Quantum $E(2)$ group and its Pontryagin dual

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Abstract

The quantum deformation of the group of motions of the plane and its Pontryagin dual are described in details. It is shown that the Pontryagin dual is a quantum deformation of the group of transformations of the plane generated by translations and dilations. An explicite expression for the unitary bicharacter describing the Pontryagin duality is found. The Heisenberg commutation relations are written down.

0 Introduction

The quantum deformation of the (two-fold covering of the) group of motions of the Euclidean plane was introduced and investigated in [7] (von Neumann level), [9] (C^*-level) and [3] (Hopf algebra level only). We follow the approach of [9].

The paper is organized in the following way. In Section 1 we remind the basic definitions concerning the quantum $E(2)$ group. In particular the algebra A of all "continuous functions on $E_\mu(2)$ vanishing at infinity" is introduced and the existence of the comultiplication $\Phi \in \mathrm{Mor}(A, A \otimes A)$ is proved. In Section 2 we investigate the structure of the unitary representations of $E_\mu(2)$. The main result (Theorem 2.1) shows that any representation is determined by a pair of operators $(\widetilde{N}, \widetilde{b})$ satisfying certain commutation relations. An explicite formula expressing the representation in terms of $\widetilde{N}$ and $\widetilde{b}$ is given. The formula contains the special function $F_\mu(\cdot)$ introduced in [10]. The proof of Theorem 2.1 heavily depends on the results of this paper.

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In Section 3 we construct the universal C^*-algebra B "generated" by two elements N and b satisfying the commutation relations discovered in Section 2. According to the general framework described in [8] B is the algebra of "continuous functions on $\widehat{E}_\mu(2)$ vanishing at infinity", where $\widehat{E}_\mu(2)$ denotes the Pontryagin dual of $E_\mu(2)$.

The comultiplication $\widehat{\Phi} \in \mathrm{Mor}(B, B \otimes B)$ is introduced in Section 4. This way $\widehat{E}_\mu(2)$ is endowed with the group structure. We compute $\widehat{\Phi}(N)$ and $\widehat{\Phi}(b)$ and use the results to identify $\widehat{E}_\mu(2)$ with a quantum deformation of the group of transformation of the Euclidean plane generated by translations and dilations.

Let $w \in M(B \otimes A)$ be the universal bicharacter describing the $\widehat{E}_\mu(2)$ — $E_\mu(2)$ duality. It should be pointed out that we have an explicite formula (25) expressing w in terms of the "generators" of B and A. No such formula is known for $S_\mu U(2)$ case (The Pontryagin dual of $S_\mu U(2)$ is described in [6]). We strongly belive that reaching the better understanding of the contraction procedure of [3] and using (25) one obtains the corresponding formula for $S_\mu U(2)$.

In Section 5 we derive the Heisenberg commutation relations related to the $E_\mu(2)$ -group.

A few remarks about the notation. Any C^*-algebra considered in this paper is concrete i.e. embeded into $B(H)$, where H is a Hilbert space. Consequently elements of the algebra and the elements affiliated with it are operators on H. In what follows, $C^*(H)$ denotes the set of all C^*-algebras embeded (in a non-degenerate way) into $B(H)$. The C^*-algebra of all compact operators acting on H will be denoted by $CB(H)$:$CB(H) \in C^*(H)$.

Let $A \in C^*(H)$ and T be a closed operator acting on H. We recall (cf [1],[9]) that $T\eta A$ (T is affiliated with A) if and only if $T(I + T^*T)^{-1/2} \in M(A)$ and $(I + T^*T)^{-1/2}A$ is dense in A. $M(A)$ denotes the multiplier algebra (cf [5], Chapter 3.12):

$$M(A) = \left\{ x \in B(H) : \begin{array}{c} xa, ax \in A \\ \text{for any} a \in A \end{array} \right\}.$$

The natural topology on $M(A)$ is that of the almost uniform convergence. We say that a sequence $(x_n)_{n\in\mathbf{Z}}$ of elements of $M(A)$ converges almost uniformly to 0 if for every $a \in A$ we have $\| x_n a \| \to 0$ and $\| a x_n \| \to 0$. Whenever in this paper we refer to topological properties of a multiplier algebra (e.g. considering continuous mappings from or into $M(A)$) we always have in mind the topology of almost uniform convergence. Let us notice that on $B(H) = M(CB(H))$ this topology coincides with *-strong operator topology.

Let $A_1 \in C^*(H_1)$, $A_2 \in C^*(H_2)$ and π be a representation of A_1 acting on H_2. Then (cf [8]) $\pi \in \mathrm{Mor}(A_1, A_2)$ if and only if $\pi(A_1) \subset M(A_2)$ and $\pi(A_1)$ contains an approximate unity for A_2. Any $\pi \in \mathrm{Mor}((A_1, A_2)$ admits the natural extension to the set of all elements affiliated with A_1: For any $T\eta A_1$ we have $\pi(T)\eta A_2$. Due to this fact the morphisms may be composed and we may speak about the C^*-algebra category ([8]). One may show that $\pi \in \mathrm{Mor}(A_1, A_2)$ if and only if π is a continuous mapping $M(A_1) \to M(A_2)$ preserving the algebraic structure (including the unity and the hermitian conjugation)

For any C^*-algebras A and B, $A \otimes B$ will denote the minimal tensor product of A and B. If $A \in C^*(H)$ and $B \in C^*(K)$ then in a natural way $A \otimes B \in C^*(H \otimes K)$.

We shall freely use the functional calculus of strongly commuting normal operators (see e.g. [11]). If n is a normal operator acting on a Hilbert space H then its spectral measure

$E_n(\cdot)$ is supported by $\mathrm{Sp}\, n$ and for any measurable function f on $\mathrm{Sp}\, n$ with values in $B(K)$ (where K is another Hilbert space) we set

$$\int_{\mathrm{Sp}\, n} f(\lambda) \otimes dE_n(\lambda).$$

Clearly[1] $f(n) \in B(K \otimes H)$.

