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Radiative corrections in systems near imperfectly reflecting boundaries are investigated. As an example, the self-energy of an unbound electron close to a single surface is calculated at one-loop level. The surface is modeled by a nondispersive dielectric half-space of a constant refractive index n. In contrast to previous, perfectly reflecting models, the evanescent modes in the optically thinner medium are taken into account and are found to play a physically very important role. The Feynman propagator of the photon field is determined and given in two alternative representations, which include the evanescent modes either as a separate contribution or through analytic continuation and deformation of the integration path for the normal component of the complex wave vector $\mathbf{k}$. The evaluation of the self-energy diagram encounters a number of problems that are specific to the boundary dependence and to the imperfect reflection at the boundary. These problems and methods for their resolution are discussed in depth.

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I. INTRODUCTION

Quantum electrodynamics (QED) is liable to corrections if the electromagnetic environment of the system under consideration is different from free space. For example, the Lamb shift in an atom changes if the atom is located not in free space but near a reflecting surface [1,2]. This and similar boundary-dependent effects are the subject of cavity QED [3]. Under most circumstances cavity QED effects are nonrelativistic in nature and hence the techniques employed in the theory of cavity QED are chiefly nonrelativistic. They rely mostly on a comparatively simple mode expansion of the electromagnetic field and on first-quantized theory for the remaining part of the system under investigation (cf. e.g. [4]). However, there are a few examples of systems that are not inherently nonrelativistic, the simplest being a free electron. By virtue of being free it lacks an in-built energy scale that could limit its virtual excitations and thus its interactions with the electromagnetic field to nonrelativistic energies. Other effects that require a fully second-quantized theory are, e.g., radiative corrections to the Casimir force between reflecting planes [5] and the Scharnhorst effect of faster-than-c light propagation in between and perpendicular to parallel plates [6]. To date all such field-theoretical calculations have been done for cavity walls that are perfectly reflecting. While it is obvious that no real material can ever really be perfectly reflecting, the model of perfect reflectivity seems to capture all the essential physics of the boundary without going wrong by anything other than a minor numerical prefactor. Its great attraction is that it is comparatively simple; for example, the photon propagator between two parallel perfectly reflecting plates can be written as a sum of the photon propagator in free space and a small boundary-dependent correction, and loop calculations using it are manageable, though not trivial because of the loss of translation invariance perpendicular to the plates [5].

However, we recently discovered that the assumption of perfect reflectivity for the cavity walls is in fact not justified for systems, such as a free electron, that admit low-frequency excitations, i.e. whose excitation spectrum has, unlike an atom's, no natural IR cutoff [7]. This is because, crudely speaking, the electron's virtual excitations of arbitrarily long wavelengths interact with evanescent electromagnetic field modes originating inside a cavity wall, and the chief defect of the perfect-reflector model is that it ignores all such evanescent modes. To show this we have modeled an imperfect reflector by a nondispersive dielectric and have calculated the self-energy of an electron in front of a dielectric half-space. We have found that taking the limit of perfect reflectivity in the result disagrees with the corresponding calculation that assumes a perfectly reflecting wall from the outset. The two results differ by a factor of 2 in one direction and even by sign in the other. While the effect as such can already be seen in a nonrelativistic calculation, we felt that there was a need for a proper second-quantized calculation, mainly for three reasons: (i) the nonrelativistic calculation yields different results for the two models but gives no clue as to the origin of this discrepancy; (ii) there is nothing that *a priori* restricts the electron's motion to nonrelativistic energy scales and, in fact, the interaction energy is being integrated up to infinity, which could potentially lead to errors that pass by unnoticed in a purely nonrelativistic calculation; and (iii) there are other effects, as e.g. the boundary-dependent $g-2$ correction to the electron's anomalous magnetic moment, which might be affected and whose calculation requires field-theoretical methods. Furthermore, in the face of a major discrepancy it seems wise to check all possibilities.

In this paper we establish the major building blocks of a full QED theory near imperfect reflectors. We concentrate on a nondispersive and nonabsorbing dielectric as a good model for an imperfectly reflecting material. Thus the medium is characterized by a single parameter, its refractive index n, which is real and the same for all frequencies. For technical simplicity we restrict ourselves to a single reflecting surface, i.e. we consider a dielectric half-space, which we take to occupy the region $z > 0$, while the region $z < 0$ is vacuum (cf. Fig. 1). For this setup the dielectric function is a single step function

$$\epsilon(\mathbf{r}) = 1 + (n^2 - 1)\theta(z),$$

which makes the solution of Maxwell's equations comparatively simple. For piecewise constant dielectric functions like this, it is advantageous to work in the generalized Coulomb gauge

$$\boldsymbol{\nabla} \cdot (\epsilon \mathbf{A}) = 0, \tag{1}$$

which we shall do in this paper. For a general coordinate dependent dielectric function $\epsilon(\mathbf{r})$ the generalized Coulomb gauge may be a very awkward choice, but for a piecewise constant $\epsilon(\mathbf{r})$ this gauge is so convenient because anywhere except right on the boundary (or boundaries) of the dielectric ($z = 0$ in our case), it is equivalent to the Coulomb gauge $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$. Thus one can work almost as if in the Coulomb gauge and just needs to make sure that the physical fields satisfy the appropriate matching conditions at the boundary, i.e. that

$$\mathbf{E}_{\parallel} \text{ continuous}, \qquad \mathbf{D}_{\perp} \text{ continuous}. \tag{2}$$

Since our model material is just a dielectric and has a magnetic permeability $\mu = 1$, the magnetic field strengths $\mathbf{B}$ and $\mathbf{H}$ are also continuous everywhere. We note that quantum mechanics in the generalized Coulomb gauge is related to that in the true Coulomb gauge by canonical transformation [8]; the two differ by surface charges leading to an electrostatic image potential in the Hamiltonian.

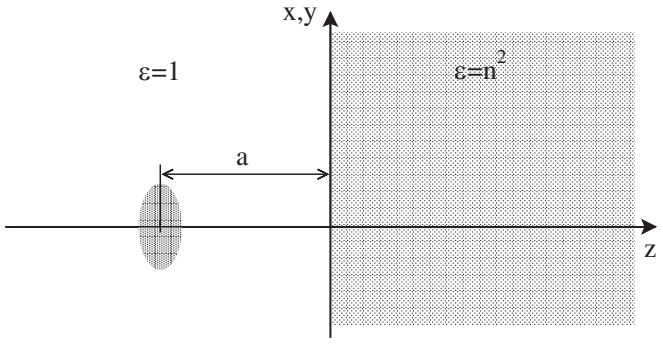

FIG. 1. A sketch of the simplest system in cavity QED, consisting of a quantum object, e.g. an atom or an electron, that is located a distance a away from a single reflecting wall. If the wall is perfectly reflecting then the electromagnetic field is excluded from the region $z > 0$. A more realistic but still simple model is to describe the wall by a constant, frequency-independent refractive index n.

If one wanted to work in a gauge that resembles the radiation gauge, one ought to choose the gauge

$$\epsilon^2 \frac{\partial \Phi}{\partial t} + \boldsymbol{\nabla} \cdot (\epsilon \mathbf{A}) = 0.$$

In this gauge Maxwell's equations for the scalar potential Φ and the vector potential $\mathbf{A}$ read

$$\epsilon \frac{\partial^2 \Phi}{\partial t^2} - \nabla^2 \Phi = \frac{\rho}{\epsilon} + \frac{1}{\epsilon}(\boldsymbol{\nabla}\Phi) \cdot \boldsymbol{\nabla}\epsilon,$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) - \nabla^2 \mathbf{A} + \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \epsilon \boldsymbol{\nabla}\left[\frac{1}{\epsilon^2} \boldsymbol{\nabla} \cdot (\epsilon \mathbf{A})\right] = \mathbf{j}.$$

It is important that they separate and that for the piecewise constant $\epsilon(\mathbf{r})$ they differ from the standard wave equations for Φ and $\mathbf{A}$ only by surface terms. However, in the present paper we shall not pursue any calculations in this gauge but work in the generalized Coulomb gauge (1).

By describing a material solely through its dielectric function or its reflectivity for electromagnetic radiation, one of course ignores a whole range of physical properties that would become important if interactions other than electromagnetic take place. For example, an electron that is very close to the surface of the material would see the microscopic structure of the surface and be subject to exchange interaction (Pauli repulsion) with the electrons in the material. Thus, for cavity QED, where one wants to investigate just the impact of changes to the electromagnetic environment of a localized electron or other quantum system, one needs to demand that this quantum system is located well away from the material. In our case this means that we shall assume that the electron is at least a few Compton wavelengths away from the surface on the vacuum side, so that there is no direct wave-function overlap between the electron and the surface and the only interaction between them is electromagnetic.

In Sec. II we calculate the full photon propagator in the presence of a nondispersive dielectric half-space starting from the normal modes of the radiation field which we briefly discuss in the Appendix. In Sec. III we use this photon propagator to determine the self-energy of a free electron located outside and a distance a away from the dielectric. In Sec. IV we do a careful asymptotic analysis of the expression for the self-energy for nonrelativistic mean energies. In Sec. IV D we compare the results for the electron's radiative self-energy in front of an imperfectly or of a perfectly reflecting surface and discuss the reasons for the disagreement between the calculation for a nondispersive dielectric and for a "perfect reflector." Section V summarizes our final results.

Throughout this paper we set $c = 1 = \hbar$ and use Heaviside-Lorentz units for electromagnetic quantities,

$\epsilon_0 = 1 = \mu_0$. Thus the fine structure constant is $e^2/4\pi \approx 1/137$.

II. CALCULATION OF THE PHOTON PROPAGATOR

A. Wightman function

The Green's functions in free quantum electrodynamics are vacuum expectation values of products of field operators. Let us first consider the Wightman functions [9]

$$D^{\mu\nu-}(x,x') = -i\langle 0|A^\mu(x)A^\nu(x')|0\rangle. \tag{3}$$

Inserting the normal modes (A2) and (A5) into (3) and taking the vacuum expectation values of the bilinear products of photon annihilation and creation operators, we obtain

$$D^{\mu\nu-}(x,x') = \sum_\sigma e^\mu_\sigma(\partial_x)e^{\nu*}_\sigma(\partial_{x'})g^{\sigma\sigma}[{}^L D^-_\sigma(x,x') + {}^R D^-_\sigma(x,x')], \tag{4}$$

with

$$\begin{aligned}{}^R D^-_\sigma(x,x') = i\int\frac{d^2k_\parallel}{(2\pi)^3}\int_{-\infty}^0 dk^d_z\frac{1}{2\omega}e^{-i(\tilde{x}-\tilde{x}')\tilde{k}}\Big\{&\theta(-z)\theta(-z')[e^{(ik_z z-ik^*_z z')}T^R_\sigma T^{R*}_\sigma] + \theta(z)\theta(z')\frac{1}{n^2}[e^{ik^d_z(z-z')}\\ &+ e^{ik^d_z(z+z')}R^R_\sigma(k)^* + e^{-ik^d_z(z+z')}R^R_\sigma(k) + e^{i(k^d_z(z-z')}R^R_\sigma R^{R*}_\sigma] + \theta(-z)\theta(z')\frac{1}{n}[e^{(ik_z z-ik^d_z z')}T^R_\sigma\\ &+ e^{ik_z z+ik^d_z z')}T^R_\sigma R^{R*}_\sigma] + \theta(z)\theta(-z')\frac{1}{n}[e^{(ik^d_z z-ik^*_z z')}T^{R*}_\sigma + e^{-ik^d_z z-ik^*_z z')}R^R_\sigma T^{R*}_\sigma]\Big\},\end{aligned} \tag{5}$$

and

$$\begin{aligned}{}^L D^-_\sigma(x,x') = i\int\frac{d^2k_\parallel}{(2\pi)^3}\int_0^\infty dk_z\frac{1}{2\omega}e^{-i(\tilde{x}-\tilde{x}')\tilde{k}}\Big\{&\theta(z)\theta(z')\frac{1}{n^2}e^{ik^d_z(z-z')}T^L_\sigma T^{L*}_\sigma + \theta(-z)\theta(-z')[e^{ik_z(z-z')} + e^{ik_z(z+z')}R^L_\sigma(k)^*\\ &+ e^{-ik_z(z+z')}R^L_\sigma(k) + e^{-ik_z(z-z')}R^L_\sigma(k)R^L_\sigma(k)^*] + \theta(-z)\theta(z')\frac{1}{n}[e^{(ik_z z-ik^d_z z')}T^{L*}_\sigma + e^{(-ik_z z-ik^d_z z')}T^{L*}_\sigma R^L_\sigma]\\ &+ \theta(z)\theta(-z')\frac{1}{n}[e^{(ik^d_z z-ik_z z')}T^L_\sigma + e^{(ik^d_z z+ik_z z')}T^L_\sigma R^{L*}_\sigma]\Big\}.\end{aligned} \tag{6}$$

