

# Quantum field theory on fractal spacetime: a new regularisation method

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**Abstract.** Spacetime is modelled as a fractal subset of  $\mathbb{R}^n$ . Analysis on homogeneous sets with non-integer Hausdorff dimension is applied to the low-order perturbative renormalisation of quantum electrodynamics. This new regularisation method implements the Dirac matrices and tensors in  $\mathbb{R}^4$  without difficulties, is gauge invariant, covariant and differs from dimensional regularisation in some aspects.

## 1. Introduction

The basis of a new regularisation method proposed here is a fractal spacetime support of the quantised fields. By 'fractal spacetime' a subset  $X$  of  $\mathbb{R}^n$  ( $n$  presumably 4) with zero  $n$ -dimensional Lebesgue measure is understood. Earlier applications of the Hausdorff measure to the spacetime dimensional problem can be found in Stillingner [1] and Barrow [2]. In two recent papers by Zeilinger and Svozil [3], operationalisation of these notions has been proposed, followed by further phenomenological analysis by Jarlskog and Yndurain as well as Müller and Schäfer [4]. Independently there have been considerations with regards to quantum foam by Isham and, more recently, by Crane and Smolin [5].

Heuristically, the fractal support  $X$  will be thought of as a uniformly distributed point subset of  $\mathbb{R}^n$ , which is assumed unbounded and closed.  $X$  inherits the metric of  $\mathbb{R}^n$ . For the time being, no physical reasoning such as quantum foam [5] (cutting out regions in spacetime by black holes) will be given, nor will there be any concrete fractal set envisaged. No scaling of the dimensional parameter will be considered (such that  $D = 4$  for macroscopic and  $D < 4$  for microscopic events), and  $D = \text{constant}$  will be assumed.

The behaviour of the measure (volume, area, length, etc) under variations of scale is essential for a definition of the Hausdorff measure and an associated dimension. To illustrate this, the measure  $\mu$  of an arbitrary but finite spacetime region  $E$  of  $X$  will be considered. In an actual measurement,  $\mu$  of  $E$ , among others, will depend on two quantities: (i) on the resolution of the experiment  $\delta$ , and (ii) on a dimensional parameter  $d$ .  $\mu$  needs to be independent of the resolution  $\delta$ , a quite reasonable condition for the volume, which otherwise would not be defined unambiguously. But then, for two resolutions  $\delta$  and  $\delta'$ ,

$$\mu(\delta, d) = \mu(\delta', d). \quad (1.1)$$

It is not evident that this condition is satisfied for arbitrary  $d$ . In fact, as can be expected, with an appropriate definition of the measure, there exists a unique number  $D$ , called *dimension*, for which the above relation is satisfied. This number can be measured only with finite accuracy, such that for actual measurements, relation (1.1) will never yield the dimension parameter  $D$  exactly.

The Hausdorff and other types of measure originate in *Carathéodory's construction*, which is defined as follows: for each metric space  $X$ , each set  $F$  of subsets  $E_i$  of  $X$ , and each positive function  $\zeta$ , such that  $0 \leq \zeta(E_i) \leq \infty$  whenever  $E_i \in F$ , a preliminary measure  $\phi_\delta$  can be constructed corresponding to  $0 < \delta \leq \infty$ , and then a final measure  $\mu$ , as follows: for every subset  $E \subset X$ , the preliminary measure  $\phi_\delta(E)$  is defined by

$$\phi_\delta(E) = \inf_{\{E_i\}} \left\{ \sum_i \zeta(E_i) : E \subset \bigcup E_i, (\text{diam } E_i) \leq \delta \right\}. \quad (1.2)$$

Since  $\phi_\delta \geq \phi_\sigma$  for  $0 < \delta < \sigma \leq \infty$ , the limit

$$\mu(E) = \lim_{\delta \rightarrow 0^+} \phi_\delta(E) = \sup_{\delta > 0} \phi_\delta(E) \quad (1.3)$$

exists for all  $E \subset X$ . In this context,  $\mu$  can be called the result of Carathéodory's construction from  $\zeta$  on  $F$ .  $\phi_\delta$  can be referred to as the size  $\delta$  approximating measure.

Let there be, for instance,

$$\zeta(E_i, d) = \omega(d)(\text{diam } E_i)^d \quad (1.4)$$

for non-empty subsets  $E_i$  of  $X$ . The geometrical factor  $\omega(d)$  depends on the geometry of the sets  $E_i$  used for covering.

