm{coev}_{T(X)} \otimes \mathrm{id}_{Y \otimes X}),$$

$$= \left(\widetilde{\mathrm{ev}}_Y(\mathrm{id}_Y \otimes s_Y^r) \otimes \mathrm{id}_{T(X) \otimes T(Y)}\right)$$
$$(\mathrm{id}_Y \otimes R_{X,T(Y)^\vee} \otimes \mathrm{id}_{T(Y)})(\mathrm{id}_{Y \otimes X} \otimes \widetilde{\mathrm{coev}}_{T(Y)}),$$

where s^l and s^r are the left and right antipodes of T.

There is a natural interpretation of R-matrices for a Hopf monad T in terms of braidings on the category of T- modules:

Theorem 2.11 ([BV07]). *Let T be a Hopf monad on an autonomous category $\mathcal{C}$. Then any R-matrix R for T defines a braiding τ on the category T-$\mathcal{C}$ as follows:*

$$\tau_{(M,r),(N,s)} = (s \otimes t)R_{M,N} \colon (M,r) \otimes (N,s) \to (N,s) \otimes (M,r).$$

This assignment is a bijection between R-matrices for T and braidings on T-$\mathcal{C}$.

Remark 2.12. In Section 8.6, we define R-matrices for a Hopf algebra A in a braided autonomous category $\mathcal{B}$ admitting a coend C. These R-matrices are morphisms $\mathfrak{r} \colon C \otimes C \to A \otimes A$ which encode R-matrices for the Hopf monads $? \otimes A$ and $A \otimes ?$. They generalize the usual R-matrices for finite-dimensional Hopf algebras.

2.8. **Morphisms of Hopf monads.** A *morphism of monads* between two monads (T, μ, η) and (T', μ', η') on a category $\mathcal{C}$ is a natural transformation $f \colon T \to T'$ such that, for every object X of $\mathcal{C}$,

$$f_X \mu_X = \mu'_X f_{T'(X)} T(f_X) \quad \text{and} \quad f_X \eta_X = \eta'_X.$$

According to [BV07, Lemma 1.7], a morphism of monads $f \colon T \to T'$ yields a functor $f^* \colon T'\text{-}\mathcal{C} \to T\text{-}\mathcal{C}$ defined by $f^*(M,r) = (M, rf_M)$. Moreover, the mapping $f \mapsto f^*$ is a bijective correspondence between: (i) morphisms of monads $f \colon T \to T'$, and (ii) functors $F \colon T'\text{-}\mathcal{C} \to T\text{-}\mathcal{C}$ such that $U_T F = U_{T'}$.

A *morphism of bimonads* between two bimonads T and T' on a monoidal category $\mathcal{C}$ is a morphism of monads $f \colon T \to T'$ which is comonoidal, that is,

$$T'_2(X,Y)f_{X \otimes Y} = (f_X \otimes f_Y)T_2(X,Y) \quad \text{and} \quad T'_0 f_{\mathbb{1}} = T_0.$$

According to [BV07, Lemma 2.9], the associated functor $f^*\colon T'\text{-}\mathcal{C} \to \mathcal{C} \to T\text{-}\mathcal{C}$ is then monoidal strict. Moreover, the mapping $f \mapsto f^*$ is a bijective correspondence between: (i) morphisms of bimonads $f\colon T \to T'$, and (ii) monoidal functors $F\colon T'\text{-}\mathcal{C} \to T\text{-}\mathcal{C}$ such that $U_T F = U_{T'}$ as monoidal functors.

A *morphism of Hopf monads* is a morphism of bimonads between Hopf monads.

Example 2.13. Let A be a Hopf algebra in a braided autonomous category $\mathcal{B}$, with braiding τ. Recall that A^{op} is a Hopf algebra in the mirror $\overline{\mathcal{B}}$ of $\mathcal{B}$. The Hopf monad $? \otimes A^{\mathrm{op}}$ on $\overline{\mathcal{B}}$ may be seen as a Hopf monad on $\mathcal{B}$. Then

$$\tau_{A,?}\colon A \otimes ? \to ? \otimes A^{\mathrm{op}}$$

is an isomorphism of Hopf monads and

$$(\tau_{A,?})^*\colon \overline{\mathcal{B}}_{A^{\mathrm{op}}} = (? \otimes A^{\mathrm{op}})\text{-}\mathcal{B} \to (A \otimes ?)\text{-}\mathcal{B} = {}_A\mathcal{B}$$

is an isomorphism of monoidal categories. Likewise, since $(A^{\mathrm{op}})^{\mathrm{op}} = A$ as Hopf algebras in $\mathcal{B}$, $\tau_{?,A}$ induces isomorphisms $? \otimes A \to A^{\mathrm{op}} \otimes ?$ and ${}_{A^{\mathrm{op}}}\overline{\mathcal{B}} \to \mathcal{B}_A$.

3. Hopf monads, comonoidal adjunctions, and coends

Monads and adjunctions are closely related. This relationship extends naturally to Hopf monads and comonoidal adjunctions between autonomous categories. We show that the forgetful functor of a Hopf monad creates and preserves coends. Lastly, we define the pushforward of a Hopf monad under an adjunction and, as a special case, the cross product of Hopf monads.

3.1. **Adjunctions.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Recall that an *adjunction* is a pair of functors $(F\colon \mathcal{C} \to \mathcal{D}, U\colon \mathcal{D} \to \mathcal{C})$ endowed with a bijection

$$\mathrm{Hom}_{\mathcal{D}}\big(F(X), Y\big) \simeq \mathrm{Hom}_{\mathcal{C}}\big(X, U(Y)\big)$$

which is natural in both $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$. The functor F is then called a *left adjoint of U* and the functor U a *right adjoint of F*. Note that a left (resp. right) adjoint of a given functor, if it exists, is unique up to a unique natural isomorphism.

An adjunction (F, U) is entirely determined by two natural transformations $\eta\colon 1_{\mathcal{C}} \to UF$ and $\varepsilon\colon FU \to 1_{\mathcal{D}}$ satisfying

$$U(\varepsilon)\,\eta_U = \mathrm{id}_U \quad \text{and} \quad \varepsilon_F F(\eta) = \mathrm{id}_F.$$

These transformations η and ε are respectively called the *unit* and *counit* of the adjunction, and collectively the *adjunction morphisms*.

Adjunctions may be composed: given two adjunctions $(F\colon \mathcal{C} \to \mathcal{D}, U\colon \mathcal{D} \to \mathcal{C})$ and $(F'\colon \mathcal{D} \to \mathcal{E}, U'\colon \mathcal{E} \to \mathcal{D})$, the pair $(F'F\colon \mathcal{C} \to \mathcal{E}, UU'\colon \mathcal{E} \to \mathcal{C})$ is an adjunction called the *composite* of (F, U) and (F', U').

Adjunctions and monads are closely related. Indeed, if T is a monad on a category $\mathcal{C}$, then the forgetful functor $U_T\colon T\text{-}\mathcal{C} \to \mathcal{C}$ has a left adjoint $F_T\colon \mathcal{C} \to T\text{-}\mathcal{C}$, defined by $F_T(X) = (T(X), \mu_X)$ for any object X of $\mathcal{C}$ and $F_T(f) = T(f)$ for any morphism f in $\mathcal{C}$. The unit of the adjunction (F_T, U_T) is the unit $\eta\colon 1_{\mathcal{C}} \to T = U_T F_T$ of the monad T, and the counit $\varepsilon\colon F_T U_T \to 1_{T\text{-}\mathcal{C}}$ of (F_T, U_T) is the T-action, that is, $\varepsilon_{(M,r)} = r$ for any T-module (M, r).

Moreover, if $(F\colon \mathcal{C} \to \mathcal{D}, U\colon \mathcal{D} \to \mathcal{C})$ is a pair of adjoint functors, with adjunction morphisms $\eta\colon 1_{\mathcal{C}} \to UF$ and $\varepsilon\colon FU \to 1_{\mathcal{D}}$, then $T = UF$ is a monad on $\mathcal{C}$ with a product $\mu = U(\varepsilon_F)\colon T^2 \to T$ and unit η. The monad (T, μ, η) is called the *monad*

of the adjunction (F, U). In addition, there exists a unique functor $K \colon \mathcal{D} \to T\text{-}\mathcal{C}$ such that $U_T K = U$ and $KF = F_T$. The functor K is called the *comparison functor* and is given by $K(D) = \big(U(D), U(\varepsilon_D)\big)$ for any object D of $\mathcal{D}$.

Note that if T is a monad on $\mathcal{C}$, then T is the monad of the adjunction (F_T, U_T) and the comparison functor is the identity functor. In general, however, the comparison functor of an adjunction need not be an equivalence.

3.2. **Monadic adjunctions.** An adjunction is *monadic* if its comparison functor (see Section 3.1) is an equivalence. Remark that the composite adjunction of two monadic adjunctions need not be monadic.

A functor $U \colon \mathcal{D} \to \mathcal{C}$ is *monadic* if it admits a left adjoint $F \colon \mathcal{C} \to \mathcal{D}$ and the adjunction (F, U) is monadic. If such is the case, the monad $T = UF$ of the adjunction (F, U) is called *the monad of* U. It is well-defined up to a unique isomorphism of monads (as the left adjoint F is unique up to a unique natural isomorphism).

For example, if T is a monad on a category $\mathcal{C}$, the forgetful functor $U_T \colon T\text{-}\mathcal{C} \to \mathcal{C}$ is monadic with monad T.

Remark 3.1. Let $U \colon \mathcal{D} \to \mathcal{C}$ be a functor. If there exist a monad T on $\mathcal{C}$ and an isomorphism of categories $K \colon \mathcal{D} \to T\text{-}\mathcal{C}$ such that $U = U_T K$, then $F = K^{-1} F_T$ is left adjoint to U and the adjunction (F, U) is monadic with monad T and comparison functor K.

3.3. **Hopf monads and comonoidal adjunctions.** A *comonoidal adjunction* is an adjunction $(F \colon \mathcal{C} \to \mathcal{D}, U \colon \mathcal{D} \to \mathcal{C})$, where $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories, such that the right adjoint $U \colon \mathcal{D} \to \mathcal{C}$ is strong monoidal. In that case, the functor F is comonoidal, with comonoidal structure given by

$$F_2(X, Y) = \varepsilon_{FX \otimes FY} FU_2(FX, FY) F(\eta_X \otimes \eta_Y) \quad \text{and} \quad F_0 = \varepsilon_{\mathbb{1}} F(U_0),$$

and the adjunction unit $\eta \colon 1_\mathcal{C} \to UF$ and counit $\varepsilon \colon FU \to 1_\mathcal{D}$ are comonoidal natural transformations (viewing U as a strong comonoidal functor).

For example, if T is a bimonad on a monoidal category $\mathcal{C}$, then the adjunction (F_T, U_T) is comonoidal.

Remark 3.2. Comonoidal adjunctions are called monoidal adjunctions in [BV07]. However, it is preferable to call them comonoidal adjunctions since they are nothing but adjunctions (F, U) such that F and U are comonoidal functors and the adjunction unit $\eta \colon 1_\mathcal{C} \to UF$ and counit $\varepsilon \colon FU \to 1_\mathcal{D}$ are comonoidal natural transformations. Indeed, in that case, U is a strong comonoidal functor, with $U_2(X, Y)^{-1} = U(\varepsilon_X \otimes \varepsilon_Y) UF_2(U(X), U(Y)) \eta_{U(X) \otimes U(Y)}$, which we may view as a strong monoidal functor.

The monad of a comonoidal adjunction between monoidal categories (resp. autonomous categories) is a bimonad (resp. a Hopf monad). More precisely:

Theorem 3.3 ([BV07])**.** *Let* $(F \colon \mathcal{C} \to \mathcal{D}, U \colon \mathcal{D} \to \mathcal{C})$ *be a comonoidal adjunction between monoidal categories. Denote by* $T = UF$ *the monad of this adjunction. Then the functor* F *is comonoidal and* T *is a bimonad on* $\mathcal{C}$. *The comparison functor* $K \colon \mathcal{D} \to T\text{-}\mathcal{C}$ *is strong monoidal and satisfies* $U_T K = U$ *as monoidal functors and* $KF = F_T$ *as comonoidal functors. If the categories* $\mathcal{C}$ *and* $\mathcal{D}$ *are furthermore autonomous, then the bimonad* T *is a Hopf monad.*

Remark 3.4. Let (F, U) be a comonoidal adjunction between autonomous categories, with unit η and counit ε. Let $T = UF$ be the Hopf monad associated with this comonoidal adjunction (see Theorem 3.3). Then the comonoidal structure and antipodes of T are:

$$T_2(X, Y) = U_2(F(X), F(Y))^{-1} U(\varepsilon_{F(X) \otimes F(Y)}) UF\big(U_2(F(X), F(Y))(\eta_X \otimes \eta_Y)\big),$$
$$T_0 = U_0^{-1} U(\varepsilon_{\mathbb{1}}) UF(U_0),$$
$$s_X^l = {}^\vee \eta_X \, U_1^l(F(X))^{-1} U(\varepsilon_{{}^\vee F(X)}) UF\big(U_1^l(F(X))\big),$$
$$s_X^r = \eta_X^\vee \, U_1^r(F(X))^{-1} U(\varepsilon_{F(X)^\vee}) UF\big(U_1^r(F(X))\big),$$

where $U_1^l(Y) \colon {}^\vee U(Y) \to U({}^\vee Y)$ and $U_1^r(Y) \colon U(Y)^\vee \to U(Y^\vee)$ are the compatibility isomorphisms of U with duals (see [BV07, Section 3.2]).

3.4. **Hopf monads and right adjoints.** If $F \colon \mathcal{C} \to \mathcal{D}$ is a functor between autonomous categories, denote by $F^! \colon \mathcal{C} \to \mathcal{D}$ the functor defined by $F^!(X) = F({}^\vee X)^\vee$ and $F^!(f) = F({}^\vee f)^\vee$ for all objects X and morphisms f in $\mathcal{C}$.

Lemma 3.5. *Let* $U \colon \mathcal{D} \to \mathcal{C}$ *be a strong monoidal functor between autonomous categories. If* $F \colon \mathcal{C} \to \mathcal{D}$ *is a left adjoint for* U, *then* $F^!$ *is a right adjoint for* U.

Proof. Since U is strong monoidal, we have $U({}^\vee X) \simeq {}^\vee U(X)$ for any object X of $\mathcal{C}$. Hence we have the following isomorphisms:

$$\mathrm{Hom}_{\mathcal{C}}\big(U(X), Y\big) \simeq \mathrm{Hom}_{\mathcal{C}}\big({}^\vee Y, {}^\vee U(X)\big) \simeq \mathrm{Hom}_{\mathcal{C}}\big({}^\vee Y, U({}^\vee X)\big)$$
$$\simeq \mathrm{Hom}_{\mathcal{D}}\big(F({}^\vee Y), {}^\vee X\big) \simeq \mathrm{Hom}_{\mathcal{D}}\big(X, F({}^\vee Y)^\vee\big) = \mathrm{Hom}_{\mathcal{D}}\big(X, F^!(Y)\big)$$

which are natural in both $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$. $\square$

Proposition 3.6. *Let* T *be a Hopf monad on an autonomous category* $\mathcal{C}$. *Then:*

   (a) *The endofunctor* $T^!$ *of* $\mathcal{C}$ *is a right adjoint of* T.
   (b) *The functor* $F_T^! \colon \mathcal{C} \to T\text{-}\mathcal{C}$ *is a right adjoint of the forgetful functor* U_T.

Proof. Part (a) is [BV07, Corollary 3.12]. Part (b) is Lemma 3.5 applied to the comonoidal adjunction (F_T, U_T). $\square$

Remark 3.7. If T is a Hopf monad on an autonomous category $\mathcal{C}$, then the adjunction morphisms $e \colon TT^! \to 1_{\mathcal{C}}$ and $h \colon 1_{\mathcal{C}} \to T^! T$ are given by $e_X = s_{{}^\vee X}^r$ and $h_X = (s_X^l)^\vee$, where s^l and s^r denote the left and right antipodes of T.

   Recall that a functor $G \colon \mathcal{D} \to \mathcal{C}$ *preserves colimits* if the image under G of a colimit in $\mathcal{D}$ is a colimit in $\mathcal{C}$. A functor $G \colon \mathcal{D} \to \mathcal{C}$ *creates colimits* if, for any functor $F \colon I \to \mathcal{D}$ such that $GF \colon I \to \mathcal{C}$ has a colimit, this colimit lifts uniquely to a colimit of F. See [Mac98] for more precise definitions.

   Since the forgetful functor of a monad which preserves colimits creates colimits (by [Bor94, Proposition 4.3.2]), Proposition 3.6 admits the following corollary:

Corollary 3.8. *A Hopf monad* T *on an autonomous category* $\mathcal{C}$ *preserves colimits, and its forgetful functor* $U_T \colon T\text{-}\mathcal{C} \to \mathcal{C}$ *creates and preserves colimits.*

3.5. **Coends and Hopf monads.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ be a functor. A *dinatural transformation* $d\colon F \to Z$ from F to an object Z of $\mathcal{D}$ is a family $d = \{d_X\colon F(X,X) \to Z\}_{X \in \mathrm{Ob}(\mathcal{C})}$ of morphisms in $\mathcal{D}$ satisfying the dinaturality condition

$$d_Y F(\mathrm{id}_Y, f) = F(f, \mathrm{id}_X) d_X$$

for every morphism $f\colon X \to Y$ in $\mathcal{C}$. We denote by $\mathrm{DINAT}(F,Z)$ the set of dinatural transformations from F to Z.

A *coend* of a functor $F\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$ consists of an object C of $\mathcal{C}$ and a dinatural transformation $i\colon F \to C$ which is universal in the sense that, for every dinatural transformation $d\colon F \to Z$, there exists a unique morphism $r\colon C \to Z$ such that $d_X = r \circ i_X$ for all $X \in \mathrm{Ob}(\mathcal{C})$. In other words, the map

$$\begin{cases} \mathrm{Hom}_{\mathcal{D}}(C,Z) & \to & \mathrm{DINAT}(F,Z) \\ \qquad\quad r & \mapsto & ri \end{cases}$$

is a bijection. The dinatural transformation i is then called a *universal dinatural transformation* for F. A coend of F, if it exists, is unique up to unique isomorphism. Following [Mac98], we denote it by $\int^{X \in \mathcal{C}} F(X,X)$.

Coends are well-behaved under adjunction:

Lemma 3.9. *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be categories, $(F\colon \mathcal{C} \to \mathcal{D}, U\colon \mathcal{D} \to \mathcal{C})$ be an adjunction, and $G\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{E}$ be a functor. We have*

$$\int^{X \in \mathcal{C}} G(F(X), X) \simeq \int^{Y \in \mathcal{D}} G(Y, U(Y)),$$

meaning that if either coend exists, then both exist and they are naturally isomorphic.

Proof. Denote by $\eta\colon 1_{\mathcal{C}} \to UF$ and $\varepsilon\colon FU \to 1_{\mathcal{D}}$ the adjunction morphisms. The lemma results from the existence of a bijection:

$$\psi\colon \mathrm{DINAT}\big(G(F \times 1_{\mathcal{C}}), E\big) \to \mathrm{DINAT}\big(G(1_{\mathcal{D}^{\mathrm{op}}} \times U), E\big)$$

which is natural in $E \in \mathrm{Ob}(\mathcal{E})$. It is defined by $\psi(d) = d_U G(\varepsilon, \mathrm{id}_U)$, and its inverse by $\psi^{-1}(t) = t_F G(\mathrm{id}_F, \eta)$. □

Coends are special cases of colimits (see [Mac98]), and in particular, a functor which preserves (resp. creates) colimits preserves (resp. creates) coends. Hence, by Corollary 3.8:

Proposition 3.10. *Let T be a Hopf monad on an autonomous category $\mathcal{C}$ and $F\colon \mathcal{D}^{\mathrm{op}} \times \mathcal{D} \to T\text{-}\mathcal{C}$ be a functor. Then the coend $C = \int^{Y \in \mathcal{D}} U_T F(Y,Y)$ exists if and only if the coend $\int^{Y \in \mathcal{D}} F(Y,Y)$ exists. Moreover, given a coend $C = \int^{Y \in \mathcal{D}} U_T F(Y,Y)$ with universal dinatural transformation $i_Y\colon U_T F(Y,Y) \to C$, there exists a unique action $r\colon T(C) \to C$ of T on C such that $i_Y\colon F(Y,Y) \to (C,r)$ is T-linear. We then have $(C,r) = \int^{Y \in \mathcal{D}} F(Y,Y)$ with universal dinatural transformation i. The morphism $r\colon T(C) \to C$ is characterized by*

$$rT(i_Y) = i_Y \alpha_Y, \quad \textit{where} \quad F(Y,Y) = (U_T F(Y,Y), \alpha_Y),$$

as $T(i)$ is a universal dinatural transformation.

3.6. **Pushforward of a monad under an adjunction.** Let $(F\colon \mathcal{C} \to \mathcal{D}, U\colon \mathcal{D} \to \mathcal{C})$ be an adjunction and Q be an endofunctor of $\mathcal{D}$. The endofunctor UQF of $\mathcal{C}$ is called the *pushforward of Q under the adjunction (F,U)* and is denoted by $(F,U)_*Q$.

If Q is a monad, then $(F,U)_*Q$ is a monad: it is the monad of the composite adjunction $(F_Q F, UU_Q)$ of (F,U) and (F_Q, U_Q).

If Q is comonoidal and (F,U) is comonoidal, then $(F,U)_*Q$ is comonoidal with comonoidal structure the composition of the comonoidal structures of U_T, Q, and F_T.

By Theorem 3.3, if the adjunction (F,U) is comonoidal and Q is a bimonad, then $(F,U)_*Q$ is a bimonad (since the composite of comonoidal adjunctions is a comonoidal adjunction).

