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Quantum Field Theory of Fermions and Parafermions Constructed from Quantum Bose Fields. I

——*Simplification and Generalization*——

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In previous papers a Bose representation of fermions and parafermions was constructed in such a way that (1) the only fundamental quantum fields were Bose fields and (2) no property of the Fermi or para-Fermi systems (as, e.g., the Pauli principle for fermions) be lost. The use of that construction and also the proofs of the theorems were made difficult by the complexity of many of the mathematical expressions we used previously.

In the present paper we simplify much the above-mentioned construction. Moreover no results (concerning the Bose representation) proved in the previous papers are lost in the present reformulation, but all new results are more general: In the previous papers we made the construction of the Bose representation of fermions for the case of pure Fock representations of Quantum Field Theory. The present work is consistent with Fock as well as with non-Fock representations of the Fermi anticommutation relations.

§ 1. Introduction

Let $f_\xi(z)$, $f_\xi^+(z)$, $\xi=1, 2, \cdots, T$, respectively, be the annihilation and creation parts of a Fermi or para-Fermi quantum field. Whereas for the Fermi case the quantum rules are

$$[f_\xi(z), f_{\xi'}^+(z')]_+ = \delta_{\xi\xi'}\delta(z-z')I \tag{1·1a}$$

and

$$[f_\xi(z), f_{\xi'}(z')]_+ = 0, \tag{1·1b}$$

those for the para-Fermi case are[1),2)]

$$[[f_\xi^+(z), f_{\xi'}(z')]_-, f_{\xi''}^+(z'')]_- = 2\delta_{\xi'\xi''}\delta(z'-z'')f_\xi^+(z) \tag{1·2a}$$

and

$$[[f_\xi(z), f_{\xi'}(z')]_-, f_{\xi''}(z'')]_- = 0. \tag{1·2b}$$

Let $b_\zeta(x)$, $b_\zeta^+(x)$, $\zeta=1, 2, \cdots, R$, respectively, be the annihilation and creation parts of a Fock representation of a Bose field, then

$$[b_\zeta(x), b_{\zeta'}^+(x')]_- = \delta_{\zeta\zeta'}\delta(x-x')I \tag{1·3a}$$

and

$$[b_\zeta(x), b_{\zeta'}(x')]_- = 0, \tag{1·3b}$$

so that a unique no-Bose particle state $|0\rangle^{\mathscr{B}}$ exists:

$$b_\zeta(\boldsymbol{x})|0\rangle^{\mathscr{B}}=0 \text{ for all } \boldsymbol{x},\zeta. \tag{1·4}$$

We shall prove that there exist c-number coefficients $F_{\xi\zeta\zeta'}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x}')$ *and* $F^{+}_{\xi\zeta\zeta'}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x}')$ *as well as suitable Bose state vector spaces denoted by* $\mathscr{B}_1$ *and* $\mathscr{B}$ *such that the formulae*

$$f_\xi(\boldsymbol{z})=\int d^3x\int d^3x'\sum_{\zeta\zeta'=1}^{B}F_{\xi\zeta\zeta'}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x}')\,b_\zeta^{+}(\boldsymbol{x})\,b_{\zeta'}(\boldsymbol{x}') \tag{1·5a}$$

and

$$f_\xi^{+}(\boldsymbol{z})=\int d^3x\int d^3x'\sum_{\zeta\zeta'=1}^{B}F^{+}_{\xi\zeta\zeta'}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x}')\,b_\zeta^{+}(\boldsymbol{x})\,b_{\zeta'}(\boldsymbol{x}') \tag{1·5b}$$

respectively provide a Bose representation of the Fermi and para-Fermi fields. Fermi fields are obtained when $f_\xi(\boldsymbol{z})$ and $f_\xi^{+}(\boldsymbol{z})$ act on $\mathscr{B}_1$; para-Fermi fields are obtained when $f_\xi(z)$ and $f_\xi^{+}(z)$ act on $\mathscr{B}$. Therefore, the Bose constructed fields (1·5) satisfy Eqs. (1·1) on $\mathscr{B}_1$ and (1·2) on $\mathscr{B}$.

The above results were previously established by Kademova[3] and Kademova and Kálnay[4] for finite dimensional second quantization and later on extended by Kademova, Mac Cotrina and Kálnay[5] for quantum field theory. However, the proofs of most of the theorems involved an embarrassing use of binary arithmetic, so that in spite of its elementary character, most of the demonstrations were quite difficult to read. Moreover, several of the main results were obscured because of the appearance in some formulae of complicated binary arithmetical symbols.