If $n \eta A$ (where $A \in C^*(H)$) then there exists unique $\phi_n \in \mathrm{Mor}(C_\infty(\mathrm{Sp}\, n), A)$ such that $n = \phi_n(\zeta_{\mathrm{Sp}\, n})$ where $\zeta_{\mathrm{Sp}\, n} \eta C_\infty(\mathrm{Sp}\, n)$ is introduced by the formula $\zeta_{\mathrm{Sp}\, n}(t) = t$ for any $t \in \mathrm{Sp}\, n$ (we remind that the elements affiliated with $C_\infty(\mathrm{Sp}\, n)$ are just continuous complex valued functions on $\mathrm{Sp}\, n$). If moreover the values of $f(\cdot)$ belong to $M(B)$ (where $B \in C^*(K)$) and the mapping $f : \mathrm{Sp}\, n \to M(B)$ is continuous then $f \in M(B \otimes C_\infty(\mathrm{Sp}\, n))$ and

$$f(n) = (\mathrm{id} \otimes \phi_n) f$$

It shows that $f(n) \in M(B \otimes A)$.

Let N be a selfadjoint operator with the integer spectrum and v be a unitary operator. Then (cf the functional calculus of many strongly commuting normal operators)

$$(I \otimes v)^{N \otimes I} = \int_{\mathbf{Z} \times S^1} z^s dE_N(s) \otimes dE_v(z)$$

where $dE_N(\cdot)$ and $dE_v(\cdot)$ are spectral measures of N and v resp. The reader should notice that $(I \otimes v)^{N \otimes I} = f_1(v)$ where $F_1(z) = z^N$ for any $z \in S^1$ (and (cf the footnote) $(I \otimes v)^{N \otimes I} = f_2(N)$, where $f_2(s) = v^s$ for any $s \in \mathbf{Z}$).

In our proofs we often refer to the results of [10] leaving to the reader the verification of the assumptions of the theorems being used. In this place we recall two basic definitions:

$$\overline{\mathbf{C}}^\mu = \{ t \in \mathbf{C} : t = 0 \text{ or } \mid t \mid \in \mu^{\mathbf{Z}} \}$$

$$F_\mu(t) = \prod_{k=0}^{\infty} \frac{1 + \mu^{2k} \overline{t}}{1 + \mu^{2k} t}$$

For $t = -\mu^{-2k}$ $(k = 0, 1, \ldots)$ we set $F_\mu(t) = -1$. Then $F_\mu \in C(\overline{\mathbf{C}}^\mu)$.

We shall also use the leg numbering notation. It is used to embed objects belonging to a tensor product of algebras into the tensor product involving more factors. For example if $w \in A \otimes A$ then $w_{12} = \phi_{12}(w)$, $w_{13} = \phi_{13}(w)$ and $w_{23} = \phi_{23}(w)$ where ϕ_{12}, ϕ_{13} and ϕ_{23} are elements of $\mathrm{Mor}(A \otimes A, A \otimes A \otimes A)$ introduced by the formulae $\phi_{12}(a \otimes b) = a \otimes b \otimes I$, $\phi_{13}(a \otimes b) = a \otimes I \otimes b$ and $\phi_{23}(a \otimes b) = I \otimes a \otimes b$ where $a, b \in A$.

For the other notation not explained above we refer to our previous papers.

[1] Sometimes $f(n)$ denotes $\int_{\mathrm{Sp}\, n} dE_n(\lambda) \otimes f(\lambda)$. Then $f(n) \in B(H \otimes K)$. In a more complicated situation when $K = K_1 \otimes K_2$, $f(n)$ may denote (cf leg numbering notation) $\int_{\mathrm{Sp}\, n} f(\lambda)_{13} dE_n(\lambda)_2$. In this case $f(n) \in B(K_1 \otimes H \otimes K_2)$. It is always clear from the context, which notation is actually used.

1 Quantum $E(2)$ group

Let us fix the deformation parameter $\mu \in]0,1[$. The (non-commutative) C^*-algebra A of all "continuous functions on $E_\mu(2)$ vanishing at infinity" may be described in the following way:

Let $(e_{kl})_{k,l-\text{integers}}$ be the canonical basis in $H = l^2(\mathbf{Z}\times\mathbf{Z})$. For any $k,l \in \mathbf{Z}$ we set:

$$\left.\begin{array}{rcl} ve_{kl} & = & e_{k-1,l} \\ ne_{kl} & = & \mu^k e_{k,l+1} \end{array}\right\} \tag{1}$$

Then v is a unitary and n is a normal operator acting on H and $\mathrm{Sp}\,(n) \subset \overline{\mathbf{C}}^\mu$. By definition A is the norm closure of the set of all operators of the form $\sum v^k f_k(n)$ where k runs over a finite subset of $\mathbf{Z}$ and $f_k \in C_\infty(\overline{\mathbf{C}}^\mu)$. Then A is a non-unital C^*-algebra and $v, n \eta A$. Moreover

$$vnv^* = \mu n \tag{2}$$

The reader easily verifies that A has the following *universality* property:

Theorem 1.1 *1. Let π be a representation of A in a Hilbert space $\widetilde{H}$ and*

$$\left.\begin{array}{rcl} \widetilde{v} & = & \pi(v) \\ \widetilde{n} & = & \pi(n) \end{array}\right\} \tag{3}$$

Then

$$\left.\begin{array}{l} \widetilde{v} \textit{ is unitary} \\ \widetilde{n} \textit{ is normal and } \mathrm{Sp}\,\widetilde{n} \subset \overline{\mathbf{C}}^\mu \\ \widetilde{v}\widetilde{n}\widetilde{v}^* = \mu\widetilde{n} \end{array}\right\} \tag{4}$$

2. Any pair of operators $(\widetilde{v},\widetilde{n})$ acting on a Hilbert space $\widetilde{H}$ and satisfying the relations (4) is of the form (3), where π is a representation of A in $\widetilde{H}$. The representation π is uniquely determined by $(\widetilde{v},\widetilde{n})$.