Finally, if $\mathcal{C}$ and $\mathcal{D}$ are autonomous, (F,U) is comonoidal, and Q is a Hopf monad, then $(F,U)_*Q$ is a Hopf monad.

Remark 3.11. The structural morphisms of $(F,U)_*Q$ can be expressed using those of Q and the adjunction morphisms of (F,U) (by applying Remark 3.4).

3.7. **Cross products.** Let T be a monad on a category $\mathcal{C}$ and Q be an endofunctor of $T\text{-}\mathcal{C}$. Denote by η and ε the unit and counit of (U_T, F_T). The pushforward of Q under the adjunction (F_T, U_T) is called the *cross product of Q by T* and denoted by $Q \rtimes T$. Recall that $Q \rtimes T = U_T Q F_T$ as an endofunctor of $\mathcal{C}$.

If (Q, q, v) is a monad on $T\text{-}\mathcal{C}$, then $Q \rtimes T$ is a monad on $\mathcal{C}$ with product p and unit e given by

$$p = q_{F_T} Q(\varepsilon_{QF_T}) \quad \text{and} \quad e = v_{F_T}\eta.$$

If T is a bimonad and Q is comonoidal, then $Q \rtimes T$ is a comonoidal with comonoidal structure given by

$$(Q \rtimes T)_2(X,Y) = Q_2\big(F_T(X), F_T(Y)\big)\, Q\big(\varepsilon_{F_T(X) \otimes F_T(Y)} F_T(\eta_X \otimes \eta_Y)\big),$$
$$(Q \rtimes T)_0 = Q_0\, Q(\varepsilon_{(\mathbb{1}, T_0)}).$$

If T and Q are bimonads, then $Q \rtimes T$ is a bimonad. If T and Q are Hopf monads, then $Q \rtimes T$ is a Hopf monad with left and right antipodes given by

$$a_X^l = {}^\vee\eta_X S^l_{{}^\vee F_T(X)} Q(\varepsilon_{{}^\vee Q F_T(X)}) \quad \text{and} \quad a_X^r = \eta_X^\vee S^r_{F_T^\vee(X)} Q(\varepsilon_{Q F_T(X)^\vee}),$$

where S^l and S^r are the antipodes of Q.

Example 3.12. Let H be a bialgebra over a field $\Bbbk$ and A be an H-module algebra, that is, an algebra in the monoidal category ${}_H\mathrm{Vect}_{\Bbbk}$ of left H-modules. In this situation, the cross product $A \rtimes H$ (see [Maj95, Proposition 1.6.6]) is the $\Bbbk$-algebra $A \rtimes H = A \otimes_{\Bbbk} H$ with unit $1_A \otimes 1_H$ and product given by

$$(a \otimes h)(b \otimes k) = a(h_{(1)} \cdot b) \otimes h_{(2)} \cdot k.$$

Recall that $H \otimes ?$ is a monad on $\mathrm{Vect}_{\Bbbk}$ and $A \otimes ?$ is a monad on ${}_H\mathrm{Vect}_{\Bbbk}$. Then

$$(A \otimes ?) \rtimes (H \otimes ?) = (A \rtimes H) \otimes ?$$

as monads. Moreover, if H is a quasitriangular bialgebra and A is a H-module bialgebra, that is, a bialgebra in the braided category ${}_H\mathrm{Vect}_{\Bbbk}$, then $A \rtimes H$ is a

$\Bbbk$-bialgebra with counit $\varepsilon_A \otimes \varepsilon_H$ and coproduct given by

$$\Delta(a \otimes h) = \sum_i a_{(1)} \otimes t_i \cdot h_{(1)} \otimes r_i \cdot a_{(2)} \otimes h_{(2)},$$

where $R = \sum_i r_i \otimes t_i$ is the R-matrix of H. In that case $(A \otimes ?) \rtimes (H \otimes ?) = (A \rtimes H) \otimes ?$ as bimonads.

4. Distributive laws and liftings

Given two monads P and T on a category $\mathcal{C}$, when is the composition PT a monad? How can one lift P to a monad on the category $T\text{-}\mathcal{C}$? Beck's theory of distributive law [Bec69] answers these questions. In this section, we recall the basic results of this theory and extend them to Hopf monads.

4.1. **Distributive laws between algebras.** Let (A, m, u) and (B, μ, η) be two algebras in a monoidal category $\mathcal{C}$. Given a morphism $\Omega \colon B \otimes A \to A \otimes B$ in $\mathcal{C}$, set

$$p = (m \otimes \mu)(\mathrm{id}_A \otimes \Omega \otimes \mathrm{id}_B) \colon (A \otimes B) \otimes (A \otimes B) \to (A \otimes B).$$

Then $(A \otimes B, p, u \otimes \eta)$ is an algebra in $\mathcal{C}$ if and only if Ω satisfies

$$\Omega(\mathrm{id}_B \otimes m) = (m \otimes \mathrm{id}_B)(\mathrm{id}_A \otimes \Omega)(\Omega \otimes \mathrm{id}_A), \qquad \Omega(\mathrm{id}_B \otimes u) = u \otimes \mathrm{id}_B,$$

$$\Omega(\mu \otimes \mathrm{id}_A) = (\mathrm{id}_A \otimes \mu)(\Omega \otimes \mathrm{id}_B)(\mathrm{id}_B \otimes \Omega), \qquad \Omega(\eta \otimes \mathrm{id}_A) = \mathrm{id}_A \otimes \eta.$$

If such is the case, we say that Ω is a *distributive law of B over A*. The algebra $(A \otimes B, p, u \otimes \eta)$ is then denoted by $A \otimes_\Omega B$. Note that $i = (\mathrm{id}_A \otimes \eta) \colon A \to A \otimes_\Omega B$ and $j = (u \otimes \mathrm{id}_B) \colon B \to A \otimes_\Omega B$ are algebra morphisms and that the *middle unitary law* holds:

$$p(\mathrm{id}_A \otimes \eta \otimes u \otimes \mathrm{id}_A) = \mathrm{id}_{A \otimes B}.$$

In other words, we have $p(i \otimes j) = \mathrm{id}_{A \otimes B}$.

Remark 4.1. Let (C, p, e) be an algebra in $\mathcal{C}$ and $i \colon A \to C$, $j \colon B \to C$ be two algebra morphisms such that $\Theta = p(i \otimes j) \colon A \otimes B \to C$ is an isomorphism in $\mathcal{C}$. Then there exists a unique distributive law Ω of B over A such that Θ is an algebra isomorphism from $A \otimes_\Omega B$ to C. Moreover:

$$\Omega = \Theta^{-1} p(j \otimes i), \quad i = \Theta(\mathrm{id}_A \otimes \eta) \quad \text{and} \quad j = \Theta(u \otimes \mathrm{id}_B).$$

Remark 4.2. If a distributive law $\Omega \colon B \otimes A \to A \otimes B$ of B over A is an isomorphism, then Ω^{-1} is a distributive law of A over B and $\Omega \colon B \otimes_{\Omega^{-1}} A \to A \otimes_\Omega B$ is an isomorphism of algebras.

Example 4.3. Let A and B be bialgebras in a braided category $\mathcal{B}$. A distributive law of B over A is *comultiplicative* if it satisfies

$$(\mathrm{id}_A \otimes \tau_{A,B} \otimes \mathrm{id}_B)(\Delta_A \otimes \Delta_B)\Omega = (\Omega \otimes \Omega)(\mathrm{id}_B \otimes \tau_{B,A} \otimes \mathrm{id}_A)(\Delta_B \otimes \Delta_A),$$

$$(\varepsilon_A \otimes \varepsilon_B)\Omega = \varepsilon_B \otimes \varepsilon_A,$$

where τ is the braiding of $\mathcal{B}$. A comultiplicative distributive law is nothing but a distributive law between algebras in the monoidal category of coalgebras in $\mathcal{B}$. Let Ω be a comultiplicative distributive law of B over A. Then $A \otimes_\Omega B$ is a bialgebra

in $\mathcal{B}$. Furthermore, if A and B are Hopf algebras, then $A \otimes_\Omega B$ is a Hopf algebra with structural morphisms:

$$m_{A\otimes_\Omega B} = (m_A \otimes m_B)(\mathrm{id}_A \otimes \Omega \otimes \mathrm{id}_B), \qquad u_{A\otimes_\Omega B} = u_A \otimes u_B,$$
$$\Delta_{A\otimes_\Omega B} = (\mathrm{id}_A \otimes \tau_{A,B} \otimes \mathrm{id}_B)(\Delta_A \otimes \Delta_B), \qquad \varepsilon_{A\otimes_\Omega B} = \varepsilon_A \otimes \varepsilon_B,$$
$$S_{A\otimes_\Omega B} = S_A \otimes S_B,$$

where m_C, u_C, Δ_C, ε_C, S_C denote respectively the product, unit, coproduct, counit, and antipode of a Hopf algebra C.

4.2. **Lifting monads and bimonads.** Let (P, m, u) be a monad on a category $\mathcal{C}$ and $U \colon \mathcal{D} \to \mathcal{C}$ be a functor. A *lift of the monad P to $\mathcal{D}$* is a monad $(\tilde{P}, \tilde{m}, \tilde{u})$ on $\mathcal{D}$ such that $PU = U\tilde{P}$, $m_U = U(\tilde{m})$, and $u_U = U(\tilde{u})$.

Let P be a bimonad on a monoidal category $\mathcal{C}$ and $U \colon \mathcal{D} \to \mathcal{C}$ be a strong monoidal functor. A *lift of the bimonad P to $\mathcal{D}$* is bimonad $\tilde{P}$ on $\mathcal{D}$ which is a lift of the monad P to $\mathcal{D}$ such that $U\tilde{P} = PU$ as comonoidal functors.

4.3. **Distributive laws between monads.** Let (T, μ, η) and (P, m, u) be monads on a category $\mathcal{C}$. Following Beck [Bec69], a *distributive law of T over P* is a natural transformation $\Omega \colon TP \to PT$ verifying

$$\Omega_X T(m_X) = m_{T(X)} P(\Omega_X) \Omega_{P(X)}, \qquad \Omega_X T(u_X) = u_{T(X)},$$
$$\Omega_X \mu_{P(X)} = P(\mu_X) \Omega_{T(X)} T(\Omega_X), \qquad \Omega_X \eta_{P(X)} = P(\eta_X),$$

for all object X of $\mathcal{C}$.

Remark 4.4. Viewing the monads T and P as algebras in the monoidal category of endofunctors of $\mathcal{C}$ (with monoidal product $\circ$ and unit object $1_\mathcal{C}$), the above definition of a distributive law agrees with that given in Section 4.1.

Let Ω be a distributive law of T over P. First, Ω defines a monad structure on the endofunctor PT of $\mathcal{C}$, with product p and unit e given by

$$p_X = m_{T(X)} P^2(\mu_X) P(\Omega_{T(X)}) \quad \text{and} \quad e_X = u_{T(X)} \eta_X.$$

The monad (PT, p, e) is denoted by $P \circ_\Omega T$. Second, Ω defines a lift $(\tilde{P}^\Omega, \tilde{m}, \tilde{u})$ of the monad P to the category $T\text{-}\mathcal{C}$ as follows:

$$\tilde{P}^\Omega(M, r) = \big(P(M), P(r)\Omega_M\big), \quad \tilde{m}_{(M,r)} = m_M, \quad \text{and} \quad \tilde{u}_{(M,r)} = u_M.$$

Furthermore, there is a canonical isomorphism of categories

$$K \colon \begin{cases} \tilde{P}^\Omega\text{-}(T\text{-}\mathcal{C}) & \longrightarrow & (P \circ_\Omega T)\text{-}\mathcal{C} \\ \big((M,r),s\big) & \longmapsto & \big(M, U_T(s)P(r)\big) \end{cases}$$

with inverse

$$K^{-1} \colon \begin{cases} (P \circ_\Omega T)\text{-}\mathcal{C} & \longrightarrow & \tilde{P}^\Omega\text{-}(T\text{-}\mathcal{C}) \\ (A, \alpha) & \longmapsto & \big((A, \alpha\, u_{T(A)}), \alpha P(\eta_A)\big) \end{cases}.$$

In fact K is the comparison functor of the composite adjunction

$$\tilde{P}^\Omega\text{-}(T\text{-}\mathcal{C}) \underset{F_{\tilde{P}^\Omega}}{\overset{U_{\tilde{P}^\Omega}}{\rightleftarrows}} T\text{-}\mathcal{C} \underset{F_T}{\overset{U_T}{\rightleftarrows}} \mathcal{C}.$$

Thus this composite adjunction is monadic with monad $P \circ_\Omega T$.

The assignments $\Omega \mapsto P \circ_\Omega T$ and $\Omega \mapsto \tilde{P}^\Omega$ are one-to-one in the following sense:

Theorem 4.5 ([Bec69])**.** *Let (T, μ, η) and (P, m, u) be monads on a category $\mathcal{C}$. We have bijective correspondences between:*

(i) *distributive laws $\Omega \colon TP \to PT$ of T over P;*
(ii) *products $p \colon PTPT \to PT$ for which:*
   (a) *$(PT, p, u_T\eta)$ is a monad on $\mathcal{C}$;*
   (b) *$u_T \colon T \to PT$ and $P(\eta) \colon P \to PT$ are morphisms of monads;*
   (c) *the middle unitary law $p_X P(\eta_{PT(X)} u_{T(X)}) = \mathrm{id}_{PT(X)}$ holds;*
(iii) *lifts of the monad P on $\mathcal{C}$ to a monad $\tilde{P}$ on T-$\mathcal{C}$.*

4.4. **Distributive laws between bimonads.** Let T and P be bimonads on a monoidal category $\mathcal{C}$. Recall that TP and PT are comonoidal endofunctors of $\mathcal{C}$. A distributive law $\Omega \colon TP \to PT$ of T over P is *comonoidal* if it is comonoidal as a natural transformation, that is, if it satisfies

$$(PT)_2(X, Y)\Omega_{X \otimes Y} = (\Omega_X \otimes \Omega_Y)(TP)_2(X, Y) \quad \text{and} \quad (PT)_0 \Omega_{\mathbb{1}} = (TP)_0.$$

Remark 4.6. Viewing the bimonads T and P as algebras in the monoidal category of comonoidal endofunctors of $\mathcal{C}$ (see Remark 2.2), a comonoidal distributive law is a distributive law in the sense of Section 4.1.

Beck's Theorem 4.5 was generalized by Street [Str72] to monads in a 2-category. Applying this theorem to the case of the 2-category of monoidal categories and comonoidal functors, we obtain:

Theorem 4.7. *Let (T, μ, η) and (P, m, u) be bimonads on a monoidal category $\mathcal{C}$. We have bijective correspondences between:*

(i) *comonoidal distributive laws $\Omega \colon TP \to PT$ of T over P;*
(ii) *products $p \colon PTPT \to PT$ for which:*
   (a) *$(PT, p, u_T\eta)$ is a bimonad on $\mathcal{C}$;*
   (b) *$u_T \colon T \to PT$ and $P(\eta) \colon P \to PT$ are morphisms of bimonads;*
   (c) *the middle unitary law $p_X P(\eta_{PT(X)} u_{T(X)}) = \mathrm{id}_{PT(X)}$ holds;*
(iii) *Lifts of the bimonad P on $\mathcal{C}$ to a bimonad $\tilde{P}$ on T-$\mathcal{C}$.*

Also, if Ω is a comonoidal distributive law of T over P, the canonical isomorphism of categories $\tilde{P}^\Omega$-$(T$-$\mathcal{C}) \simeq (P \circ_\Omega T)$-$\mathcal{C}$ is strict monoidal.

Example 4.8. Let $\mathcal{B}$ be a braided category, A and B be two bialgebras in $\mathcal{B}$, and $\Omega \colon B \otimes A \to A \otimes B$ be a morphism in $\mathcal{B}$. Then the following conditions are equivalent:

(i) $\Omega \otimes \mathrm{id}_{1_\mathcal{B}}$ is a comonoidal distributive law of $B \otimes ?$ over $A \otimes ?$;
(ii) $\mathrm{id}_{1_\mathcal{B}} \otimes \Omega$ is a comonoidal distributive law of $? \otimes A$ over $? \otimes B$;
(iii) Ω is a comultiplicative distributive law of B over A (see Example 4.3).

If such is the case, we have the following equalities of bimonads:

$$(A \otimes ?) \circ_{(\Omega \otimes \mathrm{id}_{1_\mathcal{B}})} (B \otimes ?) = (A \otimes_\Omega B) \otimes ?,$$
$$(? \otimes B) \circ_{(\mathrm{id}_{1_\mathcal{B}} \otimes \Omega)} (? \otimes A) = ? \otimes (A \otimes_\Omega B).$$

Remark 4.9. Let $\Omega \colon TP \to PT$ be a distributive law between monads on a category $\mathcal{C}$. Then $\tilde{P}^\Omega \rtimes T = P \circ_\Omega T$ as monads, where $\rtimes$ denotes the cross product (see Section 3.7). Moreover, if $\mathcal{C}$ is monoidal, T and P are bimonads, and Ω is comonoidal, then $\tilde{P}^\Omega \rtimes T = P \circ_\Omega T$ as bimonads.

4.5. **Distributive laws and antipodes.** We show here that if $\Omega\colon TP \to PT$ is a comonoidal distributive law between Hopf monads, then the composition $P \circ_\Omega T$ and the lift $\tilde{P}^\Omega$ are Hopf monads:

Proposition 4.10. *Let T and P be bimonads on a monoidal category $\mathcal{C}$ and let $\Omega\colon TP \to PT$ be a comonoidal distributive law of T over P. Then:*

(a) *If $\mathcal{C}$ is left autonomous, T has a left antipode s^l, and P has a left antipode S^l, then the bimonads $P \circ_\Omega T$ and $\tilde{P}^\Omega$ have left antipodes, denoted by a^l and $\tilde{S}^l$ respectively, given by*

$$a_X^l = S_X^l P(s_{P(X)}^l) PT({}^\vee\Omega_X)\colon PT\big({}^\vee PT(X)\big) \to {}^\vee X,$$

$$\tilde{S}_{(M,r)}^l = S_M^l\colon \tilde{P}^\Omega\big({}^\vee\tilde{P}^\Omega(M,r)\big) \to {}^\vee(M,r).$$

(b) *If $\mathcal{C}$ is right autonomous, T has a right antipode s^r, and P has a right antipode S^r, then the bimonads $P\circ_\Omega T$ and $\tilde{P}^\Omega$ have right antipodes, denoted by a^r and $\tilde{S}^r$ respectively, given by*

$$a_X^r = S_X^r P(s_{P(X)}^r) PT(\Omega_X^\vee)\colon PT\big(PT(X)^\vee\big) \to X^\vee,$$

$$\tilde{S}_{(M,r)}^r = S_M^r\colon \tilde{P}^\Omega\big(\tilde{P}^\Omega(M,r)^\vee\big) \to (M,r)^\vee.$$

Proof. Let us prove part (a). One first checks that a_X^l satisfies the axioms of a left antipode, that is,

$$(PT)_0 PT(\mathrm{ev}_X) PT\big({}^\vee(u_{T(X)}\eta_X) \otimes \mathrm{id}_X\big)$$
$$= \mathrm{ev}_{PT(X)}(a_{PT(X)}^l PT({}^\vee p_X) \otimes \mathrm{id}_{PT(X)})(PT)_2({}^\vee PT(X), X),$$
$$(u_{T(X)}\eta_X \otimes \mathrm{id}_{\vee X})\mathrm{coev}_X (PT)_0$$
$$= (p_X \otimes a_X^l)(PT)_2(PT(X), {}^\vee PT(X)) PT(\mathrm{coev}_{PT(X)}).$$

This can be done by applying the axioms for the left antipodes s^l and S^l of T and P and the axioms for the distributive law Ω. By Theorem 2.7(b), this implies that $(P \circ_\Omega T)\text{-}\mathcal{C}$ is left autonomous. Now recall that

$$K\colon \begin{cases} \tilde{P}^\Omega\text{-}(T\text{-}\mathcal{C}) & \xrightarrow{\ \sim\ } & (P \circ_\Omega T)\text{-}\mathcal{C} \\ \big((M,r),s\big) & \longmapsto & \big(M, sP(r)\big) \end{cases}$$

is a strict monoidal isomorphism of categories (see Section 4.3). Therefore $\tilde{P}^\Omega\text{-}(T\text{-}\mathcal{C})$ is left autonomous and so, by Theorem 2.7(b), $\tilde{P}^\Omega$ has a left antipode $\tilde{S}^l$. Furthermore, given a $\tilde{P}^\Omega$-module $((M,r),s)$, we have

$$K^{-1}\big({}^\vee K((M,r),s)\big) = \big(({}^\vee M, s_M^l T(r)), \rho\big),$$

where $U_T(\rho) = a_M^l PT({}^\vee P(r)^\vee s) P(\eta_M) = S_M^l P({}^\vee s)$. Hence $\tilde{S}_{(M,r)}^l = S_M^l$.

Part (b) results from part (a) applied to the coopposite Hopf monads

$$(P \circ_\Omega T)^{\mathrm{cop}} = P^{\mathrm{cop}} \circ_\Omega T^{\mathrm{cop}} \quad \text{and} \quad (\tilde{P}^\Omega)^{\mathrm{cop}} = \big(\widetilde{P^{\mathrm{cop}}}\big)^\Omega,$$

see Remark 2.3. $\qquad\square$

From Proposition 4.10, Theorem 4.7, and Remark 4.9, we deduce:

Corollary 4.11. *If T and P are Hopf monads on an autonomous category $\mathcal{C}$ and $\Omega\colon TP \to PT$ is a comonoidal distributive law, then $P \circ_\Omega T$ is a Hopf monad on $\mathcal{C}$, $\tilde{P}^\Omega$ is a Hopf monad on $T\text{-}\mathcal{C}$, and $\tilde{P}^\Omega \rtimes T = P \circ_\Omega T$ as Hopf monads.*

4.6. **Invertible distributive laws.** Let T, P be two monads on a category $\mathcal{C}$ and $\Omega\colon TP \to PT$ be an invertible distributive law of T over P. Then $\Omega^{-1}\colon PT \to TP$ is a distributive law of P over T and Ω is a isomorphism of monads from $T \circ_{\Omega^{-1}} P$ to $P \circ_\Omega T$.