The first purpose of the present paper is to offer much simpler proofs and more transparent results by recognizing that the use of binary arithmetic can be completely avoided. Then we expect that it would be simpler for practical purposes to use the Bose representation of Fermions.

In the paper by Kademova, Mac Cotrina and Kálnay,[5] (called (A) in what follows), only the Fock representation of the Fermi fields algebra was constructed in Bose terms.

The second purpose of this research is to generalize (A) *extending its results to the case of general representations of the commutation relations* (*1·1*) *and* (*1·2*). *In this way, Fock as well as non-Fock representations of Fermi and para-Fermi fields will be constructed in terms of quantum Bose fields.* This is relevant to the present interest for non-Fock representations. (See, e.g., Refs. 6)~9).) The underlying Bose field $b_\zeta(\boldsymbol{x})$ will belong to the Fock representation.

One part of the results obtained in Refs. 3) and 4) as well as in (A) used para-Bose fields as the fundamental entities. Greater generality was obtained in this way, though the proofs were more involved.

The third purpose of the present paper is to reformulate these proofs in terms of pure Bose fields, obtaining a further simplification of the *formalism.*

For the discussion of the physical meaning of the Bose representation of fermions and parafermions, see § 6 of (A) as well as Ref. 10), where implications for physics are suggested. Here we only recall that[4] *"a possibility is given for reformulating physical theories in equivalent ones without fermions (and parafermions)"*.

The transformation properties of the Bose constructed Fermi fields are given in Ref. 11) where it is shown that there is no conflict between the tensor transformation law of the $b_\zeta(\boldsymbol{x})$ and the spinor transformation law of the $f_\xi(\boldsymbol{z})$. The Fermi evolution equations can be retrieved.[11] In the Appendix of (A) we have shown, as an example, how to obtain the evolution in time of a Bose constructed Fermi field according to the Dirac equation. How to avoid non-orthodox Bose hamiltonians is shown in Ref. 12). The c-number limit of the theory is given in Ref. 13) and comparison between our approach and former attempts to represent fermions in Bose terms is done in § 6 of (A). Ókubo obtained recently a new Bose representation of fermions: See § 2 of Ref. 14). See also Refs. 15) and 16).

The organization of the present paper is as follows: In § 2 we introduce the notation and define and discuss the c-number coefficients $F_{\xi\zeta\zeta'}(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{x}')$. In § 3 we state and prove the theorems for the Fermi case and in § 4 we do the same for the para-Fermi case. We separated both cases to allow the reader to avoid parastatistics if he is only interested in ordinary statistics. However, we stress that the para-Fermi case may be quite interesting for elementary particle physics.[10] (A short introduction to the Fock representation of parastatistics is given in § 3 of (A), where the notation is almost similar to that used in the present paper.) Finally, the difference of the point of view between Ref. 10) and the present paper is shown in § 5.

§ 2. Discussion and notation

We call $\mathscr{B}_n$ the n-boson subspace of the state vector $\mathscr{B}$ of a Fock representation of the Bose commutation relations (1·3):

$$\mathscr{B} = \bigoplus_{n=0}^{\infty} \mathscr{B}_n \,. \tag{2·1}$$

A basis in $\mathscr{B}_n$ is well known as

$$|x_1, \zeta_1;\ x_2, \zeta_2;\ \cdots;\ x_n, \zeta_n\rangle^{\mathscr{B}} = (n!)^{-1/2} b^+_{\zeta_1}(x_1)\, b^+_{\zeta_2}(x_2) \cdots b^+_{\zeta_n}(x_n)|0\rangle^{\mathscr{B}}. \tag{2·2}$$

We denote by $|\ \ \rangle^{\mathscr{B}} \in \mathscr{B}$ the Bose kets in contraposition to the para-Fermi kets $|\ \ \rangle^{\mathscr{F}} \in \mathscr{F}$. When restricted to a Fock representation of the para-Fermi algebra and to an irreducible representation of order p of para-Fermi statistics, we shall denote $|\ \ \rangle^{\mathscr{F}p} \in \mathscr{F}^p$. In particular $|\ \ \rangle^{\mathscr{F}1}$ will be the Fermi kets and $\mathscr{F}^1$ their Fock state vector space, because in a Fock representation the para-Fermi systems of order one are the Fermi systems.[1),2)]

