

QUANTUM DYNAMICAL SEMIGROUP AND ITS ASYMPTOTIC BEHAVIORS

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ABSTRACT. In this study we consider quantum dynamical semigroup with a normal faithful invariant state. A quantum dynamical semigroup $\alpha = \{\alpha_t\}_{t \geq 0}$ is a class of linear normal identity-preserving mappings on a von Neumann algebra \mathcal{M} with semigroup property and some positivity condition. We investigate the asymptotic behaviors of the semigroup such as ergodicity or mixing properties in terms of their eigenvalues under the assumption that the semigroup satisfies positivity. This extends the result of [13] which is obtained under the assumption that the semigroup satisfy 2-positivity.

1. Introduction and preliminaries

Let \mathcal{M} be a von Neumann algebra over a separable Hilbert space H . We define a quantum dynamical semigroup $\alpha = \{\alpha_t\}_{t \geq 0}$ as a σ -weakly continuous one parameter semigroup of normal positive unital mappings on \mathcal{M} . Then a positive unital mapping α_t satisfies $\|\alpha_t\| = 1$ and $\alpha_t(A^*) = \alpha_t(A)^*$ for $A \in \mathcal{M}$. We note that σ -weak continuity means that for each $\rho \in \mathcal{M}_*$ and $A \in \mathcal{M}$, the function $t \mapsto \rho(\alpha_t(A))$ is continuous. This is equivalent to $\rho \circ \alpha_t \rightarrow \rho$ as $t \rightarrow 0+$ for each $\rho \in \mathcal{M}_*$. Here \mathcal{M}_* is the predual of \mathcal{M} .

Semigroups of this type have been studied under various positivity requirement such as complete positivity. In this paper, we investigate the asymptotic properties of the semigroup under weaker positivity condition than complete positivity.

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Let \mathbb{C} be the algebra of all complex numbers and M_n be the algebra of all $n \times n$ matrices over \mathbb{C} , and I_n the identity map on M_n . $\mathcal{M} \otimes M_n$ denotes the algebra of all $n \times n$ matrices over \mathcal{M} .

A mapping $\alpha_t : \mathcal{M} \rightarrow \mathcal{M}$ is said to be n -positive if $\alpha_t \circ I_n$ is positive, i.e., for all $A_1, A_2, \dots, A_n, B_1, B_2, \dots, B_n \in \mathcal{M}$,

$$\sum_{i,j=1}^n B_i^* \alpha_t(A_i^* A_j) B_j \geq 0.$$

If the above inequality holds for all $n = 1, 2, \dots$, α_t is said to be completely positive.

The semigroup $\alpha = \{\alpha_t\}_{t \geq 0}$ is said to satisfy Schwarz inequality if

$$\alpha_t(A^* A) \geq \alpha_t(A)^* \alpha_t(A) \quad \text{for all } t \geq 0.$$

2-positivity implies Schwarz inequality and Schwarz inequality implies positivity(positive preserving).

We assume that the semigroup admits a faithful, normal, α -invariant state ω on \mathcal{M} . Invariance means

$$\omega \circ \alpha_t = \omega \quad \text{for all } t \geq 0.$$

The triplet $(\mathcal{M}, \alpha, \omega)$ is called a quantum dynamical system or quantum Markov system.

We can consider the following construction. By GNS representation we may assume ω is a vector state from a cyclic and separating vector ξ_0 , that is,

$$\omega(A) = (\xi_0, A\xi_0)$$

and so $H = [\mathcal{M}\xi_0]$. For any subset of \mathcal{L} of H , $[\mathcal{L}]$ denotes the closed linear subspace of H generated by \mathcal{L} . If we assume that a dynamical system $(\mathcal{M}, \alpha, \omega)$ satisfies Schwarz inequality, then we can construct a strongly continuous contraction semigroup $T = \{T_t\}_{t \geq 0}$ on $H = [\mathcal{M}\xi_0]$ such that

$$T_t A \xi_0 = \alpha_t(A) \xi_0 \quad \text{for } A \in \mathcal{M} \text{ and } t \geq 0.$$

Since

$$\begin{aligned} \|T_t A \xi_0\|^2 &= \|\alpha_t(A) \xi_0\|^2 \\ &= \langle \xi_0, \alpha_t(A)^* \alpha_t(A) \xi_0 \rangle = \omega(\alpha_t(A)^* \alpha_t(A)) \\ &\leq \omega(\alpha_t(A^* A)) = \omega(A^* A) = \|A \xi_0\|^2 \end{aligned}$$

for $A \in \mathcal{M}$, T_t is a contraction on $\mathcal{M}\xi_0$ and so can be extended on H . In the fourth inequality, we used the fact that α_t satisfies Schwarz inequality.