When  $F$  is the set of all non-empty subsets of  $X$ , the resulting measure  $\mu_H$  is called the  $d$ -dimensional Hausdorff measure over  $X$ ; in particular, when  $F$  is the set of all (closed or open) balls in  $X$ ,  $\omega(d) = \pi^{d/2} 2^{-d} / \Gamma(d/2 + 1)$ . Other types of measures are discussed in Federer [6, p 169ff]. In what follows, the Hausdorff measure  $\mu_H$  is considered, reproducing the usual results for length, area and volume, etc, of integer-dimensional point sets. There is no forcing reason to apply  $\mu_H$  *a priori*; it is not unique and other measures do not seem ambiguous or ill suited for application in calculus.

Of course, principally one could resist speaking about  $D$  as the 'dimension' of a set. Rather, this parameter could be considered as a heuristic criterion for the packing density of the elements of the set (note, however, that although the set of rationals  $\mathbb{Q}$  is dense,  $D(\mathbb{Q}) = 0$ , as for all countable point sets). The dimensional parameter could then be ascribed to the topological dimension  $D_T$ , which *per definitionem* is always integer. However, the concept of topological dimension turns out to be operationally unrealisable (see [3] for details). The same can be said from a definition of dimension as the number of independent vectors of a manifold.

Throughout this paper, the need for a generalised calculus based on non-integer-dimensional topological spaces is strongly felt. The stakes are high, not only in physics. This paper intends to develop some of the perspectives, but it should be recognised that some of the approaches are conjectural in nature. In § 2, a short introduction to the calculus on non-integer-dimensional sets with their points uniformly distributed is given. In § 3 follows a treatment of the Fourier-Stieltjes transformation needed for the evaluation of Feynman rules in momentum space. Finally, some low-order radiative corrections to free quantum electrodynamics are evaluated for  $D < 4$ . The investigation is completed by two appendices, one containing explicit calculations of low-order radiative corrections, the other containing a brief introduction to measure theory.

## 2. Integration on measurable metric sets

Consider a measurable metric set  $[X, \mu_H]$  with  $X \subset \mathbb{R}^n, n \geq 4$ . The elements of  $X$  are denoted by  $x, y, z, \dots$ , and represented by  $n$ -tuples of real numbers

$$x = (x_1, x_2, \dots, x_n) \tag{2.1}$$

such that  $X$  is embedded in  $\mathbb{R}^n$ .  $X$  shall be further restricted by the following stringent conditions.

- (i)  $X$  is closed.
- (ii)  $X$  is unbounded.
- (iii)  $X$  is regular (homogeneous, uniform) with its points randomly distributed.

The importance of these requirements, as well as their basic definitions, will be explained below.

The metric  $d(x, y)$  as a function of two points  $x$  and  $y \in X$  is defined via the Kronecker  $\delta$  function in  $n$  dimensions

$$d(x, y) = \sum_{i,j} [\delta_{ij}(y_i - x_i)(y_j - x_j)]^{1/2} \tag{2.2a}$$

and the diameter of a subset  $E \subset X$  is defined by

$$d(E) = (\text{diam } E) = \sup\{d(x, y) : x, y \in E\} \tag{2.2b}$$

respectively.  $\zeta(E_i, d)$  is defined in (1.4).

The Hausdorff measure  $\mu_H$  of a subset  $E \subset X$  with the associated Hausdorff dimension  $D \in \mathbb{R}$  is defined by

$$\mu_H(E, D) = \lim_{\epsilon \rightarrow 0} \inf_{\{E_i\}} \left\{ \sum_i \zeta(E_i, D) : E \subset \bigcup_i E_i, d(E_i) < \epsilon \forall i \right\} \tag{2.3}$$

$$D = \sup\{d \in \mathbb{R} : d > 0, \mu_H(E, d) = \infty\}. \tag{2.4}$$

In the following I shall study generalisations of simple functions on  $X, f(x) = \sum_i \beta_i \chi_{E_i}(x)$ , where  $\chi_E$  is the characteristic function of  $E$ : continuous functions  $\lim_{x \rightarrow y} f(x) = f(y)$  whenever  $\lim_{x \rightarrow y} d(x, y) = 0$ , are considered. The Lebesgue-Stieltjes integral over continuous functions can be evaluated as the limit of infinitesimal covering diameter: when  $\{E_i\}$  is a disjointed covering and  $x_i \in E_i$ , then insertion into definition (A2.13) yields

$$\int_X f(x) d\mu_H(x) = \lim_{d(E_i) \rightarrow 0} \sum_{\bigcup_i E_i = X} f(x_i) \inf_{\{E_j\} \text{ with } \bigcup_j E_j = E_i} \sum_j \zeta(E_j, D). \tag{2.5}$$