3. Let π, $\widetilde{H}$, $\widetilde{v}$, $\widetilde{n}$ be as in Statement 1 and $\widetilde{A} \in C^*(\widetilde{H})$. Then

$$(\widetilde{v}, \widetilde{n} \eta \widetilde{A}) \Longleftrightarrow (\pi \in \mathrm{Mor}(A, \widetilde{A})).$$

We shall use this theorem to introduce the comultiplication on A. Let $\widetilde{v} = v \otimes v$ and $\widetilde{n} = v \otimes n \dot{+} n \otimes v^*$. Clearly $\widetilde{v}$ is a unitary element of $M(A\otimes A)$. Using Theorems 2.1 and 5.1 of [10] one can easily verify that $\widetilde{n}$ is a normal element affiliated with $A \otimes A$ and that $\mathrm{Sp}\,\widetilde{n} \subset \overline{\mathbf{C}}^\mu$. Obviously $\widetilde{v}\widetilde{n}\widetilde{v}^* = \mu\widetilde{n}$. By virtue of Theorem 1.1 we get

Theorem 1.2 *There exists unique $\Phi \in \mathrm{Mor}(A, A\otimes A)$ such that*

$$\left.\begin{array}{rcl} \Phi(v) & = & v \otimes v \\ \Phi(n) & = & v \otimes n \dot{+} n \otimes v^* \end{array}\right\} \tag{5}$$

One can easily verify (cf [9], Section 3) that Φ is coassociative: $(\Phi\otimes\mathrm{id})\Phi = (\mathrm{id}\otimes\Phi)\Phi$. The quantum space $E_\mu(2)$ endowed with the associative binary operation induced by Φ is the quantum group investigated in this paper.

Let $z\in S^1$. By virtue of Theorem 1.1 there exists unique $\varphi_z\in\mathrm{Mor}(A,\mathbf{C})$ such that

$$\left.\begin{array}{rcl}\varphi_z(v) & = & z\\ \varphi_z(n) & = & 0\end{array}\right\} \tag{6}$$

Using (5) one can easily verify that

$$\varphi_z*\varphi_{z'}=\varphi_{zz'} \tag{7}$$

for any $z,z'\in S^1$. For $z=1$ we have $\varphi_1*v=v*\varphi_1=v$ and $\varphi_1*n=n*\varphi_1=n$. Therefore $\varphi_1*a=a*\varphi_1=a$ for any $a\eta A$. It means that φ_1 is the counit of $E_\mu(2)$. For $z=-1$ we have $\varphi_{-1}*v=-v=v*\varphi_{-1}$ and $\varphi_{-1}*n=-n=n*\varphi_{-1}$. Therefore

$$\varphi_{-1}*a=a*\varphi_{-1} \tag{8}$$

for any $a\eta A$. It means that φ_{-1} is a central element of the convolution algebra of $E_\mu(2)$.

Proposition 1.3 *Let $\widetilde{B}$ be a C^*-algebra and $\widetilde{w}\in M(\widetilde{B}\otimes A)$. Then the mapping $S^1\ni z\mapsto(\mathrm{id}\otimes\varphi_z)\widetilde{w}\in M(\widetilde{B})$ is continuous.*

Proof: Let us notice that the right hand sides of (6) depend continuously on z. Therefore $\varphi\in\mathrm{Mor}(A,C(S^1))$ and $(\mathrm{id}\otimes\varphi)\in M(\widetilde{B}\otimes C(S^1))$. On the other hand it is known that $M(\widetilde{B}\otimes C(S^1))$ consists of all continuous mappings $S^1\to M(\widetilde{B})$.

Q.E.D

For any $z\in S^1$ and $a\in A$ we set

$$\theta_z(a)=\varphi_z*a*\varphi_z \tag{9}$$

Then $(\theta_z)_{z\in S^1}$ is a one parameter group of automorphisms of A; $\theta_z(v)=z^2v$ and $\theta_z(n)=n$. If f is a continuous bounded complex valued function on $\overline{\mathbf{C}}^\mu$then $f(n)\in M(A)$ and $\theta_z(f(n))=f(n)$. Conversely if $a\in M(A)$ and $\theta_z(a)=a$ then $a=f(n)$ where $f\in C_{\mathrm{bounded}}(\overline{\mathbf{C}}^\mu)$. We have even stronger result:

Proposition 1.4 *Let $\widetilde{B}$ be a C^*-algebra and $a\in M(\widetilde{B}\otimes A)$. Then the following two conditions are equivalent:*

1. For all $z\in S^1$

$$(\mathrm{id}\otimes\theta_z)(a)=a$$

2. The element a is of the form $a=f(n)$, where $f:\overline{\mathbf{C}}^\mu\to M(\widetilde{B})$ is a continuous bounded mapping.