Now we introduce the notion of Jordan product. This makes us to replace Schwarz inequality condition by simple positivity. A linear space $\mathcal{A} \subset \mathbb{B}(\mathcal{H})$ is called a JW^* -algebra if it is σ -weakly closed, contains the identity operator and is closed w.r.t. the Jordan product \circ ;

$$A \circ B = AB + BA \quad \text{for } A, B \in \mathcal{A}.$$

A linear map $\phi : \mathcal{M} \rightarrow \mathcal{M}$ is called a Jordan morphism if

$$\begin{aligned} \phi(A^*) &= \phi(A)^* \\ \phi(A \circ B) &= \phi(A) \circ \phi(B) \end{aligned}$$

for $A, B \in \mathcal{M}$.

We consider the following construction.

Define an inner product on \mathcal{M} by

$$(A, B)_0 = \omega(A^* \circ B) \quad \text{for } A, B \in \mathcal{M}.$$

Let $\|\cdot\|_0$ be the induced norm and \mathcal{H} be the Hilbert space completion of \mathcal{M} w.r.t. this norm. Define the operator U_t on \mathcal{M} by

$$U_t A = \alpha_t(A) \quad \text{for } A \in \mathcal{M}.$$

Then we have

$$\|U_t A\|_0^2 = \omega(\alpha_t(A)^* \circ \alpha_t(A)) \leq \omega(\alpha_t(A^* \circ A)) = \omega(A^* \circ A) = \|A\|_0^2.$$

Second inequality holds since every element in \mathcal{M} is normal element w.r.t. the Jordan product \circ ([1] Proposition 3.2.4). Thus U_t is a contraction on \mathcal{M} and so can be extended to the whole space \mathcal{H} . Denote it also by U_t . Since $(U_t(A), B)_0 = \omega(\alpha_t(A)^* \circ B)$ for $A, B \in \mathcal{M}$ and the map $t \rightarrow \alpha_t(A)$ is σ -weakly continuous, $U = \{U_t\}_{t \geq 0}$ is a weakly continuous contraction semigroup on H_0 .

2. Eigenvalues and eigenspaces

In 1982, Watanabe [13] investigate the asymptotic behaviors of semigroups α and T in terms of their eigenvalues under the assumption that it satisfies Schwarz inequality. We obtain the corresponding analogue under simple positivity introducing Jordan product.

We introduce the notions of eigenvalues and eigenvectors of semigroups $U = \{U_t\}_{t \geq 0}$ and $\alpha = \{\alpha_t\}_{t \geq 0}$, respectively.

For nonzero η in \mathcal{H} , if there exists a complex number $\chi(t)$ such that

$$U_t(\eta) = \chi(t)\eta \quad \text{for all } t \geq 0$$

then $\chi(t) = e^{izt}$ for some $z \in \mathbb{C}^+ = \{x + iy | y \geq 0\}$ since U_t is a contraction.

For each $z \in \mathbb{C}^+$, let

$$\mathcal{H}_z = \{\eta \in \mathcal{H} | U_t(\eta) = e^{izt}\eta \quad \text{for all } t \geq 0\}.$$

Then it is a closed linear subspace of \mathcal{H} . If $\mathcal{H}_z \neq 0$, then z is called an eigenvalue of T and each $\eta \in \mathcal{H}_z$, an eigenvector of T corresponding to z and \mathcal{H}_z , an eigenspace corresponding to z . We concentrate on real eigenvalues. Let $E(U)$ be the set of all real eigenvalues of U and let $P_z : \mathcal{H} \rightarrow \mathcal{H}_z$ be the orthogonal projection of \mathcal{H} onto \mathcal{H}_z .