We shall show, regardless of being or not being in a Fock representation of the para-Fermi algebra, that the Bose state vector space $\mathcal{B}$ will contain the state vector space of the para-Fermi algebra: $\mathcal{F}\subseteq\mathcal{B}$. Moreover, when restricted to Fock representations, $\mathcal{B}_p$ will be identical to a Bose representation of an irreducible representation state vector space $\mathcal{F}^p$ of order p of the para-Fermi statistics: $\mathcal{B}_p=\mathcal{F}^p$. *In particular, in a Fock representation of the algebras, the one-boson state vector space* $\mathcal{B}_1$ *will be the Bose representation of the whole Fermi state vector space* $\mathcal{F}^1:\mathcal{B}_1=\mathcal{F}_1$. *Notice that we are not asserting that the one-fermion states* fill $\mathcal{B}_1$: We are stating that the one-boson subspace $\mathcal{B}_1$ contains not only the one-fermion states but also the no-fermion state $|0\rangle^{\mathcal{F}1}$ as well as the multi-fermion states consistent with the Pauli principle.

Let $F_\xi(\boldsymbol{z})$ be a matrix representation of a Fermi algebra whose $(\boldsymbol{x}\zeta,\boldsymbol{x}'\zeta')$ elements are called $F_{\xi\zeta\zeta'}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x}')$, and $F_\xi^+(\boldsymbol{z})$ the hermitian conjugate matrix of elements denoted by

$$F^+_{\xi\zeta\zeta'}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x}')=F^*_{\xi\zeta'\zeta}(\boldsymbol{z},\boldsymbol{x}',\boldsymbol{x}), \tag{2·3}$$

where the star denotes complex conjugation. We have

$$\begin{aligned}([F_\xi(\boldsymbol{z}),F^+_{\xi''}(\boldsymbol{z}'')]_+)_{\boldsymbol{x}\zeta,\boldsymbol{x}''\zeta''}=\sum_{\zeta'=1}^{B}\int d^3x'\,[F_{\xi\zeta\zeta'}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x}')F^+_{\xi''\zeta'\zeta''}(\boldsymbol{z}'',\boldsymbol{x}',\boldsymbol{x}'')\\ +F^+_{\xi''\zeta\zeta'}(\boldsymbol{z}'',\boldsymbol{x},\boldsymbol{x}')F_{\xi\zeta'\zeta''}(\boldsymbol{z},\boldsymbol{x}',\boldsymbol{x}'')]=\delta_{\xi\xi''}\delta_{\zeta\zeta''}\delta(\boldsymbol{z}-\boldsymbol{z}'')\delta(\boldsymbol{x}-\boldsymbol{x}'')\end{aligned} \tag{2·4a}$$

and similarly

$$([F_\xi(\boldsymbol{z}),F_{\xi''}(\boldsymbol{z}'')]_+)_{\boldsymbol{x}\zeta,\boldsymbol{x}''\zeta''}=0\,. \tag{2·4b}$$

In the above the detail was given in order to fix the notation.

As the complex numbers $F_{\xi\zeta\zeta'}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x}')$ are the same as those which enter into Eq. (1·5), *the following doubt arises immediately: Are we constructing the quantum fermions* $f_\xi(\boldsymbol{z})$ *from pure quantum bosons* $b_\zeta(\boldsymbol{x})$, *or from quantum bosons* $b_\zeta(\boldsymbol{x})$ *plus quantum fermions* $F_\xi(\boldsymbol{z})$*? The answer is respectively yes and no*, because the $F_{\xi\zeta\zeta'}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{x}')$ enter as *c*-number coefficients in Eqs. (1·5); the only field operators in our formalism are the Bose operators $b_\zeta(\boldsymbol{x})$ and the Fermi (or para-Fermi) operators $f_\xi(\boldsymbol{z})$; only they act on the state vector spaces of our formalism, $\mathcal{B}$ and $\mathcal{F}$. The $F_\xi(\boldsymbol{z})$ are certainly Fermi operators, but act on a vector space which is not one of the state vector spaces (as $\mathcal{B}$ and $\mathcal{F}$) of our formalism; The $F_\xi(\boldsymbol{z})$ acts on the space of the vectors w of components $w_{\boldsymbol{x}\zeta}$. Let us illustrate this with an example: According to the results to be proved in § 3 for a Fock representation, the no-fermion state $|0\rangle^{\mathcal{F}1}$ will be a certain one-boson state

$$|0\rangle^{\mathcal{F}1}=\int d^3x\sum_{\zeta=1}^{B}\mathcal{O}_\zeta(\boldsymbol{x})b_\zeta^+(\boldsymbol{x})|0\rangle^{\mathcal{B}}\in\mathcal{B}_1\,, \tag{2·5}$$

where for each value of $\boldsymbol{x},\zeta$, the $\mathcal{O}_\zeta(\boldsymbol{x})$ are complex numbers to be later specified. The one-fermion states with quantum numbers $\boldsymbol{z},\xi$ will be, according to Eqs.