Proof: We shall use the notion of crossed product algebra (cf [4] and [5]). Let σ_μ is the action of $\mathbf{Z}$ on $C_\infty(\overline{\mathbf{C}}^\mu)$ introduced by the formula

$$(\sigma_{\mu n}f)(t)=f(\mu^{-n}t)$$

One can easily verify that

$$\widetilde{B} \otimes A = C_\infty(\overline{\mathbf{C}}^\mu, \widetilde{B}) \times_{\mathrm{id}\otimes\sigma_\mu} \mathbf{Z}$$

and that $S^1 \ni z^2 \mapsto \mathrm{id} \otimes \theta_z$ is the corresponding dual action. Moreover, since $\mathbf{Z}$ is a discrete group, $C_\infty(\overline{\mathbf{C}}^\mu, \widetilde{B})$ is a subalgebra of $\widetilde{B} \otimes A$. The embeding is given by

$$C_\infty(\overline{\mathbf{C}}^\mu, \widetilde{B}) \ni f \mapsto f(n) \in \widetilde{B} \otimes A$$

By virtue of Theorem 7.8.8 of [5], $\left\{ f(n) : f \in C_\infty(\overline{\mathbf{C}}^\mu, \widetilde{B}) \right\}$ coincides with the set of all elements of $\widetilde{B} \otimes A$ that are $(\mathrm{id} \otimes \theta_z)$-invariant.Using this fact and remembering that $M(C_\infty(\overline{\mathbf{C}}^\mu, \widetilde{B})) = C_{\mathrm{bounded}}(\overline{\mathbf{C}}^\mu, M(\widetilde{B}))$ one may show that $\left\{ f(n) : f \in C_{\mathrm{bounded}}(\overline{\mathbf{C}}^\mu, M(\widetilde{B})) \right\}$ coicides with the set of all elements of $M(\widetilde{B} \otimes A)$ that are $(\mathrm{id} \otimes \theta_z)$-invariant.

Q.E.D

2 Unitary representations of $E_\mu(2)$

Let $\widetilde{K}$ be a Hilbert space and $\widetilde{w}$ be a unitary element of $M(CB(\widetilde{K}) \otimes A)$. We say that $\widetilde{w}$ is a (strongly continuous) representations of $E_\mu(2)$ acting on $\widetilde{K}$ if

$$(\mathrm{id} \otimes \Phi)\widetilde{w} = \widetilde{w}_{12}\widetilde{w}_{13} \tag{10}$$

The main result of the paper is contained in the following

Theorem 2.1 *Let $\widetilde{K}$ be a Hilbert space and*

$$\widetilde{w} = F_\mu(\mu^{\widetilde{N}/2}\widetilde{b} \otimes vn)(I \otimes v)^{\widetilde{N} \otimes I} \tag{11}$$

where $\widetilde{N}$ and $\widetilde{b}$ are closed operators acting on $\widetilde{K}$ such that

(i) $\widetilde{N}$ is selfadjoint

(ii) $\widetilde{b}$ is normal

(iii) $\widetilde{N}$ and $|\,\widetilde{b}\,|$ strongly commute

(iv) On $(\ker \widetilde{b})^\perp$

$$(\mathrm{Phase}\, \widetilde{b})^* \widetilde{N} (\mathrm{Phase}\, \widetilde{b}) = \widetilde{N} + 2I$$

(v) The joint spectrum

$$\mathrm{Sp}\,(\widetilde{N}, |\,\widetilde{b}\,|) \subset \overline{\Sigma_\mu}$$

where $\overline{\Sigma_\mu}$ is the closure of the set

$$\Sigma_\mu = \{(s, \mu^r) : s, r - \frac{s}{2} \in \mathbf{Z}\} \tag{12}$$

Then

1. $\widetilde{w}$ is a unitary representation of $E_\mu(2)$ acting on $\widetilde{K}$

2. Any unitary representation of $E_\mu(2)$ is of this form

3. $\widetilde{N}$ and $\widetilde{b}$ are uniquely determined by $\widetilde{w}$. Moreover for any $\widetilde{B} \in C^*(\widetilde{K})$

$$(\widetilde{w} \in M(\widetilde{B} \otimes A)) \Longleftrightarrow (\widetilde{N}, \widetilde{b}\, \eta \widetilde{B}) \tag{13}$$

Proof: Ad 1. If $\widetilde{b}=0$ then $\mu^{\widetilde{N}/2}\widetilde{b}\otimes vn=0$, $\widetilde{w}=(I\otimes v)^{\widetilde{N}\otimes I}$ and the statement is trivial. Therefore we may assume that $\ker\widetilde{b}=0$ and $(\ker\widetilde{b})^{\perp}=\widetilde{K}$.

Let us notice that the right hand sides of the following equations

$$\begin{array}{rcl}\mathrm{Phase}\,(\mu^{\widetilde{N}/2}\widetilde{b}\otimes vn) & = & (\mathrm{Phase}\,\widetilde{b})\otimes v(\mathrm{Phase}\,n)\\ \mid\mu^{\widetilde{N}/2}\widetilde{b}\otimes vn\mid & = & \mu\,\mu^{\widetilde{N}/2}\mid\widetilde{b}\mid\otimes\mid n\mid\end{array}$$

are affiliated with $CB(\widetilde{K})\otimes A$ and mutually commute. Moreover Sp $\mid\mu^{\widetilde{N}/2}\widetilde{b}\otimes vn\mid\subset\mu^{\mathbf{Z}}\cup\{0\}$. Therefore $\mu^{\widetilde{N}/2}\widetilde{b}\otimes vn$ is a normal element affiliated with $CB(\widetilde{K})\otimes A$ and its spectrum is contained in $\overline{\mathbf{C}}^{\mu}$. Consequently $\widetilde{w}$ is a unitary element of $M(CB(\widetilde{K})\otimes A)$.