The notations for the wave vectors are as defined in Eq. (A4), and in addition we have introduced the new variables $\tilde{k} = (k^0, \mathbf{k}_\parallel)$ and $\tilde{x} = (x^0, \mathbf{x}_\parallel)$ in 2 + 1 dimensional Minkowski space. The sum of ${}^L D^-_\sigma(x,x')$ and ${}^R D^-_\sigma(x,x')$ can be simplified by taking into account that the Fresnel coefficients R and T are real functions of the wave vectors and by using various relations (A7) and (A8) between them and their products. We obtain two equivalent expressions:

$$\begin{aligned}{}^L D^-_\sigma(x,x') + {}^R D^-_\sigma(x,x') = i\int\frac{d^2k_\parallel}{(2\pi)^3}\frac{1}{2\omega}e^{-i(\tilde{x}-\tilde{x}')\tilde{k}}\Big\{&\theta(z)\theta(z')\frac{1}{n^2}\int_{-\infty}^\infty dk^d_z[e^{ik^d_z(z-z')} + e^{-ik^d_z(z+z')}R^R_\sigma]\\ &+ \theta(-z)\theta(-z')\Big[\int_{-\infty}^\infty dk_z e^{ik_z(z-z')} + \int_{-\infty}^\infty dk_z e^{-ik_z(z+z')}R^L_\sigma + \int_{-\Gamma}^0 dk^d_z e^{ik_z z-ik^*_z z'}T^{R*}_\sigma T^R_\sigma\Big]\\ &+ \theta(-z)\theta(z')\frac{1}{n}\int_{-\infty}^\infty dk^d_z e^{ik_z z-ik^d_z z'}T^R_\sigma + \theta(z)\theta(-z')\frac{1}{n}\int_{-\infty}^\infty dk_z e^{-ik_z z'+ik^d_z z}T^{R*}_\sigma\Big\}\end{aligned} \tag{7}$$

and

$$\begin{aligned}{}^L D^-_\sigma(x,x') + {}^R D^-_\sigma(x,x') = i\int\frac{d^2k_\parallel}{(2\pi)^3}\frac{1}{2\omega}e^{-i(\tilde{x}-\tilde{x}')\tilde{k}}\Big\{&\theta(z)\theta(z')\frac{1}{n^2}\int_{\infty}^\infty dk^d_z[e^{ik^d_z(z-z')} + e^{-ik^d_z(z+z')}R^R_\sigma]\\ &+ \theta(-z)\theta(-z')\Big[\int_{-\infty}^\infty dk_z e^{ik_z(z-z')} + \int_C dk_z e^{ik_z(z+z')}R^L_\sigma\Big]\\ &+ \theta(-z)\theta(z')\frac{1}{n}\int_{-\infty}^\infty dk^d_z e^{ik_z z-k^d_z z'}T^R_\sigma + \theta(z)\theta(-z')\frac{1}{n}\int_{-\infty}^\infty dk_z e^{-ik_z z'+ik^d_z z}T^{R*}_\sigma\Big\},\end{aligned} \tag{8}$$

with $\Gamma = ((n^2-1)k^2_\parallel)^{1/2}$. The difference between these two expressions is how the contributions from evanescent waves are included. In the first expression (7) they appear in the second line as a separate integral over k^d_z from $-\Gamma$ to 0. In the second expression (8) they are included in the integration along the path C in the complex k_z plane shown in Fig. 2: it runs along the real axis from $-\infty$ to 0, then down the negative imaginary axis from 0 to $-i\Gamma/n$ to the

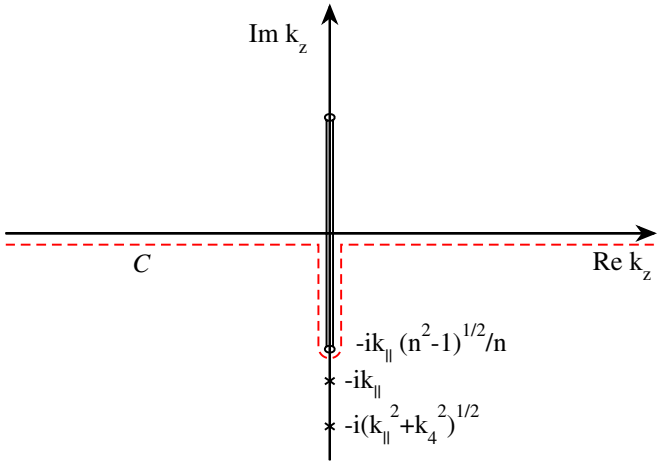

FIG. 2 (color online). The integration path C in the complex k_z plane.

left of the square root cut, back up to the origin to the right of the cut, and then along the real axis from 0 to $+\infty$. The cut is due to $k_z^d = (n^2k_z^2 + (n^2-1)k_\parallel^2)^{1/2}/n$ and extends from $k_z = +i\Gamma/n$ to $k_z = -i\Gamma/n$. The part of C that runs left and right of the cut is identical to the integral over k_z^d in the second line of (7), i.e. it gives the contribution of the evanescent waves. This works because

$$R^L|_{k_z=-i\kappa,k_z^d=-K} - R^L|_{k_z=-i\kappa,k_z^d=K} = \frac{k_z}{k_z^d} n^2 T^R T^{R*}|_{k_z=-i\kappa,k_z^d=-K}. \tag{9}$$

Note that the $\omega = 0$ singularity at $k_z = \pm ik_\parallel$ does not come into play if one chooses the cut of $\omega = (k_z^2 + k_\parallel^2)^{1/2}$ along $-i\infty \ldots - ik_\parallel$ and $ik_\parallel \ldots i\infty$.

B. Feynman propagator

For the calculation of radiative corrections we need the Feynman propagator, which can be reconstructed from the Wightman functions (4), (7), and (8) according to

$$\begin{aligned} D^{\mu\nu c}(x,x') &= -i\langle 0|TA^\mu(x)A^\nu(x')|0\rangle \\ &= \theta(x^0 - x'^0)D^{\mu\nu -}(x,x') \\ &\quad + \theta(x'^0 - x^0)D^{\nu\mu -}(x',x). \end{aligned}$$

Thus $D^{\mu\nu c}(x,x')$ can be written in the same way as the Wightman function in (4),

$$D^{\mu\nu c}(x,x') = \sum_\sigma e_\sigma^\mu(\partial_x)e_\sigma^{\nu*}(\partial_{x'})g^{\sigma\sigma}D_\sigma^c(x,x').$$

The sum over the polarizations is gauge dependent. In the Coulomb gauge the sum runs over $\sigma = TE, TM$ only. In the covariant, i.e. Feynman gauge the two unphysical polarizations G, C have to be included. Proceeding from the simplified expressions (7) and (8) for the Wightman functions, we obtain for the polarization component σ of the Feynman propagator

$$\begin{aligned} D_\sigma^c(x,x') = -\int \frac{d^3\tilde{k}}{(2\pi)^4} e^{-i(\tilde{x}-\tilde{x}')\tilde{k}} \Big\{ &\theta(z)\theta(z')\int_{-\infty}^{\infty} dk_z^d [e^{ik_z^d(z-z')} + e^{-ik_z^d(z+z')}R_\sigma^R] \frac{1}{n^2k_0^2 - k_p^2 - k_z^{d2} + i\epsilon} \\ &+ \theta(-z)\theta(-z')\Big[\int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} + \int_C dk_z e^{ik_z(z+z')}R_\sigma^L\Big]\frac{1}{k^2 + i\epsilon} \\ &+ \theta(-z)\theta(z')\frac{1}{n}\int_{-\infty}^{\infty} dk_z^d e^{ik_z z - ik_z^d z'} T_\sigma^R \frac{1}{k_0^2 - (k_p^2 + k_z^{d2})/n^2 + i\epsilon} \\ &+ \theta(z)\theta(-z')\frac{1}{n}\int_{-\infty}^{\infty} dk_z^d e^{-ik_z z' + ik_z^d z} T_\sigma^{R*} \frac{1}{k_0^2 - (k_p^2 + k_z^{d2})/n^2 + i\epsilon}\Big\}. \end{aligned} \tag{10}$$

In an alternative formulation one can replace the second line of (10) by

$$\theta(-z)\theta(-z')\Big[\int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} + \int_{-\infty}^{\infty} dk_z e^{ik_z(z+z')}R_\sigma^L + \int_{-\Gamma}^{0} dk_z^d e^{ik_z z - ik_z^* z'} T_\sigma^{R*} T_\sigma^R\Big]\frac{1}{k^2 + i\epsilon}. \tag{11}$$

Note that the property $e_\lambda^\mu(\partial_x)e_\lambda^{\nu*}(\partial_{x'}) = e_\lambda^{\mu*}(\partial_x)e_\lambda^\nu(\partial_{x'})$ follows directly from the definition (A1) of the polarization vectors. Note also that k_0 in Eq. (10) is a free integration variable and is not fixed to ω as it was in the case of the Wightman functions (7) and (8).

The propagator differs from its equivalent in free space by not being translation invariant along z. However, it is still symmetric in its arguments, $D_\sigma^c(x,x') = D_\sigma^c(x',x)$. This is easily shown by a change of variables $k \to -k$. For real k_z this is straightforward; for imaginary $k_z = -i\kappa$ (with $\kappa > 0$) one needs the relation

$$T^R(k_z = -i\kappa, -k_z^d) = T^{R*}(k_z = -i\kappa, k_z^d),$$

for which one has to take care to stay on the same sheet in the complex plane.

The wave equations that the Feynman propagator satisfies are

$$(\partial_{x_0}^2 - \Delta_x) D_\sigma^c(x,x') = \delta^{(4)}(x-x'), \quad \text{for } z<0,$$
$$(n^2\partial_{x_0}^2 - \Delta_x) D_\sigma^c(x,x') = \delta^{(4)}(x-x'), \quad \text{for } z>0.$$

Checking the wave equation, for example, for $z<0$ leads to

$$\begin{aligned}(\partial_{x_0}^2 - \Delta_x) D_\sigma^c(x,x') &= \theta(-z)\theta(-z')\delta^{(4)}(x-x') \\ &+ \int \frac{d^3\tilde{k}}{(2\pi)^4} e^{-i(\tilde{x}-\tilde{x}')\tilde{k}} \Big\{ \theta(-z)\theta(-z') \\ &\times \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} R_\sigma^L \\ &+ \theta(-z)\theta(z')\frac{1}{n} \\ &\times \int_{-\infty}^{\infty} dk_z^d e^{ik_z z - ik_z^d z'} T_\sigma^R \Big\}.\end{aligned}$$

The integral should vanish, which it does—because the integration path can be closed in the lower k_z plane as the Fresnel coefficients do not have poles there.

A direct check of the Feynman propagator is the reconstruction of the Wightman functions (7) and (8) by performing the k_0 integration. For $x_0 > 0$ the integration path can be closed in the lower half-plane

$$\int_{-\infty}^{\infty} \frac{\exp(-ik_0 x^0)}{k_0^2 + \mathbf{k}^2 + i\epsilon} = -\frac{i}{2|\mathbf{k}|} \exp(-i|\mathbf{k}|x^0)\theta(x_0), \quad (12)$$

and we thus reproduce (7) and (8) for $x_0 - x_0' > 0$.