(1·5b) and (2·5),

$$f_\xi^+(\mathbf{z})|0\rangle^{\mathcal{F}1} = \sum_{\zeta\zeta'=1}^{R} \int d^3x \int d^3x' [F^+_{\xi\zeta\zeta'}(\mathbf{z}, \mathbf{x}, \mathbf{x}') \mathcal{O}_{\zeta'}(\mathbf{x}')] [b_\zeta^+(\mathbf{x})|0\rangle^{\mathcal{B}}] \in \mathcal{B}_1 . \quad (2\cdot 6)$$

It is clear that the first bracket on the last line is a c-number with respect to the kets of the theory, while the second bracket stands for the action of the quantum creation field $b_\zeta(\mathbf{x})$ on the no-boson ket. Here the c-number rôle of the $F_{\xi\zeta\zeta'}(\mathbf{z}, \mathbf{x}, \mathbf{x}')$ is quite clear.

On the other hand, it was suggested in Ref. 11) that the $F_{\xi\zeta\zeta'}(\mathbf{z}, \mathbf{x}, \mathbf{x}')$ could be interpreted as classical fields. We have two mathematically equivalent descriptions of the facts:

(i) The quantum Fermi field $f_\xi(\mathbf{z})$ is constructed from a quantum Bose field $b_\zeta(\mathbf{x})$ and a trilinear classical (in the sense of c-number) field $F_{\xi\zeta\zeta'}(\mathbf{z}, \mathbf{x}, \mathbf{x}')$.

(ii) The quantum Fermi field $f_\xi(\mathbf{z})$ is constructed from a quantum Bose field $b_\zeta(\mathbf{x})$; the $F_{\xi\zeta\zeta'}(\mathbf{z}, \mathbf{x}, \mathbf{x}')$ are c-number coefficients which appear in linear combinations, which contribute to wave packets like the (2·6).*)

The first description may be more suggestive according to the transformation laws studied in Ref. 11). Both points of view have physical sense and are not mutually exclusive. In both cases, *the only quantum fundamental entities* are the Bose field $b_\zeta(\mathbf{x})$ and its state vector space $\mathcal{B}$.

Turning back to the notation again, the one which we used in the present paper is the same as that of (A) with the following simplifications: (a) As we do not consider here the case in which the underlying field $b_\zeta(\mathbf{x})$ is a higher order para-Bose field, the upper indices on the b-s and on the $\mathcal{B}$-s are suppresed. (The upper indices were used in (A) to indicate higher order parastatistics.) Here $b_\zeta(\mathbf{x})$ is always a Bose field, and $\mathcal{B}$ always a Bose Fock space. Our $b_\zeta(\mathbf{x})$ and $\mathcal{B}$ would be the $b_\zeta^1(\mathbf{x})$ and $\mathcal{B}^1$ of (A). (b) For simplicity, we have dropped the upper index 1 on $F_{\xi\zeta\zeta'}(\mathbf{z}, \mathbf{x}, \mathbf{x}')$, which was used in (A) to remind us that $F_{\xi\zeta\zeta'}(\mathbf{z}, \mathbf{x}, \mathbf{x}')$ is a Fermi matrix, and *not* a higher order para-Fermi matrix.

§ 3. Bose representation of fermions

A. *Properties valid for general representations of the Fermi anticommutation relations*

Theorem 3.1: The entities $f_\xi(\mathbf{z})$, $f_\xi^+(\mathbf{z})$ defined in Eqs. (1·5) act on the one-boson subspace $\mathcal{B}_1$, according to a Bose representation of the Fermi creation and annihilation fields. The Bose subspace $\mathcal{B}_1$ contains a Bose representation of the whole Fermi state vector space.

Proof: That the theorem is right is seen by reading the 'second proof' proposed in (A) for the Theorem called 4.4 in that reference and trivially adapting it to

*) We are indebted to Dr. M. García Sucre for clarifying observation on this subject. We owe to him the wave packet point of view. (M. García Sucre, private communication.)