On the other hand, for each $z \in \mathbb{C}^+$,

$$\mathcal{M}_z = \{A \in \mathcal{M} | \alpha_t(A) = e^{izt}A \quad \text{for all } t \geq 0\}$$

is a σ -weakly closed linear subspace of \mathcal{M} and in particular, \mathcal{M}_0 is JW^* subalgebra of \mathcal{M} . This follows from Schwarz-type inequality of α w.r.t. Jordan product and the faithfulness of ω . If $\mathcal{M}_z \neq 0$, then z is called an eigenvalue of α and each $A \in \mathcal{M}_z$, an eigenvector of α corresponding to z and \mathcal{M}_z an eigenspace corresponding to z . Note that the set $E(\alpha)$ of all real eigenvalues of α is the symmetric set.

Let \mathcal{N} be the σ -weak closure of the linear span of all $\mathcal{M}_\lambda, \lambda \in E(\alpha)$. Then \mathcal{N} is a JW^* -algebra. We call it the eigenalgebra of α . Let \mathcal{K} be the closed linear space of \mathcal{H} generated by $\{\mathcal{H}_\lambda : \lambda \in E(T)\}$, i.e.,

$$\mathcal{N} = \overline{\cup_{\lambda \in E(\alpha)} \mathcal{M}_\lambda}^{\sigma\text{-weak}}, \quad \mathcal{K} = [\cup_{\lambda \in E(T)} \mathcal{H}_\lambda]_0$$

where $[\mathcal{L}]_0$ means the completion of \mathcal{L} w.r.t. $(\cdot, \cdot)_0$.

If we let δ and L be the infinitesimal generators of α and T , respectively, then the eigenvalues defined above coincide with the usual ones of the (unbounded) operators δ and L . However the present definition is more convenient for our discussions. Note that eigenvalues may be defined for any dynamical semigroup.

As in the case of Hilbert space, we can construct the projection of \mathcal{M} onto \mathcal{M}_λ ($\lambda \in E(\alpha)$).

PROPOSITION 2.1 ([7]). *Let $(\mathcal{M}, \alpha, \omega)$ be a quantum dynamical system. Then, for $\lambda \in E(\alpha)$, there exists a normal norm one projection ϵ_λ of \mathcal{M} onto \mathcal{M}_λ which has the following properties:*

- (1) For $A \in \mathcal{M}$, $\epsilon_\lambda(A) = P_\lambda A$.
- (2) $\epsilon_\lambda \circ \alpha_t = \alpha_t \circ \epsilon_\lambda = e^{i\lambda t} \epsilon_\lambda$ ($t \geq 0$).
- (3) $\omega \circ \epsilon_\lambda = 0$ ($\lambda \neq 0$).
- (4) $\epsilon_\lambda \circ \epsilon_\mu = 0$ ($\lambda \neq \mu$).

$$(5) \ \epsilon_\lambda(A)^* = \epsilon_{-\lambda}(A^*) \quad (A \in \mathcal{M}).$$

We give two results that show the relation of eigenvalues and eigenspaces of α and U . One is proved in [8] and the other is our result.

PROPOSITION 2.2. *Let $(\mathcal{M}, \alpha, \omega)$ be a quantum dynamical system. Then we have*

- (1) $E(\alpha) = E(U)$
- (2) $[\mathcal{M}_\lambda]_0 = \mathcal{H}_\lambda$.

THEOREM 2.3. *Let $(\mathcal{M}, \alpha, \omega)$ be a quantum dynamical system. Then we have*

$$[\mathcal{N}]_0 = [\mathcal{K}].$$

Proof. Since $\mathcal{N} \subset \mathcal{K}$ is clear, choose $h \in \mathcal{K}$ and $h \perp [\mathcal{N}]_0$. It suffices to show that $(h, A)_0 = 0$ for all $A \in \mathcal{M}$. Since $A \in \mathcal{H}$, A is decomposed as $A = A' + A''$, where $A' \in \mathcal{K}$ and $A'' \in \mathcal{K}^\perp$. If we let $P : \mathcal{H} \rightarrow \mathcal{K}$ be a projection, then $A' = PA$. Since there exists $\lambda \in \mathbb{R}$ such that $\|PA - P_\lambda A\|_0 < \epsilon$ for any $\epsilon > 0$ and $P_\lambda A = \epsilon_\lambda(A) \in \mathcal{M}_\lambda \subset \mathcal{N}$, we get $A' = PA \in \mathcal{N}$. Thus $(h, A')_0 = 0$ and so $(h, A)_0 = (h, A')_0 + (h, A'')_0 = 0$. \square

We have a result that says α_t 's act as Jordan morphisms on M_μ under the simple positivity.