From now on,  $X$  is assumed metrically *unbounded*, i.e. for every  $x \in X$  and  $r > 0$  there exists a point  $y$  such that  $d(x, y) > r$ . The assumption that  $D$  is uniquely defined in all of  $X$  requires  $X$  to be *regular (homogeneous, uniform)* with respect to the measure, i.e.  $\mu_H(B_r(x), D) = \mu_H(B_r(y), D)$  for all elements  $x, y \in X$  and (convex) balls  $B_r(x)$  and  $B_r(y) \subset X$  of the form  $B_{r>0}(x) = \{y : d(x, y) \leq r, x, y \in X\}$ . In particular [7]

$$\varliminf_{r \rightarrow 0} \mu_H(B_r(x), D) = \overline{\varliminf}_{r \rightarrow 0} \mu_H(B_r(x), D) \tag{2.6}$$

for all points  $x \in X$ . The upper and lower limits of a sequence of sets  $\{E_j\}$  are defined by  $\varliminf_{j \rightarrow \infty} E_j = \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} E_j$  and  $\overline{\varliminf}_{j \rightarrow \infty} E_j = \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} E_j$ . This implies that  $D$  is constant in  $X$  (for ball coverings,  $\omega(B_r(x), D) = \pi^{D/2} 2^{-D} / \Gamma(D/2 + 1)$ ). Uniformity plays an essential role for further explicit evaluation of the formal expressions. It seems to be also a dominant pattern in nature, from Brownian motion to percolation.

In the limit  $d(E_i) \rightarrow 0$ , the infimum is satisfied by the requirement that the variation over all coverings  $\{E_{ji}\}$  is replaced by one single covering  $E_i$ , such that  $\bigcup_j E_{ji} \rightarrow E_i \supset x_i$ . One finds for the integral (2.5)

$$\int_X f(x) d\mu_H(x) = \lim_{d(E_i) \rightarrow 0} \sum_{E_i} f(x_i) \zeta(E_i, D). \tag{2.7}$$

$X$  can be tessellated into (regular) polyhedra; in particular it is always possible to divide  $\mathbb{R}^n$  into parallelepipeds of the form

$$E_{i_1, \dots, i_n} = \{(x_1, \dots, x_n) \in X : x_j = (i_j - 1)\Delta x_j + \alpha_j, 0 \leq \alpha_j \leq \Delta x_j, j = 1, \dots, n\}. \tag{2.8}$$

For  $n = 2$ ,  $E_{i_1, i_2}$  is shown in figure 1. Since  $X$  is uniform

$$d\mu_H(x) = \lim_{d(E_{i_1, \dots, i_n}) \rightarrow 0} \zeta(E_{i_1, \dots, i_n}, D) = \lim_{d(E_{i_1, \dots, i_n}) \rightarrow 0} \prod_{j=1}^n (\Delta x_j)^{D/n} =: \prod_{j=1}^n d^{D/n} x_j. \tag{2.9}$$

The range of integration  $X$  may also be parametrised by polar coordinates with  $r = d(x, 0)$  and angle  $\Omega$ .  $E_{r, \Omega}$  can be thought of as spherically symmetric covering around a centre at the origin (see figure 2 for the two-dimensional case). In the limit, the function  $\zeta(E_{r, \Omega}, D)$  is given by

$$d\mu_H(r, \Omega) = \lim_{d(E_{r, \Omega}) \rightarrow 0} \zeta(E_{r, \Omega}, D) = d\Omega^{D-1} r^{D-1} dr. \tag{2.10}$$

When  $s(x)$  is symmetric with respect to some centre  $x_0 \in X$ , i.e.  $s(x) = \text{constant}$  for all  $x$  satisfying  $d(x, x_0) = r$  for arbitrary values of  $r$ ; then a transformation

$$X \rightarrow Z : x \rightarrow z = x - x_0 \tag{2.11}$$

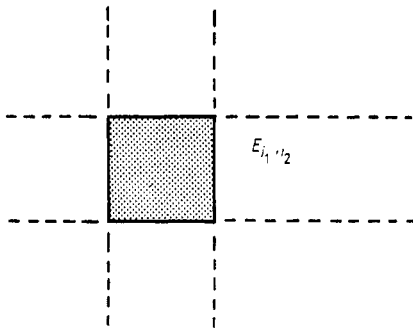


Figure 1. The parallelepiped covering for  $D = 2$ .