By virtue of (iii) and (iv) we have

$$\begin{array}{rl} & (\mu^{\widetilde{N}/2}\widetilde{b}\otimes v^2\otimes vn)(I\otimes v\otimes I)^{\widetilde{N}\otimes I\otimes I}\\ = & (I\otimes v\otimes I)^{(\widetilde{N}-2I)\otimes I\otimes I}(\mu^{\widetilde{N}/2}\widetilde{b}\otimes v^2\otimes vn)\\ = & (I\otimes v\otimes I)^{\widetilde{N}\otimes I\otimes I}(\mu^{\widetilde{N}/2}\widetilde{b}\otimes I\otimes vn)\end{array}$$

Therefore

$$F_\mu(\mu^{\widetilde{N}/2}\widetilde{b}\otimes v^2\otimes vn)(I\otimes v\otimes I)^{\widetilde{N}\otimes I\otimes I}=(I\otimes v\otimes I)^{\widetilde{N}\otimes I\otimes I}F_\mu(\mu^{\widetilde{N}/2}\widetilde{b}\otimes I\otimes vn) \qquad (14)$$

We compute

$$(\mathrm{id}\otimes\Phi)(\widetilde{w})=F_\mu(\mu^{\widetilde{N}/2}\otimes\Phi(vn))(I\otimes\Phi(v))^{\widetilde{N}\otimes\Phi(I)}$$

By virtue of (5), $\Phi(vn)=v^2\otimes vn\dotplus vn\otimes I$. Therefore

$$(\mathrm{id}\otimes\Phi)(\widetilde{w})=F_\mu(\mu^{\widetilde{N}/2}\widetilde{b}\otimes vn\otimes I)F_\mu(\mu^{\widetilde{N}/2}\widetilde{b}\otimes v^2\otimes vn)(I\otimes v\otimes v)^{\widetilde{N}\otimes I\otimes I}$$

and finally taking into account (14) we get

$$(\mathrm{id}\otimes\Phi)(\widetilde{w})=\widetilde{w}_{12}\widetilde{w}_{13}$$

where

$$\begin{array}{rcl}\widetilde{w}_{12} & = & F_\mu(\mu^{\widetilde{N}/2}\widetilde{b}\otimes vn\otimes I)(I\otimes v\otimes I)^{\widetilde{N}\otimes I\otimes I}\\ \widetilde{w}_{13} & = & F_\mu(\mu^{\widetilde{N}/2}\widetilde{b}\otimes I\otimes vn)(I\otimes I\otimes v)^{\widetilde{N}\otimes I\otimes I}\end{array}$$

It shows that $\widetilde{w}$ is a representation of $E_\mu(2)$ acting on $\widetilde{K}$.

Ad 2. Let $\widetilde{w}$ be a unitary representation of $E_\mu(2)$ acting on a Hilbert space $\widetilde{K}$. Then for any $z\in S^1$,$(\mathrm{id}\otimes\varphi_z)\widetilde{w}$ is a unitary operator acting on $\widetilde{K}$; according to Prop 1.3 the mapping $S^1\ni z\mapsto(\mathrm{id}\otimes\varphi_z)\widetilde{w}\in B(\widetilde{K})$ is *-strongly continuous and using (7) one can verify that $(\mathrm{id}\otimes\varphi_{zz'})\widetilde{w}=(\mathrm{id}\otimes\varphi_z)\widetilde{w}(\mathrm{id}\otimes\varphi_{z'})\widetilde{w}$. Therefore (Stone‘s theorem) there exists selfadjoint operator $\widetilde{N}$ acting on $\widetilde{K}$ such that Sp $\widetilde{N}\subset\mathbf{Z}$ and

$$(\mathrm{id}\otimes\varphi_z)\widetilde{w}=z^{\widetilde{N}}$$

for any $z \in S^1$.

Let $P = \frac{1}{2}(I - (-1)^{\widetilde{N}})$, $M_+ = \frac{1}{2}(\widetilde{N} + P)$ and $M_- = \frac{1}{2}(\widetilde{N} - P)$. Then M_+ and M_- are selfadjoint strongly commuting operators, $\operatorname{Sp} M_+$,$\operatorname{Sp} M_- \subset \mathbf{Z}$ and $\widetilde{N} = M_+ + M_-$. Using (8) one can check that $(-1)^{\widetilde{N}} \otimes I$ commutes with $\widetilde{w}$. Therefore

$$(P \otimes I)\widetilde{w} = \widetilde{w}(P \otimes I) \tag{15}$$

By virtue of (9)

$$(\mathrm{id} \otimes \theta_z)\widetilde{w} = (z^{\widetilde{N}} \otimes I)\widetilde{w}(z^{\widetilde{N}} \otimes I)$$

On the other hand remembering that $\theta_z v = z^2 v$ we have

$$\begin{aligned} (\mathrm{id} \otimes \theta_z)(I \otimes v)^{M_+ \otimes I} &= (z^{2M_+} \otimes I)(I \otimes v)^{M_+ \otimes I} \\ (\mathrm{id} \otimes \theta_z)(I \otimes v)^{M_- \otimes I} &= (I \otimes v)^{M_- \otimes I}(z^{2M_-} \otimes I) \end{aligned}$$

Comparing the last three relations and using (15) we see that the element $(I \otimes v)^{-(M_+ \otimes I)}\widetilde{w}(I \otimes v)^{-(M_- \otimes I)}$ is $(\mathrm{id} \otimes \theta_z)$-invariant. Therefore (cf Proposition 1.4)

$$\widetilde{w} = (I \otimes v)^{M_+ \otimes I} f(n)(I \otimes v)^{M_- \otimes I} \tag{16}$$

where $f : \overline{\mathbf{C}}^{\mu} \to B(\widetilde{K})$ is a *-strongly continuous mapping and $f(t)$ is unitary for all $t \in \overline{\mathbf{C}}^{\mu}$ (otherwise $\widetilde{w}$ would not be unitary).