the continuum case. However we shall present it again in detail for completeness. Moreover, we remark that this proof does not require at all restricting to the Fock representation of the Fermi field, such restriction was stated in the Theorem 4.4 of (A). It is sufficient to prove that the entities (1·5) satisfy the Fermi anticommutation relations (1·1) when $f_\xi(z)$ and $f_\xi^+(z)$ act on the basis (2·2) with $n=1$. From Eqs. (1·3)～(1·5) and (2·2) we have

$$f_\xi(z)f_{\hat\xi}^+(\hat z)|x'',\zeta''\rangle^{\mathscr{B}}=\int d^3x\int d^3x'\int d^3\hat x\int d^3\hat x'\sum_{\zeta\zeta'\hat\zeta\hat\zeta'=1}^{R}$$

$$F_{\xi\zeta\zeta'}(z,x,x')F^+_{\hat\xi\hat\zeta\hat\zeta'}(\hat z,\hat x,\hat x')b_\zeta^+(x)b_{\zeta'}(x')b_{\hat\zeta}^+(\hat x)b_{\hat\zeta'}(\hat x')b_{\zeta''}^+(x'')|0\rangle^{\mathscr{B}}$$

$$=\int d^3x\sum_{\zeta=1}^{R}[F_\xi(z)F_{\hat\xi}^+(\hat z)]_{x\zeta,x''\zeta''}|x,\zeta\rangle^{\mathscr{B}}. \tag{3·1a}$$

Similary,

$$f_{\hat\xi}^+(\hat z)f_\xi(z)|x'',\zeta''\rangle^{\mathscr{B}}=\int d^3x\sum_{\zeta=1}^{R}[F_{\hat\xi}^+(\hat z)F_\xi(z)]_{x\zeta,x''\zeta''}|x,\zeta\rangle^{\mathscr{B}} \tag{3·1b}$$

so that, by combining with Eq. (2·4a),

$$[f_\xi(z),f_{\hat\xi}^+(\hat z)]_+\mathscr{B}_1=\delta_{\xi\hat\xi}\delta(z-\hat z)\mathscr{B}_1. \tag{3·2a}$$

With similar calculations,

$$[f_\xi(z),f_{\hat\xi}(\hat z)]_+\mathscr{B}_1=0. \tag{3·2b}$$

That

$$\mathscr{F}^1\subseteq\mathscr{B}_1 \tag{3·3}$$

is seen by observing that the algebra $\mathscr{F}^1$ is generated by the quantum fields (1·5) which act in a closed form in $\mathscr{F}^1$. □

Theorem 3.2: The c-number Fermi fields $F_{\xi\zeta\zeta'}(z,x,x')$, $F^+_{\xi\zeta\zeta'}(z,x,x')$ can be retrieved from quantum Fermi fields $f_\xi(z)$, $f_\xi^+(z)$ according to the formulae

$$F_{\xi\zeta\zeta'}(z,x,x')={}^{\mathscr{B}}\langle x\zeta|f_\xi(z)|x'\zeta'\rangle^{\mathscr{B}} \tag{3·4a}$$

and

$$F^+_{\xi\zeta\zeta'}(z,x,x')={}^{\mathscr{B}}\langle x\zeta|f_\xi^+(z)|x'\zeta'\rangle^{\mathscr{B}}. \tag{3·4b}$$

Proof: Trivial by using Eqs. (1·3)～(1·5) and (2·2) particularized to $n=1$. □

Notice, however, that in Theorem 3.1 we said nothing about which of the representations of the Fermi commutation relations is the one to which $F_\xi(z)$ belongs. From Theorem 3.2 we have now the following consequence:

Theorem 3.3: The quantum field $f_\xi(z)$ and the Fermi matrix $F_\xi(z)$ belong to the same representation of the Fermi anticommutation relations.

Corollary 3.4: All the representations of the Fermi anticommutation relations which accept a matrix representation $[F_\xi(z)]_{x\zeta,x'\zeta'}$ are consistent with a Bose representation like the one under consideration.

Proof: Given a representation of the Fermi anticommutation relations which accept such a matrix form, substitute that matrix in Eqs. (1·5), let the resulting operator act on $\mathscr{B}_1$, and using Theorem 3.3, find that the Bose representation is of the same class as the original matrix representation. □

Notation 3.5: If a normalized (but not necessarily unique) vector exists such that it is annihilated by all the Fermi matrices $F_\xi(z)$, then we call it $\mathcal{O}$ and we denote by $\mathcal{O}_\zeta(x)$ its components. Provided $\mathcal{O}$ exists, then we have

$$F_\xi(z)\mathcal{O}=0 \quad \text{for all} \quad z,\xi, \tag{3·5a}$$

i.e.,

$$\sum_{\zeta'=1}^{B}\int d^3x\, F_{\xi\zeta\zeta'}(z,x,x')\mathcal{O}_{\zeta'}(x')=0 \quad \text{for all } z,x,\xi,\zeta\,. \tag{3·5b}$$