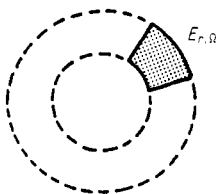


Figure 2. The spherical covering  $E_{r, \Omega}$  used for the definition of the infinitesimal volume element for  $D = 2$ .

can be performed to shift the centre of symmetry to the origin (since  $X$  is not a linear space, (2.11) need not be a map of  $X$  onto itself; (2.11) is measure preserving). The integral over a  $D$ -dimensional metric space is then given by

$$\int_X s \, d\mu_H = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int_0^\infty s(r)r^{D-1} \, dr. \tag{2.12}$$

The remaining integral is known in the theory of the Weyl fractional calculus (see, for instance, [8]). There, the Weyl fractional integral  $W^{-D}$  is given by

$$W^{-D}f(x) = \frac{1}{\Gamma(D)} \int_0^\infty (t-x)^{D-1}f(t) \, dt.$$

When  $W^{-D}f(0)$  is considered, the integral in (2.12) is reproduced (the Weyl fractional differentiation can be defined likewise). In particular, for  $S(x) = (x^2 + l^2)^{-n}$ , and for the Gaussian  $\sigma(x) = \exp(-sx^2)$ ,

$$\int_X S(x) \, d\mu_H(x) = \frac{\pi^{D/2}l^{D-2n}\Gamma(n - D/2)}{\Gamma(n)} \tag{2.13a}$$

$$\int_X \sigma(x) \, d\mu_H(x) = \pi^{D/2}s^{-D/2}[W^{-D/2} \exp(-x)]_{x=0} = \pi^{D/2}s^{-D/2}. \tag{2.13b}$$

The integral defined in (2.7) satisfies the following conditions.

(i) Linearity:

$$\int_X (af_1 + bf_2) \, d\mu_H = a \int_X f_1 \, d\mu_H + b \int_X f_2 \, d\mu_H \tag{2.14}$$

where  $f_1$  and  $f_2$  are arbitrary functions and  $a$  and  $b$  are arbitrary constants.

(ii) Translational invariance:

$$\int_X f(x + x_0) \, d\mu_H(x) = \int_X f(x) \, d\mu_H(x) \tag{2.15}$$

since  $d\mu_H(x - x_0) = d\mu_H(x)$  as a consequence of homogeneity (uniformity).

(iii) Scaling property:

$$\int_X f(ax) \, d\mu_H(x) = a^{-D} \int_X f(x) \, d\mu_H(x) \tag{2.16}$$

since  $d\mu_H(x/a) = a^{-D} \, d\mu_H(x)$ .

By evaluating the integral of a generating function of the form

$$g(x, s, q) = \exp(-sx^2 + xq) \tag{2.17}$$

it has been shown by Wilson [9] that conditions (2.14)-(2.16) define the integral up to normalisation:

$$\int_X \exp(-sx^2 + xq) \, d\mu_H(x) = \pi^{D/2}s^{-D/2} \exp(q^2/4s). \tag{2.18}$$

Any function of the form  $g(x^2, xq_1, \dots, xq_n)$  can be generated by using differentiation with respect to  $s$  and  $q$  and summation of  $g(x, s, q)$ . Notice that, for  $q = 0$ , (2.18) is identical to (2.13*b*), which was obtained directly without conditions (2.14)–(2.16).

The generalised  $\delta^{(D)}$  function for sets with non-integer Hausdorff dimension can be defined via

$$\int_X f(t)\delta^{(D)}(x-t) d\mu_H(t) = f(x). \tag{2.19}$$

### 3. Fourier–Stieltjes transformation and Feynman rules

In this section, the Fourier–Stieltjes transform  $\tilde{f}(k)$  of a continuous function  $f(x)$  on an unbounded measurable metric space  $[X, \mu_H]$  will be considered. As a starting point, the  $n$ -dimensional linear vector space  $\mathbb{R}^n$ ,  $n \geq 4$ ,  $n \in \mathbb{N}$  is chosen. As has been already assumed in the preceding section,  $X \subset \mathbb{R}^n$  is embedded in  $\mathbb{R}^n$ . Then the Fourier–Stieltjes transformation is given by

$$f(x) = \frac{1}{V} \int_{\tilde{X} \subset \mathbb{R}^n} \tilde{f}(k) \exp(ikx) d\mu_H(k) \tag{3.1}$$

$$\tilde{f}(k) = \int_{X \subset \mathbb{R}^n} f(x) \exp(-ikx) d\mu_H(x) \tag{3.2}$$

where the points  $x = (x_1, \dots, x_n)$  and  $k = (k_1, \dots, k_n)$  are represented by  $n$ -tuples of real numbers and  $\tilde{X}$  is the space of the  $k$ .