Inserting (16) into left hand side of (10) we obtain

$$(I \otimes v \otimes v)^{M_+ \otimes I \otimes I} f(v \otimes n \dot{+} n \otimes v^*)(I \otimes v \otimes v)^{M_- \otimes I \otimes I} = \widetilde{w}_{12}\widetilde{w}_{13} \tag{17}$$

Applying $(\mathrm{id} \otimes \varphi_z \otimes \mathrm{id})$ and $(\mathrm{id} \otimes \mathrm{id} \otimes \varphi_z)$ to the both sides of this relation we get

$$\begin{aligned} (z^{\widetilde{N}} \otimes I)\widetilde{w} &= (I \otimes zv)^{M_+ \otimes I} f(zn)(I \otimes zv)^{M_- \otimes I} \\ \widetilde{w}(z^{\widetilde{N}} \otimes I) &= (I \otimes zv)^{M_+ \otimes I} f(n\overline{z})(I \otimes zv)^{M_- \otimes I} \end{aligned} \tag{18}$$

Inserting v instead of z we get

$$\begin{aligned} (I \otimes v \otimes I)^{\widetilde{N} \otimes I \otimes I}\widetilde{w}_{13} &= (I \otimes v \otimes v)^{M_+ \otimes I \otimes I} f(v \otimes n)(I \otimes v \otimes v)^{M_- \otimes I \otimes I} \\ \widetilde{w}_{12}(I \otimes I \otimes v)^{\widetilde{N} \otimes I \otimes I} &= (I \otimes v \otimes v)^{M_+ \otimes I \otimes I} f(n \otimes v^*)(I \otimes v \otimes v)^{M_- \otimes I \otimes I} \end{aligned}$$

Comparing the last two relations with (17) we have

$$f(v \otimes n \dot{+} n \otimes v^*) = f(n \otimes v^*)f(v \otimes n)$$

By virtue of Theorem 4.2 of [10] there exists a normal operator b' acting on $\widetilde{K}$ with the spectrum $\operatorname{Sp} b' \subset \overline{\mathbf{C}}^{\mu}$ such that $f(t) = F_\mu(b't)$ for all $t \in \overline{\mathbf{C}}^{\mu}$. Consequently $f(n) = F_\mu(b' \otimes n)$,

$$\widetilde{w} = (I \otimes v)^{M_+ \otimes I} F_\mu(b' \otimes n)(I \otimes v)^{M_- \otimes I} \tag{19}$$

and $f(n\overline{z}) = F_\mu(\overline{z}b' \otimes n)$ for any $z \in S^1$. Inserting these data into (18) and performing simple computations we obtain

$$F_\mu(\overline{z}b' \otimes n) = (z^{-M_+} \otimes I)F_\mu(b' \otimes n)(z^{M_+} \otimes I)$$

.

Remembering that $\mathrm{Sp}\, n = \overline{\mathbf{C}}^\mu$ and that the family of functions $\{F_\mu(t\cdot) : t \in \overline{\mathbf{C}}^\mu\}$ separates points of $\overline{\mathbf{C}}^\mu$ we conclude that

$$z^{-M_+}b'z^{M_+} = \overline{z}b' \tag{20}$$

for any $z \in S^1$. Replacing z by v^* we get

$$(I \otimes v)^{M_+\otimes I}(b' \otimes I)(I \otimes v)^{-(M_+\otimes I)} = b' \otimes v$$

On the other hand using (2) we get

$$(I \otimes v)^{M_+\otimes I}(I \otimes n)(I \otimes v)^{-(M_+\otimes I)} = \mu^{M_+} \otimes n$$

and combining the two formulae we obtain

$$\begin{aligned}(I \otimes v)^{M_+\otimes I}(b' \otimes n)(I \otimes v)^{-(M_+\otimes I)} &= \mu^{M_+}b' \otimes nv \\ &= \mu^{\widetilde{N}/2}\widetilde{b} \otimes vn\end{aligned}$$

where $\widetilde{b} = \mu^{\frac{1}{2}P-I}b'$ and (cf (19)) formula (11) follows.

Due to (15), P commutes with all operators on $\widetilde{K}$ considered in this proof. In particular

$$b'\mu^{\frac{1}{2}P} = \mu^{\frac{1}{2}P}b'.$$

Remembering that $\widetilde{N} = 2M_+ - P$ and using (20) we get

$$z^{\widetilde{N}}b'z^{-\widetilde{N}} = z^2b'$$

for all $z \in S^1$. Finally we know that $\mathrm{Sp}\, b' \subset \overline{\mathbf{C}}^\mu$. Using these facts one can easily show that $\widetilde{N}$ and $\widetilde{b} = \mu^{\frac{1}{2}P-I}b'$ satisfy the conditions (i) — (v) listed in Theorem 2.1

Ad 3. Assume now that $\widetilde{w}$ is given by (10) where operators $\widetilde{N}$ and $\widetilde{b}$ satisfy all the five conditions (i) — (v). Then one can easily verify that

$$(\mathrm{id} \otimes \varphi_z)\widetilde{w} = z^{\widetilde{N}} \tag{21}$$

for any $z \in S^1$. It shows that $\widetilde{N}$ is uniquely determined by $\widetilde{w}$. Moreover denoting by M_+, M_- and P the operators related to $\widetilde{N}$ in the way described in the second part of this proof we have

$$(I \otimes v)^{-(M_+\otimes I)}\widetilde{w}(I \otimes v)^{-(M_-\otimes I)} = F_\mu(b' \otimes n) \tag{22}$$

where $b' = \widetilde{b}\mu^{I-\frac{1}{2}P}$ is a normal operator with the spectrum contained in $\overline{\mathbf{C}}^\mu$. Remembering that $\mathrm{Sp}\, n = \overline{\mathbf{C}}^\mu$ and that the family of functions $\{F_\mu(t\cdot) : t \in \overline{\mathbf{C}}^\mu\}$ separates points of $\overline{\mathbf{C}}^\mu$ we see that b' is uniquely determined by $\widetilde{w}$. So is $\widetilde{b}$.