Next we consider two special cases. The simplest situation is the limit $n \to 1$, when there is no medium. Correspondingly all reflection coefficients R tend to zero, the transmission coefficients T approach one, and $\Gamma \to 0$, so that complex values do not appear in the integration over k_z. For all polarizations σ the propagator components equal the massless scalar propagator

$$D_\sigma^c(x,x') = -\int \frac{d^4k}{(2\pi)^4} e^{-i(x-x')k} \frac{1}{k^2 + i\epsilon}. \quad (13)$$

Thus, according to (A9) our Feynman propagator (10) coincides in the limit $n \to 1$ with the standard free-space propagators: either with the covariant propagator in the Feynman gauge or with the propagator in the Coulomb gauge, depending on which modes have been included in the sum over polarizations.

In the limit $n \to \infty$ only the left-incident modes survive, so that, according to (A10), the Wightman functions (4) and their components (7) simplify greatly.

$$\begin{aligned}D_\sigma^-(x,x') = i \int \frac{d^2k_\parallel}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_z}{2\omega} e^{-i(\tilde{x}-\tilde{x}')\tilde{k}} \theta(-z)\theta(-z') \\ \times \left[e^{ik_z(z-z')} + e^{ik_z(z+z')} R_\sigma^L(k) \right] \end{aligned} \quad (14)$$

with

$$R_{TE}^L = R_C^L = R_G^L = -1, \qquad R_{TM}^L = 1.$$

A corresponding representation follows for the Feynman propagator.

Finally we need to establish the electrostatic Green function, which corresponds to the Coulomb interaction. One way to start is from the retarded Green function for the Coulomb mode C. The retarded propagator differs from the Feynman propagator (10) by the $i\epsilon$ prescription in the denominator, such that $i\epsilon \to i\epsilon k_0$. Since we are looking for a static Green's function we need to calculate

$$\begin{aligned}G_C(\vec{x},\vec{x}') &= \int dx^0 D_C^{\rm ret}(x,x') \\ &= \int \frac{d^2k_\parallel}{(2\pi)^3} e^{i(\mathbf{x}_\parallel - \mathbf{x}_\parallel')\cdot\mathbf{k}_\parallel} \frac{1}{\mathbf{k}^2} \Big\{ \theta(-z)\theta(-z') \int_{-\infty}^{\infty} dk_z \Big(e^{ik_z(z-z')} + e^{ik_z(z+z')} \frac{1-n^2}{1+n^2} \Big) \\ &+ \theta(z)\theta(z') \int_{-\infty}^{\infty} dk_z \Big(e^{ik_z(z-z')} + e^{-ik_z(z+z')} \frac{n^2-1}{n^2+1} \Big) \\ &+ \theta(-z)\theta(z') \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \frac{2n}{1+n^2} + \theta(z)\theta(-z') \int_{-\infty}^{\infty} dk_z e^{ik_z(z-z')} \frac{2n}{1+n^2} \Big\}. \end{aligned} \quad (15)$$

In our conventions $A^0(x) = \Phi^0(x)$ for $z<0$ and $A^0(x) = n\Phi^0(x)$ for $z>0$, so that we obtain for the electrostatic Green's function

$$\begin{aligned}G_\Phi(x,x') &= -i\langle 0|\Phi^0(x)\Phi^0(x')|0\rangle \\ &= \int \frac{d^2k_\parallel}{(2\pi)^3} \int_{-\infty}^{\infty} dk_z e^{i(\mathbf{x}_\parallel - \mathbf{x}_\parallel')\cdot\mathbf{k}_\parallel} \frac{1}{\mathbf{k}^2} \Big\{ \theta(-z)\theta(-z') \Big(e^{ik_z(z-z')} + e^{ik_z(z+z')} \frac{1-n^2}{1+n^2} \Big) \\ &+ \theta(z)\theta(z') \frac{1}{n^2} \Big(e^{ik_z(z-z')} + e^{-ik_z(z+z')} \frac{n^2-1}{n^2+1} \Big) + \theta(-z)\theta(z') e^{ik_z(z-z')} \frac{2}{1+n^2} + \theta(z)\theta(-z') e^{ik_z(z-z')} \frac{2}{1+n^2} \Big\}. \end{aligned} \quad (16)$$

This result agrees with the classical electrostatic Green's function, which can be derived easily, so that we have an additional check.

III. THE SELF-ENERGY OF THE ELECTRON

The energy shift of the electron can be determined by considering the electron propagator and its radiative corrections due to the coupling to the photon field [10]. We start by considering the free electron propagator

$$S^{(0)}_{\alpha\beta}(x,y) = \langle 0|\mathsf{T}\psi_\alpha(x)\overline{\psi}_\beta(y)|0\rangle. \tag{17}$$

Using canonical quantization one can write the spinor

$$\psi_\alpha(\mathbf{r},t) = \int \frac{d^3\mathbf{p}}{(2\pi)^{3/2}}\sqrt{\frac{m}{E}}\sum_{i=1,2}[b_{i,\mathbf{p}}u^{(i)}_\alpha(\mathbf{p})e^{-iEt+i\mathbf{p}\cdot\mathbf{r}} + d^\dagger_{i,\mathbf{p}}v^{(i)}_\alpha(\mathbf{p})e^{iEt-i\mathbf{p}\cdot\mathbf{r}}], \tag{18}$$

where $b_{i,\mathbf{p}}$ annihilates an electron of helicity i and momentum $\mathbf{p}$ and $d^\dagger_{i,\mathbf{p}}$ creates a positron of helicity i and momentum $\mathbf{p}$. The particle eigenspinors $u^{(i)}_\alpha(\mathbf{p})$ are solutions of the Dirac equation,

$$u^{(i)}(\mathbf{p}) = \sqrt{\frac{p^0+m}{2m}}\begin{pmatrix}\phi^{(i)}\\ \frac{\sigma\cdot\mathbf{p}}{p_0+m}\phi^{(i)}\end{pmatrix}, \tag{19}$$

with $\phi^{(i)}$ two orthogonal and normalized two-spinors. Thus the normalization of $u^{(i)}(\mathbf{p})$ is

$$\overline{u}^{(i)}(\mathbf{p})u^{(j)}(\mathbf{p}) = \delta_{ij}.$$

The antiparticle eigenspinors $v^{(i)}(\mathbf{p})$ are similar to (19), except with upper and lower components interchanged and normalized to -1. Inserting the mode expansion (18) and its conjugate into the expression for the propagator (17) and Fourier transforming to go from the time variable $(x_0 - y_0)$ to the energy E one finds

$$S^{(0)}_{\alpha\beta}(\mathbf{x},\mathbf{y},E) = -\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{m}{E(\mathbf{p})}\sum_{i=1,2}\frac{u^{(i)}(\mathbf{p})\overline{u}^{(i)}(\mathbf{p})}{E(\mathbf{p})-E-i\epsilon}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + \int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{m}{E(\mathbf{p})}\sum_{i=1,2}\frac{\overline{v}^{(i)}(\mathbf{p})v^{(i)}(\mathbf{p})}{E(\mathbf{p})+E-i\epsilon}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}. \tag{20}$$

This shows that the propagator has positive energy poles at the particle energies $E(\mathbf{p})$ and negative energy poles at the antiparticle energies $-E(\mathbf{p})$. Radiative corrections represent a perturbation and cause shifts in the particle and antiparticle eigenfunctions and in their energies. For small perturbations these shifts are small and expressions can be linearized in them. Linearizing the change of the propagator $\delta S = S - S^{(0)}$, one obtains a term that is linear in the energy shift and has a double pole at the particle or antiparticle energy,

$$\delta S^{(0)}_{\alpha\beta}(\mathbf{x},\mathbf{y},E) \simeq \int\frac{d^3\mathbf{p}}{(2\pi)^3} \times\frac{m}{E(\mathbf{p})}\sum_{i=1,2}\left\{\frac{u^{(i)}(\mathbf{p})\overline{u}^{(i)}(\mathbf{p})}{[E(\mathbf{p})-E-i\epsilon]^2}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - \frac{\overline{v}^{(i)}(\mathbf{p})v^{(i)}(\mathbf{p})}{[E(\mathbf{p})+E-i\epsilon]^2}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right\}\delta E(\mathbf{p}) + (\text{other terms without double poles}). \tag{21}$$

Further terms that are linear in the shifts of the particle and antiparticle eigenfunctions do not give rise to terms with double poles since the eigenfunctions appear only in the numerator of Eq. (20).

The energy shift $\delta E(\mathbf{p})$ can now be determined by comparing the expression (21) to the change of the electron propagator as determined from standard Feynman perturbation theory. At one-loop level, i.e. to order e^2, the radiative correction to the propagator is

$$\delta S^{(0)}_{\alpha\beta}(\mathbf{x},\mathbf{y},E) = \int d^3\mathbf{z}\int d^3\mathbf{z}' S^{(0)}_{\alpha\gamma}(\mathbf{x},\mathbf{z},E) \times\Sigma_{\gamma\kappa}(\mathbf{z},\mathbf{z}',E)S^{(0)}_{\kappa\beta}(\mathbf{z}',\mathbf{y},E), \tag{22}$$

where $\Sigma_{\alpha\beta}(\mathbf{x},\mathbf{x}',E)$ is obtained from the standard electron self-energy

$$\Sigma_{\alpha\beta}(x,x') = -ie^2\gamma^\mu_{\alpha\kappa}S^{(0)}_{\kappa\lambda}(x-x')\gamma^\nu_{\lambda\beta}D^c_{\mu\nu}(x,x') \tag{23}$$

by Fourier transformation from the time into the energy domain. The electron propagator $S^{(0)}(x-x')$ is the same as in free space and thus translation invariant in all 4 directions, but the photon propagator $D^c_{\mu\nu}(x,x')$ is affected by the presence of the dielectric medium and therefore not translation invariant in the x_3 direction. Substituting the representation (20) in terms of eigenfunctions into Eq. (22) one obtains

$$\delta S^{(0)}_{\alpha\beta}(\mathbf{x},\mathbf{y},E) = \int d^3\mathbf{z}\int d^3\mathbf{z}'\int\frac{d^3\mathbf{p}}{(2\pi)^3}\int\frac{d^3\mathbf{p}'}{(2\pi)^3}\frac{m}{E(\mathbf{p})} \times\frac{m}{E(\mathbf{p}')}\sum_{i=1,2}\sum_{j=1,2}\Bigg[e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{z})+i\mathbf{p}'\cdot(\mathbf{z}'-\mathbf{y})} \times\frac{u^{(i)}_\alpha(\mathbf{p})\overline{u}^{(i)}_\gamma(\mathbf{p})}{E(\mathbf{p})-E-i\epsilon}\Sigma_{\gamma\kappa}(\mathbf{z},\mathbf{z}',E) \times\frac{u^{(j)}_\kappa(\mathbf{p}')\overline{u}^{(j)}_\beta(\mathbf{p}')}{E(\mathbf{p}')-E-i\epsilon}+\dots\Bigg]. \tag{24}$$

Further terms all contain antiparticle operators and at least one negative energy pole. Since we are interested in the energy shift of a particle rather than an antiparticle, we need to focus only on terms with two particle poles. For $\mathbf{p}=\mathbf{p}'$ Eq. (24) has the same double pole as Eq. (21), and thus a simple comparison of the coefficients of those double-pole terms should yield an expression for the en-

ergy shift $\delta E(\mathbf{p})$ in (21). However, a mathematically clean comparison is possible only if one introduces a quantization volume L^3 with periodic boundary conditions so as to discretize the momentum $\mathbf{p}$. Then all integrals over momenta turn into sums according to the prescription

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \to \frac{1}{L^3}\sum_{\mathbf{p}}.$$

Only the term with $\mathbf{p} = \mathbf{p}'$ in the double sum over momenta in Eq. (24) gives rise to a double pole in the energy, and comparison with the double-pole term in Eq. (21) therefore yields

$$\delta E(\mathbf{p}) = \frac{1}{L^3}\int d^3\mathbf{z}\int d^3\mathbf{z}'\frac{m}{E(\mathbf{p})} \times e^{-i\mathbf{p}\cdot(\mathbf{z}-\mathbf{z}')}\overline{u}^{(i)}_{\gamma}(\mathbf{p})\Sigma_{\gamma\kappa}(\mathbf{z}, \mathbf{z}', E)u^{(i)}_{\kappa}(\mathbf{p}). \quad (25)$$

It is advantageous to work with the Fourier representation of Σ. Because of the lack of translation invariance in the x_3 direction, the Fourier transform of the self-energy with respect to $\mathbf{x} - \mathbf{x}'$ has a residual dependence on $x_3 + x_3'$,

$$\Sigma_{\alpha\beta}(x, x') = \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-x')}\Sigma_{\alpha\beta}(q, x_3 + x_3'). \quad (26)$$

In a finite quantization volume the integral over $d^3\mathbf{q}$ again turns into a sum, and we can rewrite Eq. (25) as

$$\delta E(\mathbf{p}) = \frac{1}{L^6}\int d^3\mathbf{z}\int d^3\mathbf{z}'\sum_{\mathbf{q}}\frac{m}{E(\mathbf{p})} \times e^{i(\mathbf{q}-\mathbf{p})\cdot(\mathbf{z}-\mathbf{z}')}\overline{u}^{(i)}_{\gamma}(\mathbf{p})\Sigma_{\gamma\kappa}(q, z_3 + z_3')u^{(i)}_{\kappa}(\mathbf{p}). \quad (27)$$

Since we want to work out the energy shift of a particle as a function of its distance from the dielectric, we need to form localized wave packets in the x_3 direction—so that the concept of a certain distance between the electron and the surface of the dielectric at all makes sense. If the center of the packet is at $x_3 = -a$ (see Fig. 1) then we can approximate $z_3 + z_3' \simeq -2a$ and carry out the $\mathbf{z}$ and $\mathbf{z}'$ integrations in (27). The result simplifies to

$$\delta E(\mathbf{p}) = \frac{m}{E(\mathbf{p})}\overline{u}^{(i)}_{\gamma}(\mathbf{p})\Sigma_{\gamma\kappa}(p, -2a)u^{(i)}_{\kappa}(\mathbf{p}). \quad (28)$$

Note that, while $\Sigma(q, x_3 + x_3')$ in Eq. (26) is, in general, off shell, it is on the mass shell in Eqs. (27) and (28) because q and p are on shell.