If $\mathcal{C}$ is monoidal, P, T are bimonads, and $\Omega\colon TP \to PT$ is a comonoidal distributive law of T on P, then $\Omega^{-1}\colon PT \to TP$ is a comonoidal distributive law of P over T and Ω is a isomorphism of bimonads from $T \circ_{\Omega^{-1}} P$ to $P \circ_\Omega T$.

Proposition 4.12. *Let P, T be Hopf monads on an autonomous category $\mathcal{C}$. Then any comonoidal distributive law $\Omega\colon TP \to PT$ of T over P is invertible. Furthermore, for any object X of $\mathcal{C}$, we have*

$$\Omega_X^{-1} = S^r_{\vee TP(X)} P\big(s^r_{P(^\vee TP(X))}\big) PT\big(\Omega^\vee_{\vee TP(X)}\big) PT\big(P(s^l_{P(X)})^\vee\big) PT\big(S^{l\vee}_X\big),$$

where s^l, s^r, S^l, S^r denote left and right antipodes of T and P, respectively.

Proof. The functors T, P and PT are Hopf monads by assumption and Corollary 4.11. Therefore, by Proposition 3.6, the functors $T^!$, $P^!$, and $(PT)^!$ are right adjoints for T, P, and PT, respectively. On the other hand, by composition of adjunctions, $P^! \circ T^! = (PT)^!$ is a right adjoint for TP. As a left adjoint is unique up to a unique natural isomorphism, we obtain a canonical isomorphism $\alpha\colon PT \to TP$. Denoting $e\colon TT^! \to 1_\mathcal{C}$, $h\colon 1_\mathcal{C} \to T^!T$, $e'\colon PP^! \to 1_\mathcal{C}$, $h'\colon 1_\mathcal{C} \to P^!P$, $E\colon PT(PT)^! \to 1_\mathcal{C}$, and $H\colon 1_\mathcal{C} \to (PT)^!PT$ the adjunction morphisms, we have $\alpha = E_{TP}PTP^!(h_P)PT(h')$. Now the adjunction morphisms can be expressed in terms of the antipodes, see Remark 3.7. Therefore, using Proposition 4.10, we get that, for any object X of $\mathcal{C}$,

$$\alpha_X = S^r_{\vee TP(X)} P(s^r_{P(^\vee TP(X))}) PT(\Omega^\vee_{\vee TP(X)}) PT(P(s^l_{P(X)})^\vee) PT(S^{l\vee}_X).$$

Furthermore, $E_X = S^r_{\vee X} P(s^r_{P(^\vee X)}) PT(\Omega^\vee_X)$ and $H_X = (PT)^!(\Omega_X) P(s^l_{P(X)})^\vee S^{l\vee}_X$. Hence

$$
\begin{aligned}
\mathrm{id}_{PT(X)} &= E_{PT(X)} PT(H_X)\\
&= E_{PT(X)} PT(PT)^!(\Omega_X) PT(P(s^l_{P(X)})^\vee) PT(S^{l\vee}_X)\\
&= \Omega_X E_{TP(X)} PT(P(s^l_{P(X)})^\vee) PT(S^{l\vee}_X) \quad \text{by functoriality of } E\\
&= \Omega_X S^r_{\vee TP(X)} P(s^r_{P(^\vee TP(X))}) PT(\Omega^\vee_{\vee TP(X)}) PT(P(s^l_{P(X)})^\vee) PT(S^{l\vee}_X)\\
&= \Omega_X \alpha_X.
\end{aligned}
$$

This shows that Ω, as the inverse of the isomorphism α, is an isomorphism. $\qquad\square$

Remark 4.13. Let $\Omega\colon B \otimes A \to A \otimes B$ be a distributive law between two Hopf algebras A and B in a braided autonomous category $\mathcal{B}$ with braiding τ. Then, applying Proposition 4.12 to the distributive law of Example 4.8, we find that Ω is invertible and its inverse is given by

$$\Omega^{-1} = (S_B^{-1} \otimes S_A^{-1})\tau_{B,A}^{-1} \Omega\, \tau_{A,B}(S_A \otimes S_B),$$

where S_A and S_B are the antipodes of A and B.

5. The centralizer of a Hopf monad

In this section, we introduce the notion of centralizer of a Hopf monad and interpret its category of modules as the categorical center relative to the Hopf monad.

5.1. **Centralizers of endofunctors.** Let $\mathcal{C}$ be a monoidal category and T be an endofunctor of $\mathcal{C}$.

A *centralizer of T at an object X of $\mathcal{C}$* is a pair (Z, δ), where $Z \in \mathrm{Ob}(\mathcal{C})$ and

$$\delta = \{\delta_Y \colon X \otimes Y \to T(Y) \otimes Z\}_{Y \in \mathrm{Ob}(\mathcal{C})} \colon X \otimes 1_{\mathcal{C}} \to T \otimes Z$$

is a natural transformation, verifying the following universal property: for every object W of $\mathcal{C}$ and every natural transformation $\xi \colon X \otimes 1_{\mathcal{C}} \to T \otimes W$, there exists a unique morphism $r \colon Z \to W$ in $\mathcal{C}$ such that $\xi = (\mathrm{id}_T \otimes r)\delta$. Note that a centralizer of T at X, if it exists, is unique up to a unique isomorphism.

Remark 5.1. The notion of a centralizer is not invariant under left/right symmetry. We should properly call it a 'left-handed' centralizer. We can as well define a 'right-handed' centralizer of T at X to be a pair (Z', δ'), with

$$\delta' = \{\delta'_Y \colon Y \otimes X \to Z' \otimes T(Y)\}_{Y \in \mathrm{Ob}(\mathcal{C})} \colon 1_{\mathcal{C}} \otimes X \to Z' \otimes T$$

satisfying the relevant universal property. Note that this is equivalent to saying that (Z', δ') is a 'left-handed' centralizer of T at X in the monoidal category $\mathcal{C}^{\otimes \mathrm{op}}$. By left/right symmetry, all notions and results concerning 'left-handed' centralizers can be adapted to the 'right-handed' version.

The endofunctor T is said to be *centralizable at an object X of $\mathcal{C}$* if it admits a centralizer at X.

A *centralizer* of T is a pair (Z_T, ∂), where Z_T is an endofunctor of $\mathcal{C}$ and

$$\partial = \{\partial_{X,Y} \colon X \otimes Y \to T(Y) \otimes Z_T(X)\}_{X,Y \in \mathrm{Ob}(\mathcal{C})} \colon \otimes \to (T \otimes Z_T)\sigma_{\mathcal{C},\mathcal{C}}$$

is a natural transformation such that $(Z_T(X), \partial_{X,1_{\mathcal{C}}})$ is a centralizer of T at X for every object X of $\mathcal{C}$.

The endofunctor T is said to be *centralizable* if it admits a centralizer. An endofunctor of $\mathcal{C}$ is centralizable if and only if it is centralizable at every object of $\mathcal{C}$. In that case, its centralizer is essentially unique. More precisely:

Lemma 5.2. *Let T be an endofunctor of a monoidal category $\mathcal{C}$. We have:*

(a) *Given a centralizer $(Z_T(X), \partial_X)$ of T at every object X of $\mathcal{C}$, the assignment $Z_T \colon X \mapsto Z_T(X)$ admits a unique structure of functor such that*

$$\partial = \{\partial_{X,Y} = (\partial_X)_Y \colon X \otimes Y \to T(Y) \otimes Z_T(X)\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$$

is a natural transformation. The pair (Z_T, ∂) is then a centralizer of T.

(b) *If (Z, ∂) and (Z', ∂') are centralizers of T, then there exists a unique natural isomorphism $\alpha \colon Z \to Z'$ such that $\partial' = (\mathrm{id}_T \otimes \alpha)\partial$.*

Proof. For each morphism $f \colon X \to X'$ in $\mathcal{C}$, by the universal property of centralizers there exists a unique morphism $Z_T(f) \colon Z_T(X) \to Z_T(X')$ such that

$$(\mathrm{id}_T \otimes Z_T(f))\, \partial_{X,1_{\mathcal{C}}} = \partial_{X',1_{\mathcal{C}}}\, (f \otimes 1_{\mathcal{C}}),$$

and this assignment defines the only structure of functor on Z_T such that ∂ is a natural transformation. $\square$

5.2. **Centralizers and coends.** In this section, we give a characterization of centralizable endofunctors in a left autonomous category in terms of coends.

Proposition 5.3. *Let $\mathcal{C}$ be a left autonomous category, T be an endofunctor of $\mathcal{C}$, and X be an object of $\mathcal{C}$. Then T is centralizable at X if and only if the coend*

$$Z_T(X) = \int^{Y \in \mathcal{C}} {}^{\vee}T(Y) \otimes X \otimes Y$$

exists. If such is the case, denoting by i the universal dinatural transformation of the coend and setting

$$(\partial_X)_Y = \big(\mathrm{id}_{T(Y)} \otimes i_Y\big)(\mathrm{coev}_{T(Y)} \otimes \mathrm{id}_{X \otimes Y}),$$

the pair $(Z_T(X), \partial_X)$ is a centralizer of T at X.

Proof. Let $F \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ be the functor defined by $F(Y, Z) = {}^{\vee}T(Y) \otimes X \otimes Z$ and $F(f, g) = {}^{\vee}T(f) \otimes X \otimes g$. By duality, we have a bijection

$$\psi \colon \mathrm{DINAT}(F, Z) \to \mathrm{HOM}(X \otimes 1_{\mathcal{C}}, T \otimes Z)$$

which is natural in $Z \in \mathrm{Ob}(\mathcal{C})$. It is defined by

$$\psi(j)_Y = \big(\mathrm{id}_{T(Y)} \otimes j_Y\big)(\mathrm{coev}_{T(Y)} \otimes \mathrm{id}_{X \otimes Y}) \colon X \otimes Y \to T(Y) \otimes Z$$

and its inverse by

$$\psi^{-1}(\delta)_Y = (\mathrm{ev}_{T(Y)} \otimes \mathrm{id}_Z)(\mathrm{id}_{{}^{\vee}T(Y)} \otimes \delta_Y) \colon {}^{\vee}T(Y) \otimes X \otimes Y \to Z.$$

Therefore T is centralizable at X if and only if F admits a coend and, if so, the centralizer of T at X is canonically isomorphic to the coend of F. $\square$

5.3. **Extended factorization property of the centralizer.** Let T be a centralizable endofunctor of a monoidal category $\mathcal{C}$ and (Z_T, ∂) be a centralizer of T. For any non-negative integer n, let

$$\partial^n \colon \otimes_{n+1} \to (T^{\otimes n} \otimes Z_T^n)\sigma_{\mathcal{C}, \mathcal{C}^n}$$

be the natural transformation defined by the following diagram:

In other words, the morphisms

$$\partial^n_{X, Y_1, \ldots, Y_n} \colon X \otimes Y_1 \otimes \cdots \otimes Y_n \to T(Y_1) \otimes \cdots \otimes T(Y_n) \otimes Z_T^n(X)$$

are defined inductively by $\partial^0_X = \mathrm{id}_X$ and

$$\partial^{n+1}_{X, Y_1, \ldots, Y_{n+1}} = (\mathrm{id}_{T(Y_1) \otimes \cdots \otimes T(Y_n)} \otimes \partial_{Z_T^n(X), Y_{n+1}})(\partial^n_{X, Y_1, \ldots, Y_n} \otimes \mathrm{id}_{Y_{n+1}}).$$

Notice $\partial^1 = \partial$ and $\partial^{p+q} = (\mathrm{id}_{T^{\otimes p}} \otimes \partial^q)(\partial^p \otimes \mathrm{id}_{\otimes_q})$ for all non-negative integers p, q.

Lemma 5.4. *Assume $\mathcal{C}$ is left autonomous. Let $\mathcal{D}$ be category and $K, L \colon \mathcal{D} \to \mathcal{C}$ be two functors. For every non-negative integer n and every natural transformation $\xi \colon K \otimes \otimes_n \to (T^{\otimes n} \otimes L)\sigma_{\mathcal{D}, \mathcal{C}^n}$, there exists a unique natural transformation*

FIGURE 1. Structural morphisms of Z_T

$r\colon Z_T^n K \to L$ such that

$$(\mathrm{id}_{T(Y_1)\otimes\cdots\otimes T(Y_n)} \otimes r_X)\partial^n_{X,Y_1,\ldots,Y_n} = \xi_{X,Y_1,\ldots,Y_n},$$

that is,
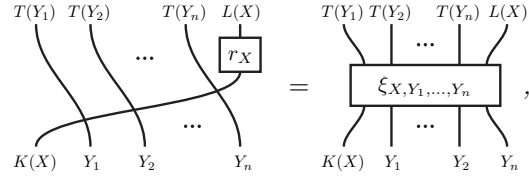
for all $X \in \mathrm{Ob}(\mathcal{D})$ and $Y_1,\ldots,Y_n \in \mathrm{Ob}(\mathcal{C})$.

Remark 5.5. We will often write the equality defining r in Lemma 5.4 as

$$(\mathrm{id}_{T^{\otimes n}} \otimes r)\partial^n_{K,1_{\mathcal{C}^n}} = \xi.$$

Strictly speaking, it should be $(\mathrm{id}_{T^{\otimes n}} \otimes r)\sigma_{\mathcal{D},\mathcal{C}^n}\partial^n_{K,1_{\mathcal{C}^n}} = \xi$. However, in this kind of formula, we will usually omit the permutation σ, as it can easily be recovered from the context.

Proof of Lemma 5.4. The lemma can be verified by induction on n using the Parameter Theorem and Fubini Theorem for coends (see [Mac98]) and the fact that, by Proposition 5.3, we have $Z_T K(X) = \int^{Y\in\mathcal{C}} {}^\vee T(Y)\otimes K(X)\otimes Y$ for all $X \in \mathrm{Ob}(\mathcal{C})$. $\square$

5.4. **Structure of centralizers.** In this section, we show that the centralizer Z_T of a Hopf monad T is a Hopf monad. The structural morphisms of Z_T are defined as in Figure 1 using the extended factorization property of Z_T given in Lemma 5.4. More precisely:

Theorem 5.6. *Let T be a centralizable endofunctor of a left autonomous category $\mathcal{C}$ and let (Z_T,∂) be its centralizer. We have:*

(a) *If T is comonoidal, then Z_T is a monad on $\mathcal{C}$, with product $m\colon Z_T^2 \to Z_T$ and unit $u\colon 1_{\mathcal{C}} \to Z_T$ defined by*

$$(\mathrm{id}_{T^{\otimes 2}} \otimes m)\partial^2 = (T_2 \otimes \mathrm{id}_{Z_T})\partial_{1_{\mathcal{C}},\otimes} \quad and \quad u = (T_0 \otimes \mathrm{id}_{Z_T})\partial_{1_{\mathcal{C}},\mathbb{1}}.$$

(b) *If (T,μ,η) is a monad, then Z_T is comonoidal, with comonoidal structure defined by*

$$\big(\mathrm{id}_T \otimes (Z_T)_2\big)\partial_{\otimes,1_{\mathcal{C}}} = (\mu \otimes \mathrm{id}_{Z_T^{\otimes 2}})(\partial_{1_{\mathcal{C}},T} \otimes \mathrm{id}_{Z_T})(\mathrm{id}_{1_{\mathcal{C}}} \otimes \partial),$$
$$\big(\mathrm{id}_T \otimes (Z_T)_0\big)\partial_{\mathbb{1},1_{\mathcal{C}}} = \eta.$$

(c) *If T is a bimonad, then Z_T is a bimonad on $\mathcal{C}$, with the monad structure of part (a) and the comonoidal structure of part (b).*

(d) *If $\mathcal{C}$ is autonomous, T is a bimonad, and T has a right antipode s^r, then the bimonad Z_T has a left antipode S^l defined by*

$$(\mathrm{id}_T \otimes S^l)\partial_{\vee Z_T,1_{\mathcal{C}}} = {}^{\vee}\!\big((s^r \otimes \mathrm{id}_{Z_T})\partial_{1_{\mathcal{C}},T^{\vee}}\big).$$

(e) *If $\mathcal{C}$ is autonomous, T is a bimonad, and T has a left antipode s^l, then the bimonad Z_T has a right antipode S^r defined by*

$$(\mathrm{id}_T \otimes S^r)\partial_{Z_T^{\vee},1_{\mathcal{C}}} = \big((s^l \otimes \mathrm{id}_{Z_T})\partial_{1_{\mathcal{C}},{}^{\vee}T}\big)^{\vee}.$$

In particular, if $\mathcal{C}$ is autonomous and T is a Hopf monad, then Z_T is a Hopf monad.

Remark 5.7. The centralizer construction $T \mapsto Z_T$ is functorial, contravariant in T. More precisely, let $\mathcal{C}$ be a left autonomous category and T, T' be two centralizable endofunctors of $\mathcal{C}$, with centralizers (Z_T,∂) and $(Z_{T'},\partial')$, respectively. Then, for each natural transformation $f\colon T \to T'$, there exists a unique natural transformation $Z_f\colon Z_{T'} \to Z_T$ such that

$$(\mathrm{id}_{T'} \otimes Z_f)\partial' = (f \otimes \mathrm{id}_{Z_T})\partial.$$

We have $Z_{fg} = Z_g Z_f$ and $Z_{\mathrm{id}_T} = \mathrm{id}_{Z_T}$. Moreover, if f is comonoidal, then Z_f is a morphism of monads. If f if a morphism of monads, then Z_f is comonoidal. Thus, if f is a morphism of bimonads or Hopf monads, so is Z_f.

Remark 5.8. Let T be a centralizable Hopf monad on an autonomous category $\mathcal{C}$, with centralizer (Z_T,∂). Set

$$\partial'_{X,Y} = \left.\begin{matrix} Z_T(X) \;\; T(Y) \\ \boxed{s_Y^r} \\ Y \quad X \end{matrix}\right/ : Y \otimes X \to Z_T(X) \otimes T(Y)$$

and $Z_{T^{\mathrm{cop}}} = (Z_T)^{\mathrm{cop}}$. Then $(Z_{T^{\mathrm{cop}}},\partial')$ is a centralizer of T^{cop} in $\mathcal{C}^{\otimes\mathrm{op}}$. Moreover, $Z_{T^{\mathrm{cop}}} = (Z_T)^{\mathrm{cop}}$ as Hopf monads when Z_T and $Z_{T^{\mathrm{cop}}}$ are equipped with the Hopf monad structure of Theorem 5.6. In the language of Remark 5.1, 'left centralizability' and 'right centralizability' are equivalent for a Hopf monad T, and a 'left-handed' centralizer $Z'_T = (Z_{T^{\mathrm{cop}}})^{\mathrm{cop}}$ can be identified with a 'right-handed' centralizer Z_T in a manner preserving the Hopf monad structures.

Proof of Theorem 5.6. To simplify notation, set $Z = Z_T$. Let us prove part (a). By definition of the product m and unit u of Z, we have

$$\big(\mathrm{id}_{T^{\otimes 3}} \otimes mZ(m)\big)\partial^3 = \big(T_2 \otimes \mathrm{id}_T \otimes m\big)\partial^2_{1_{\mathcal{C}},\otimes,1_{\mathcal{C}}}$$
$$= \big((T_2 \otimes \mathrm{id}_T)T_2 \otimes \mathrm{id}_Z\big)\partial_{1_{\mathcal{C}},\otimes_2}$$
$$= \big((\mathrm{id}_T \otimes T_2)T_2 \otimes \mathrm{id}_Z\big)\partial_{1_{\mathcal{C}},\otimes_2}$$
$$= \big(\mathrm{id}_T \otimes T_2 \otimes m\big)\partial^2_{1_{\mathcal{C}},1_{\mathcal{C}},\otimes}$$
$$= \big(\mathrm{id}_{T^{\otimes 3}} \otimes mm_Z\big)\partial^3.$$

Therefore $mZ(m) = mm_Z$ by the uniqueness assertion of Lemma 5.4. Likewise, since

$$\big(\mathrm{id}_T \otimes mZ(u)\big)\partial = (\mathrm{id}_T \otimes m)\partial_{Z,1_{\mathcal{C}}}(u \otimes \mathrm{id}_{1_{\mathcal{C}}})$$
$$= (T_0 \otimes \mathrm{id}_T \otimes m)\partial^2_{1_{\mathcal{C}},\mathbb{1},1_{\mathcal{C}}}$$
$$= \big((T_0 \otimes \mathrm{id}_T)T_2(\mathbb{1},-) \otimes \mathrm{id}_Z\big)\partial$$
$$= (\mathrm{id}_T \otimes \mathrm{id}_Z)\partial$$

and

$$(\mathrm{id}_T \otimes mu_Z)\partial = (\mathrm{id}_T \otimes T_0 \otimes m)\partial^2_{1_{\mathcal{C}},1_{\mathcal{C}},\mathbb{1}}$$
$$= \big((\mathrm{id}_T \otimes T_0)T_2(-,\mathbb{1}) \otimes \mathrm{id}_Z\big)\partial$$
$$= (\mathrm{id}_T \otimes \mathrm{id}_Z)\partial,$$

we get $mZ(u) = \mathrm{id}_Z = mu_Z$. Hence (Z, m, u) is a monad on $\mathcal{C}$.