Although in principle,  $X$  may be embedded in a vector space of higher dimension than four, for the moment a coordinate frame is chosen such that for all points  $x \in X$ ,  $x_i = 0$  for  $i > 4$ ; and  $X$  and  $\tilde{X} \subset \mathbb{R}^4$ . Then the function  $f(x)$  in (3.2) is proportional to  $2\pi\delta(x_i)$  for all  $i > 4$ , and the coordinates can be represented by  $x = (x_1, \dots, x_4)$  and  $k = (k_1, \dots, k_4)$ .

The main questions remaining to be answered are: (i) what is the functional form of the Fourier–Stieltjes transforms  $\tilde{f}$ , and (ii) what is  $\tilde{X}$  and its Hausdorff measure  $\mu_H(\tilde{X})$ ? The following conjectures will be stated below.

(i)  $D(X) = D(\tilde{X})$ .

(ii)  $\tilde{X}$  inherits all properties of  $X$ , in particular its closedness, unboundedness and regularity.

(iii) The functional form of the Fourier–Stieltjes transform  $\tilde{f}(k)$  depends on  $D(X)$ ; in particular when  $X = \mathbb{R}^n$ ,  $\tilde{f}(k)$  is the ordinary Fourier transform  $f(k)$  in  $n$  dimensions.

Next,  $\tilde{g}(k)$  will be derived for a class of functions of the form  $g(x^2, xq_1, \dots, xq_n)$  useful for evaluation of Feynman integrals in perturbative quantum field theory [9]. These functions can be produced with the help of the generating function  $g(x, s, q) = \exp(-sx^2 + xq)$  introduced in (2.17)

$$\tilde{g}(k, s, q) = \int_X \exp(-sx^2 + xq - ikx) d\mu_H(x). \tag{3.3}$$

A translation  $x \rightarrow q/2s - x$  yields

$$\tilde{g}(k, s, q) = \exp(q^2/4s - ikq/2s) \int_x \exp(-sx^2 + ikx) d\mu_H(x). \tag{3.4}$$

When  $\pi(x)$  is the orthogonal projection along the  $k$  axis and  $D > 1$ , the remaining integral can be split up [1] into a  $(D - 1)$ -dimensional subspace  $X \setminus \pi(X)$  and a one-dimensional integration along  $\pi(X)$ , since  $D(\pi(X)) = 1$  (see also Falconer [7, p 75ff] for details):

$$\int_{-\infty}^{\infty} \exp(-sp^2 + ikp) dp \int_{X \setminus \pi(X)} \exp(-sx^2) d\mu_H(x). \tag{3.5}$$

The first integral on the right-hand side is standard and the second integral is evaluated in (2.18). Hence the Fourier-Stieltjes transform  $\tilde{g}(k, s, q)$  is given by

$$\tilde{g}(k, s, q) = \pi^{D/2} s^{-D/2} \exp[-ikq/2s + (q^2 - k^2)/4s]. \tag{3.6}$$

For  $q = 0$ ,  $g(x, s, 0)$  reduces to a Gaussian distribution and  $\tilde{g}(k, s, 0)$  reduces to the results of Stillinger [1].

Substitution of  $s$  by  $1/4s$  yields the inverse transformation (3.2) of  $\tilde{g}(k, s, 0) = (\pi/s)^{d/2} \exp(-k^2/4s)$

$$g(x, s, 0) = \frac{1}{V} \left(\frac{\pi}{s}\right)^{D/2} \int_{\tilde{X}} \exp(ikx - k^2/4s) d\mu_H(k). \tag{3.7}$$

If and only if  $V = (2\pi)^D$ ,  $D(X) = D(\tilde{X})$ , and  $\tilde{X}$  is homogeneous (uniform) such that  $d\mu_H(k)$  can be parametrised as in (2.10), then an identical calculation yields

$$g(x, s, 0) = \exp(-sx^2). \tag{3.8}$$

The same argument holds for the generating function (2.17) for the calculation of Feynman diagrams. Therefore, for this specific functional class it has been shown that  $\tilde{X}$  inherits all properties of  $X$ , in particular its dimension, homogeneity (uniformity) and measure.