Further we need to remark that in the Coulomb gauge the energy shift is not wholly due to the radiative self-energy (23): we have to add to (28) the electrostatic energy

$$\delta E_{\text{Coulomb}} = \frac{e^2}{2}\int d^3x\int d^3x' \times \langle{:}\overline{\psi}(x)\gamma^0\psi(x){:}\overline{G}_{\Phi}(x, x'){:}\overline{\psi}(x')\gamma^0\psi(x'){:}\rangle, \quad (29)$$

where $\overline{G}_{\Phi}$ is the part of the electrostatic Green's function (16) that depends on the presence of the dielectric. The electrostatic shift is easy to evaluate, which we shall do in Sec. IV E.

We are interested in the self-energy corrections for an electron located well outside the dielectric. Because of the electron's localization, its direct interaction with the dielectric medium is completely negligible, i.e. there is no wave-function overlap between the electron and the microscopic constituents of the dielectric. That is why, for $x_3 < 0$ and $x_3' < 0$, we can work with the standard free electron propagator,

$$S^{(0)}(x - x') = (i\gamma^{\mu}\partial_{x^{\mu}} + m)D^c_m(x - x'),$$
$$D^c_m(x - x') = -\int \frac{d^4q}{(2\pi)^4} e^{-iq(x-x')}\frac{1}{q^2 - m^2 + i\epsilon}. \quad (30)$$

The impact of the dielectric medium onto the self-energy of the electron is consequently just due to the electromagnetic interaction, i.e. due to the fact that the photon propagator (10) depends on the presence and electromagnetic properties of the medium. Since we are interested only in the energy shift due to the presence of the dielectric, we split the photon propagator into the free photon propagator and a medium-dependent part,

$$D^c(x, x') = D^c_{(0)}(x, x') + \overline{D}(x, x'),$$

and take only the medium-dependent part $\overline{D}(x, x')$ for calculating the self-energy (23) and the energy shift (28). This also means that we do not have to deal with regularization and renormalization; these have been done in the free part of the photon field, and we work with already renormalized quantities. As in previous calculations of boundary-dependent corrections to free-space quantities (cf. e.g. [4,11]), all medium-dependent corrections will then automatically be finite. In the formalism, this is evident from the fact that the medium-dependent part of the photon propagator depends not on $(z - z')$ but only on $(z + z')$. Thus divergences, which normally arise in loop calculations at coinciding arguments $z = z'$, could appear only when $z + z' = 0$ as well, i.e. at $z = 0 = z'$ when the electron touches the surface of the dielectric. However, at close proximity of the electron to the surface, our macroscopic model of the dielectric medium would be invalid anyway, so that z and z' must be well away from 0 on physical grounds.

If, however, one were to consider a coupled field theory with a Dirac field that was not excluded from the surface of the dielectric on such physical grounds, then one would need additional renormalization terms to counter the divergences arising on the boundary between the vacuum and dielectric. This renormalization would supplement the standard free-space renormalization but would not interfere with it. The details of such a renormalization scheme are beyond the scope of this paper, as this is not a question affecting the issues and physical model investigated here.

From (10) we see that for $z<0$ and $z'<0$, i.e. outside the dielectric, the medium-dependent part of the photon propagator is

$$\overline{D}^{\mu\nu c}(x,x')|_{z,z'<0} = -\sum_\sigma e^\mu_\sigma(\partial_x)e^{\nu *}_\sigma(\partial_{x'})g^{\sigma\sigma}\overline{D}^c_\sigma(x,x'),$$

$$\overline{D}^c_\sigma(x,x') = \int\frac{d^3\tilde{k}}{(2\pi)^4}\int_C dk_z e^{-i\tilde{k}(\tilde{x}-\tilde{x}')+ik_z(z+z')} \times R^L_\sigma\frac{1}{k^2+i\epsilon}. \quad (31)$$

The derivatives of the polarization vectors (A1) act on a plane wave and its reflection. We obtain

$$e^\mu_\sigma(\partial_x)e^{\nu *}_\sigma(\partial_{x'})g^{\sigma\sigma}e^{-i\tilde{k}(\tilde{x}-\tilde{x}')+ik_z(z+z')} = g^{\mu\nu}_\sigma(k)e^{-i\tilde{k}(\tilde{x}-\tilde{x}')+ik_z(z+z')}$$

with

$$g^{\mu\nu}_{TE}(k) = g^{mn}-\frac{k_mk_n}{-k^2_\parallel} = -\left(\delta_{mn}-\frac{k_mk_n}{k^2_\parallel}\right),$$

$$g^{\mu\nu}_{TM}(k) = \left\{g^{mn}=\frac{k_mk_nk_z^2}{\mathbf{k}^2k^2_\parallel}, g^{m3}=\frac{k_mk_z}{\mathbf{k}^2}, g^{3m}=-\frac{k_z^*k_m}{\mathbf{k}^2}, g^{33}=-\frac{k^2_\parallel}{\mathbf{k}^2}\right\}, \quad (32)$$

$$g^{\mu\nu}_C(k) = g^{00},$$

$$g^{\mu\nu}_G(k) = \left\{g^{mn}=-\frac{k_mk_n}{\mathbf{k}^2}, g^{33}=\frac{k_z^2}{\mathbf{k}^2}, g^{m3}=\frac{k_zk_m}{\mathbf{k}^2}=-g^{3m}\right\}$$

where $m,n=1,2$. For evanescent waves, one has $k_z=-i\kappa$, $k_z^2=-\kappa^2$, and $\mathbf{k}^2=k^2_\parallel-\kappa^2$.

Inserting the electron propagator (30) and the photon propagator (31) into the expression for the self-energy (23), we have to multiply several γ matrices. Using

$$\gamma^\mu\gamma^\lambda\gamma^\nu = s^{\alpha\mu\lambda\nu}\gamma_\alpha - i\epsilon^{\alpha\mu\lambda\nu}\gamma_5\gamma_\alpha, \qquad \epsilon_{0123}=+1,$$
$$s^{\alpha\mu\lambda\nu} = g^{\alpha\mu}g^{\lambda\nu}+g^{\alpha\nu}g^{\lambda\mu}-g^{\alpha\lambda}g^{\nu\mu}, \quad (33)$$

we encounter the following γ valued invariants:

$$I^\sigma_1 = mg^{\mu\nu}g^\sigma_{\mu\nu}, \qquad I^\sigma_{15}=\frac{m}{2}[\gamma^\mu,\gamma^\nu]g^\sigma_{\mu\nu},$$
$$I^\sigma_{25} = \epsilon^{\alpha\mu\lambda\nu}\gamma_5\gamma_\alpha g^\sigma_{\mu\nu}q_\lambda,$$
$$I^\sigma_2 = s^{\alpha\mu\lambda\nu}\gamma_\alpha g^\sigma_{\mu\nu}q_\lambda = g^{\mu\nu}_\sigma\gamma_\mu q_\nu + g^{\nu\mu}_\sigma\gamma_\mu q_\nu - (\gamma q)I^\sigma_1/m,$$

with q being the Fourier variable in the electron propagator (30) and $g^\sigma_{\mu\nu}$ having the k dependence as in (32). In terms of those the distance-dependent part of the self-energy is

$$\Sigma=\sum_\sigma\Sigma^\sigma, \qquad \Sigma^\sigma=[\Sigma^\sigma_1+\Sigma^\sigma_{15}+\Sigma^\sigma_2+\Sigma^\sigma_{25}] \quad (34)$$

with

$$\Sigma^\sigma_\theta(x,x') = -ie^2\int\frac{d^4q}{(2\pi)^4}\int\frac{d^3\tilde{k}}{(2\pi)^4}\int_C dk_z\frac{e^{-iq(x-x')}}{q^2-m^2+i\epsilon} \times I^\sigma_\theta(q,k)\frac{e^{-i\tilde{k}(\tilde{x}-\tilde{x}')+ik_z(z+z')}}{k^2+i\epsilon}R^L_\sigma. \quad (35)$$

In the same way as for the total self-energy in Eq. (26) we perform a Fourier transformation for the components in the sum (34), which again retain a dependence on $z+z'$ due to the broken translation invariance in the z direction,

$$\Sigma^\sigma_\theta(x,x') = \int\frac{d^4p}{(2\pi)^4}e^{-ip(x-x')}\Sigma^\sigma_\theta(p,z+z').$$

Making the variable replacements $\tilde{q}=\tilde{p}-\tilde{k}$, $q_z=p_z$ in Eq. (35) we obtain

$$\Sigma^\sigma_\theta(p,z+z') = -ie^2\int\frac{d^3\tilde{k}}{(2\pi)^4} \times\int_C dk_z\frac{1}{(\tilde{p}-\tilde{k})^2-p_z^2-m^2+i\epsilon} \times I^\sigma_\theta(\tilde{p}-\tilde{k},p_z,k)\frac{e^{ik_z(z+z')}}{\tilde{k}^2-k_z^2+i\epsilon}R^L_\sigma. \quad (36)$$

This expression looks much simpler than it is to evaluate. The loss of translation invariance perpendicular to the surface of the dielectric is one source of complications, and the interference of incident and reflected waves is another. In order to evaluate the self-energy components (36) we need the explicit expressions for the invariants I^σ_θ, which depend on the mode σ. For the two physical modes we find

$$I^{TE}_1=m, \qquad I^{TE}_{15}=0, \qquad I^{TE}_{25}=0,$$
$$I^{TE}_2 = -2\left[\gamma_\parallel\cdot(\mathbf{p}_\parallel-\mathbf{k}_\parallel)-\gamma_\parallel\cdot\mathbf{k}_\parallel\frac{\mathbf{k}_\parallel\cdot(\mathbf{p}_\parallel-\mathbf{k}_\parallel)}{k^2_\parallel}\right] + \gamma_\parallel\cdot(\mathbf{p}_\parallel-\mathbf{k}_\parallel)+\gamma_3p_z-\gamma_0(p_0-k_0), \quad (37)$$

$$I^{TM}_1 = m\left(1-2\frac{k_z^2}{\mathbf{k}^2}\right), \quad I^{TM}_{15}=-m\frac{1}{\mathbf{k}^2}[\gamma^3,\gamma^n]k_zk^n,$$
$$I^{TM}_{25} = 2\epsilon^{\alpha m\lambda 3}\gamma_5\gamma_\alpha(p_\lambda-k_\lambda)\frac{k_mk_z}{\mathbf{k}^2},$$
$$I^{TM}_2 = 2\left\{\frac{(\gamma_\parallel\cdot\mathbf{k}_\parallel)[\mathbf{k}_\parallel\cdot(\mathbf{p}_\parallel-\mathbf{k}_\parallel)]}{k^2_\parallel\mathbf{k}^2}k_z^2-p_z\gamma_3\frac{k^2_\parallel}{\mathbf{k}^2}\right\} + [\gamma_\parallel\cdot(\mathbf{p}_\parallel-\mathbf{k}_\parallel)+\gamma_3p_z-\gamma_0(p_0-k_0)]\left(1-2\frac{k_z^2}{\mathbf{k}^2}\right). \quad (38)$$

IV. ASYMPTOTIC ANALYSIS OF THE SELF-ENERGY

A. General approach and approximations

Our aim is to determine the energy shift of an electron that is localized in the z direction. The shift will depend on the distance a of the electron from the surface of the dielectric, and without localization the notion of this dis-

tance would not make sense. Physically the localization could be realized by sending a tightly focused beam parallel to the surface or by confining the electron by means of magnetic and/or electric fields. So, we will in fact not be working directly with momentum eigenstates (19) but we will use them to form with wave packets that peak at $z = a$ and whose average momentum in the z direction is p_z. The wave packet may move as a whole, which is why we have not approximated $\mathbf{z} - \mathbf{z}'$ in Eq. (27); but we shall assume the electromagnetic field to be the same across the packet, which corresponds to the dipole approximation in atomic physics, and which is why we have set $z_3 + z_3' \simeq -2a$ in Eq. (28).