Let us prove part (b). By definition of the natural transformation Z_2, we have

$$\big(\mathrm{id}_T \otimes (\mathrm{id}_Z \otimes Z_2)Z_2\big)\partial_{\otimes_3,1_{\mathcal{C}}}$$
$$= (\mu T(\mu) \otimes \mathrm{id}_{Z^{\otimes 3}})(\partial_{T^2,1_{\mathcal{C}}} \otimes \mathrm{id}_{Z^{\otimes 2}})(\mathrm{id}_{1_{\mathcal{C}}} \otimes \partial_{T,1_{\mathcal{C}}} \otimes \mathrm{id}_Z)(\mathrm{id}_{\otimes} \otimes \partial)$$
$$= (\mu\mu_T \otimes \mathrm{id}_{Z^{\otimes 3}})(\partial_{T^2,1_{\mathcal{C}}} \otimes \mathrm{id}_{Z^{\otimes 2}})(\mathrm{id}_{1_{\mathcal{C}}} \otimes \partial_{T,1_{\mathcal{C}}} \otimes \mathrm{id}_Z)(\mathrm{id}_{\otimes} \otimes \partial)$$
$$= \big(\mathrm{id}_T \otimes (Z_2 \otimes \mathrm{id}_Z)Z_2\big)\partial_{\otimes_3,1_{\mathcal{C}}},$$

and so $(\mathrm{id}_Z \otimes Z_2)Z_2 = (Z_2 \otimes \mathrm{id}_Z)Z_2$ by Lemma 5.4. Likewise, since

$$\big(\mathrm{id}_T \otimes (\mathrm{id}_Z \otimes Z_0)Z_2(-,\mathbb{1})\big)\partial = (\mu \otimes \mathrm{id}_Z \otimes Z_0)(\partial_{1_{\mathcal{C}},T} \otimes \mathrm{id}_{Z(\mathbb{1})})(\mathrm{id}_{1_{\mathcal{C}}} \otimes \partial_{\mathbb{1},1_{\mathcal{C}}})$$
$$= (\mu \otimes \mathrm{id}_Z)\partial_{1_{\mathcal{C}},T}(\mathrm{id}_{1_{\mathcal{C}}} \otimes \eta) = (\mu T(\eta) \otimes \mathrm{id}_Z)\partial = \partial$$

and

$$\big(\mathrm{id}_T \otimes (Z_0 \otimes \mathrm{id}_Z)Z_2(\mathbb{1},-)\big)\partial = (\mu \otimes Z_0 \otimes \mathrm{id}_Z)(\partial_{\mathbb{1},T} \otimes \mathrm{id}_Z)\partial$$
$$= (\mu\eta_T \otimes \mathrm{id}_Z)\partial = \partial,$$

we get $(\mathrm{id}_Z \otimes Z_0)Z_2(1_{\mathcal{C}},\mathbb{1}) = \mathrm{id}_Z = (Z_0 \otimes \mathrm{id}_Z)Z_2(\mathbb{1},1_{\mathcal{C}})$. Hence Z is a comonoidal functor.

Let us prove part (c). We have to show that m and u are comonoidal morphisms. Since μ and η are comonoidal, we have

$$(\mathrm{id}_{T^{\otimes 2}} \otimes Z_2 m)\partial^2_{\otimes,1_{\mathcal{C}},1_{\mathcal{C}}} = (T_2\mu \otimes \mathrm{id}_{Z^{\otimes 2}})(\partial_{1_{\mathcal{C}},T} \otimes \mathrm{id}_Z)(\mathrm{id}_{1_{\mathcal{C}}} \otimes \partial)$$
$$= \big((\mu \otimes \mu)T_2T(T_2) \otimes \mathrm{id}_{Z^{\otimes 2}}\big)(\partial_{1_{\mathcal{C}},T} \otimes \mathrm{id}_Z)(\mathrm{id}_{1_{\mathcal{C}}} \otimes \partial)$$
$$= \big(\mathrm{id}_{T^{\otimes 2}} \otimes (m \otimes m)Z_2Z(Z_2)\big)\partial^2_{\otimes,1_{\mathcal{C}},1_{\mathcal{C}}}$$

and $(\mathrm{id}_{T^{\otimes 2}} \otimes Z_0 m_\mathbb{1})\partial^2_{\mathbb{1},1_\mathcal{C},1_\mathcal{C}} = T_2\eta = \eta \otimes \eta = \big(\mathrm{id}_{T^{\otimes 2}} \otimes Z_0 Z(Z_0)\big)\partial^2_{\mathbb{1},1_\mathcal{C},1_\mathcal{C}}$. Therefore $Z_2 m = (m \otimes m)Z_2 Z(Z_2)$ and $Z_0 m_\mathbb{1} = Z_0 Z(Z_0)$ by Lemma 5.4, that is, m is comonoidal. Moreover,

$$Z_2 u = (T_0 \otimes Z_2)\partial_{\otimes,\mathbb{1}} = (T_0\mu_\mathbb{1} \otimes \mathrm{id}_{Z^{\otimes 2}})\partial^2_{1_\mathcal{C},1_\mathcal{C},\mathbb{1}}$$
$$= (T_0 T(T_0) \otimes \mathrm{id}_{Z^{\otimes 2}})\partial^2_{1_\mathcal{C},1_\mathcal{C},\mathbb{1}} = u \otimes u$$

and $Z_0 u_\mathbb{1} = (T_0 \otimes Z_0)\partial_{\mathbb{1},\mathbb{1}} = T_0\eta_\mathbb{1} = \mathrm{id}_\mathbb{1}$. Hence u is comonoidal.

   Parts (d) and (e) can be proved in a similar way, but we will instead deduce them in Section 5.8 from the next theorem, Theorem 5.12. $\qquad\square$

5.5. Categorical center relative to a Hopf monad.
Let T be a comonoidal endofunctor of a monoidal category $\mathcal{C}$. The *center of $\mathcal{C}$ relative to T*, or for short the *T-center of $\mathcal{C}$*, is the category $\mathcal{Z}_T(\mathcal{C})$ defined as follows: objects are pairs (M,σ), where M is an object of $\mathcal{C}$ and $\sigma\colon M \otimes 1_\mathcal{C} \to T \otimes M$ is a natural transformation, such that
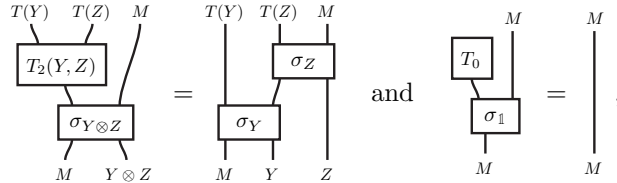
that is,

$$(T_2(Y,Z) \otimes \mathrm{id}_M)\sigma_{Y \otimes Z} = (\mathrm{id}_{T(Y)} \otimes \sigma_Z)(\sigma_Y \otimes \mathrm{id}_Z) \quad \text{for all } Y,Z \in \mathrm{Ob}(\mathcal{C}),$$
$$(T_0 \otimes \mathrm{id}_M)\sigma_\mathbb{1} = \mathrm{id}_M.$$

A morphism $f\colon (M,\sigma) \to (M',\sigma')$ is a morphism $f\colon M \to M'$ in $\mathcal{C}$ such that

$$(\mathrm{id}_{T(Y)} \otimes f)\sigma_Y = \sigma'_Y(f \otimes \mathrm{id}_Y)$$

for every object Y of $\mathcal{C}$. The composition and identities are inherited from $\mathcal{C}$.

   Let $\mathcal{U}_T\colon \mathcal{Z}_T(\mathcal{C}) \to \mathcal{C}$ be the forgetful functor defined by

$$\mathcal{U}_T(M,\sigma) = M \quad \text{and} \quad \mathcal{U}_T(f) = f.$$

If $\mathcal{C}$ is autonomous and T is a Hopf monad, then $\mathcal{Z}_T(\mathcal{C})$ is autonomous. More precisely:

Proposition 5.9. *Let (T,μ,η) be a bimonad on a monoidal category $\mathcal{C}$. Then $\mathcal{Z}_T(\mathcal{C})$ is monoidal, with monoidal product $(M,\sigma) \otimes (N,\gamma) = (M \otimes N, \rho)$ and unit object $(\mathbb{1},\eta)$, where*

$$\rho_Y = (\mu_Y \otimes \mathrm{id}_{M \otimes N})(\sigma_{T(Y)} \otimes \mathrm{id}_N)(\mathrm{id}_M \otimes \gamma_Y) = $$

and the forgetful functor $\mathcal{U}_T\colon \mathcal{Z}_T(\mathcal{C}) \to \mathcal{C}$ is strict monoidal. Now assume $\mathcal{C}$ is autonomous. If T has a right antipode s^r, then $\mathcal{Z}_T(\mathcal{C})$ is left autonomous and left

duals are given by $^\vee(M,\sigma) = (^\vee M, \sigma^l)$, *where*

$$\sigma^l_Y = {}^\vee\big((s^r_Y \otimes \mathrm{id}_M)\sigma_{T(Y)^\vee}\big) = $$

with left evaluation ev_M *and left coevaluation* coev_M. *If T has a left antipode s^l, then $\mathcal{Z}_T(\mathcal{C})$ is right autonomous and right duals are given by* $(M,\sigma)^\vee = (M^\vee, \sigma^r)$, *where*

$$\sigma^r_Y = \big((s^l_Y \otimes \mathrm{id}_M)\sigma_{{}^\vee T(Y)}\big)^\vee = $$

with right evaluation $\widetilde{\mathrm{ev}}_M$ *and right coevaluation* $\widetilde{\mathrm{coev}}_M$. *In particular, if T is a Hopf monad, then the category $\mathcal{Z}_T(\mathcal{C})$ is autonomous.*

Proof. The fact that $(M \otimes N, \rho)$ and $(\mathbb{1}, \eta)$ are objects of $\mathcal{Z}_T(\mathcal{C})$ results from the comonoidality of μ and η. The associativity and unitarity of the monoidal product of $\mathcal{Z}_T(\mathcal{C})$ follows from the associativity of μ and unitarity of η, and its functoriality from the definition of morphisms of $\mathcal{Z}_T(\mathcal{C})$.

Assume $\mathcal{C}$ is autonomous and T has a right antipode s^r. The fact that $(^\vee M, \sigma^l)$ is an object of $\mathcal{Z}_T(\mathcal{C})$ results from the anti-comultiplicativity of s^r (see Section 2.3). The axioms of a right antipode imply that the left evaluation ev_M and left coevaluation coev_M are morphisms of $\mathcal{Z}_T(\mathcal{C})$. Thus $(^\vee M, \sigma^l)$ is a left dual of (M, σ). Likewise, if T has a left antipode, $(M^\vee, \sigma^r)$ is a right dual of (M, σ). $\qquad\square$

Remark 5.10. If $\mathcal{C}$ is an autonomous category, then $\mathcal{Z}_{1_{\mathcal{C}}}(\mathcal{C})$ coincides with the usual center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$ (see Section 1.10).

Remark 5.11. The definition of the category $\mathcal{Z}_T(\mathcal{C})$ is not left/right symmetric. One may also consider the category $\mathcal{Z}'_T(\mathcal{C}) = \mathcal{Z}_{T^{\mathrm{cop}}}(\mathcal{C}^{\otimes\mathrm{op}})^{\otimes\mathrm{op}}$ whose objects are pairs (M, σ), where M is an object of $\mathcal{C}$ and $\sigma\colon 1_{\mathcal{C}} \otimes M \to M \otimes T$ is a natural transformation satisfying the obvious conditions. If $\mathcal{C}$ is autonomous and T is a Hopf monad, then the category $\mathcal{Z}'_T(\mathcal{C})$ is autonomous and isomorphic to $\mathcal{Z}_T(\mathcal{C})$ via the strict monoidal functor $\mathcal{Z}_T(\mathcal{C}) \to \mathcal{Z}'_T(\mathcal{C})$ defined by $(M, \sigma) \mapsto (M, \sigma')$, where

$$\sigma'_Y = (\widetilde{\mathrm{ev}}_Y(\mathrm{id}_Y \otimes s^r_Y) \otimes \mathrm{id}_{M \otimes T(Y)})(\mathrm{id}_Y \otimes \sigma_{T(Y)^\vee} \otimes \mathrm{id}_{T(Y)})(\mathrm{id}_{Y \otimes M} \otimes \widetilde{\mathrm{coev}}_{T(Y)}).$$

In particular, $\mathcal{Z}'_{1_{\mathcal{C}}}(\mathcal{C}) = \mathcal{Z}'(\mathcal{C})$, see Remark 1.2.

5.6. **Monadicity of centers.** In this section, we show that the center relative to a centralizable Hopf monad is monoidally equivalent to the category of modules of its centralizer.

Theorem 5.12. *Let T be a centralizable comonoidal endofunctor of a left autonomous category $\mathcal{C}$, with centralizer (Z_T, ∂). The functor $E\colon Z_T\text{-}\mathcal{C} \to \mathcal{Z}_T(\mathcal{C})$, defined by*

$$E(M, r) = \big(M, (\mathrm{id}_T \otimes r)\partial_{M,1_{\mathcal{C}}}\big) \quad and \quad E(f) = f,$$

is an isomorphism of categories such that the following triangle commutes:

$$
\begin{array}{ccc}
Z_T\text{-}\mathcal{C} & \xrightarrow{\quad E \quad} & \mathcal{Z}_T(\mathcal{C}) \\
& \circlearrowleft & \\
\scriptstyle{U_{Z_T}} \searrow & & \swarrow \scriptstyle{\mathcal{U}_T} \\
& \mathcal{C} &
\end{array}
$$

Furthermore, if T is a bimonad (which implies that Z_T is a bimonad and $\mathcal{Z}_T(\mathcal{C})$ is monoidal), then E is strict monoidal (and so $\mathcal{U}_T E = U_{Z_T}$ as monoidal functors).

We prove Theorem 5.12 in Section 5.7.

Remark 5.13. The functor $\mathcal{F}_T = E F_{Z_T} \colon \mathcal{C} \to \mathcal{Z}_T(\mathcal{C})$ is left adjoint to $\mathcal{U}_T$ and the adjunction $(\mathcal{F}_T, \mathcal{U}_T)$ is monadic with monad Z_T (see Remark 3.1). If T is a bimonad, this adjunction is comonoidal and Z_T is its associated bimonad (see Theorem 3.3).

A monoidal category $\mathcal{C}$ is said to be *centralizable* if its identity endofunctor $1_{\mathcal{C}}$ is centralizable. In such a case, the centralizer of $1_{\mathcal{C}}$ is called the *centralizer* of $\mathcal{C}$. In view of Remark 5.10, we have:

Corollary 5.14. *Let $\mathcal{C}$ be a centralizable autonomous category with centralizer (Z, ∂). Then the forgetful functor $\mathcal{U} \colon \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ is monadic with monad Z. In fact Z is a Hopf monad and the functor $Z\text{-}\mathcal{C} \to \mathcal{Z}(\mathcal{C})$, defined by*

$$
(M, r) \mapsto \big(M, (\mathrm{id}_{1_{\mathcal{C}}} \otimes r)\partial_{M,1_{\mathcal{C}}}\big) \quad \text{and} \quad f \mapsto f,
$$

is a strict monoidal isomorphism of categories.

Remark 5.15. The monadicity assertion of Corollary 5.14 is a consequence of [DS07, Theorem 4.3].

Remark 5.16. We will see in Section 6.2 that $R = (u \otimes \mathrm{id})\partial$ is an R-matrix for Z (where u denotes the unit of Z), making the isomorphism of Corollary 5.14 an isomorphism of braided categories.

5.7. **Proof of Theorem 5.12.** Throughout this section, let T be a centralizable endofunctor of a left autonomous category $\mathcal{C}$, with centralizer (Z_T, ∂). Recall that Z_T is a monad by Theorem 5.6(a). Denote by m and u its product and unit.

Remark first that, by Lemma 5.4, for any object M of $\mathcal{C}$, we have a bijection:

$$
\left\{
\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}}(Z_T(M), M) & \to & \mathrm{HOM}(M \otimes 1_{\mathcal{C}}, T \otimes M) \\
r & \mapsto & \sigma_{(M,r)} = \{(\mathrm{id}_{T(Y)} \otimes r)\partial_{M,Y}\}_{Y \in \mathrm{Ob}(\mathcal{C})}
\end{array}
\right. .
$$

Lemma 5.17. *Let M be an object of $\mathcal{C}$ and $r \colon Z_T(M) \to M$ be a morphism in $\mathcal{C}$. Then (M, r) is a Z_T-module if and only if $(M, \sigma_{(M,r)})$ is an object of $\mathcal{Z}_T(\mathcal{C})$.*

Proof. By definition of the multiplication m of Z_T, we have

$$
(T_2 \otimes \mathrm{id}_M)(\sigma_{(M,r)})_{\otimes} = (T_2 \otimes r)\partial_{M,\otimes} = (\mathrm{id}_{T^{\otimes 2}} \otimes rm_M)\partial^2_{M,1_{\mathcal{C}},1_{\mathcal{C}}}.
$$

Moreover,

$$
(\mathrm{id}_T \otimes \sigma_{(M,r)})(\sigma_{(M,r)} \otimes \mathrm{id}_{1_{\mathcal{C}}}) = \big(\mathrm{id}_{T^{\otimes 2}} \otimes rZ_T(r)\big)\partial^2_{M,1_{\mathcal{C}},1_{\mathcal{C}}}.
$$

Therefore, by Lemma 5.4, $(T_2 \otimes \mathrm{id}_M)(\sigma_{(M,r)})_{\otimes} = (\mathrm{id}_T \otimes \sigma_{(M,r)})(\sigma_{(M,r)} \otimes \mathrm{id}_{1_{\mathcal{C}}})$ if and only if $rm_M = rZ_T(r)$. Also, since $(T_0 \otimes \mathrm{id}_M)(\sigma_{(M,r)})_{\mathbb{1}} = (T_0 \otimes r)\partial_{M,\mathbb{1}} = ru_M$, we have $(T_0 \otimes \mathrm{id}_M)(\sigma_{(M,r)})_{\mathbb{1}} = \mathrm{id}_M$ if and only if $ru_M = \mathrm{id}_M$. $\square$

Lemma 5.18. *Let (M, r) and (N, s) be two Z_T-modules. Let $f\colon M \to N$ be a morphism in $\mathcal{C}$. Then f is Z_T-linear if and only if it is a morphism from $(M, \sigma_{(M,r)})$ to $(N, \sigma_{(N,s)})$ in $\mathcal{Z}_T(\mathcal{C})$.*

Proof. We have $(\mathrm{id}_T \otimes f)\sigma_{(M,r)} = (\mathrm{id}_T \otimes fr)\partial_{M,1_c}$ and

$$\sigma_{(N,s)}(f \otimes \mathrm{id}_T) = (\mathrm{id}_T \otimes s)\partial_{N,1_c}(f \otimes \mathrm{id}_{1_c}) = (\mathrm{id}_T \otimes sZ_T(f))\partial_{M,1_c}.$$

Therefore, by Lemma 5.4, we obtain $(\mathrm{id}_T \otimes f)\sigma_{(M,r)} = \sigma_{(N,s)}(f \otimes \mathrm{id}_T)$ if and only if $fr = sZ_T(f)$. $\qquad\qquad\square$

Using Lemmas 5.17 and 5.18, one sees that the functor $E\colon Z_T\text{-}\mathcal{C} \to \mathcal{Z}_T(\mathcal{C})$, given by $E(M, r) = (M, \sigma_{(M,r)})$ and $E(f) = f$, is a well-defined isomorphism of categories. Furthermore it clearly satisfies $\mathcal{U}_T E = U_{Z_T}$.

Assume now that (T, μ, η) is a bimonad. Then Z_T is a bimonad by Theorem 5.6(c) and the category $\mathcal{Z}_T(\mathcal{C})$ is monoidal by Proposition 5.9. Since, for any Z_T-modules (M, r) and (N, s), we have

$$\begin{aligned}
E(M, r) \otimes E(N, s) &= (M, \sigma_{(M,r)}) \otimes (N, \sigma_{(N,s)}) \\
&= \big(M \otimes N, (\mu \otimes r \otimes s)(\partial_{M,T} \otimes \mathrm{id}_{Z_T(N)})(\mathrm{id}_M \otimes \partial_{N,1_c})\big) \\
&= \big(M \otimes N, (\mathrm{id}_T \otimes (r \otimes s)(Z_T)_2(M, N))\partial_{M \otimes N,1_c}\big) \\
&= E\big((M, r) \otimes (N, s)\big)
\end{aligned}$$

and $E\big(\mathbb{1}, (Z_T)_0\big) = \big(\mathbb{1}, (\mathrm{id}_T \otimes (Z_T)_0)\partial_{\mathbb{1},1_c}\big) = (\mathbb{1}, \eta)$, the functor E is strict monoidal. Finally, we have $\mathcal{U}_T E = U_{Z_T}$ as monoidal functors because the forgetful functors $U_{Z_T}\colon Z_T\text{-}\mathcal{C} \to \mathcal{C}$ and $\mathcal{U}_T\colon \mathcal{Z}_T(\mathcal{C}) \to \mathcal{C}$ are strict monoidal.