Theorem 3.6: If a normalized (but not necessarily unique) one-boson state $|0\rangle^{\mathscr{F}1}$ exists such that $f_\xi(z)$ defined in Eq. (1·5a) satisfies

$$f_\xi(z)|0\rangle^{\mathscr{F}1}=0 \quad \text{for all} \quad z,\xi, \tag{3·6}$$

then

$$^{\mathscr{B}}\langle 0|b_\zeta(x)|0\rangle^{\mathscr{F}1}$$

is an $\mathcal{O}$ vector and we can write

$$\mathcal{O}_\zeta(x)={}^{\mathscr{B}}\langle 0|b_\zeta(x)|0\rangle^{\mathscr{F}1}. \tag{3·7}$$

Proof: From Eqs. (3·6) we have

$$b_{\hat{\zeta}}(\hat{x})f_\xi(z)|0\rangle^{\mathscr{F}1}=0 \quad \text{for all} \quad \hat{x},z,\hat{\zeta},\xi, \tag{3·8}$$

from which, with the help of Eq. (1·5) and of $\mathscr{F}^1\subseteq\mathscr{B}_1$ we deduce

$$\sum_{\zeta'=1}^{B}\int d^3x' F_{\xi\zeta\zeta'}(z,x,x')b_{\zeta'}(x')|0\rangle^{\mathscr{F}1}=0\,. \tag{3·9}$$

From this Eq. (3·7) follows. Now we can prove that

$$^{\mathscr{F}1}\langle 0|0\rangle^{\mathscr{F}1}=1 \quad \text{implying} \quad \|\mathcal{O}\|=1\,. \tag{3·10}$$

□

Theorem 3.7: If a normalized (but not necessarily unique) $\mathcal{O}$ vector exists, then the right-hand side of Eq. (2·5) is annihilated by $f_\xi(z)$ so that Eq. (2·5) is correct.

Proof: From Eqs. (3·5) and (1·3)～(1·5) we see that the right-hand side of Eq. (2·5) is annihilated by $F_\xi(z)$. Now, it is obtained that

$$\|\mathcal{O}\|=1 \quad \text{implies} \quad {}^{\mathscr{F}1}\langle 0|0\rangle^{\mathscr{F}1}=1\,. \tag{3·11}$$

□

From Theorems 3.6 and 3.7 or directly from Theorem 3.3 we deduce:

Corollary 3.8: The set of the $\mathcal{O}$ (modulo a phase factor) is in a one-to-one correspondence with the set of the $|0\rangle^{\mathcal{F}1}$ (modulo a phase factor). In particular, if one of the sets is empty, the same happens to the other one. In the case they are not empty, and in fact for any representation of the anticommutation relations, to each $\mathcal{O}$ corresponds a $|0\rangle^{\mathcal{F}1}$ according to Eq. (2·5).

B. *Properties valid for the Fock representation of the Fermi anticommutation relations*

From Corollary 3.8 we have, as a particular case:

Corollary 3.9: The Bose constructed Fermi field (1·5) belongs to a Fock representation if and only if the Fermi matrix $F_\xi(\boldsymbol{z})$ belongs to a Fock representation.

Assumption 3.10: In the remainder of this section the matrix $F_\xi(\boldsymbol{z})$ will belong to an irreducible Fock representation of the Fermi anticommutation relations, so that an unique $\mathcal{O}$ vector exists.

Then, from Corollary 3.8 and Theorem 3.3 we have:

Corollary 3.11: The unique no-Fermi particle state of the Fermi field $f_\xi(\boldsymbol{z})$ is given by Eq. (2·5). In

$$\mathcal{F}^1 = \mathcal{B}_1 \tag{3·12}$$

the Bose constructed Fermi fields (1·5) generate an irreducible Fock representation of the Fermi algebra.

This Corollary is one of the main results obtained in (A), which in turn were suggested by similar results for finite second quantization obtained in Refs. 3) and 4).

Theorem 3.12: The basis

$$|\boldsymbol{z}_1, \xi_1; \cdots; \boldsymbol{z}_n \xi_n\rangle^{\mathcal{F}1} = (n!)^{-1/2} f_{\xi_1}^+(\boldsymbol{z}_1) \cdots f_{\xi_n}^+(\boldsymbol{z}_n)|0\rangle^{\mathcal{F}1} \tag{3·13a}$$

can be expressed as

$$|\boldsymbol{z}_1, \xi_1; \cdots; \boldsymbol{z}_n, \xi_n\rangle^{\mathcal{F}1} = (n!)^{-1/2} \int d^3x \sum_{\zeta=1}^{B} [F_{\xi_1}^+(\boldsymbol{z}_1) \cdots F_{\xi_n}^+(\boldsymbol{z}_n)\mathcal{O}]_{x\zeta} b_\zeta^+(\boldsymbol{x})|0\rangle^{\mathcal{B}} . \tag{3·13b}$$