The extent of the wave packet must be small compared with the distance a from the surface, but otherwise the details of the wave packet are not relevant. This implies $am \gg 1$, i.e. that the distance a must be very much larger than the Compton wavelength $\lambda\!\!\!^{-}_C = \hbar/(mc)$. Thus, we shall aim for an expansion in $1/(ap_0)$.

In order to proceed with the calculation of the self-energy (36), we want to perform a Wick rotation $k_0 = ik_4$. By design, the poles of the photon propagator lie in the right position for this. However, the poles of the term that originates from the electron propagator may interfere; they lie at $k_0 = p_0 \pm [\sqrt{(\mathbf{p}_\parallel - \mathbf{k}_\parallel)^2 + p_z^2 + m^2} - i\epsilon]$. There is no problem if they come to lie in the 2nd and 4th quadrant of the complex k_0 plane, but for $2\mathbf{p}_\parallel \cdot \mathbf{k}_\parallel > \mathbf{k}_\parallel^2$, one of the poles lies in the 1st rather than the 2nd quadrant, if we take p to be on shell, and a pole in the 1st quadrant interferes with the Wick rotation. There are several ways of dealing with this problem. One could work with a strongly deformed integration path and then carry along the separate contribution from the pole, or one could go off shell to move the pole out of the 1st quadrant and then do an analytic continuation to a result for on-shell p, or one could avoid the problem altogether by approximating $p_\parallel \approx 0$ in the denominator of (36). We have decided on the last approach because it is straightforward and we are not interested in the ultrarelativistic motion of the electron.

Thus we set $p_\parallel = 0$ in the denominator of (36) but leave $p_\parallel$ untouched elsewhere, i.e. retain it in I_θ^σ (37) and (38). Then we can perform the Wick rotation $k_0 = ik_4$ without problems and obtain

$$\Sigma_\theta^\sigma(p, z + z') = e^2 \int \frac{d^2k_\parallel}{(2\pi)^4} \int_{-\infty}^{\infty} dk_4 \frac{1}{k_4^2 + 2ip_0k_4 + \mathbf{k}_\parallel^2} \times \int_C dk_z I_\theta^\sigma \frac{e^{ik_z(z+z')}}{k_4^2 + \mathbf{k}_\parallel^2 + k_z^2} R_\sigma^L.$$

The integration over the three-dimensional (Euclidean) space $(\mathbf{k}_\parallel, k_4)$ can be carried out in spherical polar coordinates by defining $k_4 = \rho\cos\vartheta$, $k_x = \rho\sin\vartheta\cos\varphi$, $k_y = \rho\sin\vartheta\sin\varphi$. We find

$$\Sigma_\theta^\sigma(p, z + z') = \frac{e^2}{(2\pi)^3} \int_0^\infty d\rho \int_{-1}^{1} d(\cos\vartheta) \times \frac{\rho^2}{\rho^2 + 2ip_0\rho\cos\vartheta} \times \int_C dk_z \frac{e^{ik_z(z+z')}}{\rho^2 + k_z^2} R_\sigma^L \frac{1}{2\pi} \int_0^{2\pi} d\varphi I_\theta^\sigma. \tag{39}$$

The only φ dependence is in the invariants I_θ^σ; carrying out the integration and using the fact that p is on shell, one gets

$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi (I_1^{TE} + I_2^{TE}) = \gamma_0 i\rho\cos\vartheta - \gamma_\parallel \cdot \mathbf{p}_\parallel,$$
$$\frac{1}{2\pi}\int_0^{2\pi} d\varphi (I_1^{TM} + I_2^{TM}) = -\gamma_0 i\rho\cos\vartheta + \gamma_\parallel \cdot \mathbf{p}_\parallel + \frac{\mathbf{k}_\parallel^2}{\mathbf{k}^2}(2\gamma_0 i\rho\cos\vartheta - 2\gamma_3 p_z - \gamma_\parallel \cdot \mathbf{p}_\parallel),$$
$$\int_0^{2\pi} d\varphi I_{15}^{TM} = 0, \qquad \int_0^{2\pi} d\varphi I_{25}^{TM} = 0.$$

The next step in the evaluation of Eq. (39) is to carry out the integration over k_z by means of contour integration. The remaining two-dimensional integral over ρ and $\cos\vartheta$ can then be calculated asymptotically for $|z + z'|$ very much larger than the Compton wavelength. Since the technical details differ between the TE and TM polarizations, we consider their contributions one after the other. While the calculation to follow is perfectly general for all values of z and z', provided $|z + z'|p_0 \gg 1$, we now simplify the notation and set $z + z' = -2a$, as this is the value at which we need to evaluate the self-energy in Eq. (28) for the radiative shift.

B. TE contributions to the self-energy

For the TE polarization the integrand of (39) has only one pole in the lower k_z plane, and that is at $k_z = -i\rho = -i(\mathbf{k}_\parallel^2 + k_4^2)^{1/2}$ (cf. Fig. 2). The k_z integration can thus easily be carried out by deforming the contour C and evaluating the residue at $-i\rho$. We emphasize that when evaluating the reflection coefficient R_{TE}^L, Eq. (A3), at this point, one must take great care that the branch cut of the square root in k_z^d is indeed taken to run as shown in Fig. 2. Renaming $\cos\vartheta = t$, we can write the result of the contour integration as

$$\Sigma^{TE} = \frac{e^2}{8\pi^2}\int_0^\infty d\rho \int_{-1}^{1} dt \frac{\gamma_0 i\rho t - \gamma_\parallel \cdot \mathbf{p}_\parallel}{\rho + 2ip_0 t} R_{TE}^L(t) e^{-2a\rho} = \frac{e^2}{4\pi^2}\int_0^\infty d\rho \int_0^1 dt \rho \frac{2\gamma_0 p_0 t^2 - \gamma_\parallel \cdot \mathbf{p}_\parallel}{\rho^2 + 4p_0^2 t^2} R_{TE}^L(t) e^{-2a\rho}$$

with

$$R^L_{TE}(t) = \frac{1-\sqrt{(n^2-1)t^2+1}}{1+\sqrt{(n^2-1)t^2+1}}. \tag{40}$$

Next we scale the integration variable $\rho = 2p_0 t\xi$. In terms of the new variable ξ the integral reads

$$\Sigma^{TE} = \frac{e^2}{4\pi^2}\int_0^\infty d\xi \int_0^1 dt\xi \times \frac{2\gamma_0 p_0 t^2 - \gamma_\parallel\cdot\mathbf{p}_\parallel}{\xi^2+1} R^L_{TE}(t) e^{-4p_0 at\xi}. \tag{41}$$

This integral can be evaluated asymptotically for large values of $p_0 a$. A standard method of obtaining an asymptotic expansion for integrals with an exponentially damped integrand is repeated integration by parts. However, in two-dimensional integrals like the one above, this method generally fails because integration by parts in one variable generates inverse powers of the other variable and the resulting integral diverges at the lower limit. For a general discussion of this problem and its remedy, we refer the reader to Ref. [12]. Here we observe that $R^L_{TE}(t)$ and hence the integrand of (41) actually behave as $O(t^2)$ for $t\to 0$. Thus we can integrate by parts in the ξ integral twice without jeopardizing the convergence of the t integral. In this way we find to leading order in $1/(p_0 a)$

$$\Sigma^{TE} \approx \frac{e^2}{4\pi^2}\frac{1}{(4p_0 a)^2}\int_0^1 dt \frac{2\gamma_0 p_0 t^2 - \gamma_\parallel\cdot\mathbf{p}_\parallel}{t^2} R^L_{TE}(t). \tag{42}$$

The t integral in this expression is elementary.

C. TM contributions to the self-energy

The TM polarization is more difficult to deal with, since the invariants I^{TM}_θ introduce a factor $1/\mathbf{k}^2$ into the integrand of (39), which leads to an additional pole in the lower k_z plane at $k_z = -i\rho\sin\vartheta = -ik_\parallel$ (cf. Fig. 2). Thus, closing the contour in the lower k_z plane picks up two residues, one at $-i\rho$ and one at $-i\rho\sin\vartheta$. The result is

$$\begin{aligned}\Sigma^{TM} &= \frac{e^2}{8\pi^2}\int_0^\infty d\rho\int_{-1}^1 dt\frac{1}{\rho+2ip_0 t}\left\{\left[\gamma_\parallel\cdot\mathbf{p}_\parallel - \gamma_0 i\rho t + \frac{1-t^2}{t^2}(\gamma_\parallel\cdot\mathbf{p}_\parallel - 2\gamma_0 i\rho t + 2\gamma_3 p_z)\right]R^L_{TM}(t)e^{-2a\rho}\right.\\ &\left. - \frac{\sqrt{1-t^2}}{t^2}(\gamma_\parallel\cdot\mathbf{p}_\parallel - 2\gamma_0 i\rho t + 2\gamma_3 p_z)R^L_{TM}(0)e^{-2a\rho\sqrt{1-t^2}}\right\}\\ &= \frac{e^2}{4\pi^2}\int_0^\infty d\rho\int_0^1 dt\frac{\rho}{\rho^2+4p_0^2t^2}\left\{\left[\frac{1}{t^2}\gamma_\parallel\cdot\mathbf{p}_\parallel - 2\gamma_0 p_0(2-t^2) + \frac{1-t^2}{t^2}2\gamma_3 p_z\right]R^L_{TM}(t)e^{-2a\rho}\right.\\ &\left. - \frac{\sqrt{1-t^2}}{t^2}(\gamma_\parallel\cdot\mathbf{p}_\parallel - 4\gamma_0 p_0 t^2 + 2\gamma_3 p_z)R^L_{TM}(0)e^{-2a\rho\sqrt{1-t^2}}\right\}\end{aligned} \tag{43}$$

where we have again renamed $\cos\vartheta = t$ and abbreviated

$$R^L_{TM}(t) = \frac{n^2-\sqrt{(n^2-1)t^2+1}}{n^2+\sqrt{(n^2-1)t^2+1}}.$$

As before, we are interested in an asymptotic result for Σ^{TM} for large values of $p_0 a$. To be able to do asymptotic analysis one needs to separate the terms with different arguments in the exponential. However, doing this simple-mindedly leads to two divergent integrals because their integrands each behave as $O(t^{-2})$ for $t\to 0$. That is why we add and subtract the same term and subdivide the integral as follows,

$$\Sigma^{TM} = \Sigma^{TM}_A + \Sigma^{TM}_B + \Sigma^{TM}_C + \Sigma^{TM}_D, \tag{44}$$

with

$$\begin{aligned}\Sigma^{TM}_A &= \frac{e^2}{4\pi^2}\int_0^\infty d\xi\int_0^1 dt\frac{\xi}{\xi^2+1}[2\gamma_0 p_0(t^2-2) - 2\gamma_3 p_z]R^L_{TM}(t)e^{-4p_0 at\xi},\\ \Sigma^{TM}_B &= \frac{e^2}{4\pi^2}(2\gamma_3 p_z + \gamma_\parallel\cdot\mathbf{p}_\parallel)\int_0^\infty d\xi\int_0^1 dt\frac{\xi}{\xi^2+1}\frac{1}{t^2}[R^L_{TM}(t) - R^L_{TM}(0)]e^{-4p_0 at\xi},\\ \Sigma^{TM}_C &= \frac{e^2}{4\pi^2}(2\gamma_3 p_z + \gamma_\parallel\cdot\mathbf{p}_\parallel)R^L_{TM}(0)\int_0^\infty d\xi\int_0^1 dt\frac{\xi}{\xi^2+1}\frac{1}{t^2}(e^{-4p_0 at\xi} - \sqrt{1-t^2}e^{-4p_0 at\sqrt{1-t^2}\xi}),\\ \Sigma^{TM}_D &= \frac{e^2}{4\pi^2}4\gamma_0 p_0 R^L_{TM}(0)\int_0^\infty d\xi\int_0^1 dt\frac{\xi}{\xi^2+1}\sqrt{1-t^2}e^{-4p_0 at\sqrt{1-t^2}\xi}\end{aligned}$$

where we have again rescaled $\rho = 2p_0 t\xi$.