5.8. **End of the proof of Theorem 5.6.** Let us prove part (d) of Theorem 5.6. Let (T, μ, η) be a centralizable bimonad on an autonomous category $\mathcal{C}$, with centralizer (Z_T, ∂). By Theorem 5.6(c), Z_T is a bimonad. By Theorem 5.12, the functor $E\colon Z_T\text{-}\mathcal{C} \to \mathcal{Z}_T(\mathcal{C})$, defined by

$$E(M, r) = \big(M, (\mathrm{id}_T \otimes r)\partial_{M,1_c}\big) \quad \text{and} \quad E(f) = f,$$

is a strict monoidal isomorphism. Assume T admits a right antipode s^r. Then the category $\mathcal{Z}_T(\mathcal{C})$ is left autonomous by Proposition 5.9. Hence the category $Z_T\text{-}\mathcal{C}$ is left autonomous, and so Z_T admits a left antipode by Theorem 2.7(b). Denote by m the product of Z_T, u its unit, and S^l its right antipode. Let X be an object of $\mathcal{C}$. In the category $Z_T\text{-}\mathcal{C}$, we have a duality

$$\big({}^{\vee}(Z_T(X), m_X), (Z_T(X), m_X), \mathrm{ev}_{Z_T(X)}, \mathrm{coev}_{Z_T(X)}\big),$$

where ${}^{\vee}(Z_T(X), m_X) = \big({}^{\vee}Z_T(X), S^l_{Z_T(X)}Z_T({}^{\vee}m_T)\big)$. Hence, E being strict monoidal, we have a duality in the category $\mathcal{Z}_T(\mathcal{C})$:

$$\big(E({}^{\vee}(Z_T(X), m_X)), E(Z_T(X), m_X), \mathrm{ev}_{Z_T(X)}, \mathrm{coev}_{Z_T(X)}\big),$$

where $E\big({}^{\vee}(Z_T(X), m_X)\big) = \big({}^{\vee}Z_T(X), (\mathrm{id}_T \otimes S^l_{Z_T(X)}Z_T({}^{\vee}m_T))\partial_{{}^{\vee}Z_T(X),1_c}\big)$. Now, by Proposition 5.9, we also have the following duality in $\mathcal{Z}_T(\mathcal{C})$:

$$\big({}^{\vee}E(Z_T(X), m_X), E(Z_T(X), m_X), \mathrm{ev}_{Z_T(X)}, \mathrm{coev}_{Z_T(X)}\big),$$

where ${}^{\vee}E(Z_T(X), m_X) = \big({}^{\vee}Z_T(X), {}^{\vee}((s^r \otimes m_X)\partial_{Z_T(X),T^{\vee}})\big)$. Hence, by uniqueness of duals up to unique isomorphism,

$$\big(\mathrm{id}_T \otimes S^l_{Z_T(X)}Z_T({}^{\vee}m_T)\big)\partial_{{}^{\vee}Z_T(X),1_c} = {}^{\vee}\big((s^r \otimes m_X)\partial_{Z_T(X),T^{\vee}}\big).$$

Composing on the left with $(\mathrm{id}_T \otimes {}^\vee u_X) = {}^\vee(u_X \otimes \mathrm{id}_{T^\vee})$, we get

$$(\mathrm{id}_T \otimes S_X^l)\partial_{{}^\vee Z_T(X),1_\mathcal{C}} = {}^\vee\big((s^r \otimes \mathrm{id}_{Z_T(X)})\partial_{X,T^\vee}\big),$$

which is the defining relation of Theorem 5.6(d). Hence we have proved part (d) of Theorem 5.6. Part (e) can be shown similarly.

6. The double of a Hopf monad

Given a centralizable Hopf monad T on an autonomous category $\mathcal{C}$, we construct the canonical distributive law Ω of T over its centralizer Z_T, which serves two purposes. First, Ω gives rise to a new Hopf monad $D_T = Z_T \circ_\Omega T$, called the double of T. The double D_T is actually quasitriangular and $Z(T\text{-}\mathcal{C}) \simeq D_T\text{-}\mathcal{C}$ as braided categories, see Section 6.2. Second, Ω defines a lift of the Hopf monad Z_T to a Hopf monad $\tilde{Z}_T^\Omega$ on T-$\mathcal{C}$, which turns out to be the centralizer of the category T-$\mathcal{C}$, and so $\tilde{Z}_T^\Omega(\mathbb{1}, T_0)$ is the coend of T-$\mathcal{C}$, see Section 6.3.

Most of the results of this section are special cases of results of Section 7. We state them here for convenience.

6.1. **The canonical distributive law.** Let T be a centralizable Hopf monad on an autonomous category $\mathcal{C}$ and (Z_T, ∂) be its centralizer.

Recall (see Proposition 5.3) that $Z_T(X) = \int^{Y \in \mathcal{C}} {}^\vee T(Y) \otimes X \otimes Y$, with universal dinatural transformation

$$i_{X,Y} = (\mathrm{coev}_{T(Y)} \otimes \mathrm{id}_{Z_T(X)})(\mathrm{id}_{{}^\vee T(X)} \otimes \partial_{X,Y}),$$

which is natural in X and dinatural in Y. Since $T(i)$ is a universal dinatural transformation (see Section 3.5), we can define a natural transformation $\Omega \colon TZ_T \to Z_T T$ by

$$\Omega_X T(i_{X,Y}) = i_{T(X),T(Y)}\big({}^\vee\mu_Y s_{T(Y)}^l T({}^\vee\mu_Y) \otimes \mathrm{id}_{T(X)\otimes T(Y)}\big)T_3\big({}^\vee T(Y), X, Y\big),$$

where μ and s^l are the product and left antipode of T and where $T_3 \colon T\otimes_3 \to T^{\otimes 3}$ is defined as in Section 1.5.

Theorem 6.1. *The natural transformation $\Omega \colon TZ_T \to Z_T T$ is an invertible comonoidal distributive law.*

We call Ω the *canonical distributive law of T*. We prove Theorem 6.1 in Section 7.4.

The inverse $\Omega^{-1} \colon Z_T T \to TZ_T$ of the distributive law Ω is the natural transformation defined by

$$\Omega_X^{-1} i_{T(X),Y} = \big(\mathrm{ev}_{T(Y)}(\mathrm{id}_{{}^\vee T(Y)} \otimes \mu_Y T(\mu_Y)) \otimes T(i_{X,T(Y)}) \otimes \mathrm{ev}_Y(s_Y^l \otimes \mathrm{id}_Y)\big)$$
$$\circ T_3(T^2(Y), {}^\vee T^2(Y) \otimes X \otimes T(Y), {}^\vee T(Y))T(\mathrm{coev}_{T^2(Y)} \otimes \mathrm{id}_X \otimes \mathrm{coev}_{T(Y)}).$$

Remark 6.2. The canonical distributive law of T is the only natural transformation $\Omega \colon TZ_T \to Z_T T$ satisfying

$$(\mu \otimes \Omega)T_2 T(\partial) = (\mu \otimes \mathrm{id}_{Z_T T})\partial_{T,T}T_2.$$

Remark 6.3. One can show that R-matrices for T correspond bijectively with morphisms of Hopf monads $f \colon Z_T \to T$ satisfying $\mu T(f) = \mu f_T \Omega$. The R-matrix associated with such a morphism f is $R = (\mathrm{id}_T \otimes f)\partial$.

6.2. **The double of a Hopf monad.** Let T be a centralizable Hopf monad on an autonomous category $\mathcal{C}$, with centralizer (Z_T, ∂). Let $\Omega\colon TZ_T \to Z_T T$ be the canonical distributive law of T. By Corollary 4.11,

$$D_T = Z_T \circ_\Omega T$$

is a Hopf monad on $\mathcal{C}$. Denote by η and u the units of T and Z_T, respectively.

Theorem 6.4. *The natural transformation $R = \{R_{X,Y}\}_{X,Y \in \mathrm{Ob}(\mathcal{C})}$, defined by*

$$R_{X,Y} = \big(u_{T(Y)} \otimes Z_T(\eta_X)\big)\partial_{X,Y}\colon X \otimes Y \to D_T(Y) \otimes D_T(X),$$

is an R-matrix for the Hopf monad D_T.

The quasitriangular Hopf monad D_T is called the *double of T*. This terminology is justified by the fact that the braided categories $Z(T\text{-}\mathcal{C})$ and $D_T\text{-}\mathcal{C}$ coincide. More precisely, let $\mathcal{U}\colon \mathcal{Z}(T\text{-}\mathcal{C}) \to \mathcal{C}$ be the strict monoidal forgetful functor defined by

$$\mathcal{U}\big((M,r),\sigma\big) = M \quad \text{and} \quad \mathcal{U}(f) = f.$$

Let $I\colon D_T\text{-}\mathcal{C} \to \mathcal{Z}(T\text{-}\mathcal{C})$ be the functor defined by $I(f) = f$ and

$$I(M,r) = \big((M, ru_{T(M)}),\sigma\big) \quad \text{with} \quad \sigma_{(N,s)} = (s \otimes rZ_T(\eta_M))\partial_{M,N}.$$

Theorem 6.5. *The functor I is a strict monoidal isomorphism of braided categories such that the following triangle of monoidal functors commutes:*

$$
\begin{array}{ccc}
D_T\text{-}\mathcal{C} & \xrightarrow{\quad I \quad} & \mathcal{Z}(T\text{-}\mathcal{C}) \\
& \circlearrowleft & \\
{\scriptstyle U_{D_T}} \searrow & & \swarrow {\scriptstyle \mathcal{U}} \\
& \mathcal{C} &
\end{array}
$$

We prove Theorems 6.4 and 6.5 in Section 7.5.

Remark 6.6. The functor $\mathcal{F} = IF_{D_T}\colon \mathcal{C} \to \mathcal{Z}(T\text{-}\mathcal{C})$ is left adjoint to $\mathcal{U}$, and the adjunction $(\mathcal{F},\mathcal{U})$ is monadic with monad D_T (see Remark 3.1). Moreover, D_T is the Hopf monad associated with this comonoidal adjunction (see Theorem 3.3).

Remark 6.7. According to Remark 5.1, the construction of the double of a Hopf monad T admits a 'right-handed' version: if Z_T' is a 'right-handed' centralizer of T, there exists a 'right-handed' canonical law Ω' of T over Z_T', and hence a Hopf monad $D_T' = Z_T' \circ_{\Omega'} T$ endowed with an R-matrix R' such that $D_T'\text{-}\mathcal{C} \simeq \mathcal{Z}'(T\text{-}\mathcal{C})$ as braided categories. If we identify Z_T' to Z_T as in Remark 5.8, then $\Omega' = \Omega$, $D_T' = D_T$ as Hopf monads, and $R' = R^{*-1}$.

Remark 6.8. Let T be a centralizable Hopf monad on an autonomous category $\mathcal{C}$ and (Z_T, ∂) be its centralizer. Denote by η and u the units of T and Z_T, respectively. Assuming $u_T\colon T \to Z_T T$ is a monomorphism, one can show that the canonical distributive law of T is the only comonoidal distributive law $\Omega\colon TZ_T \to Z_T T$ such that

$$R = \big(u_T \otimes Z_T(\eta)\big)\partial$$

is an R-matrix for the Hopf monad $Z_T \circ_\Omega T$. This generalizes Drinfeld's original characterization of the double of a finite-dimensional Hopf algebra.

6.3. **The centralizer and the coend of a category of modules.** Let T be a centralizable Hopf monad on an autonomous category $\mathcal{C}$. Let (Z_T, ∂) be the centralizer of T and $\Omega \colon TZ_T \to Z_T T$ be the canonical distributive law of T. By Corollary 4.11, $\tilde{Z}_T^\Omega$ is Hopf monad which is a lift of the Hopf monad Z_T to $T\text{-}\mathcal{C}$. Recall that

$$\tilde{Z}_T^\Omega(M, r) = (Z_T(M), Z_T(r)\Omega_M) \quad \text{and} \quad \tilde{Z}_T^\Omega(f) = Z_T(f).$$

For any T-modules (M, r) and (N, s), set

$$\tilde{\partial}_{(M,r),(N,s)} = (s \otimes \mathrm{id}_{Z_T(M)})\partial_{M,N} \colon (M, r) \otimes (N, s) \to (N, s) \otimes \tilde{Z}_T^\Omega(M, r).$$

Theorem 6.9. *The pair $(\tilde{Z}_T^\Omega, \tilde{\partial})$ is a centralizer of the category $T\text{-}\mathcal{C}$.*

We prove Theorem 6.9 in Section 7.6.
Recall that

$$Z_T(\mathbb{1}) = \int^{Y \in \mathcal{C}} {}^\vee T(Y) \otimes Y$$

with universal dinatural transformation $i_Y = (\mathrm{ev}_{T(Y)} \otimes \mathrm{id}_{Z_T(\mathbb{1})})\partial_{\mathbb{1},Y}$. Denote by $\alpha = Z_T(T_0)\Omega_{\mathbb{1}}$ the T-action of $\tilde{Z}_T^\Omega(\mathbb{1}, T_0)$. It is characterized by

$$\alpha T(i_Y) = i_{T(Y)}\big({}^\vee\mu_Y s^l_{T(Y)} T({}^\vee\mu_Y) \otimes \mathrm{id}_{T(Y)}\big)T_2\big({}^\vee T(Y), Y\big).$$

By Theorem 6.9 and Proposition 5.3, $\tilde{Z}_T^\Omega(\mathbb{1}, T_0) = (Z_T(\mathbb{1}), \alpha)$ is the coend of $T\text{-}\mathcal{C}$. That is,

$$(Z_T(\mathbb{1}), \alpha) = \int^{(M,r) \in T\text{-}\mathcal{C}} {}^\vee(M, r) \otimes (M, r),$$

with universal dinatural transformation $\tilde{i}_{(M,r)} = i_M({}^\vee r \otimes M)$.

The coend $(Z_T(\mathbb{1}), \alpha)$ of $T\text{-}\mathcal{C}$ is a coalgebra in $T\text{-}\mathcal{C}$, with coproduct and counit given by

$$\Delta = (Z_T)_2(\mathbb{1}, \mathbb{1}) \colon Z_T(\mathbb{1}) \to Z_T(\mathbb{1}) \otimes Z_T(\mathbb{1}) \quad \text{and} \quad \varepsilon = (Z_T)_0 \colon Z_T(\mathbb{1}) \to \mathbb{1}.$$

Assume now that T is furthermore quasitriangular, with R-matrix R, so that the autonomous category $T\text{-}\mathcal{C}$ is braided. Then the coalgebra $\big((Z_T(\mathbb{1}), \alpha), \Delta, \varepsilon\big)$ becomes a Hopf algebra in $T\text{-}\mathcal{C}$ endowed with a self-dual Hopf pairing (see Section 8.3). Its unit is

$$u = (T_0 \otimes \mathrm{id}_{Z_T(\mathbb{1})})\partial_{\mathbb{1},\mathbb{1}} \colon \mathbb{1} \to Z_T(\mathbb{1}).$$

Its product m, antipode S, and Hopf pairing ω are given in Figure 2.

Remark 6.10. In Sectgion 9.3, we treat the case of the centralizer of a fusion category $\mathcal{F}$ (which is a quasitriangular Hopf monad by Theorem 6.5) to get a convenient description of the coend of $\mathcal{Z}(\mathcal{F})$.
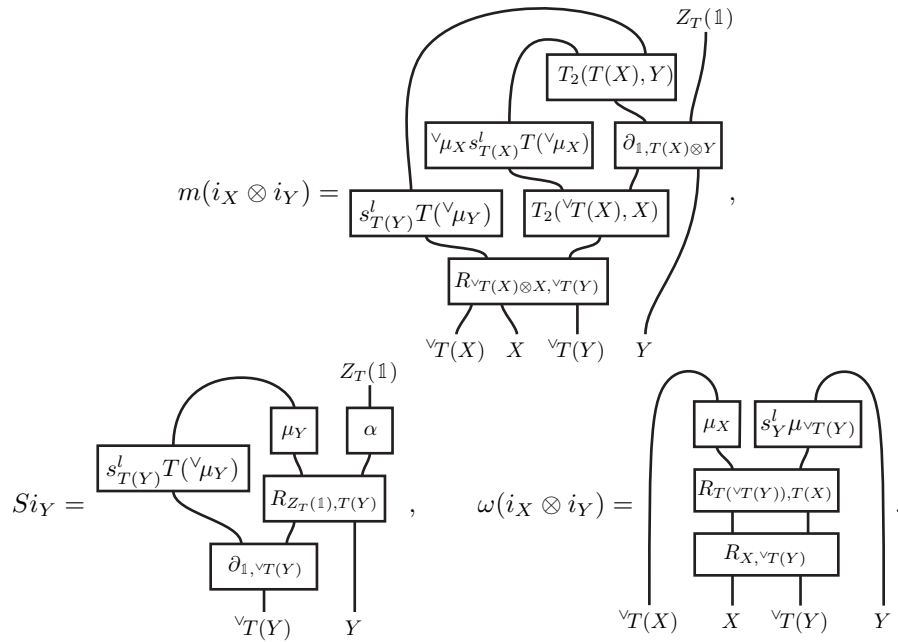
FIGURE 2. Hopf algebra structure of the coend of $T\text{-}\mathcal{C}$

7. THE CENTRALIZER OF A HOPF MONAD ON A CATEGORY OF MODULES

In this section, we study the centralizer of a Hopf monad Q on the category $T\text{-}\mathcal{C}$ of modules over a Hopf monad T on an autonomous category $\mathcal{C}$. We show that it is centralizable whenever the cross product $Q \rtimes T$ is centralizable. In that case, the centralizer of $Q \rtimes T$ lifts naturally to a centralizer of Q, which also turns out to be a lift of Hopf monads. Hence we have a canonical distributive law Ω of T over $Z_{Q \rtimes T}$ and a Hopf monad $D_{Q,T} = Z_{Q \rtimes T} \circ_\Omega T$ on $\mathcal{C}$. We interpret the category of $D_{Q,T}$ modules as the center of $T\text{-}\mathcal{C}$ relative to Q.

7.1. Centralizability on categories of modules. In this section, given a Hopf monad T on an autonomous category $\mathcal{C}$, we give a criterion for an endofunctor Q of $T\text{-}\mathcal{C}$ to be centralizable in terms of the centralizability of the cross product $Q \rtimes T$ on $\mathcal{C}$ (see Section 3.7 for the definition of cross products).

Proposition 7.1. *Let T be a Hopf monad on an autonomous category $\mathcal{C}$ and let Q be an endofunctor of $T\text{-}\mathcal{C}$. Let (M, r) be a T-module. Then:*

(a) *The endofunctor Q is centralizable at (M, r) if and only if $Q \rtimes T$ is centralizable at $U_T(M, r) = M$.*

(b) *Assume $Q \rtimes T$ is centralizable at M, with centralizer (Z, δ). Then Q admits a unique centralizer $(\tilde{Z}, \tilde{\delta})$ at (M, r) such that*

$$U_T(\tilde{Z}) = Z \quad and \quad \tilde{\delta}_{(N,s)} = (Q(s) \otimes \mathrm{id}_Z)\delta_N$$

for any T-module (N, s).

Remark 7.2. In the second formula of Proposition 7.1(b), $Q(s)$ makes sense because $s\colon (T(N), \mu_N) \to (N, s)$ is a morphism in T-$\mathcal{C}$. This formula can be written as

$$\tilde{\delta} = (Q(\varepsilon) \otimes \mathrm{id}_Z)\delta,$$

where ε denotes the counit of the adjunction (U_T, F_T).

Proof of Proposition 7.1. Let us prove part (a). Fix a T-module (M, r). By Proposition 5.3, Q is centralizable at (M, r) if and only if the coend

$$\int^{(N,s)\in T\text{-}\mathcal{C}} {}^\vee Q(N, s) \otimes (M, r) \otimes (N, s)$$

exists. Since the functor U_T creates and preserves coends (see Section 3.5) and is strict monoidal, this is equivalent to the existence of the coend

$$\int^{(N,s)\in T\text{-}\mathcal{C}} U_T\big({}^\vee Q(N, s) \otimes (M, r) \otimes (N, s)\big)$$

$$= \int^{(N,s)\in T\text{-}\mathcal{C}} {}^\vee U_T Q(N, s) \otimes M \otimes U_T(N, s).$$

By Lemma 3.9, this is equivalent to the existence of the coend

$$\int^{Y\in\mathcal{C}} {}^\vee U_T Q F_T(Y) \otimes M \otimes Y = \int^{Y\in\mathcal{C}} {}^\vee Q \rtimes T(Y) \otimes M \otimes Y,$$

and so, by Proposition 5.3, to the fact that $Q \rtimes T$ is centralizable at M.

Let us prove part (b). By Proposition 5.3, we have

$$Z = \int^{Y\in\mathcal{C}} {}^\vee Q \rtimes T(Y) \otimes M \otimes Y,$$

with universal dinatural transformation $i_Y = (\mathrm{ev}_{Q\rtimes T(Y)} \otimes \mathrm{id}_Z)(\mathrm{id}_{{}^\vee Q\rtimes T(Y)} \otimes \delta_Y)$. By Lemma 3.9, we also have

$$Z = \int^{(N,s)\in T\text{-}\mathcal{C}} {}^\vee U_T Q(N, s) \otimes M \otimes U_T(N, s)$$

with universal dinatural transformation $j_{(N,s)} = i_N({}^\vee U_T Q(s) \otimes \mathrm{id}_{M\otimes N})$. Set

$$\tilde{\delta}_{(N,s)} = (\mathrm{id}_{Q(N,s)} \otimes j_{(N,s)})(\mathrm{coev}_{Q(N,s)} \otimes \mathrm{id}_M).$$

By Proposition 3.10, there exists a unique T-action $\alpha\colon T(Z) \to Z$ such that $j_{(N,s)}$, or equivalently $\tilde{\delta}_{(N,s)}$, is T-linear for all T-modules (N, s). Furthermore we have

$$(Z, \alpha) = \int^{(N,s)\in T\text{-}\mathcal{C}} {}^\vee Q(N, s) \otimes (M, r) \otimes (N, s)$$

with universal dinatural transformation j. Set $\tilde{Z} = (Z, \alpha)$. By Proposition 5.3, $(\tilde{Z}, \tilde{\delta})$ is a centralizer of Q at (M, r). By construction, we have $U_T(\tilde{Z}) = Z$ and $\tilde{\delta}_{(N,s)} = (U_T Q(s) \otimes \mathrm{id}_Z)\delta_N$ for every T-module (N, s). Furthermore, since α is the only action of T on $Z = U_T(\tilde{Z})$ such that every $\tilde{\delta}_{(N,s)}$ is T-linear, $(\tilde{Z}, \tilde{\delta})$ is the only centralizer of Q at (M, r) satisfying the conditions of part (b). □

Applying Lemma 5.2 and Proposition 7.1(a), we immediately deduce:

Corollary 7.3. *Let T be a Hopf monad on an autonomous category $\mathcal{C}$ and let Q be an endofunctor of T-$\mathcal{C}$. Then Q is centralizable if and only if, for any T-module (M, r), the endofunctor $Q \rtimes T$ of $\mathcal{C}$ is centralizable at M.*

7.2. **Lifting centralizers.** In this section, given a centralizable Hopf monad T on an autonomous category $\mathcal{C}$ and an endofunctor Q of T-$\mathcal{C}$, we show that a centralizer of $Q \rtimes T$ lifts uniquely to a centralizer of Q. Furthermore, if Q is comonoidal (resp. a bimonad), then it is also a lift as a monad (resp. a bimonad).