THEOREM 2.4. *For $A \in \mathcal{M}_\mu$ and for all $t \geq 0$, $\alpha_t(A^* \circ A) = \alpha_t(A^*) \circ \alpha_t(A)$ holds.*

Proof. Note that every element in \mathcal{M} is normal w.r.t. Jordan product. Then

$$\begin{aligned} \alpha_t(A^* \circ A) &\geq \alpha_t(A)^* \circ \alpha_t(A) \\ &= (e^{i\mu s} A)^* \circ e^{i\mu s} A \\ &= A^* \circ A. \end{aligned}$$

On the other hand, because of α -invariance of ω , $\omega(\alpha_t(A^* \circ A) - A^* \circ A) = 0$.

By the faithfulness of ω ,

$$\begin{aligned} \alpha_t(A^* \circ A) &= A^* \circ A \\ &= (e^{i\mu s} A)^* \circ e^{i\mu s} A \\ &= \alpha_t(A)^* \circ \alpha_t(A). \end{aligned}$$

\square

Now we can extend the algebra on which α_t 's act as Jordan morphism. In the proof we adopted a slight modification of those which is given in [8].

PROPOSITION 2.5. *Let*

$$\mathcal{B} = \{A \in \mathcal{M} | \alpha_t(A^* \circ A) = \alpha_t(A^*) \circ \alpha_t(A) \text{ for all } t \geq 0\}.$$

Then

- (1) \mathcal{B} is a JW^* -subalgebra of \mathcal{M} .
- (2) $A \in \mathcal{B}$ implies $\alpha_t(A) \in \mathcal{B}$ for each $t \geq 0$.
- (3) $\alpha_t|_{\mathcal{B}}$ is an injective Jordan morphism.

Proof. (1) Let ϕ be a normal positive unital mapping on \mathcal{M} and

$$\mathcal{B}_\phi = \{A \in \mathcal{M} : \phi(A^* \circ A) = \phi(A^*) \circ \phi(A)\}.$$

Since \mathcal{B} is the intersection of \mathcal{B}_ϕ 's with $\phi = \alpha_t$ w.r.t. all $t \geq 0$, it suffices to show that \mathcal{B}_ϕ is a JW^* -algebra of \mathcal{M} .

For $A, B \in \mathcal{B}_\phi$,

$$(1) \quad \begin{aligned} \phi[(A+B)^* \circ (A+B)] &= \phi(A^*) \circ \phi(A) + \phi(A^* \circ B) \\ &\quad + \phi(B^* \circ A) + \phi(B^*) \circ \phi(B), \end{aligned}$$

$$(2) \quad \begin{aligned} \phi[(A+B)^*] \circ \phi(A+B) &= \phi(A^*) \circ \phi(A) + \phi(A^*) \circ \phi(B) \\ &\quad + \phi(B^*) \circ \phi(A) + \phi(B^*) \circ \phi(B). \end{aligned}$$

From Schwarz-type inequality for Jordan product \circ , we obtain

$$\phi(A^* \circ B) + \phi(B^* \circ A) \geq \phi(A^*) \circ \phi(B) + \phi(B^*) \circ \phi(A).$$

Also, replacing A by $-A$ in the above inequality, we get the reverse inequality and so we have

$$\phi(A^* \circ B) + \phi(B^* \circ A) = \phi(A^*) \circ \phi(B) + \phi(B^*) \circ \phi(A).$$

Again, from this inequality, (1) and (2),

$$\phi[(A+B)^* \circ (A+B)] = \phi[(A+B)^*] \circ \phi(A+B)$$

which implies $A+B \in \mathcal{B}_\phi$.

Since, for every $A = A^* \in \mathcal{B}_\phi$,

$$\phi(A^2) = [\phi(A)]^2 \quad \text{and} \quad \phi(A^4) = [\phi(A)]^4,$$

we get, for any $A \in \mathcal{M}$,

$$\phi(A^2 \circ A^2) = 2[\phi(A)]^4 = [\phi(A)]^2 \circ [\phi(A)]^2 = \phi(A^2) \circ \phi(A^2).$$

This means $A^2 \in \mathcal{B}_\phi$. Since $A \circ B = (A+B)^2 - A^2 - B^2$, \mathcal{B}_ϕ is closed w.r.t. Jordan product.