Proof: Use Eqs. (1·3)～(1·5) and (2·5). □

In Eq. (2·2·7) of (A) it was stated that given a Fermi state $|\psi\rangle^{\mathcal{F}1}$, a complex valued function $g_\zeta^\psi(\boldsymbol{x})$ exists such that

$$|\psi\rangle^{\mathcal{F}1} = \int d^3x \sum_{\zeta=1}^{B} g_\zeta^\psi(\boldsymbol{x}) b_\zeta^+(\boldsymbol{x})|0\rangle^{\mathcal{B}} .$$

The Theorem 3·12 shows us how to construct the function $g_\zeta^\psi(\boldsymbol{x})$. On the other hand, Eqs. (3·13) take the rôle of Eq. (2·2·10) of (A).

Let us discuss the antisymmetry and the Pauli principle. Clearly they are consequence of the anticommutation relations (1·1), but it is interesting to visualize this in the frame of the Bose construction. We see from Eq. (3·13b) that in the Bose construction we still have antisymmetry for fermions $f_\xi(\boldsymbol{z})$ because Eqs. (2·4) are satisfied by the $F_\xi(\boldsymbol{z})$. It is remarkable that the antisymmetry of the Bose constructed fermions $f_\xi(\boldsymbol{z})$ can be obtained in spite of the symmetry associated to bosons.

§ 4. Bose representation of parafermions

A. *Properties valid for general representations of the para-Fermi commutation relations*

Theorem 4.1: The entities $f_\xi(\boldsymbol{z})$, $f_\xi^+(\boldsymbol{z})$ defined in Eqs. (1·5) act on the whole Bose state vector space $\mathscr{B}$ according to a Bose representation of the para-Fermi creation and annihilation fields. The Bose state vector space contains a Bose representation of the whole para-Fermi state vector space $\mathscr{F}$.

Proof: Compute the left-hand side of Eq. (1·2) with the fields $f_\xi(\boldsymbol{z})$ replaced by their expressions (1·5) and use Eqs. (1·3). □

A similar theorem was previously shown for the first time by Kademova for the case of second quantization with two generators only, for the Fermi algebra (see Ref. 3)). For the extensions to general finite second quantization and to field theory, see respectively Ref. 4) and Theorem 5.2 of (A). In those papers quoted the para-Fermi fields were constructed more generally in terms of para-Bose fields, not necessarily bosons. Of course, as bosons are particular cases of parabosons,[1),2)] the present Theorem 4.1 directly follows from Theorem 5.2 of (A). Here we have presented a much more elementary proof which does not involve parastatistics.

Remark 4.2: We stress that the formulae (1·5) are the same for the Fermi case (§ 3) as well as for the para-Fermi case (present section): The c-numbers $F_{\xi\zeta\zeta'}(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{x}')$, are matrix elements of a *Fermi* algebra (2·4) in *both* cases.

Remark 4.3: For the para-Fermi case, the identity (3·4) (being $F_{\xi\zeta\zeta'}(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{x}')$ a Fermi matrix element and $f_\xi(\boldsymbol{z})$ a para-Fermi field) is no longer correct. Therefore, we have no properties for the para-Fermi case like those expressed in Theorems 3.2 and 3.3 and Corollary 3.4. We do not intend that (1·5) provides all the representations of a non-Fock para-Fermi algebra.

We retain without modification the Notation 3.5; but, we do not have a Theorem like 3.6. According to Bracken and Gray[7)] we call *reservoir states* those which, in a general representation, are annihilated by the destruction part of a para-Fermi field:

$$f_\xi(\boldsymbol{z})|0\rangle^{\mathscr{F}}=0\,. \tag{4·1}$$

Theorem 4.4: Given an $\mathcal{O}$ state, we can construct in Bose terms, reservoir states of the para-Fermi algebra: One state for each value of $n=1, 2, 3, \cdots$,

$$|0\rangle^{\mathscr{F}}=\int d^3x_1\cdots\int d^3x_n\sum_{\zeta_1\cdots\zeta_n=1}^{R}\mathcal{O}_{\zeta_1}(\boldsymbol{x}_1)\cdots\mathcal{O}_{\zeta_n}(\boldsymbol{x}_n)b^+_{\zeta_1}(\boldsymbol{x}_1)\cdots b^+_{\zeta_n}(\boldsymbol{x}_n)|0\rangle^{\mathscr{B}}\in\mathscr{B}_n\,. \tag{4·2}$$

Proof: Use Eqs. (1·3)~(1·5) and (3·5). □

Remark 4.5: Other reservoir states than (4·2) may perhaps exist in a general representation of the commutation relations.