Theorem 7.4. *Let T be a Hopf monad on an autonomous category $\mathcal{C}$ and let Q be an endofunctor of T-$\mathcal{C}$. Assume $Q \rtimes T$ is centralizable, with centralizer $(Z_{Q \rtimes T}, \partial)$. Then:*

(a) *The centralizer of $Q \rtimes T$ lifts uniquely to a centralizer of Q. More precisely, Q admits a unique centralizer $(Z_Q, \tilde{\partial})$ such that $U_T Z_Q = Z_{Q \rtimes T} U_T$ and*

$$\tilde{\partial}_{(M,r),(N,s)} = (Q(s) \otimes \mathrm{id}_{Z_{Q \rtimes T}(M)}) \partial_{M,N}$$

for all T-modules (M, r) and (N, s).

(b) *If Q is comonoidal, the monad Z_Q is a lift of the monad $Z_{Q \rtimes T}$ to T-$\mathcal{C}$.*

(c) *If Q is a bimonad, the bimonad Z_Q is a lift of the bimonad $Z_{Q \rtimes T}$ to T-$\mathcal{C}$.*

Proof. Part (a) is a direct consequence of Lemma 5.2 and Proposition 7.1(b). Let $(Z_Q, \tilde{\partial})$ be the centralizer of Q given by part (a).

Assume Q is comonoidal. Then $Q \rtimes T$ is comonoidal by Section 3.7. Therefore both Z_Q and $Z_{Q \rtimes T}$ are monads by Theorem 5.6(a). Denote by η and ε the unit and counit of the adjunction (U_T, F_T). By part (a), we have

$$U_T(\tilde{\partial}) = (U_T Q(\varepsilon) \otimes \mathrm{id}_{Z_{Q \rtimes T} U_T}) \partial_{U_T, U_T}.$$

By definition of the product $\tilde{m}$ of Z_Q, we have

$$(\mathrm{id}_{Q F_T \otimes Q F_T} \otimes \tilde{m}) \tilde{\partial}^2_{1_{T\text{-}\mathcal{C}}, F_T, F_T} = \big(Q_2(F_T, F_T) \otimes \mathrm{id}_{Z_Q}\big) \tilde{\partial}_{1_{T\text{-}\mathcal{C}}, F_T \otimes F_T}.$$

Hence, we get

$$\big(U_T Q(\varepsilon_{F_T}) \otimes U_T Q(\varepsilon_{F_T}) \otimes U_T(\tilde{m})\big) \partial^2_{U_T, T, T}$$
$$= \big(U_T \big(Q_2(F_T, F_T) Q(\varepsilon_{F_T \otimes F_T})\big) \otimes \mathrm{id}_{U_T Z_Q}\big) \partial_{U_T, T \otimes T}.$$

Composing on the right with $(\mathrm{id}_{U_T} \otimes \eta \otimes \eta)$ and then using the expression of the comonoidal structure of $Q \rtimes T$ (see Section 3.7) and the identity $\varepsilon_{F_T} F_T(\eta) = \mathrm{id}_{F_T}$, we obtain

$$\big(\mathrm{id}_{(Q \rtimes T)^{\otimes 2}} \otimes U_T(\tilde{m})\big) \partial^2_{U_T, 1_{\mathcal{C}}, 1_{\mathcal{C}}} = \big((Q \rtimes T)_2 \otimes \mathrm{id}_{Z_{Q \rtimes T} U_T}\big) \partial_{U_T, \otimes},$$

and so, by definition of the product m of $Z_{Q \rtimes T}$, we have $U_T(\tilde{m}) = m_{U_T}$. Moreover, denoting by $\tilde{u}$ and u the units of Z_Q and $Z_{Q \rtimes T}$, respectively, we have

$$U_T(\tilde{u}) = U_T\big((Q_0 \otimes \mathrm{id}_{Z_Q}) \tilde{\partial}_{1_{T\text{-}\mathcal{C}}, (\mathbb{1}, T_0)}\big)$$
$$= \big(U_T(Q_0) U_T Q(T_0) \otimes \mathrm{id}_{U_T Z_Q}\big) \partial_{U_T, \mathbb{1}}$$
$$= \big((Q \rtimes T)_0 \otimes \mathrm{id}_{Z_{Q \rtimes T} U_T}\big) \partial_{U_T, \mathbb{1}} = u_{U_T}.$$

Hence we have proved part (b).

Suppose now Q is a bimonad. Then $Q \rtimes T$ is a bimonad (see Section 3.7). Therefore both Z_Q and $Z_{Q \rtimes T}$ are bimonads by Theorem 5.6(c). By definition of the morphism $(Z_Q)_2$, we have

$$\big(\mathrm{id}_{Q F_T} \otimes (Z_Q)_2\big) \tilde{\partial}_{\otimes, F_T} = (q_{F_T} \otimes \mathrm{id}_{Z_Q^{\otimes 2}}) (\tilde{\partial}_{1_{T\text{-}\mathcal{C}}, Q F_T} \otimes \mathrm{id}_{Z_Q}) (\mathrm{id}_{1_{T\text{-}\mathcal{C}}} \otimes \tilde{\partial}_{1_{T\text{-}\mathcal{C}}, F_T}),$$

where q is the product of Q. Thus

$$\big(U_T Q(\varepsilon_{F_T}) \otimes U_T((Z_Q)_2)\big)\partial_{U_T \otimes U_T, T} = \big(U_T(q_{F_T} Q(\varepsilon_{QF_T})) \otimes \mathrm{id}_{(U_T Z_Q)^{\otimes 2}}\big)$$
$$\circ \big(\partial_{U_T, Q \rtimes T} \otimes \mathrm{id}_{U_T Z_Q}\big)\big(\mathrm{id}_{U_T} \otimes (U_T Q(\varepsilon_{F_T}) \otimes \mathrm{id}_{U_T Z_Q})\partial_{U_T, T}\big).$$

Composing on the right with $(\mathrm{id}_{U_T} \otimes \mathrm{id}_{U_T} \otimes \eta)$, since the product of $Q \rtimes T$ is given by $p = U_T\big(q_{F_T} Q(\varepsilon_{QF_T})\big)$, we obtain

$$\big(\mathrm{id}_{Q \rtimes T} \otimes U_T((Z_Q)_2)\big)\partial_{U_T \otimes U_T, 1_\mathcal{C}}$$
$$= \big(p \otimes \mathrm{id}_{(Z_{Q \rtimes T} U_T)^{\otimes 2}}\big)(\partial_{U_T, Q \rtimes T} \otimes \mathrm{id}_{Z_{Q \rtimes T} U_T})(\mathrm{id}_{U_T} \otimes \partial_{U_T, 1_\mathcal{C}}),$$

and so, by definition of the morphism $(Z_{Q \rtimes T})_2$, we obtain $(U_T Z_Q)_2 = (Z_{Q \rtimes T} U_T)_2$. Now, by definition of the morphism $(Z_Q)_0$, we have

$$\big(\mathrm{id}_{QF_T} \otimes (Z_Q)_0\big)\tilde{\partial}_{(\mathbb{1}, T_0), F_T} = v_{F_T}.$$

where v is the unit of Q. Applying U_T and composing with η, we get

$$\big(\mathrm{id}_{Q \rtimes T} \otimes U_T((Z_Q)_0)\big)\partial_{\mathbb{1}, 1_\mathcal{C}} = U_T(v_{F_T})\eta.$$

Since $U_T(v_{F_T})\eta$ is the unit of $Q \rtimes T$ and by definition of the morphism $(Z_{Q \rtimes T})_0$, we have $(U_T Z_Q)_0 = (Z_{Q \rtimes T} U_T)_0$. Hence $U_T Z_Q = Z_{Q \rtimes T} U_T$ as comonoidal functors, and we have proved part (c). $\qquad\square$

7.3. The canonical distributive law and the double.

Throughout this section, let T be a Hopf monad on an autonomous category $\mathcal{C}$ and Q be a comonoidal endofunctor of T-$\mathcal{C}$ such that $Q \rtimes T$ is centralizable with centralizer $(Z_{Q \rtimes T}, \partial)$.

By Theorem 7.4, the centralizer $(Z_{Q \rtimes T}, \partial)$ lifts to a centralizer $(Z_Q, \tilde{\partial})$ of Q and the monad Z_Q is a lift of the monad $Z_{Q \rtimes T}$ to T-$\mathcal{C}$. The monad

$$D_{Q,T} = Z_Q \rtimes T$$

is called the *double of the pair* (Q, T). Since lifts correspond bijectively with distributive laws (see Theorem 4.5), there exists a unique distributive law Ω of T over $Z_{Q \rtimes T}$ such that

$$Z_Q = \tilde{Z}_{Q \rtimes T}^{\Omega}.$$

This distributive law is called the *canonical distributive law of the pair* (Q, T). It provides a description of structure of the monad $D_{Q,T}$:

$$D_{Q,T} = Z_{Q \rtimes T} \circ_\Omega T.$$

Proposition 7.5.   (a) *If Q is a bimonad, then the canonical distributive law Ω is comonoidal, $D_{Q,T}$ is a bimonad, and $Z_Q = \tilde{Z}_{Q \rtimes T}^{\Omega}$ as bimonads.*
   (b) *If Q is a Hopf monad, then $D_{Q,T}$ is a Hopf monad.*

Proof. Let us prove part (a). By Theorem 7.4, Z_Q is a lift of $Z_{Q \rtimes T}$ as a bimonad. Therefore, by Theorem 4.7, Ω is comonoidal and $D_{Q,T}$ is a bimonad.

Let us prove part (b). Since $Q \rtimes T$ is a Hopf monad (see Section 3.7), so is $Z_{Q \rtimes T}$ (by Theorem 5.6). Therefore $D_{Q,T}$ is a Hopf monad (by Corollary 4.11). $\qquad\square$

Let $\mathcal{U}\colon \mathcal{Z}_Q(T\text{-}\mathcal{C}) \to \mathcal{C}$ be the functor defined as the composition of the forgetful functors $\mathcal{U}_Q\colon \mathcal{Z}_Q(T\text{-}\mathcal{C}) \to T\text{-}\mathcal{C}$ and $U_T\colon T\text{-}\mathcal{C} \to \mathcal{C}$, that is,

$$\mathcal{U}\big((M, r), \sigma\big) = M \quad \text{and} \quad \mathcal{U}(f) = f.$$

Denoting η and u the units of T and $Z_{Q \rtimes T}$, let $I \colon D_{Q,T}\text{-}\mathcal{C} \to \mathcal{Z}_Q(T\text{-}\mathcal{C})$ be the functor defined by

$$I(M,r) = \big((M, ru_{T(M)}), \sigma\big) \quad \text{and} \quad I(f) = f,$$

where $\sigma_{(N,s)} = (U_T Q(s) \otimes r Z_{Q \rtimes T}(\eta_M)) \partial_{M,N}$.

Theorem 7.6. *The functor I is an isomorphism of categories such that the following triangle commutes:*

$$
\begin{array}{ccc}
D_{Q,T}\text{-}\mathcal{C} & \xrightarrow{\ \ I\ \ } & \mathcal{Z}_Q(T\text{-}\mathcal{C}) \\
& \circlearrowleft & \\
\scriptstyle{U_{D_{Q,T}}} \searrow & & \swarrow \scriptstyle{\mathcal{U}} \\
& \mathcal{C} &
\end{array}
$$

Furthermore, if Q is a bimonad (so that $D_{Q,T}$ is a bimonad and $\mathcal{Z}_Q(T\text{-}\mathcal{C})$ is monoidal), then the functor I is strict monoidal (and so $\mathcal{U}I = U_{D_{Q,T}}$ as monoidal functors).

Remark 7.7. The functor $\mathcal{F} = IF_{D_{Q,T}} \colon \mathcal{C} \to \mathcal{Z}_Q(T\text{-}\mathcal{C})$ is left adjoint to $\mathcal{U}$ and the adjunction $(\mathcal{F}, \mathcal{U})$ is monadic with monad $D_{Q,T}$ (see Remark 3.1). If Q is a bimonad, this adjunction is comonoidal and $D_{Q,T}$ is its associated bimonad (see Theorem 3.3).

Proof. By Section 4.3, since $D_{Q,T} = Z_{Q \rtimes T} \circ_\Omega T$ and $Z_Q = \tilde{Z}^\Omega_{Q \rtimes T}$, the functor

$$
L \colon \begin{cases} D_{Q,T}\text{-}\mathcal{C} & \longrightarrow & Z_Q\text{-}(T\text{-}\mathcal{C}) \\ (M,r) & \longmapsto & \big((M, ru_{T(M)}), r Z_{Q \rtimes T}(\eta_M)\big) \end{cases}
$$

is an isomorphism of categories. By Theorem 5.12, the functor

$$
E \colon \begin{cases} Z_Q\text{-}(T\text{-}\mathcal{C}) & \longrightarrow & Z_Q(T\text{-}\mathcal{C}) \\ ((M,r),s) & \longmapsto & \big((M,r), (\mathrm{id}_Q \otimes s)\tilde{\partial}_{(M,r),1_{T\text{-}c}}\big) \end{cases}
$$

is an isomorphism of categories. Using Theorem 7.4(a), one verifies that $I = EL$. Thus I is an isomorphism of categories, and it clearly satisfies $\mathcal{U}I = U_{D_{Q,T}}$.

Assume Q is a bimonad. Then L is strict monoidal (by Theorem 4.7) and E is strict monoidal (by Theorem 5.12). Hence $I = EL$ is strict monoidal, and so $\mathcal{U}I = U_{D_{Q,T}}$ as monoidal functors (since $\mathcal{U}$ and $U_{D_{Q,T}}$ are strict monoidal). $\qquad\square$

The canonical distributive law Ω can be described explicitly as follows. By Proposition 5.3, we have

$$Z_{Q \rtimes T}(X) = \int^{Y \in \mathcal{C}} {}^\vee Q \rtimes T(Y) \otimes X \otimes Y$$

with universal dinatural transformation

$$i_{X,Y} = (\mathrm{ev}_{Q \rtimes T(Y)} \otimes \mathrm{id}_{Z_{Q \rtimes T}(X)})(\mathrm{id}_{{}^\vee Q \rtimes T(Y)} \otimes \partial_{X,Y}).$$

Recall that $T(i)$ is a universal dinatural transformation (see Proposition 3.10). Denote by s^l the left antipode of T and by ε the counit of the adjunction (F_T, U_T).

Proposition 7.8. *The canonical distributive law Ω of the pair (Q,T) is invertible, and Ω and Ω^{-1} are characterized as natural transformations by*

$$\Omega_X T(i_{X,Y}) = i_{T(X),T(Y)}\big({}^\vee b_Y s^l_Y T({}^\vee a_Y) \otimes \mathrm{id}_{T(X) \otimes T(Y)}\big) T_3({}^\vee Q \rtimes T(Y), X, Y),$$

$$\Omega^{-1}_X i_{T(X),Y} = \big(\mathrm{ev}_{Q \rtimes T(Y)} \otimes T(i_{X,T(Y)}) \otimes \mathrm{ev}_Y\big)(\mathrm{id}_{{}^\vee Q \rtimes T(Y)} \otimes E_{X,Y} \otimes \mathrm{id}_Y),$$

where $a_Y = U_T(\varepsilon_{QF_T(Y)})$, $b_Y = U_TQ(\varepsilon_{F_T(Y)})$, *and*

$$E_{X,Y} = (a_Y T(b_Y) \otimes \mathrm{id}_{T(^\vee Q \rtimes T\, T(Y) \otimes X \otimes T(Y))} \otimes s_Y^l)$$
$$\circ T_3(Q \rtimes T\, T(Y), {}^\vee Q \rtimes T\, T(Y) \otimes X \otimes T(Y), {}^\vee T(Y))$$
$$\circ T(\mathrm{coev}_{Q \rtimes T\, T(Y)} \otimes \mathrm{id}_X \otimes \mathrm{coev}_{T(Y)}).$$

Remark 7.9. In the special case $Q = 1_{T\text{-}\mathcal{C}}$, we have $1_{T\text{-}\mathcal{C}} \rtimes T = T$ and so, by Proposition 7.8, the canonical distributive law of the pair $(1_{T\text{-}\mathcal{C}}, T)$ is nothing but the canonical law of T defined in Section 6.1, and the double $D_{1_{T\text{-}\mathcal{C}},T}$ of the pair $(1_{T\text{-}\mathcal{C}}, T)$ coincides with the double D_T of T defined in Section 6.2.

Proof. Note that a_Y and $a_Y' = s_{Q \rtimes T(Y)}^l T({}^\vee a_Y)$ are the T-actions of $QF_T(Y)$ and ${}^\vee QF_T(Y)$, respectively. By adjunction we have $b_Y Q \rtimes T(\eta_Y) = \mathrm{id}_{Q \rtimes T(Y)}$.

Recall that $\tilde{Z}_{Q \rtimes T}^\Omega$ is the centralizer of Q, with universal dinatural transformation

$$j_{(M,r),(N,s)} = i_{M,N}\big({}^\vee U_TQ(\varepsilon_{(N,s)}) \otimes \mathrm{id}_M \otimes \mathrm{id}_N\big).$$

In particular, given two objects X, Y of $\mathcal{C}$, the morphism $j_{F_T(X),F_T(Y)}$ is T-linear, that is,

$$Z_{Q \rtimes T}(\mu_X)\Omega_{T(X)}T(j_{F_T(X),F_T(Y)}) = j_{F_T(X),F_T(Y)}\gamma_{X,Y},$$

where

$$\gamma_{X,Y} = \big(a_Y' \otimes \mu_X \otimes \mu_Y\big)T_3({}^\vee Q \rtimes T(Y), T(X), T(Y))$$

is the T-action of ${}^\vee QF_T(Y) \otimes F_T(X) \otimes F_T(Y)$. Composing on the right with $T(\mathrm{id}_{^\vee Q \rtimes T(Y)} \otimes \eta_X \otimes \eta_Y)$, the left-hand side becomes

$$Z_{Q \rtimes T}(\mu_X)\Omega_{T(X)}T\big(i_{T(X),T(Y)}({}^\vee b_Y \otimes \eta_X \otimes \eta_Y)\big)$$
$$= Z_{Q \rtimes T}(\mu_X)\Omega_{T(X)}TZ_{Q \rtimes T}(\eta_X)T(i_{X,Y}) = \Omega_X T(i_{X,Y}),$$

and the right-hand side becomes

$$i_{T(X),T(Y)}({}^\vee b_Y a_Y' \otimes \mu_X T(\eta_X) \otimes \mu_Y T(\eta_Y))T_3({}^\vee Q \rtimes T(Y), X, Y).$$

Hence the formula for Ω.

Let $\Omega' \colon Z_{Q \rtimes T}T \to TZ_{Q \rtimes T}$ be the natural transformation defined by

$$\Omega_X' i_{T(X),Y} = \big(\mathrm{ev}_{Q \rtimes T(Y)} \otimes T(i_{X,T(Y)}) \otimes \mathrm{ev}_Y\big)(\mathrm{id}_{^\vee Q \rtimes T(Y)} \otimes E_{X,Y} \otimes \mathrm{id}_Y).$$

Using the axioms of a left antipode, one shows that $\Omega'\Omega = \mathrm{id}_{TZ_{Q \rtimes T}}$ and $\Omega'\Omega = \mathrm{id}_{Z_{Q \rtimes T}T}$ by verifying that $\Omega_X'\Omega_X T(i_{X,Y}) = T(i_{X,Y})$ and $\Omega_X\Omega_X' i_{T(X),Y} = i_{T(X),Y}$. This is left to the reader. Note that when Q is a Hopf monad, the invertibility of Ω follows from Proposition 4.12, since in this case both T and $Z_{Q \rtimes T}$ are Hopf monads. $\square$

Remark 7.10. Let T be a Hopf monad on an autonomous category $\mathcal{C}$ and Q be a comonoidal endofunctor of $T\text{-}\mathcal{C}$ such that $Q \rtimes T$ is centralizable. Consider the following diagram:
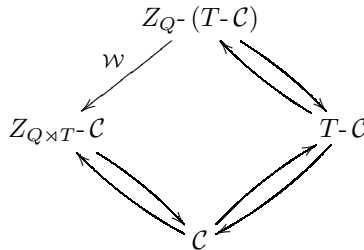
where a double arrow represents the adjunction of the corresponding monad and the functor $\mathcal{W}$ is defined by $\mathcal{W}\big((M,r),s\big) = (M,s)$ and $\mathcal{W}(f) = f$. This diagram is a distributive adjoint square in the sense of Beck [Bec69] whose distributive law is precisely the canonical distributive law Ω of the pair (Q,T). Furthermore, since Ω is invertible by Proposition 7.8, the monad T lifts to a monad $\tilde{T}^{\Omega^{-1}}$ on $Z_{Q\rtimes T}$-$\mathcal{C}$. Therefore, since $Z_Q = \tilde{Z}_{Q\rtimes T}^{\Omega}$, we have an isomorphism of categories:

$$\tilde{T}^{\Omega^{-1}}\text{-}(Z_{Q\rtimes T}\text{-}\mathcal{C}) \simeq (T \circ_{\Omega^{-1}} Z_{Q\rtimes T})\text{-}\mathcal{C} \simeq (Z_{Q\rtimes T} \circ_{\Omega} T)\text{-}\mathcal{C} \simeq Z_Q\text{-}(T\text{-}\mathcal{C}).$$

Via this isomorphism, $\mathcal{W}$ is the forgetful functor $U_{\tilde{T}^{\Omega^{-1}}}$. Hence $\mathcal{W}$ is monadic. Note that when Q is a bimonad, the four monadic adjunctions are comonoidal.