By Schwarz-type inequality w.r.t. Jordan product \circ ,

$$D[A, B] \equiv \phi(A^* \circ B) - \phi(A^*) \circ \phi(B)$$

defines a positive sesquilinear form on \mathcal{B}_ϕ . From the proof of ([9]) we have that the fact $D[A, A] = 0$ for some $A \in \mathcal{B}_\phi$ implies $D[A, B] = 0$ for all $B \in \mathcal{B}_\phi$, i.e.,

$$(3) \quad \phi(A \circ B) = \phi(A) \circ \phi(B) \quad \text{for } A, B \in \mathcal{B}_\phi .$$

Since

$$\begin{aligned} \mathcal{B}_\phi &= \{A \in \mathcal{M} : \phi(A \circ B) = \phi(A) \circ \phi(B) \quad \text{for each } B \in \mathcal{M}\} \\ &= \bigcap_{B \in \mathcal{B}_\phi} \{A \in \mathcal{M} : \phi(A \circ B) = \phi(A) \circ \phi(B)\} \\ &\equiv \bigcap_{B \in \mathcal{B}_\phi} \mathcal{B}_{\phi, B} \end{aligned}$$

and $\mathcal{B}_{\phi, B}$ is closed in σ -weak topology, \mathcal{B}_ϕ is σ -weakly closed.

(2) Let $t \geq 0$ be fixed and $A \in \mathcal{B}$. For each $s \geq 0$,

$$\begin{aligned} \alpha_s(\alpha_t(A)^* \circ \alpha_t(A)) &= \alpha_s(\alpha_t(A^* \circ A)) = \alpha_{s+t}(A^* \circ A) \\ &= \alpha_{s+t}(A^*) \circ \alpha_{s+t}(A) = \alpha_s(\alpha_t(A)^*) \circ \alpha_s(\alpha_t(A)). \end{aligned}$$

(3) If $\alpha_t(A) = 0$ for $A \in \mathcal{B}$,

$$\alpha_t(A^* \circ A) = \alpha_t(A^*) \circ \alpha_t(A) = 0.$$

So,

$$\omega(A^* \circ A) = \omega(\alpha_t(A^* \circ A)) = 0.$$

The faithfulness of ω yields $A = 0$. □

3. Ergodicity and mixing conditions

We say a dynamical system $(\mathcal{M}, \alpha, \omega)$ is ergodic if $\mathcal{M}_0 = \mathbb{C}\mathbf{1}$ and a contraction semigroup U on \mathcal{H} is ergodic if its fixed point space \mathcal{H}_0 is one dimensional.

DEFINITION 3.1. A dynamical system $(\mathcal{M}, \alpha, \omega)$ is said to be weak mixing if

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\omega(\alpha_s(A)B) - \omega(A)\omega(B)| ds = 0$$

for all $A, B \in \mathcal{M}$.

DEFINITION 3.2. A contraction semigroup $U = \{U_t\}_{t \geq 0}$ on a Hilbert space \mathcal{H} is said to be weak mixing if for any $h_1, h_2 \in \mathcal{H}$

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |(U_s h_1, h_2)_0 - (P_0 h_1, h_2)_0| ds = 0$$

where P_0 is the projection onto the fixed point space \mathcal{H}_0 of U .

It is known that the weak mixing implies ergodicity.

THEOREM 3.3. Let $(\mathcal{M}, \alpha, \omega)$ be ergodic. Then $\omega \circ \epsilon_0 = \omega$ if and only if $\epsilon_0(A) = \omega(A)\mathbf{1}$ for $A \in \mathcal{M}$.

Proof. Since α is ergodic, $\epsilon_0(A) = \gamma(A)\mathbf{1}$ for some constant $\gamma(A) \in \mathbb{C}$. Then

$$\omega(A) = \omega(\epsilon_0(A)) = \gamma(A)\omega(\mathbf{1}) = \gamma(\mathbf{A})$$

and so

$$\epsilon_0(A) = \omega(A)\mathbf{1}.$$

Conversely,

$$\omega \circ \epsilon_0(A) = \omega(\omega(A)\mathbf{1}) = \omega(\mathbf{A}).$$

□

Now we obtain the following result whose proof is motivated by those of [8] and [13].