We do the asymptotic analysis of these integrals one by one, starting with Σ_A^{TM}. In order to get an asymptotic expansion for large $p_0 a$, one would try to integrate by parts. However, the integrand of Σ_A^{TM} behaves as $O(1)$ for $t \to 0$, and thus the factor $1/t$ that one gets through integrating by parts in the ξ integral would destroy the convergence at $t = 0$. Adapting the general method of obtaining an asymptotic expansion of such two-dimensional integrals [12], we add and subtract the problematic point at $t = 0$ and write

$$\begin{aligned}\Sigma_A^{TM} = &-\frac{e^2}{4\pi^2}\int_0^\infty d\xi \int_0^1 dt \frac{\xi}{\xi^2+1}(4\gamma_0 p_0 + 2\gamma_3 p_z) \\ &\times R_{TM}^L(0) e^{-4p_0 a t\xi} + \frac{e^2}{4\pi^2}\int_0^\infty d\xi \int_0^1 dt \frac{\xi}{\xi^2+1} \\ &\times \{[2\gamma_0 p_0(t^2-2) - 2\gamma_3 p_z] R_{TM}^L(t) \\ &+ (4\gamma_0 p_0 + 2\gamma_3 p_z) R_{TM}^L(0)\} e^{-4p_0 a t\xi}.\end{aligned}$$

The first of the integrals is easy to calculate; the t integral is immediate, and the remaining integral over ξ is a well-known combination of sine and cosine integrals [13]. The integrand of the second integral now behaves as $O(t^2)$ for $t \to 0$, and we can thus integrate by parts twice without getting convergence problems at $t = 0$. In this way we find to order $1/(p_0 a)^2$

$$\begin{aligned}\Sigma_A^{TM} \approx &-\frac{e^2}{8\pi^2 p_0 a}(2\gamma_0 p_0 + \gamma_3 p_z) R_{TM}^L(0)\left(\frac{\pi}{2} - \frac{1}{4p_0 a}\right) \\ &+ \frac{e^2}{32\pi^2 p_0^2 a^2}\left\{\gamma_0 p_0 \int_0^1 \frac{dt}{t^2}[(t^2-2)R_{TM}^L(t) \right. \\ &\left. + 2R_{TM}^L(0)] - \gamma_3 p_z \int_0^1 \frac{dt}{t^2}[R_{TM}^L(t) - R_{TM}^L(0)]\right\}\end{aligned} \tag{45}$$

for which we have also made use of the known asymptotics of the sine and cosine integrals [13].

The asymptotics of Σ_B^{TM} can be calculated similarly by adding and subtracting the next term in the Taylor expansion of $R_{TM}^L(t)$, i.e. by replacing

$$\begin{aligned}\frac{1}{t^2}[R_{TM}^L(t) - R_{TM}^L(0)] = R_{TM}^{L\,\prime\prime}(0)/2 + \frac{1}{t^2}[R_{TM}^L(t) \\ - R_{TM}^L(0) - t^2 R_{TM}^{L\,\prime\prime}(0)/2].\end{aligned}$$

The first term can then be integrated exactly, and the rest behaves as $O(t^2)$ for $t \to 0$ and can thus be integrated by parts with respect to ξ twice. The result to order $1/(p_0 a)^2$ is

$$\begin{aligned}\Sigma_B^{TM} \approx &-\frac{e^2}{4\pi^2 p_0 a}(2\gamma_3 p_z + \gamma_\parallel \cdot \mathbf{p}_\parallel)\frac{1}{4p_0 a}\left\{\frac{1}{2}R_{TM}^{L\,\prime\prime}(0) \right. \\ &\times\left(\frac{\pi}{2} - \frac{1}{4p_0 a}\right) + \frac{1}{4p_0 a}\int_0^1 \frac{dt}{t^2}\left[\frac{R_{TM}^L(t) - R_{TM}^L(0)}{t^2} \right. \\ &\left.\left. - \frac{1}{2}R_{TM}^{L\,\prime\prime}(0)\right]\right\}.\end{aligned} \tag{46}$$

Next we turn our attention to the asymptotic evaluation of Σ_C^{TM}. We cannot separate the two summands in the integrand because otherwise the t integral does not converge. Thus, to manipulate just one part, we must set the lower limit of the t integral to some small positive ε and take the limit $\varepsilon \to 0$ only once we have combined all parts again. We write

$$\begin{aligned}\int_\varepsilon^1 dt \frac{\sqrt{1-t^2}}{t^2} e^{-4p_0 a t\sqrt{1-t^2}\xi} = &\int_\varepsilon^{1/\sqrt{2}} dt \frac{\sqrt{1-t^2}}{t^2} e^{-4p_0 a t\sqrt{1-t^2}\xi} \\ &+ \int_{1/\sqrt{2}}^1 dt \frac{\sqrt{1-t^2}}{t^2} e^{-4p_0 a t\sqrt{1-t^2}\xi}.\end{aligned}$$

In the integral that runs from $1/\sqrt{2}$ to 1 we make a change of variable from t to $s = \sqrt{1-t^2}$. Then renaming s into t again and ignoring terms that vanish in the limit $\varepsilon \to 0$, we find

$$\begin{aligned}\int_\varepsilon^1 dt \frac{\sqrt{1-t^2}}{t^2} e^{-4p_0 a t\sqrt{1-t^2}\xi} = &\int_\varepsilon^{1/\sqrt{2}} dt \left[\frac{\sqrt{1-t^2}}{t^2} \right. \\ &\left. + \frac{t^2}{(1-t^2)^{3/2}}\right] e^{-4p_0 a t\sqrt{1-t^2}\xi}.\end{aligned}$$

Now we substitute $v = t\sqrt{1-t^2}$ and obtain

$$\begin{aligned}&\int_\varepsilon^1 dt \frac{\sqrt{1-t^2}}{t^2} e^{-4p_0 a t\sqrt{1-t^2}\xi} \\ &\quad = \int_\varepsilon^{1/2} dv \frac{1-2v^2}{v^2\sqrt{1-4v^2}} e^{-4p_0 a v\xi}.\end{aligned}$$

Renaming the integration variable v into t again, we can use this identity to write Σ_C^{TM} as

$$\begin{aligned}\Sigma_C^{TM} = &\frac{e^2}{4\pi^2}(2\gamma_3 p_z + \gamma_\parallel \cdot \mathbf{p}_\parallel) R_{TM}^L(0) \int_0^\infty d\xi \frac{\xi}{\xi^2+1} \\ &\times\left[\int_0^{1/2} dt \frac{1}{t^2}\left(1 - \frac{1-2t^2}{\sqrt{1-4t^2}}\right) e^{-4p_0 a t\xi} \right. \\ &\left. + \int_{1/2}^1 dt \frac{1}{t^2} e^{-4p_0 a t\xi}\right].\end{aligned}$$

Integrating by parts in the ξ integral twice, we obtain to order $1/(p_0 a)^2$

$$\Sigma_C^{TM} \approx -\frac{e^2}{4\pi^2}(2\gamma_3 p_z + \gamma_\parallel \cdot \mathbf{p}_\parallel) R_{TM}^L(0) \frac{1}{48 p_0^2 a^2}. \tag{47}$$

For the asymptotic expansion of Σ_D^{TM} we apply much the same tricks as above: we split the t integration at $1/\sqrt{2}$ and in the integral from $1/\sqrt{2}$ to 1 we substitute $s = \sqrt{1-t^2}$. Then we rename s back to t, combine the two integrals again, and substitute $v = t\sqrt{1-t^2}$ to obtain

$$\begin{aligned}\Sigma_D^{TM} &= \frac{e^2}{\pi^2}\gamma_0 p_0 R_{TM}^L(0)\int_0^\infty d\xi \int_0^{1/2} dv \frac{\xi}{\xi^2+1} \\ &\quad\times \frac{1}{\sqrt{1-4v^2}} e^{-4p_0 a v\xi} \\ &= \frac{e^2}{\pi^2}\gamma_0 p_0 R_{TM}^L(0)\Bigg[\int_0^\infty d\xi \int_0^{1/2} dv \frac{\xi}{\xi^2+1} e^{-4p_0 a v\xi} \\ &\quad + \int_0^\infty d\xi \int_0^{1/2} dv \frac{\xi}{\xi^2+1}\left(\frac{1}{\sqrt{1-4v^2}}-1\right)e^{-4p_0 a v\xi}\Bigg].\end{aligned}$$

The first of those integrals can be solved in terms of known special functions [13], and in the second integral we can integrate by parts twice with respect to ξ. Thus we obtain for Σ_D^{TM} to order $1/(p_0 a)^2$

$$\Sigma_D^{TM} \approx \frac{e^2}{8\pi a}\gamma_0 R_{TM}^L(0). \tag{48}$$

According to (44) we combine the results (45)–(48) to obtain the leading-order self-energy contribution from the TM modes. It turns out to be one order larger than that from the TE modes.

$$\begin{aligned}\Sigma^{TM} \approx -\frac{e^2}{32\pi p_0 a}\Bigg[&\gamma_\parallel\cdot\mathbf{p}_\parallel \frac{n^2(n^2-1)}{(n^2+1)^2} \\ &+ 2\gamma_3 p_z \frac{2n^4-n^2-1}{(n^2+1)^2} + O(1/p_0 a)\Bigg].\end{aligned}$$

Thus, to leading order the radiative self-energy is due to just the TM modes.

$$\begin{aligned}\Sigma(p,-2a) \approx -\frac{e^2}{32\pi p_0 a}\Bigg[&\gamma_\parallel\cdot\mathbf{p}_\parallel \frac{n^2(n^2-1)}{(n^2+1)^2} \\ &+ 2\gamma_3 p_z \frac{2n^4-n^2-1}{(n^2+1)^2}\Bigg] + \Sigma^{\text{next}}.\end{aligned} \tag{49}$$

The next-to-leading term Σ^{next} is easily determined from the results (42) and (45)–(48). While all the t integrals are elementary, it is convenient to use formula manipulation software like MAPLE to evaluate and combine them. The final result for the next-to-leading-order contribution to the radiative self-energy is

$$\begin{aligned}\Sigma^{\text{next}} = -\frac{e^2}{32\pi^2(p_0 a)^2}\Bigg\{&2\gamma_0 p_0\Bigg[\frac{n^2+1}{\sqrt{n^2-1}}\ln\left(n+\sqrt{n^2-1}\right) - \frac{n^3+4n^2+n+2}{(n^2+1)(n+1)} - \frac{2n^4}{(n^2+1)^{3/2}}\operatorname{arctanh}\frac{n-1}{\sqrt{n^2+1}}\Bigg] \\ &+ \gamma_\parallel\cdot\mathbf{p}_\parallel\Bigg[\frac{n^2(2n^4+3n^2-3n+1)}{3(n+1)(n^2+1)^2} - \frac{2n^4}{(n^2-1)(n^2+1)^{5/2}}\operatorname{arctanh}\frac{n-1}{\sqrt{n^2+1}}\Bigg] \\ &+ \gamma_3 p_z\Bigg[\frac{2n^6+n^5-3n^4-4n^3-4n^2+n+1}{3(n+1)(n^2+1)^2} + \frac{4n^6}{(n^2-1)(n^2+1)^{5/2}}\operatorname{arctanh}\frac{n-1}{\sqrt{n^2+1}}\Bigg]\Bigg\}.\end{aligned}$$