B. *Properties valid for Fock representation of the para-Fermi commutation relations*

Assumption 4.6: We shall assume that the Fermi matrices $F_\xi(z)$, $F_\xi^+(z)$ generate an irreducible Fock representation of the Fermi algebra: The vector $\mathcal{O}$ will exist and be unique.

Theorem 4.7: The entities $f_\xi(z)$, $f_\xi^+(z)$ defined in Eqs. (1·5) act on the p-Bose particle subspace $\mathcal{B}_p$ according to an irreducible Bose representation of the para-Fermi algebra of order p of parastatistics. The reservoir state (4·2) with $n=p$ is a Bose representation of the no-parafermion state.

Proof: A similar theorem was proved for the numerable case as the Theorem 4.3 of (A) but by using embarrasing calculations with binary arithmetics: In the same paper the Theorem 4.3 of (A) was used to prove a similar one [Theorem 5.3 of (A)]*) for the continuum case. In other words, the previous proof of a theorem like our 4.3 is based on binary arithmetics. The present proof does not use the corresponding tedious calculations and runs as follows: Because of the Theorem by Greenberg and Messiah[2] and also by taking into account our theorem 4.1, it should only be proved that:

(i) There exists a no-parafermion state given by Eq. (4·2) with $n=p$.

(ii) This state is the unique reservoir state.

(iii) The order of parastatistics is p, i.e.,[2]

$$f_\xi(z)f_{\xi'}^+(z')|0\rangle^{\mathcal{F}p}=p\delta_{\xi\xi'}\delta(z-z')|0\rangle^{\mathcal{F}p}\,. \tag{4·3}$$

Point (i) results from Theorem 4.4 and Assumption 4.6. Point (ii) is proved as follows: Let us assume that $|0'\rangle^{\mathcal{F}p}$ is a general no-parafermion state; express it as a linear combination of the basis (2·2) with $n=p$. Then write that it be annihilated by the $f_\xi(z)$; a little algebra leads to the result that the coefficients of the linear combination are just the products $\mathcal{O}_{\zeta_1}(x_1)\cdots\mathcal{O}_{\zeta_n}(x_n)$, so that Eq. (4·2) shows that $|0\rangle^{\mathcal{F}p}=|0'\rangle^{\mathcal{F}p}$. Point (iii) results from Eqs. (4·2), (1·3) and (1·4). □

C. *Independent proof of theorems corresponding to the Fermi case*

An irreducible representation of the Fermi algebra is known to be a particular case of the irreducible representations of the para-Fermi algebra. (It is the para-Fermi algebra of order one.)[1,2] As a consequence, by putting $p=1$ in the present section, we have a separate proof of several results of § 3.

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*) Erratum in the proof of Theorem 5.3 of (A): "It follows from Theorem 5.2 and from the last Lemma", should read "It follows Theorems 4.3, 5.2 and from the last Lemma".

§ 5. Final discussion

We stress again that the only quantum entities used in Eq. (1·5) to construct a Bose representation of the Fermi algebra are the Bose fields $b_\zeta(\boldsymbol{x})$ and the Bose space $\mathscr{B}$ in the following remarks: In an equation like (2·6), the right-hand side, being a linear combination of the one-boson states $b_\zeta^+(x)|0\rangle^{\mathscr{B}}$, is a one-boson state itself, whatever the coefficients of the linear combination may be. Therefore, in spite of the mathematical fact that the coefficients $F_{\xi\zeta\zeta'}(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{x}')$ may be put in a compact form of a Fermi matrix $F_\xi(\boldsymbol{z})$, the right-hand side of Eq. (2·6) is a one-boson state so that the left-hand side of that equation is also a one-boson state.

From a mathematical viewpoint the results of § 3 are perhaps not so surprising: They are essentially reduced to the fact that the matrix elements of a Fermi field can be labeled with the quantum numbers of a basis of one-boson states. What are strange are the physical consequences: To have an arbitrary Fermi state as a linear combination of one-boson states; and to have with the same formulae (1·5) para-Fermi statistics (not only Fermi statistics) by only changing the Bose subspace on which the fields (1·5) act.

As we have presented a Bose representation of fermions, *a possibility is open to reduce the number of "elementary" particles in a given theory.* We want to express our ideas about this but they involve speculations. We want to separate them from what can be proved on a reasonable mathematical basis. We devoted the present paper to these proofs and transfer to Ref. 10) our ideas about the possible applications to physics of this formalism.

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