THEOREM 3.4. Let $(\mathcal{M}, \alpha, \omega)$ be a quantum dynamical system and let $U = \{U_t\}_{t \geq 0}$ be the associated contraction semigroup. Consider three assertions:

- (1) $(\mathcal{M}, \alpha, \omega)$ is weak mixing.
- (2) $E(\alpha) = \{0\}$ and $\mathcal{M}_0 = \mathbb{C}\mathbf{1}$.
- (3) U is weak mixing.

Then we have (1) \Rightarrow (2) \Rightarrow (3). If $\epsilon_0(A) = \omega(A)\mathbf{1}$, then (3) \Rightarrow (1) holds.

Proof. (1) \Rightarrow (2) Fix $t \geq 0$ arbitrary and take fixed element A of α_t . Then weak mixing condition implies $A \in \mathbb{C}\mathbf{1}$. i.e., we have $\{A \in \mathcal{M} | \alpha_t(A) = A\} = \mathbb{C}\mathbf{1}$ for each $t \geq 0$.

If $0 \neq \lambda \in E(\alpha)$, there exists a nonzero $A \in \mathcal{M}$ s.t. $\alpha_t(A) = e^{i\lambda t}A \quad \forall t \geq 0$. For $t = \frac{2\pi}{\lambda}$, $\alpha_{\frac{2\pi}{\lambda}}(A) = A$ and so A is a fixed point of $\alpha_{\frac{2\pi}{\lambda}}$. Then $A = k_0\mathbf{1}$ for some $k_0 \in \mathbb{C}$. But this is impossible for nonzero A since $\lambda \neq 0$. Thus $E(\alpha) = \{0\}$.

(2) \Rightarrow (3) Note that assertion (2) is equivalent to $E(U) = \{0\}$ and the corresponding eigenspace \mathcal{H}_0 is one dimensional. Let $\tilde{U} = \{\tilde{U}_t\}_{t \geq 0}$ be a minimal unitary dilation of $U = \{U_t\}_{t \geq 0}$ in a Hilbert space \tilde{H}_0 . Then it

is known that $E(U) = E(\tilde{U})$ and the corresponding eigenvectors are the same ([9]).

So we have $E(\tilde{U}) = 0$ and the fixed point space of \tilde{U} in \mathcal{H} is $\{ch_0 | c \in \mathbb{C}\}$ for some $0 \neq h_0 \in \mathcal{H}$ with $\|h_0\|_0 = 1$. Then from the classical ergodic theory for unitary operators we know

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |(\tilde{U}_s \tilde{h}_1, \tilde{h}_2)_0 - (\tilde{h}_1, h_0)_0 (h_0, \tilde{h}_2)_0| ds = 0$$

where $(\cdot, \cdot)_0$ is the inner product in $\tilde{\mathcal{H}}$ and $\tilde{h}_1, \tilde{h}_2 \in \tilde{\mathcal{H}}$. In particular, for $h_1, h_2 \in \mathcal{H}$,

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |(U_s h_1, h_2)_0 - (h_1, h_0)_0 (h_0, h_2)_0| ds = 0.$$

This is nothing but the form of weak mixing.

(3) \Rightarrow (1) Since $\epsilon_0(A) = P_0 A$ for $A \in \mathcal{M}$, we have

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |(\alpha_s(A)^*, B)_0 - (\epsilon_0(A)^*, B)_0| ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\omega(\alpha_s(A) \circ B) - \omega(\epsilon_0(A) \circ B)| ds \end{aligned}$$

for $A, B \in \mathcal{M}$. Note that the set of functionals $\{\omega(\cdot \circ B) | B \in \mathcal{M}\}$ is norm dense in \mathcal{M}_* by the faithfulness of ω .

Then we have for every $\rho \in \mathcal{M}_*$,

$$\begin{aligned} 0 &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\rho(\alpha_s(A)) - \rho(\epsilon_0(A))| ds \\ &= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\rho(\alpha_s(A)) - \omega(A)\rho(1)| ds. \end{aligned}$$

In the second equality we used the fact that weak mixing implies ergodicity. Taking $\rho(\cdot) = \omega(\cdot B)$ for arbitrary but fixed $B \in \mathcal{M}$, we get

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t |\omega(\alpha_s(A)B) - \omega(A)\omega(B)| = 0.$$

The proof is complete. □

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