D. Self-energy in the limit $n \to \infty$

In the limit of perfect reflectivity $n \to \infty$ the calculation of the self-energy simplifies considerably. All reflection coefficients go to either $+1$ or -1 [cf. Eq. (A10)], and the photon propagator takes on a much simpler form with the k_3 integration running straight along the real axis [cf. Eq. (14)]. The calculation of the self-energy can then proceed in exactly the same way as explained in Sec. IV A above. It starts to differ only with the asymptotic analysis of Eqs. (41) and (43). For TE the asymptotic analysis of (41) relied on the fact that $R_{TE}^L(t)$ behaves as $O(t^2)$ for $t \to 0$, but in the limit $n \to \infty$ we have $R_{TE}^L = -1$ for all t, which leads to a very different asymptotic behavior of the self-energy Σ^{TE}. To leading order we find

$$\Sigma_{\text{perf}}^{TE} \approx \frac{e^2}{32\pi p_0 a}\gamma_\parallel\cdot\mathbf{p}_\parallel.$$

In the case of TM, something similar happens. The integrals Σ_C^{TM}, Σ_D^{TM}, and, to leading order in $1/p_0 a$, also Σ_A^{TM} give the same with the limit $n \to \infty$ taken first as they do for finite n and with the limit $n \to \infty$ taken in the end result of the asymptotic calculation. However, the integral Σ_B^{TM} does not even appear if the limit $n \to \infty$ is taken straightaway. For finite n its asymptotically leading term depends on the second derivative $R_{TM}^L{}''(0)$, which is of course zero if the limit $n \to \infty$ has been taken first and R_{TM}^L is a constant. If one takes $n \to \infty$ first, then to leading order only Σ_A^{TM} and Σ_D^{TM} contribute, and one obtains

$$\Sigma_{\text{perf}}^{TM} \approx -\frac{e^2}{16\pi p_0 a}\gamma_3 p_z.$$

In total the radiative part of the self-energy with the perfect-reflector limit taken first is

$$\Sigma_{\text{perf}}(p,-2a) \approx \frac{e^2}{32\pi p_0 a}(\gamma_\parallel\cdot\mathbf{p}_\parallel - 2\gamma_3 p_z),$$

which clearly differs from the limit $n \to \infty$ of Eq. (49). So, mathematically not surprisingly, we find that the result for the self-energy differs depending on whether we perform the calculation for finite n and subsequently take the limit $n \to \infty$, or whether we take the limit of perfect reflectivity

$n \to \infty$ first and evaluate the integrals then. The cause of this is simply the fact that the limits $n \to \infty$ and $t \to 0$ in the Fresnel coefficients $R_\sigma^L(t)$ do not commute. The point $t = 0$ corresponds to $\cos\vartheta = 0$, i.e. to $k_0 = 0$. Thus, physically speaking, the limit of perfect reflectivity is not interchangeable with the limit $\omega \to 0$; in other words, long-wavelength excitations get treated very differently in and away from the limit of perfect reflectivity, $n \to \infty$. Indications of this can also be seen in the nonrelativistic calculation of the energy level shift [7].

The important lesson to be learned from this observation is that models that assume perfect reflectivity from the outset are bound to give the wrong answer if long-wavelength excitations play any role in the system under investigation. Luckily, most of cavity QED is concerned with atoms and other bound systems which have an inherent low-frequency cutoff (e.g. the lowest transition frequency ω_{ij} of an atom in state $|i\rangle$ to dipole-allowed states $|j\rangle$). However, for unbound or partially bound systems a perfect-reflector model is principally inadequate for describing any physically realizable system, no matter how good the reflectivity of the boundaries may be [7].

E. Electrostatic contribution

The evaluation of the electrostatic shift (29) is straightforward. The state in which the expectation value is being taken is a wave packet that is localized at around $\mathbf{x}_a = (0, 0, -a)$ and that has an average momentum $\mathbf{p}$. Here the localization parallel to the surface of the dielectric can of course be arbitrarily loose, as the system is translation invariant parallel to the surface and hence the energy shift does not depend on the transverse location of the particle. We choose to represent the localized state by a Gaussian wave packet,

$$|\varphi(\mathbf{x}_a, \mathbf{p})\rangle = \frac{1}{\pi^{3/4}\sigma^{3/2}} \int d^3\mathbf{q}\, e^{-[(\mathbf{q}-\mathbf{p})^2/2\sigma^2] - i\mathbf{q}\cdot\mathbf{x}_a} b^\dagger_{\mathbf{q},i}|0\rangle.$$

Taking the expectation value in Eq. (29) in this state and using the canonical mode expansion (18) for the spinor operators, we obtain

$$\begin{aligned}\delta E_{\text{Coulomb}} = &\frac{e^2}{2\pi^{3/2}\sigma^3}\int d^3\mathbf{x}\int d^3\mathbf{x}'\int d^3\mathbf{p}'\int\frac{d^3\mathbf{q}}{(2\pi)^3}\\ &\times\int\frac{d^3\mathbf{q}'}{(2\pi)^3}\frac{m^2}{p_0'\sqrt{q_0 q_0'}}\overline{G}_\Phi(\mathbf{x},\mathbf{x}')\\ &\times e^{i(q_0'-q_0)t - i(\mathbf{q}-\mathbf{q}')\cdot\mathbf{x}_a + i(\mathbf{q}-\mathbf{p}')\cdot\mathbf{x}' - i(\mathbf{q}'-\mathbf{p}')\cdot\mathbf{x}}\\ &\times e^{-[(\mathbf{q}-\mathbf{p})^2/2\sigma^2] - [(\mathbf{q}'-\mathbf{p})^2/2\sigma^2]}\\ &\times\sum_{j=1,2}\overline{u}^{(i)}(\mathbf{q}')\gamma^0 u^{(j)}(\mathbf{p}')\overline{u}^{(j)}(\mathbf{p}')\gamma^0 u^{(i)}(\mathbf{q}).\end{aligned}$$

Since the integrand as a function of $\mathbf{x}$ and $\mathbf{x}'$ is peaked at the location $\mathbf{x}_a$ of the wave packet, we can approximate the Green's function $\overline{G}_\Phi(\mathbf{x}, \mathbf{x}')$ by $\overline{G}_\Phi(\mathbf{x}_a, \mathbf{x}_a)$. Then the $\mathbf{x}$ and $\mathbf{x}'$ integrations are easy to carry out and give δ functions. The sum over polarizations j can also be done because the Dirac eigenspinors satisfy

$$\sum_{j=1,2} u^{(j)}(\mathbf{p}')\overline{u}^{(j)}(\mathbf{p}') = \frac{1}{2m}(\gamma^\mu p'_\mu + m).$$

Thus the above expression simplifies to

$$\begin{aligned}\delta E_{\text{Coulomb}} \simeq &\frac{e^2}{2\pi^{3/2}\sigma^3}\overline{G}_\Phi(\mathbf{x}_a, \mathbf{x}_a)\int d^3\mathbf{p}'\frac{m}{2p_0'^2}\\ &\times e^{-[(\mathbf{p}'-\mathbf{p})^2/\sigma^2]}\overline{u}^{(i)}(\mathbf{p}')(\gamma^0 p_0' + \boldsymbol{\gamma}\cdot\mathbf{p}' + m)\\ &\times u^{(i)}(\mathbf{p}').\end{aligned}$$

The integrand of this expression peaks at $\mathbf{p}' = \mathbf{p}$, so that we can approximate $\mathbf{p}'$ by $\mathbf{p}$ everywhere except in the exponential and carry out the integration. Taking into account that $\mathbf{p}$ is on shell, we then find

$$\delta E_{\text{Coulomb}} \simeq \frac{e^2}{2}\overline{G}_\Phi(\mathbf{x}_a, \mathbf{x}_a)\frac{m}{p_0}\overline{u}^{(i)}(\mathbf{p})\gamma^0 u^{(i)}(\mathbf{p}).$$

It remains the evaluation of the Green's function at the location of the particle. Since we are looking for the energy shift relative to a particle in free space, we use the difference of the Coulomb Green's function (16) and the free-space Coulomb Green's function,

$$\overline{G}_\Phi(x, x') = G_\Phi(x, x') - \int\frac{d^3k}{(2\pi)^3}e^{i(\mathbf{x}-\mathbf{x}')\cdot\mathbf{k}}\frac{1}{\mathbf{k}^2}.$$

As $\mathbf{x}_a$ is outside the dielectric, we have

$$\overline{G}_\Phi(\mathbf{x}_a, \mathbf{x}_a) = -\frac{n^2-1}{n^2+1}\frac{1}{4\pi^2}\int_0^\infty dk_\parallel\int_{-\infty}^\infty dk_z\frac{k_\parallel e^{-2iak_z}}{k_\parallel^2 + k_z^2}.$$

For the k_z integration we close the contour in the lower half-plane. The integration over $k_\parallel$ is then trivial. The result is

$$\overline{G}_\Phi(\mathbf{x}_a, \mathbf{x}_a) = -\frac{n^2-1}{n^2+1}\frac{1}{8\pi a}.$$

Thus the electrostatic energy shift is

$$\delta E_{\text{Coulomb}} \simeq -\frac{n^2-1}{n^2+1}\frac{e^2}{16\pi a}\frac{m}{p_0}\overline{u}^{(i)}(\mathbf{p})\gamma^0 u^{(i)}(\mathbf{p}). \quad (50)$$

For a particle at rest this agrees with the classical energy shift of a point particle in front of a dielectric half-space [14].

If one wishes, one can express this shift as being due to a Coulomb self-energy function. If the shift is given by Eq. (28) then we can write

$$\Sigma_{\text{Coulomb}}(p, -2a) \simeq -\frac{n^2-1}{n^2+1}\frac{e^2}{16\pi a}\gamma^0. \quad (51)$$

V. SUMMARY AND DISCUSSION OF THE RESULTS

Combining our results for the radiative self-energy (49) and for the electrostatic self-energy (51), we obtain for the total self-energy to leading order in $1/(p_0 a)$

$$\Sigma_{\text{total}}(p,-2a) \simeq -\frac{e^2}{32\pi p_0 a}\Big[\gamma_\parallel\cdot\mathbf{p}_\parallel\frac{n^2(n^2-1)}{(n^2+1)^2} + 2\gamma_3 p_z\frac{2n^4-n^2-1}{(n^2+1)^2} + 2\gamma^0 p_0\frac{n^2-1}{n^2+1}\Big]. \quad (52)$$

The total energy shift is easily determined from Eq. (28). Since spin-up and spin-down states are degenerate without the perturbation, the right-hand side of Eq. (28) is actually a matrix with the energy shifts as eigenvalues. In general, the self-energy operator (52) is not diagonal in the spin states $u^{(i)}(\mathbf{p})$, as given in Eq. (19), if spin-up and down states are defined along z, i.e. for

$$\phi^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \phi^{(2)} = \begin{pmatrix}0\\1\end{pmatrix}.$$

We find

$$\frac{m}{E}\overline{u}^{(1)}\Sigma_{\text{total}}u^{(1)} = \frac{m}{E}\overline{u}^{(2)}\Sigma_{\text{total}}u^{(2)} = -\frac{e^2}{32\pi a E^2}\Big(2\frac{2n^4-n^2-1}{(n^2+1)^2}\langle p_z^2\rangle + \frac{n^2(n^2-1)}{(n^2+1)^2}\langle p_\parallel^2\rangle + 2\frac{n^2-1}{n^2+1}E^2\Big) \quad (53)$$

and

$$\frac{m}{E}\overline{u}^{(2)}\Sigma_{\text{total}}u^{(1)} = \Big(\frac{m}{E}\overline{u}^{(1)}\Sigma_{\text{total}}u^{(2)}\Big)^* = -\frac{e^2}{32\pi a E^2}\Big(\frac{n^2(n^2-1)}{(n^2+1)^2} + 2\frac{n^2-1}{n^2+1}\frac{E}{E+m}\Big) \times\langle(p_x+ip_y)p_z\rangle. \quad (54)$$

Thus the energy shift is

$$\delta E = -\frac{e^2}{32\pi a E^2}\Big[2\frac{2n^4-n^2-1}{(n^2+1)^2}\langle p_z^2\rangle + \frac{n^2(n^2-1)}{(n^2+1)^2}\langle p_\parallel^2\rangle + 2\frac{n^2-1}{n^2+1}E^2 \pm\Big(\frac{n^2(n^2-1)}{(n^2+1)^2} + 2\frac{n^2-1}{n^2+1}\frac{E}{E+m}\Big) \times\sqrt{\langle p_x p_z\rangle^2 + \langle p_y p_z\rangle^2}\Big]. \quad (55)$$

For wave packets that are either stationary or whose motion preserves the symmetry of the problem, the nondiagonal elements (54) are zero and the energy shift is given simply by (53).