7.4. Proof of Theorem 6.1. This is a direct consequence of Remark 7.9 and Propositions 7.5 and 7.8 applied to the Hopf monad $Q = 1_{T\text{-}\mathcal{C}}$.

7.5. Proof of Theorems 6.4 and 6.5. By Theorem 7.6 applied to the Hopf monad $Q = 1_{T\text{-}\mathcal{C}}$ and Remark 7.9, the functor $I\colon D_T\text{-}\mathcal{C} \to \mathcal{Z}(T\text{-}\mathcal{C})$ of Theorem 6.5 is a strict monoidal isomorphism of monoidal categories such that $\mathcal{U}I = U_{D_T}$.

Now, by Remark 5.10, the category $\mathcal{Z}(T\text{-}\mathcal{C})$ is a braided category with braiding:

$$\tau_{\big((M,r),\gamma\big),\big((N,s),\delta\big)} = \gamma_{(N,s)}.$$

Therefore, since I is a strict monoidal isomorphism, there exists a unique braiding c on $D_T\text{-}\mathcal{C}$ such that I is braided. By Theorem 2.11, c is encoded by an R-matrix R for D_T. Let p and $e = u_T\eta$ be the product and unit of D_T. Then R is given by

$$
\begin{aligned}
R_{X,Y} &= c_{F_{D_T}(X),F_{D_T}(Y)}(e_X \otimes e_Y) \\
&= \tau_{IF_{D_T}(X),IF_{D_T}(Y)}(e_X \otimes e_Y) \\
&= \big(p_Y u_{D_T(Y)} \otimes p_X Z_T(\eta_{D_T(X)})\big)\partial_{D_T(X),D_T(Y)}(e_X \otimes e_Y) \\
&= \big(p_Y u_{D_T(Y)}T(e_Y) \otimes p_X Z_T(\eta_{D_T(X)}e_X)\big)\partial_{X,Y} \\
&= \big(p_Y D_T(e_Y)u_{T(Y)} \otimes p_X D_T(e_X)Z_T(\eta_X)\big)\partial_{X,Y} \\
&= \big(u_{T(Y)} \otimes Z_T(\eta_X)\big)\partial_{X,Y}.
\end{aligned}
$$

This concludes the proof of Theorems 6.4 and 6.5.

7.6. Proof of Theorem 6.9. This is a direct consequence of Remark 7.9 and Theorem 7.4 applied to the Hopf monad $Q = 1_{T\text{-}\mathcal{C}}$.

8. The double of a Hopf algebra in a braided category

In this section, we extend several classical notions concerning a Hopf algebra over a field to a Hopf algebra A in a braided autonomous category $\mathcal{B}$, namely, quasitriangularity and R-matrices and the double $D(A)$ of A. Our approach consists in applying the results of previous sections to the Hopf monad $? \otimes A$.

We need to assume that $\mathcal{B}$ admits a coend C. Then the Hopf monad $? \otimes A$ is centralizable, and its centralizer is of the form $? \otimes Z(A)$, where $Z(A)$ is a certain Hopf algebra in $\mathcal{B}$ called the centralizer of A. As an object of $\mathcal{B}$, $Z(A) = {}^{\vee}A \otimes C$.

We then define the *double of A* as $D(A) = A \otimes_{\Omega} Z(A) = A \otimes {}^{\vee}A \otimes C$, where Ω is an explicit distributive law. The double $D(A)$ is a quasitriangular Hopf algebra in $\mathcal{B}$ such that $D_{?\otimes A} = ? \otimes D(A)$ (as quasitriangular Hopf monads). It satisfies $\mathcal{Z}(\mathcal{B}_A) \simeq \mathcal{B}_{D(A)}$ (as braided categories). When $\mathcal{B} = \text{vect}_{\Bbbk}$, we have that $C = \Bbbk$, A

is a finite-dimensional Hopf algebra over $\Bbbk$, $Z(A) = (A^*)^{\mathrm{cop}}$, and $D(A)$ is the usual Drinfeld double of A.

8.1. Hopf monads represented by Hopf algebras.

Let $\mathcal{B}$ be a braided autonomous category and let A be a Hopf algebra in $\mathcal{B}$. A Hopf monad T on $\mathcal{B}$ is said to be *represented on the left* (resp. *on the right*) by A if it is isomorphic to the Hopf monad $A\otimes?$ (resp. $? \otimes A$) defined in Example 2.4.

More generally, let T be a Hopf monad on an autonomous category $\mathcal{C}$. If (A, σ) is a Hopf algebra in the center $\mathcal{Z}(\mathcal{C})$ of $\mathcal{C}$, then the Hopf monad T is said to be *represented on the left* by (A, σ) if it is isomorphic to the Hopf monad $A\otimes_\sigma?$ on $\mathcal{C}$ defined in Example 2.5. Likewise, if (A, σ) is a Hopf algebra in $\mathcal{Z}'(\mathcal{C})$, then the Hopf monad T is said to be *represented on the right* by (A, σ) if it is isomorphic to the Hopf monad $? \otimes_\sigma A$ on $\mathcal{C}$.

Not all Hopf monads can be so represented by Hopf algebras (see Remark 8.5 for an example).

8.2. Coends as Hopf algebras.

Let T be an endofunctor of an autonomous category $\mathcal{C}$. If $\mathcal{C}$ admits a braiding τ, then, by Proposition 5.3, T is centralizable if and only if the coend

$$C_T = \int^{Y\in\mathcal{B}} {}^\vee T(Y) \otimes Y$$

exists. Assume this is the case. By Lemma 3.9, if T is a monad, then C_T coincides with the coend $\int^{(M,r)\in T\text{-}\mathcal{C}} {}^\vee U_T(M,r) \otimes U_T(M,r)$ of U_T. According to Majid [Maj95], the (co)end of a strong monoidal functor from an autonomous category to a braided category is a Hopf algebra. In particular, if T is a Hopf monad and τ a braiding on $\mathcal{C}$, then C_T is a Hopf algebra in $\mathcal{C}$ braided by τ. In this section we recover this structure explicitly in terms of the braiding τ and the Hopf monad structure of T.

Let T be an endofunctor of an autonomous category $\mathcal{C}$ such that the coend $C_T = \int^{Y\in\mathcal{B}} {}^\vee T(Y) \otimes Y$ exists. Denote by $i_Y \colon {}^\vee T(Y) \otimes Y \to C_T$ the universal dinatural transformation of C_T, and set

$$\delta_Y = \;\begin{array}{c} {}^{T(Y)\ C_T} \\ \begin{picture}(0,0)\end{picture} \\ {}_{Y} \end{array}\; = (\mathrm{id}_{T(Y)} \otimes i_Y)(\mathrm{coev}_{T(Y)} \otimes \mathrm{id}_Y) \colon X \to T(Y)\otimes C_T, \text{ depicted as } \begin{array}{c} {}^{T(Y)\ C_T} \\ \begin{picture}(0,0)\end{picture} \\ {}_{Y} \end{array}.$$

If T is a monad on $\mathcal{C}$, then C_T is a coalgebra in $\mathcal{C}$, with coproduct Δ and counit ε defined by

where μ and η are the product and unit of T.

If T is comonoidal and τ is a braiding on $\mathcal{C}$, then C_T becomes an algebra in $\mathcal{C}$ with product m_τ and unit u defined by

$$
\begin{array}{ccc}
\text{[diagram: } T(X)\ T(Y)\ C_T \text{ with } m_\tau \text{ over } X\ Y] &=& \text{[diagram: } T(X)\ T(Y)\ C_T \text{ with } T_2(X,Y) \text{ over } X\otimes Y]
\end{array}
\quad \text{and} \quad u = \text{[diagram: } C_T \text{ with } T_0], \quad \text{where} \quad \tau_{X,Y} = \text{[crossing diagram } Y\ X \text{ over } X\ Y].
$$

If T is a bimonad and τ a braiding on $\mathcal{C}$, then $(C_T, m^\tau, u, \Delta, \varepsilon)$ is a bialgebra in $\mathcal{C}$ braided by τ. Furthermore, if T is a Hopf monad, then C_T is a Hopf algebra, whose antipode S_τ and its inverse S_τ^{-1} are defined by

$$
\text{[diagram: } T(X)\ C_T,\ S_\tau,\ X] = \text{[diagram: } T(X)\ C_T,\ s^l_X,\ X] \quad \text{and} \quad \text{[diagram: } T(X)\ C_T,\ S_\tau^{-1},\ X] = \text{[diagram: } T(X)\ C_T,\ s^r_X,\ X].
$$

We denote this Hopf algebra by C_T^τ.

8.3. **The coend of a braided autonomous category.** Let $\mathcal{B}$ be an autonomous category. The coend

$$
C = \int^{Y \in \mathcal{B}} {}^\vee Y \otimes Y,
$$

if it exists, is called the *coend of* $\mathcal{B}$.

Assume that $\mathcal{B}$ admits a coend C and denote by $i_Y \colon {}^\vee Y \otimes Y \to C$ its universal dinatural transformation. The *universal coaction* of C on the objects of $\mathcal{B}$ is the natural transformation δ defined by

$$
\delta_Y = (\mathrm{id}_Y \otimes i_Y)(\mathrm{coev}_Y \otimes \mathrm{id}_Y) \colon Y \to Y \otimes C, \quad \text{depicted as} \quad \delta_Y = \text{[diagram: } Y\ C \text{ over } Y].
$$

If $\mathcal{B}$ is braided, then C is a Hopf algebra in $\mathcal{B}$. This well-known fact may be viewed as a special case of the construction of Section 8.4, as $1_\mathcal{B}$ is a Hopf monad on $\mathcal{B}$ and $C = C_{1_\mathcal{B}}^\tau$, where τ is the braiding of $\mathcal{B}$. Furthermore, the morphism $\omega \colon C \otimes C \to \mathbb{1}$, defined by

$$
\text{[diagram: } X\ Y,\ \omega,\ X\ Y] = \text{[crossing diagram: } X\ Y \text{ over } X\ Y],
$$

is a Hopf pairing for C, that is, it satisfies

$$
\begin{aligned}
\omega(m \otimes \mathrm{id}_C) &= \omega(\mathrm{id}_C \otimes \omega \otimes \mathrm{id}_C)(\mathrm{id}_{C^{\otimes 2}} \otimes \Delta), & \omega(u \otimes \mathrm{id}_C) &= \varepsilon, \\
\omega(\mathrm{id}_C \otimes m) &= \omega(\mathrm{id}_C \otimes \omega \otimes \mathrm{id}_C)(\Delta \otimes \mathrm{id}_{C^{\otimes 2}}), & \omega(\mathrm{id}_C \otimes u) &= \varepsilon.
\end{aligned}
$$

These axioms imply $\omega(S \otimes \mathrm{id}_C) = \omega(\mathrm{id}_C \otimes S)$. Moreover, the pairing ω satisfies the self-duality condition $\omega\tau_{C,C}(S \otimes S) = \omega$.

In this section, the structural morphisms of C are drawn in grey and the Hopf pairing $w \colon C \otimes C \to \mathbb{1}$ is depicted as

$$
\omega = \text{[cap diagram]}_{C\ C}.
$$

Remark 8.1. The category $\mathcal{B}$ is symmetric if and only if $\omega = \epsilon \otimes \epsilon$. In particular, this is the case when $C = \mathbb{1}$.

Remark 8.2. The universal coaction of the coend on itself can be expressed in terms of its Hopf algebra structure as follows:

$$\delta_C = \;\; \Big| \!\!\! \diagdown \;\; = \;\; \bigoplus \!\! \diagdown \;\; .$$

Remark 8.3. The coend of the mirror $\overline{\mathcal{B}}$ of $\mathcal{B}$ is the Hopf algebra C^{op}, with self-dual pairing $\omega(S \otimes \mathrm{id}_C)$.

8.4. Centralizers in braided categories. Let T be an endofunctor of a braided autonomous category $\mathcal{B}$, with braiding τ. Assume that the coend

$$C_T = \int^{Y \in \mathcal{B}} {}^{\vee}T(Y) \otimes Y$$

exists. Set

$$\partial_{X,Y} = \;\; \diagdown\!\!\!\diagup\!\!\!\diagdown \;\; = (\tau_{X,T(Y)} \otimes \mathrm{id}_{C_T})(\mathrm{id}_X \otimes \delta_Y) \colon X \otimes Y \to T(Y) \otimes X \otimes C_T.$$

Then $(? \otimes C_T, \partial)$ is a centralizer of T. Likewise, set

$$\partial'_{X,Y} = \;\; \diagdown\!\!\!\diagup\!\!\!\diagup \;\; = (\delta_Y \otimes \mathrm{id}_X)\tau_{Y,X}^{-1} \colon X \otimes Y \to T(Y) \otimes C_T \otimes X.$$

Then $(C_T \otimes ?, \partial')$ is also a centralizer of T.

Assume furthermore that T is a Hopf monad. By Section 8.2, the object C_T is endowed with two Hopf algebra structures in $\mathcal{B}$, namely, C_T^τ and $(C_T^{\overline{\tau}})^{\mathrm{op}}$, where $\overline{\tau}$ is the mirror of τ. One verifies that the Hopf monad structure on $? \otimes C_T$ (resp. $C_T \otimes ?$) given by Theorem 5.6 coincides with that induced by the Hopf algebra C_T^τ (resp. $(C_T^{\overline{\tau}})^{\mathrm{op}}$). Thus:

Theorem 8.4. *Let T be a Hopf monad on a braided autonomous category $\mathcal{B}$, with braiding τ. Then T is centralizable if and only if the coend*

$$C_T = \int^{Y \in \mathcal{B}} {}^{\vee}T(Y) \otimes Y$$

exists. If such is the case, the centralizer of T is represented on the right by the Hopf algebra C_T^τ and on the left by the Hopf algebra $(C_T^{\overline{\tau}})^{\mathrm{op}}$.

Remark 8.5. In general, the centralizer Z_T of a Hopf monad T on an autonomous category $\mathcal{C}$ is isomorphic neither to $Z_T(\mathbb{1}) \otimes ?$ nor to $? \otimes Z_T(\mathbb{1})$ as an endofunctor of $\mathcal{C}$, see Remark 9.2 for a counterexample. In particular, it cannot be represented on the left by a Hopf algebra of $\mathcal{Z}(\mathcal{C})$ or on the right by a Hopf algebra of $\mathcal{Z}'(\mathcal{C})$, in the sense of Section 8.1.

8.5. **Centralizers of Hopf algebras.** Let $\mathcal{B}$ be a braided autonomous category, with braiding τ, admitting a coend $C = \int^{Y \in \mathcal{B}} {}^\vee Y \otimes Y$. Recall that C is a Hopf algebra in $\mathcal{B}$ endowed with a Hopf pairing, and denote by δ the universal coaction of C (see Section 8.3).

Let A be a Hopf algebra in $\mathcal{B}$. Set $Z(A) = {}^\vee A \otimes C$. Then

$$Z(A) = \int^{Y \in \mathcal{B}} {}^\vee(Y \otimes A) \otimes Y,$$

with universal dinatural transformation given by

$$i_Y = \begin{array}{c} {}^\vee A \quad C \\ \text{[diagram]} \\ {}^\vee A \ {}^\vee Y \ Y \end{array} : {}^\vee(Y \otimes A) \otimes Y = {}^\vee A \otimes {}^\vee Y \otimes Y \to {}^\vee A \otimes C.$$

The object $Z(A)$ is endowed with the Hopf algebra structure $C^\tau_{? \otimes A}$ defined in Section 8.2, and is called the *centralizer of A*. Explicitly, the structural morphisms of $Z(A)$ are

$$u_{Z(A)} = \begin{array}{c} {}^\vee A \quad C \\ \text{[diagram]} \end{array}, \qquad \varepsilon_{Z(A)} = \begin{array}{c} \text{[diagram]} \\ {}^\vee A \quad C \end{array}.$$

Example 8.6. Let H be a finite-dimensional Hopf algebra over a field $\Bbbk$. Note that $\Bbbk$ is the coend of the category $\mathrm{vect}_\Bbbk$ of finite-dimensional vector spaces. Then the centralizer of H is $Z(H) = (H^*)^{\mathrm{cop}}$.

By Theorem 8.4, $(? \otimes Z(A), \partial)$ is a centralizer of $? \otimes A$, where

$$\partial_{X,Y} = \begin{array}{c} Y \ A \ X \ {}^\vee A \ C \\ \text{[diagram]} \\ X \qquad Y \end{array},$$

and the Hopf monad structure on $? \otimes Z(A)$ given by Theorem 5.6 is that induced by the Hopf algebra $Z(A)$. Hence, by Theorem 5.12, we have

$$\mathcal{Z}_{? \otimes A}(\mathcal{B}) \simeq (? \otimes Z(A))\text{-}\mathcal{B} = \mathcal{B}_{Z(A)}$$

as monoidal categories.

Remark 8.7. One can show that the Hopf algebra $Z(A)$ represents on the left the centralizer $Z_{A \otimes ?}$ of $A \otimes ?$, and so

$$\mathcal{Z}_{A \otimes ?}(\mathcal{B}) \simeq (Z(A) \otimes ?)\text{-}\mathcal{B} = {}_{Z(A)}\mathcal{B}$$

as monoidal categories.

8.6. R-matrices for Hopf algebras in braided categories. In [Dri90], Drinfeld introduced the notion of R-matrix for a Hopf algebra H over a field $\Bbbk$. When H is finite-dimensional, R-matrices for H are in bijection with braidings on the category of finite-dimensional H-modules. The aim of this section is to extend the notion of an R-matrix to a Hopf algebra A in braided autonomous category so as to preserve this bijective correspondence. Note that the definition of an R-matrix for A as a morphism $\mathfrak{r}\colon \mathbb{1} \to A \otimes A$ by straightforward extension of Drinfeld's axioms (sometimes found in the literature) does not fulfil this objective. Recall that braidings on the autonomous category $\mathcal{B}_A = (? \otimes A)\text{-}\mathcal{B}$ are encoded by R-matrices for the Hopf monad $? \otimes A$. When $\mathcal{B}$ admits a coend, we can encode R-matrices for $? \otimes A$ in terms of A, which leads to our definition of an R-matrix for A.

Let A be a Hopf algebra in a braided autonomous category $\mathcal{B}$, with braiding τ. Assume that $\mathcal{B}$ admits a coend C. Any R-matrix $R_{X,Y}\colon X \otimes Y \to Y \otimes A \otimes X \otimes A$ for the Hopf monad $? \otimes A$ gives rise to a unique morphism $\mathfrak{r}\colon C \otimes C \to A \otimes A$ in $\mathcal{B}$, defined by

Re-writing the axioms for $R_{X,Y}$ (see Section 2.7) in terms of $\mathfrak{r}$ leads to the following definition: an R-*matrix* for A is a morphism

$$\mathfrak{r}\colon C \otimes C \to A \otimes A$$

in $\mathcal{B}$ which satisfies

Remark 8.8. For finite-dimensional Hopf algebras over a field $\Bbbk$, our definition of an R-matrix coincides with Drinfeld's definition (as the coend of $\mathrm{vect}_{\Bbbk}$ is $\Bbbk$).

An R-matrix $\mathfrak{r}$ for A defines an R-matrix for $? \otimes A$ (by definition) and so a braiding c on $\mathcal{B}_A = (? \otimes A)\text{-}\mathcal{B}$ (by Theorem 2.11):

$$c_{(M,r),(N,s)} = (s \otimes r)R_{M,N} = $$

As braidings on $_A\mathcal{B}$ are in bijective correspondence with braidings on $\mathcal{B}_A$ (see Remark 1.3), an R-matrix $\mathfrak{r}$ for A also defines a braiding c' on $_A\mathcal{B}$ as

$$c'_{(M,r),(N,s)} = $$

Furthermore, the map $\mathfrak{r} \mapsto c$ (resp. $\mathfrak{r} \mapsto c'$) is a bijection between R-matrices for A and braidings on $\mathcal{B}_A$ (resp. on $_A\mathcal{B}$).

A *quasitriangular Hopf algebra* in $\mathcal{B}$ is a Hopf algebra in $\mathcal{B}$ endowed with an R-matrix.

Remark 8.9. Let A be a quasitriangular Hopf algebra in $\mathcal{B}$. By construction, the monoidal isomorphism $F_A \colon (_A\mathcal{B})^{\otimes \mathrm{op}} \to \mathcal{B}_A$ of Remark 1.3 is braided.

Remark 8.10. Let A be a quasitriangular Hopf algebra in $\mathcal{B}$. Combining Remark 8.9 with Example 1.1, we obtain that $_A\mathcal{B}$ and $\mathcal{B}_A$ are braided isomorphic.

8.7. **The canonical distributive law of a Hopf algebra.** Let A be a Hopf algebra in a braided autonomous category $\mathcal{B}$ which admits a coend C. By Section 8.5, the centralizer of $? \otimes A$ is $Z_{?\otimes A} =? \otimes Z(A)$, where $Z(A) = {}^\vee A \otimes C$ is the centralizer of A. It turns out that the canonical distributive law of $? \otimes A$ over $Z_{?\otimes A}$ is of the form $\mathrm{id}_{1_\mathcal{B}} \otimes \Omega$, where $\Omega \colon Z(A) \otimes A \to A \otimes Z(A)$ is a comultiplicative distributive law of $Z(A)$ over A (see Example 4.3). We have

$$\Omega = \qquad \text{and} \quad \Omega^{-1} = $$

We call Ω the *canonical distributive law of A*.