If $\widetilde{w} \in M(\widetilde{B} \otimes A)$ then using (21), Proposition 1.3 and the C^*-version of the Stone's theorem [11] we see that $\widetilde{N}\eta\widetilde{B}$. In this case $M_+, M_-\eta\widetilde{B}$ and (cf (22)) $F_\mu(b'\otimes n) \in M(\widetilde{B}\otimes A)$. According to the Proposition 1.4 $F_\mu(b't) \in M(\widetilde{B}))$ and the mapping $\overline{\mathbf{C}}^\mu \ni t \mapsto F_\mu(b't) \in M(\widetilde{B})$ is continuous. Using now Proposition 5.2 of [10] we get $b'\eta\widetilde{B}$ and $\widetilde{b}\eta\widetilde{B}$ ($\mu^{\frac{1}{2}P-I}$ is an invertible element of $M(\widetilde{B})$). This way we proved the "$\Longrightarrow$" part of (13). The converse implicaton is obvious.

Q.E.D

3 The character space

In this Section we find the quantum space $\widehat{E}_\mu(2)$ of all characters on $E_\mu(2)$. The (noncommutative) C^*-algebra B of all "continuous functions on $\widehat{E}_\mu(2)$ vanishing at infinity" may be described in the following way:

Let $K = l^2(\Sigma_\mu)$, where Σ_μ is the set introduced by (12) and $(\epsilon_{s\lambda})_{(s,\lambda)\in\Sigma_\mu}$ be the canonical basis in K. For any $(s,\lambda) \in \Sigma_\mu$ we set

$$\left.\begin{array}{rcl} N\epsilon_{s\lambda} & = & s\epsilon_{s\lambda} \\ b\epsilon_{s\lambda} & = & \lambda\epsilon_{s-2,\lambda} \end{array}\right\} \tag{23}$$

Then (N, b) is a pair of operator satisfying the assumptions (i) — (v) of Theorem 2.1. Moreover $\ker b = \{0\}$ and $\mathrm{Phase}\, b$ is unitary.

By definition B is the norm closure of the set of all operators of the form $\sum(\mathrm{Phase}\, b)^k g_k(N, |\ b\ |)$, where k runs over a finite subset of $\mathbf{Z}$, $g_k \in C_\infty(\overline{\Sigma_\mu})$ and $g_k(s,0) = 0$ for $k \neq 0$. Then B is a non-unital C^*-algebra and $N, b\eta B$. The reader easily verify that the algebra B has the following *universality* property

Theorem 3.1 *1. Let ρ be a representation of B in a Hilbert space $\widetilde{B}$ and*

$$\left.\begin{array}{rcl} \widetilde{N} & = & \rho(N) \\ \widetilde{b} & = & \rho(b) \end{array}\right\} \tag{24}$$

Then $(\widetilde{N}, \widetilde{b})$ satisfies the conditions (i) — (v) of Theorem 2.1.

2. Any pair of operators $(\widetilde{N}, \widetilde{b})$ acting on a Hilbert space $\widetilde{K}$ and satisfying the conditions (i) — (v) of Theorem 2.1 is of the form (24), where ρ is a representation of B in $\widetilde{K}$. Moreover ρ is uniquely determined by $(\widetilde{N}, \widetilde{b})$.

3. Let ρ, $\widetilde{K}$, $\widetilde{N}$, $\widetilde{b}$ be as in statement 1 and $\widetilde{B} \in C^*(\widetilde{K})$. Then

$$(\widetilde{N}, \widetilde{b}\eta\widetilde{B}) \Longleftrightarrow (\rho \in \mathrm{Mor}(B, \widetilde{B}))$$

.

Combining Theorems 2.1 and 3.1 we get

Theorem 3.2 *Let*

$$w = F_\mu(\mu^{N/2} b \otimes vn)(I \otimes v)^{N\otimes I} \tag{25}$$

Then w is a unitary element of $M(B \otimes A)$ and

$$(\mathrm{id} \otimes \Phi)w = w_{12}w_{13}$$

Moreover for any pair $(\widetilde{B}, \widetilde{w})$, where $\widetilde{B}$ is a C^*-algebra and $\widetilde{w}$ is a unitary element of $M(\widetilde{B} \otimes A)$ such that $(\mathrm{id} \otimes \Phi)\widetilde{w} = \widetilde{w}_{12}\widetilde{w}_{13}$, there exists unique $\rho \in \mathrm{Mor}(B, \widetilde{B})$ such that

$$\widetilde{w} = (\rho \otimes \mathrm{id})w$$

.

4 The Pontryagin dual of $E_\mu(2)$

The group structure on $\widehat{E}_\mu(2)$ is introduced in the following

Theorem 4.1 *There exists unique $\widehat{\Phi} \in \mathrm{Mor}(B, B \otimes B)$ such that*

$$(\widehat{\Phi} \otimes \mathrm{id})w = w_{23}w_{13} \tag{26}$$

The morphism $\widehat{\Phi}$ is coassociative:

$$(\widehat{\Phi} \otimes \mathrm{id})\widehat{\Phi} = (\mathrm{id} \otimes \widehat{\Phi})\widehat{\Phi}. \tag{27}$$

Proof: The existence and uniqueness of $\widehat{\Phi}$ follows immediately from the second half of Theorem 3.2 with $(\widetilde{B}, \widetilde{w})$ replaced by $(B \otimes B, w_{23}w_{13})$. To show (27) it is sufficient to notice that $(\widehat{\Phi} \otimes \mathrm{id} \otimes \mathrm{id})(\widehat{\Phi} \otimes \mathrm{id})w = w_{34}w_{24}w_{14} = (\mathrm{id} \otimes \widehat{\Phi} \otimes \mathrm{id})(\widehat{\Phi} \otimes \mathrm{id})w$.