In the limit of perfect reflectivity $n\to\infty$ the calculation yields a result for the total self-energy that differs from the limit $n\to\infty$ of Eq. (52),

$$\Sigma^{\text{perf}}_{\text{total}}(p,-2a) \simeq -\frac{e^2}{32\pi p_0 a}(-\gamma_\parallel\cdot\mathbf{p}_\parallel + 2\gamma_3 p_z + 2\gamma^0 p_0). \quad (56)$$

Accordingly, the energy shift differs from the limit $n\to\infty$ of Eq. (55). We have discussed the reasons for this discrepancy in Sec. IV D and in Ref. [7].

Quite apart from calculating the energy shift of an electron in front of an imperfectly reflecting half-space, we have established the major building blocks for QED in the presence of a dielectric half-space. Two alternative formulations for the Feynman propagator of the electromagnetic field are given in Eqs. (10) and (11). The loss of translation invariance perpendicular to the surface of the dielectric half-space is an essential complication in loop calculations, but we have demonstrated how to tackle this at the one-loop level.

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APPENDIX A: POLARIZATION VECTORS AND NORMAL MODES

In the generalized Coulomb gauge the direction of the electromagnetic field can be described by the following choice of polarization vectors:

$$\begin{aligned} e^\mu_{TE} &= -e_{\mu TE} = (-\Delta_\parallel)^{-1/2}(0,-i\partial_y, i\partial_x, 0),\\ e^\mu_{TM} &= -e_{\mu TM} = (\Delta\Delta_\parallel)^{-1/2}(0,-\partial_x\partial_z,-\partial_y\partial_z,\Delta_\parallel),\\ e^\mu_G &= -e_{\mu G} = (-\Delta)^{-1/2}(0,-i\partial_x,-i\partial_y,-i\partial_z),\\ e^\mu_C &= e_{\mu C} = (1,0,0,0). \end{aligned} \quad (A1)$$

Here $\Delta = \partial_x^2+\partial_y^2+\partial_z^2$ is the Laplacian in three dimensions and $\Delta_\parallel = \partial_x^2+\partial_y^2$ is the one in two dimensions parallel to the surface of the dielectric. The physical polarizations are the transverse electric e^μ_{TE} and the transverse magnetic e^μ_{TM}, which have vanishing electric or magnetic, respectively, field components perpendicular to the surface. The unphysical polarizations are the longitudinal e^μ_G and the timelike e^μ_C.

Constructing the normal modes is straightforward if one proceeds from a plane incident wave which gets reflected and transmitted at the surface. The incident wave is what it would be in free space (for left-incident modes) or in a homogeneous dielectric (for right-incident modes), and the transmitted and reflected components can be derived from the continuity conditions (2).

For the vector potential of the left-incident mode one finds

$$A_L^\mu(x) = \sum_{\sigma=TE,TM,G,C} \int \frac{d^3k\theta(k_z)}{(2\pi)^{3/2}(2\omega)^{1/2}} \times \Big\{ e_\sigma^\mu \exp(-ix^0k^0) a_\sigma^L(k) \Big[\theta(-z)(\exp(i\mathbf{k}\cdot\mathbf{r}) + R_\sigma^L(k)\exp(i\mathbf{k}^r\cdot\mathbf{r})) + \theta(z)\frac{1}{n}T_\sigma^L(k)\exp(i\mathbf{k}^d\cdot\mathbf{r}) \Big] + \text{H.c.} \Big\}, \quad \text{(A2)}$$

where $a_\sigma^L(k)$ is the photon annihilation operator of the mode and the reflection and transmission coefficients are

$$R_{TE}^L(k) = \frac{k_z - k_z^d}{k_z + k_z^d}, \qquad T_{TE}^L(k) = \frac{2nk_z}{k_z + k_z^d},$$
$$R_{TM}^L(k) = \frac{n^2k_z - k_z^d}{n^2k_z + k_z^d}, \qquad T_{TM}^L(k) = \frac{2n^2k_z}{n^2k_z + k_z^d}, \quad \text{(A3)}$$
$$R_C^L(k) = R_G^L(k) = \frac{k_z - n^2k_z^d}{k_z + n^2k_z^d}, \qquad T_C^L(k) = \frac{2n^2k_z}{k_z + n^2k_z^d},$$
$$T_G^L(k) = \frac{2nk_z}{k_z + n^2k_z^d}.$$

The wave vectors are

$$\mathbf{k} = (\mathbf{k}_\parallel, k_z), \qquad \mathbf{k}^r = (\mathbf{k}_\parallel, -k_z), \qquad \mathbf{k}^d = (\mathbf{k}_\parallel, k_z^d),$$
$$k_z^d = \mathrm{sgn}(k_z)(n^2k_z^2 + (n^2-1)k_\parallel^2)^{1/2}, \qquad k_\parallel^2 = k_x^2 + k_y^2,$$
$$k_0 \equiv \omega, \qquad \omega^2 = k_z^2 + k_\parallel^2 = \frac{1}{n^2}(k_z^{d2} + k_\parallel^2). \quad \text{(A4)}$$

For the right-incident modes the vector potential reads

$$A_R^\mu(x) = \sum_{\sigma=TE,TM,G,C} \int \frac{d^3k\theta(-k_z^d)}{(2\pi)^{3/2}(2\omega)^{1/2}} \times \Big\{ e_\sigma^\mu \exp(-ix^0k^0) a_\sigma^R(k) \Big[\theta(z)\frac{1}{n}(\exp(i\mathbf{k}^d\cdot\mathbf{r}) + R_\sigma^R(k)\exp(i\mathbf{k}^{r,d}\cdot\mathbf{r})) + \theta(-z)T_\sigma^R(k)\exp(i\mathbf{k}\cdot\mathbf{r}) \Big] + \text{H.c.} \Big\}, \quad \text{(A5)}$$

with the reflection and transmission coefficients

$$R_{TE}^R(k) = \frac{k_z^d - k_z}{k_z + k_z^d}, \qquad T_{TE}^R(k) = \frac{2k_z^d}{n(k_z^d + k_z)},$$
$$R_{TM}^R(k) = \frac{k_z^d - n^2k_z}{k_z^d + n^2k_z}, \qquad T_{TM}^R(k) = \frac{2k_z^d}{k_z^d + n^2k_z}, \quad \text{(A6)}$$
$$R_C^R(k) = R_G^R(k) = \frac{n^2k_z^d - k_z}{n^2k_z^d + k_z}, \qquad T_C^R(k) = \frac{2k_z^d}{n^2k_z^d + k_z},$$
$$T_G^R(k) = \frac{2nk_z^d}{n^2k_z^d + k_z}.$$

Because these modes are right-incident the z component of the incident wave vector is negative $k_z^d < 0$, and the reflected wave has the wave vector $\mathbf{k}^{d,r} = (\mathbf{k}^\parallel, -k_z^d)$. Note that the integration over $\mathbf{k}^d$ includes imaginary values of k_z, which correspond to modes that come from inside the dielectric, suffer total internal reflection at the interface, and are evanescent on the vacuum side. One has

$$k_z = \mathrm{sgn}(k_z^d)\frac{1}{n}\sqrt{k_z^{d2} - (n^2-1)k_\parallel^2} \quad \text{for } k_z^{d2} - (n^2-1)k_\parallel^2 > 0,$$
$$= -i\frac{1}{n}\sqrt{-k_z^{d2} + (n^2-1)k_\parallel^2} \quad \text{for } k_z^{d2} - (n^2-1)k_\parallel^2 < 0.$$

We have chosen the branch of the square root such that the evanescent modes are truly evanescent, i.e. exponentially falling away from the interface on the vacuum side. This also ensures that these modes are genuinely totally reflected, i.e. that the relation $R_\sigma^R R_\sigma^{R*} = 1$ is fulfilled.

For the physical polarizations TE and TM, the above modes are well known [15]. It is easy to see that they are mutually orthogonal, but the proof of their completeness is surprisingly tricky [16]. The following relations are useful for showing the completeness of the modes and for simplifying our expressions in Sec. II A. They are valid for all polarizations so that we drop the index σ. For real k_z we have

$$\frac{k_z}{k_z^d}n^2[T^{R*}T^R](-k_z,-k_z^d) + [R^{L*}R^L](k_z,k_z^d) = 1,$$
$$\frac{k_z^d}{n^2k_z}[T^LT^{L*}](k_z,k_z^d) + [R^RR^{R*}](-k_z,-k_z^d) = 1,$$
$$\frac{k_z^d}{n^2k_z}[R^LT^{L*}](-k_z,-k_z^d) + [T^RR^{R*}](k_z,k_z^d) = 0, \quad \text{(A7)}$$
$$\frac{k_z^d}{n^2k_z}[R^{L*}T^L](k_z,k_z^d) + [R^RT^{R*}](-k_z,-k_z^d) = 0,$$
$$R^{R*}(-k_z^d) = R^R(k_z^d), \qquad \frac{k_z^d}{n^2k_z}T^L = T^R,$$

and for imaginary k_z

$$R^RR^{R*} = 1, \qquad R^{R*}(-k_z^d) = R^R(k_z^d),$$
$$[R^{R*}T^R](-k_z^d) = T^R(k_z^d), \qquad [R^RT^{R*}](-k_z^d) = T^{R*}(k_z^d). \quad \text{(A8)}$$

Applying the polarization vectors (A1) on a plane wave $e^{i\mathbf{k}\cdot\mathbf{r}}$, as in (A2) and (A5), one can express them in terms of the wave vector k. However, it is important to realize that the incident, transmitted, and reflected components all have different wave vectors and thus, according to (A1), have polarization vectors that point in different directions. All four polarization vectors form a complete and orthogonal system, and the TE and TM polarizations are complete

in the subspace of physical states [17]

$$g_{\mu\nu}e^{\mu}_{\sigma}e^{\nu}_{\sigma'} = g_{\sigma\sigma'}, \qquad \sum_{\sigma=TE,TM,C,G} g^{\sigma\sigma}e^{\mu}_{\sigma}e^{\nu}_{\sigma} = g^{\mu\nu},$$
$$\sum_{\sigma=TE,TM} g^{\sigma\sigma}e^{\mu}_{\sigma}e^{\nu}_{\sigma} = g^{\mu\nu} - \eta^{\mu}\eta^{\nu} + \hat{k}^{\mu}\hat{k}^{\nu}. \tag{A9}$$

Here $\hat{k}^{\mu} = (k^{\mu} - (k\eta)\eta^{\mu})/\sqrt{(k\eta)^2 - k^2}$ is the unit vector along the spacelike part of k^{μ}, and $\eta^{\mu} = (1, 0, 0, 0)$.

Finally, we would like to consider the limit $n \to \infty$ which is commonly thought of as corresponding to a half-space bounded by a perfectly reflecting wall. Indeed, in this limit the reflection and transmission coefficients become

$$R^L_C(k) = R^L_G(k) \to -1, \quad R^L_{TE}(k) \to -1, \quad R^L_{TM}(k) \to 1,$$
$$\frac{1}{n}R^R_{\sigma}(k) \to 0, \quad \frac{1}{n}T^L_{\sigma}(k) \to 0, \quad T^R_{\sigma}(k) \to 0, \tag{A10}$$

so that only the left-incident mode (A2) survives and gets perfectly reflected at $z = 0$.

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