Remark 8.11. By Theorem 6.9, $Z_{\mathcal{B}_A}(M,r) = \big(M \otimes Z(A), (r \otimes \mathrm{id}_{Z(A)})(\mathrm{id}_M \otimes \Omega)\big)$ is the centralizer of the category $\mathcal{B}_A$. In particular, the coend of $\mathcal{B}_A$ is $Z_{\mathcal{B}_A}(\mathbb{1}, \varepsilon_A) =$

$(Z(A), \alpha)$, where

$$\alpha = (\varepsilon_A \otimes \mathrm{id}_{Z(A)})\Omega = \quad : Z(A) \otimes A \to Z(A).$$

If A is quasitriangular, so that $\mathcal{B}_A$ is braided, then $(Z(A), \alpha)$ is a Hopf algebra in $\mathcal{B}_A$ which represents $Z_{\mathcal{B}_A}$ on the right (see Theorem 8.4). However, in this case, this Hopf algebra $(Z(A), \alpha)$ is not in general a lift to $\mathcal{B}_A$ of the Hopf algebra $Z(A)$.

8.8. **The double of a Hopf algebra in a braided category.** Let A be a Hopf algebra in a braided autonomous category $\mathcal{B}$ which admits a coend C.

Let $Z(A)$ be the centralizer of A (see Section 8.5) and Ω be the canonical distributive law of A (see Section 8.7). By Example 4.3,

$$D(A) = A \otimes_\Omega Z(A) = A \otimes {}^{\vee}A \otimes C$$

is a Hopf algebra in $\mathcal{B}$. Since $? \otimes Z(A)$ is the centralizer of $? \otimes A$ (see Section 8.5), the Hopf monad $? \otimes D(A)$ is the double of $? \otimes A$, and so admits an R-matrix by Theorem 6.5, which turns out to be encoded by the following R-matrix for $D(A)$:

$$\mathfrak{r} = \quad : C \otimes C \to D(A) \otimes D(A).$$

The quasitriangular Hopf algebra $D(A)$ is called the *double of A*.

Remark 8.12. The canonical distributive law of A is the unique comultiplicative distributive law Ω of $Z(A)$ over A such that the morphism $\mathfrak{r}$ above is an R-matrix for $A \otimes_\Omega Z(A)$, see Remark 6.8.

Theorem 8.13. *Let A be a Hopf algebra in a braided autonomous category $\mathcal{B}$ admitting a coend C and let $D(A) = A \otimes_\Omega Z(A)$ be the double of A. We have the following isomorphisms of braided categories:*

$$\mathcal{Z}(\mathcal{B}_A) \simeq \mathcal{B}_{D(A)} \simeq {}_{D(A)}\mathcal{B} \simeq \mathcal{Z}'({}_A\mathcal{B}) \simeq \overline{\mathcal{Z}({}_A\mathcal{B})}.$$

Proof. By construction, the quasitriangular Hopf monad $? \otimes D(A)$ is the double of $? \otimes A$, hence the first braided isomorphism by Theorem 6.5. By Remark 8.10, ${}_{D(A)}\mathcal{B} \simeq \mathcal{B}_{D(A)}$ as braided categories since $D(A)$ is quasitriangular. Finally, we have the following isomorphisms of braided categories:

$$\mathcal{Z}(\mathcal{B}_A) \simeq \mathcal{Z}(\mathcal{B}_A)^{\otimes \mathrm{op}} \quad \text{by Remark 1.1}$$

$$\simeq \mathcal{Z}\big(({}_A\mathcal{B})^{\otimes \mathrm{op}}\big)^{\otimes \mathrm{op}} \quad \text{by Remark 8.9}$$

$$\simeq \mathcal{Z}'({}_A\mathcal{B}) \simeq \overline{\mathcal{Z}({}_A\mathcal{B})} \quad \text{by Remark 1.2.}$$

This completes the proof of the theorem. $\square$

Remark 8.14. When $\mathcal{B} = \mathrm{vect}_{\Bbbk}$ is the category of finite-dimensional vector spaces over a field $\Bbbk$, we recover the usual Drinfeld double and the interpretation of its category of modules in terms of the center. More precisely, let H be a finite-dimensional Hopf algebra over $\Bbbk$ and (e_i) be a basis of H with dual basis (e^i).

Then $D(H) = H \otimes (H^*)^{\mathrm{cop}}$ is a quasitriangular Hopf algebra over $\Bbbk$, with R-matrix $\mathfrak{r} = \sum_i e_i \otimes \varepsilon \otimes 1_H \otimes e_i$, such that

$$\mathcal{Z}\big((\mathrm{vect}_\Bbbk)_H\big) \simeq (\mathrm{vect}_\Bbbk)_{D(H)} \simeq {}_{D(H)}(\mathrm{vect}_\Bbbk) \simeq \mathcal{Z}'\big({}_H(\mathrm{vect}_\Bbbk)\big) \simeq \overline{\mathcal{Z}\big({}_H(\mathrm{vect}_\Bbbk)\big)}$$

as braided categories.

Remark 8.15. By Remark 4.2, Ω^{-1} is a distributive law of $Z(A)$ over A and induces an isomorphism of Hopf algebras

$$D(A) = A \otimes_\Omega Z(A) \xrightarrow{\;\sim\;} Z(A) \otimes_{\Omega^{-1}} A.$$

Via this isomorphism, the R-matrix $\mathfrak{r}$ of $D(A)$ is sent to the R-matrix

$$\mathfrak{r}' = \quad \raisebox{-1.5em}{\begin{array}{c}{}^\vee A \; C \; A \; {}^\vee A \; C \; A \\ \includegraphics{figure} \\ C \; C\end{array}} \quad : C \otimes C \to \big(Z(A) \otimes_{\Omega^{-1}} A\big) \otimes \big(Z(A) \otimes_{\Omega^{-1}} A\big)$$

of $Z(A) \otimes_{\Omega^{-1}} A$.

Remark 8.16. Let $\mathcal{B}$ be a braided autonomous category which admits a coend C. Then $D(\mathbb{1}) = C$ as a Hopf algebra. Hence C is quasitriangular, with R-matrix $\mathfrak{r} = u_C \varepsilon_C \otimes \mathrm{id}_C$, and $\mathcal{Z}(\mathcal{B}) \simeq \mathcal{B}_C \simeq {}_C\mathcal{B} \simeq \overline{\mathcal{Z}(\mathcal{B})}$ as braided categories. In other words, the center of $\mathcal{B}$ is self-mirror and is the category of C-modules in $\mathcal{B}$.

9. Hopf monads and fusion categories

In this section, given a $\Bbbk$-linear Hopf monad T of a fusion category $\mathcal{F}$, we describe explicitly the centralizer of T and the canonical distributive law of T. Hence, in particular, a description of the coend of $\mathcal{Z}(\mathcal{F})$, which is used in [BV10] to show that the center of a spherical fusion category is modular and in [BV08] to provide an algorithm for computing the Reshetikhin-Turaev invariant $\mathrm{RT}_{\mathcal{Z}(\mathcal{F})}$ in terms of $\mathcal{F}$ itself.

9.1. **Fusion categories.** A *fusion category* $\mathcal{F}$ over a commutative ring $\Bbbk$ is a $\Bbbk$-linear[4] autonomous category whose monoidal product $\otimes$ is $\Bbbk$-bilinear and such that there exists a finite set I of objects of $\mathcal{F}$ satisfying:

- $\mathrm{Hom}_\mathcal{F}(i,j) = \delta_{i,j}\,\Bbbk$ for all $i, j \in I$;
- $\mathbb{1} \in I$;
- each object of $\mathcal{F}$ is a finite direct sum of elements of I.

The set I is then a representative set of scalar objects of $\mathcal{F}$ (an object X of $\Bbbk$-linear category is said to be *scalar* if $\mathrm{End}(X) = \Bbbk$). The Hom spaces in $\mathcal{F}$ are free $\Bbbk$-modules of finite rank. The *multiplicity* of $i \in I$ in an object X of $\mathcal{F}$ is defined as

$$N_X^i = \mathrm{rank}_\Bbbk \, \mathrm{Hom}_\mathcal{F}(i, X) = \mathrm{rank}_\Bbbk \, \mathrm{Hom}_\mathcal{F}(X, i).$$

For each object X of $\mathcal{F}$, we choose families of morphisms $(p_X^{i,\alpha} \colon X \to i)_{1 \le \alpha \le N_X^i}$ and $(q_X^{i,\alpha} \colon i \to X)_{1 \le \alpha \le N_X^i}$ such that

$$\mathrm{id}_X = \sum_{\substack{i \in I \\ 1 \le \alpha \le N_X^i}} q_X^{i,\alpha} p_X^{i,\alpha} \quad \text{and} \quad p_X^{i,\alpha} q_X^{j,\beta} = \delta_{i,j} \delta_{\alpha,\beta}\, \mathrm{id}_i.$$

---

[4]In particular, finite direct sums exist in $\mathcal{F}$.

9.2. **Centralizers in fusion categories.** Let $\mathcal{F}$ be a fusion category over a commutative ring $\Bbbk$ and T be a $\Bbbk$-linear endofunctor T of $\mathcal{F}$. Then T is centralizable, with centralizer (Z_T, ∂) given by

$$Z_T(X) = \bigoplus_{i \in I} {}^\vee T(i) \otimes X \otimes i$$

and

$$\partial_{X,Y} = \sum_{\substack{i \in I \\ 1 \leq \alpha \leq N_Y^i}} \left(T(q_Y^{i,\alpha}) \otimes \mathrm{id}_{{}^\vee T(i) \otimes X} \otimes p_Y^{i,\alpha} \right) \left(\mathrm{coev}_{T(i)} \otimes \mathrm{id}_{X \otimes Y} \right).$$

In particular, a fusion category is centralizable, with centralizer $Z = Z_{1_\mathcal{F}}$ given by

$$Z(X) = \bigoplus_{i \in I} {}^\vee i \otimes X \otimes i.$$

Remark 9.1. By Corollary 5.14, the centralizer Z of $\mathcal{F}$ provides in particular a left adjoint F_Z to the forgetful functor $\mathcal{U} \colon \mathcal{Z}(\mathcal{F}) \simeq Z\text{-}\mathcal{F} \to \mathcal{F}$, which is called the induction functor in [ENO05].

Remark 9.2. In general, the centralizer Z of $\mathcal{F}$ is not isomorphic (as an endofunctor of $\mathcal{F}$) to $Z(\mathbb{1}) \otimes ?$ or to $? \otimes Z(\mathbb{1})$, as shown by the following counterexample. Let G be a non-commutative finite group and let $\mathcal{F}$ be the fusion category of finite-dimensional G-graded vector spaces over a field $\Bbbk$. The elements of G form a representative set of scalar objects of $\mathcal{F}$. Then $Z(x) = \bigoplus_{g \in G} g^{-1} x g$ for $x \in G$. In particular, $Z(\mathbb{1}) = \mathbb{1}^{\#G}$. Now, if $x \in G$ is not central, $Z(x)$ is not isomorphic to $Z(\mathbb{1}) \otimes x \simeq x^{\#G} \simeq x \otimes Z(\mathbb{1})$.

Assume T is a centralizable Hopf monad. By Theorem 5.6, its centralizer Z_T is a Hopf monad on $\mathcal{F}$ and its structural morphisms can be described purely in terms of those of T and of the category $\mathcal{F}$ (that is, the p, q's and the duality morphisms). They are depicted in Figure 3, where μ, η, s^l, s^r (resp. m, u, S^l, S^r) denote the product, unit, left antipode, and right antipode of T (resp. Z_T). The canonical distributive law of T is

$$\Omega_X = \sum_{\substack{i,j \in I \\ 1 \leq \alpha \leq N_{T(i)}^j}} \left({}^\vee T(q_{T(i)}^{j,\alpha}) {}^\vee \mu_i s_{T(i)}^l T({}^\vee \mu_i) \otimes \mathrm{id}_{T(X)} \otimes p_{T(i)}^{j,\alpha} \right) T_3 \left({}^\vee T(i), X, i \right).$$

Hence, we have an explicit description of the double $D_T = Z_T \circ_\Omega T$ of T and of the lift $\tilde{Z}_T^\Omega$ of Z_T to $T\text{-}\mathcal{F}$. Note that the R-matrix of D_T is

$$R_{X,Y} = \sum_{\substack{i \in I \\ 1 \leq \alpha \leq N_Y^i}} \left({}^\vee T_0 \otimes T(q_Y^{i,\alpha}) \otimes \mathrm{id}_{{}^\vee T(i)} \otimes \eta_X \otimes p_Y^{i,\alpha} \right) \left(\mathrm{coev}_{T(i)} \otimes \mathrm{id}_{X \otimes Y} \right).$$

9.3. **The coend of the center of a fusion category.** Let $\mathcal{F}$ be a fusion category over a commutative ring $\Bbbk$, and denote by Z the centralizer of $\mathcal{F}$. Recall that Z is a quasitriangular Hopf monad on $\mathcal{F}$ such that $\mathcal{Z}(\mathcal{F}) \simeq Z\text{-}\mathcal{F}$, see Section 9.2. Since Z is $\Bbbk$-linear, it is centralizable. Denote by Z_Z its centralizer and by Ω the canonical distributive law of Z over Z_Z. Then the coend of $Z(\mathcal{F})$ is

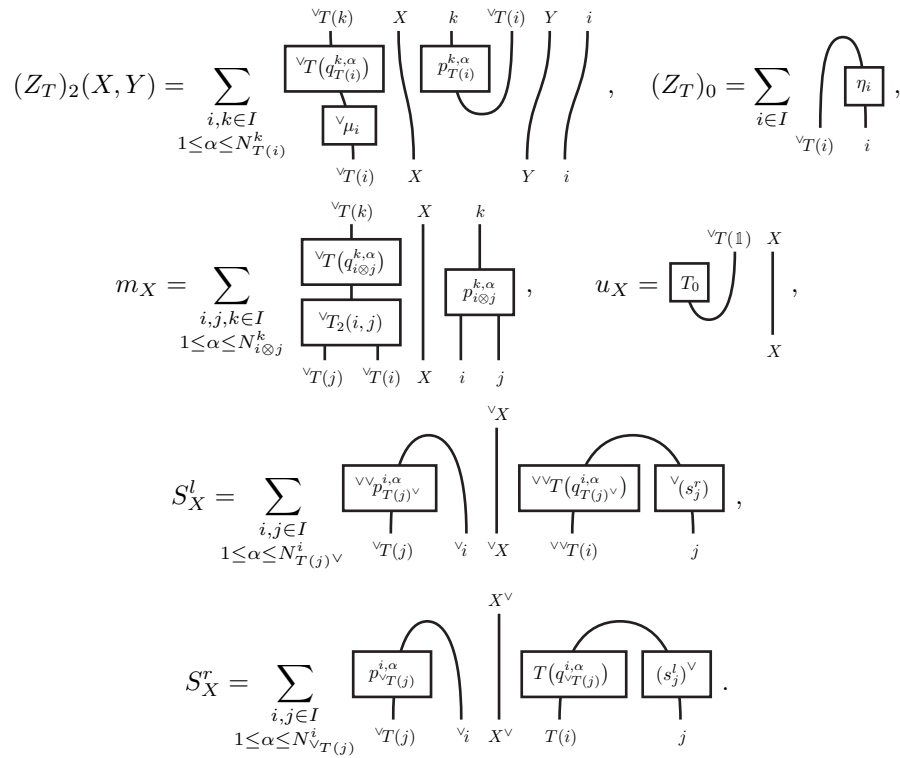$$C = \tilde{Z}_Z^\Omega(\mathbb{1}, Z_0) = \left(Z_Z(\mathbb{1}), Z_Z(Z_0)\Omega_\mathbb{1} \right).$$

$$(Z_T)_2(X,Y) = \sum_{\substack{i,k\in I \\ 1\leq\alpha\leq N^k_{T(i)}}} \quad , \qquad (Z_T)_0 = \sum_{i\in I}$$

$$m_X = \sum_{\substack{i,j,k\in I \\ 1\leq\alpha\leq N^k_{i\otimes j}}} \quad , \qquad u_X =$$

$$S^l_X = \sum_{\substack{i,j\in I \\ 1\leq\alpha\leq N^i_{T(j)^\vee}}}$$

$$S^r_X = \sum_{\substack{i,j\in I \\ 1\leq\alpha\leq N^i_{^\vee T(j)}}}$$

FIGURE 3. Structural morphisms of Z_T

Note that
$$Z_Z(\mathbb{1}) = \bigoplus_{j\in I} {}^\vee Z(j)\otimes j = \bigoplus_{i,j\in I} {}^\vee i \otimes {}^\vee j \otimes {}^{\vee\vee} i \otimes j.$$

Using the results of Section 6.3, one computes the Hopf algebra structure of C and its self-dual Hopf pairing. These are depicted in Figure 4, where the dotted lines represent the relevant canonical isomorphisms between $\mathbb{1}$ and its duals.

In [BV10], we use this explicit description of the coend of $\mathcal{Z}(\mathcal{F})$ to show that the center $\mathcal{Z}(\mathcal{F})$ of a spherical fusion category $\mathcal{F}$ is modular. In particular, this implies that if $\mathcal{F}$ is a spherical fusion category of invertible dimension over an algebraic closed field $\Bbbk$, then $\mathcal{Z}(\mathcal{F})$ is a modular ribbon fusion category (this last result was first shown in [Mü03] using different methods).

Also, this description of the coend of $\mathcal{Z}(\mathcal{F})$ leads to an explicit algorithm (involving Hopf diagrams [BV05]) for computing the Reshetikhin-Turaev invariants defined with $\mathcal{Z}(\mathcal{F})$. Moreover, this approach allows one to define these invariants over an arbitrary base ring, without assumption on the dimension of $\mathcal{F}$ (if the dimension of $\mathcal{F}$ is not invertible, this yields 'non-semisimple' invariants). See [BV08] for details.

$$\Delta_C = \sum_{\substack{i,j,k,m,n\in I \\ 1\le\alpha\le N^k_{k\otimes m} \\ 1\le\beta\le N^n_{{}^\vee k\otimes j\otimes k}}} \quad , \qquad \varepsilon_C = \sum_{j\in I} \quad ,$$

$$m_C = \sum_{\substack{i,j,k,l,m,n\in I \\ 1\le\alpha\le N^n_{{}^\vee k\otimes l\otimes k} \\ 1\le\beta\le N^i_{n\otimes k\otimes n} \\ 1\le\gamma\le N^m_{{}^{\vee\vee} n\otimes j\otimes{}^\vee n\otimes l}}} \quad , \qquad u_C = \sum_{i\in I}$$

$$S_C = \sum_{\substack{i,j,k,l\in I \\ 1\le\alpha\le N^{{}^\vee i}_{j\otimes k\otimes j^\vee} \\ 1\le\beta\le N^l_{{}^\vee j\otimes{}^\vee i\otimes{}^\vee j\otimes{}^{\vee\vee} i\otimes j}}} \quad ,$$

$$\omega_C = \sum_{\substack{i,j,k,l\in I \\ 1\le\alpha\le N^{{}^\vee k}_{i\otimes j\otimes i} \\ 1\le\beta\le N^i_{{}^\vee k\otimes{}^\vee l\otimes{}^{\vee\vee} k}}} \quad .$$
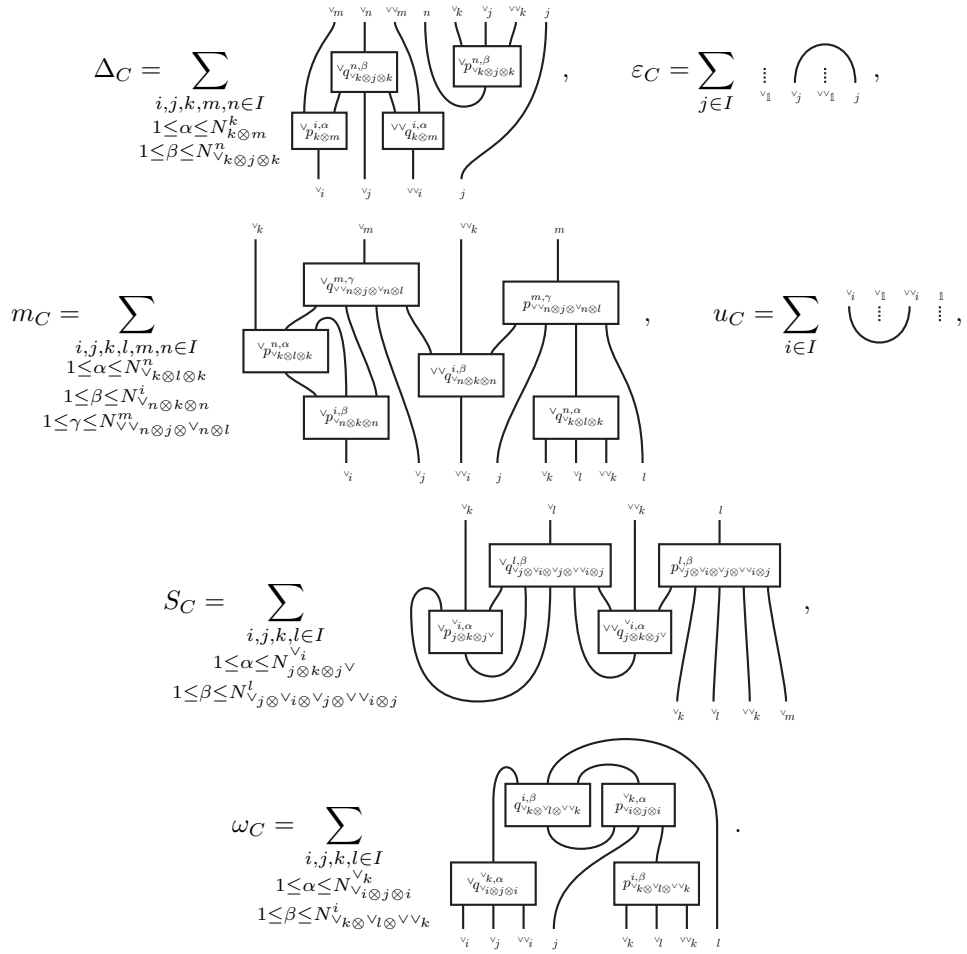
FIGURE 4. Structural morphisms of the coend of $\mathcal{Z}(\mathcal{F})$

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