Q.E.D

According to the general scheme presented in [8] the quantum space $\widehat{E}_\mu(2)$ endowed with the group structure defined by $\widehat{\Phi}$ is the Pontryagin dual of $E_\mu(2)$. We shall show that $\widehat{E}_\mu(2)$ is a quantum deformation of the three-dimensional Lie group of transformations of $\mathbf{R}^2$ generated by translations and dilations. To this end we prove

Theorem 4.2 *The action of $\widehat{\Phi}$ on the distinguished elements $N, b\eta B$ is given by the formulae*

$$\begin{aligned} \widehat{\Phi}(N) &= N \otimes I \dot{+} I \otimes N & (28) \\ \widehat{\Phi}(b) &= b \otimes \mu^{N/2} \dot{+} \mu^{-N/2} \otimes b. & (29) \end{aligned}$$

Proof: Denote by $\widetilde{N}$ and $\widetilde{b}$ the right hand sides of (28) and (29) resp. By virtue of the uniqueness stated in Theorem 2.1.3 it is sufficient to show that

$$F_\mu(\mu^{\widetilde{N}/2}\widetilde{b} \otimes vn)(I \otimes I \otimes v)^{\widetilde{N} \otimes I} = w_{23}w_{13}. \tag{30}$$

In what follows, $\widetilde{w}$ denotes the left hand side of (30). We have $\mu^{\widetilde{N}/2} = \mu^{N/2} \otimes \mu^{N/2}$. Therefore

$$\mu^{\widetilde{N}/2}\widetilde{b} \otimes vn = \mu^{N/2}b \otimes \mu^{N} \otimes vn \dot{+} I \otimes \mu^{N/2}b \otimes vn$$

and using Theorem 3.1 of [10] we get

$$F_\mu(\mu^{\widetilde{N}/2}\widetilde{b} \otimes vn) = F_\mu(I \otimes \mu^{N/2}b \otimes vn)F_\mu(\mu^{N/2}b \otimes \mu^N \otimes vn)$$

and

$$\widetilde{w} = F_\mu(I \otimes \mu^{N/2}b \otimes vn)F_\mu(\mu^{N/2}b \otimes \mu^N \otimes vn)(I \otimes I \otimes v)^{\widetilde{N}\otimes I}. \tag{31}$$

Clearly $(I \otimes I \otimes v)^{\widetilde{N}\otimes I} = (I \otimes I \otimes v)^{I\otimes N\otimes I}(I \otimes I \otimes v)^{N\otimes I\otimes I}$. Taking into account the relation (2) we obtain

$$(\mu^{N/2}b \otimes \mu^N \otimes vn)(I \otimes I \otimes v)^{I\otimes N\otimes I} = (I \otimes I \otimes v)^{I\otimes N\otimes I}(\mu^{N/2}b \otimes I \otimes vn).$$

Therefore

$$F_\mu(\mu^{N/2}b \otimes \mu^N \otimes vn)(I \otimes I \otimes v)^{I\otimes N\otimes I} = (I \otimes I \otimes v)^{I\otimes N\otimes I}F_\mu(\mu^{N/2}b \otimes I \otimes vn)$$

Combining this result with (31) we get $\widetilde{w} = w_{23}w_{13}$, where

$$\begin{aligned} w_{23} &= F_\mu(I \otimes \mu^{N/2}b \otimes vn)(I \otimes I \otimes v)^{I\otimes N\otimes I} \\ w_{13} &= F_\mu(\mu^{N/2}b \otimes I \otimes vn)(I \otimes I \otimes v)^{N\otimes I\otimes I}. \end{aligned}$$

Q.E.D

Let

$$u = \begin{pmatrix} \mu^{N/2} & , & 0 \\ b & , & \mu^{-N/2} \end{pmatrix}$$

According to (28) and (29) u is a two-dimensional (non-unitary) faithful representation of $\widehat{E}_\mu(2)$. Therefore $\widehat{E}_\mu(2)$ is a quantum deformation of the group of triangular matrices

$$\widehat{E}(2) = \left\{ \begin{pmatrix} a^{-1} & , & 0 \\ b & , & a \end{pmatrix} : \begin{array}{ccc} a \in \mathbf{R} & , & a > 0 \\ b & \in & \mathbf{C} \end{array} \right\}$$

This group coincides with the group of transformations of $\mathbf{R}^2$ generated by translations and dilations: the action of an element $g = \begin{pmatrix} a^{-1} & , & 0 \\ b & , & a \end{pmatrix} \in \widehat{E}(2)$ on $\mathbf{R}^2 = \mathbf{C}$ is described by the formula

$$\sigma_g(z) = a^2 z + ab$$

5 The Heisenberg commutation relations

Let π be a representation of A and ρ be a representation of B acting on the same Hilbert space L. Then $w_{1\pi} = (\mathrm{id}\otimes\pi\otimes\mathrm{id})w_{12}$, w_{13} and $w_{\rho 3} = (\mathrm{id}\otimes\rho\otimes\mathrm{id})w_{23}$ are well defined unitary elements of $M(B \otimes CB(L) \otimes A)$. We say that (π, ρ) satisfies the Heisenberg commutation relations if

$$w_{1\pi}w_{13}w_{\rho 3} = w_{\rho 3}w_{1\pi}$$

(cf the pentagonal equality in [2]). The very formal computations show that this is the case if and only if

$$\begin{array}{lrcl}
(i) & vNv^* & = & N - I \\
(ii) & vbv^* & = & \mu^{-\frac{1}{2}} b \\
(iii) & \multicolumn{3}{l}{N \text{ and } \mid n \mid \text{ strongly commute}} \\
(iv) & \multicolumn{3}{l}{\text{On } (\ker n)^{\perp}} \\
 & (\mathrm{Phase}\, n) N (\mathrm{Phase}\, n)^* & = & N + I \\
(v) & bn^* & = & \mu^{\frac{1}{2}} n^* b \\
(vi) & nb - \mu^{\frac{1}{2}} bn & = & \frac{1-\mu^2}{\sqrt{\mu}} \mu^{-N/2} v
\end{array}$$

where for simplicity we wrote v, n, N and b instead of $\pi(v)$, $\pi(n)$, $\rho(N)$ and $\rho(b)$ resp..

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