Finally, one obtains with  $g(x, s, q)$  from (2.17) and  $\tilde{g}(k, s, q)$  from (3.6)

$$g(x) = \frac{1}{(2\pi)^D} \int_{\tilde{X}} \tilde{g}(k) \exp(ikx) d\mu_H(k) \tag{3.9a}$$

$$\tilde{g}(k) = \int_x g(x) \exp(-ikx) d\mu_H(x). \tag{3.9b}$$

In particular, when  $\tilde{f}(k) = 1$ , insertion into (3.6) with  $q = 0$ ,  $s = 1/4\pi n^2 \rightarrow 0$  yields

$$\int_{\tilde{X}} \exp(ikx) d\mu_H(k) = (2\pi)^D \lim_{n \rightarrow \infty} n^D \exp(-\pi n^2 x^2) = (2\pi)^D \delta^{(D)}(x) \tag{3.10}$$

as can be explicitly seen by evaluating

$$\int_x \delta^{(D)}(x) f(x) d\mu_H(x) = \frac{1}{\Gamma(D/2)} f(0) \lim_{n \rightarrow \infty} \int_0^{\infty} \exp(-t) t^{D/2-1} dt = f(0)$$

where (2.13b) has been used.

Defining the quantum field theory in momentum space makes a derivation of covariant Feynman rules in  $D$ -dimensional momentum space straightforward. The

free  $n$ -point functions are not changed. The only difference to conventional Feynman rules is the substitution of the measure in the momentum integral

$$(2\pi)^{-4} d^4k \rightarrow (2\pi)^{-D} d\mu_H(k). \tag{3.11}$$

For symmetric kernels, a representation of  $\mu_H(k)$  in terms of spherical coordinates is useful:

$$(2\pi)^{-D} d\mu_H(k) = (2\pi)^{-D} d\Omega^{D-1} k^{D-1} dk. \tag{3.12}$$

When the starting point for a definition of field theory is  $X$  space, one would have to transform all expressions there to  $D$ -dimensional Fourier-Stieltjes transforms. In particular, the  $n$ -point Green functions would differ from the usual form. However, one can speculate that both approaches may yield identical results, since physical quantities should not depend on which one of the dual variables  $x$  and  $k$  is chosen as parametrisation.

It should be noted that this approach is not entirely identical with dimensional regularisation. In particular, the Minkowski metric tensor  $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$  in flat spacetime, as well as the unit tensor  $\delta_{\mu\nu} = \text{diag}(1, 1, 1, 1)$ , obey

$$g_{\mu\nu} g^{\mu\nu} = \delta_\mu^\mu = 4 \tag{3.13}$$

since they are defined in  $\tilde{X} \subset \mathbb{R}^4$ . For the same reason, the  $\gamma$  matrices as well as the  $\varepsilon$  tensor are the same as for  $\mathbb{R}^4$  and  $\gamma_5$  can be incorporated in field theory without conceptual difficulties.

#### 4. Renormalisation

In this section, an account will be given of the lowest-order contributions to radiative corrections of quantum electrodynamics with a fractal support of the fields. These contributions are defined for Hausdorff dimensions arbitrarily close but smaller than four. However, in this context, I shall not discuss contributions from overlapping radiative corrections, nor questions concerning the convergence of the perturbation expansion.

I shall start with the renormalisation of the bare two-point Green function  $S_0$  of the electron. The full propagator  $S$  can be formally written as the analytic continuation of a sum over self-energy diagrams  $S = (\gamma p - m_0 - \Sigma + i\varepsilon)^{-1}$ , where  $m_0$  is the bare electron mass and  $\Sigma$  is the proper self-energy. Substituting for  $\Sigma$  its lowest-order contribution (A1.3), and recalling (A1.4), yields

$$S(p) = \frac{Z_2}{(\gamma p - m + i\varepsilon)} [1 + Z_2(\gamma p - m)\sigma(p)]^{-1} \tag{4.1}$$

where the physical mass  $m$  and the renormalisation constant  $Z_2$  are defined by

$$m = m_0 - A \tag{4.2a}$$

$$Z_2 = 1 + B. \tag{4.2b}$$

Close to the pole, the gamma function can be expanded as  $\Gamma(x) = x^{-1} - \mathcal{C} + O(x)$  for  $x \ll 1$ , where  $\mathcal{C} \approx 0.577\ 22$  is Euler's constant. Insertion into (4.2) yields for  $4 - D \ll 1$

$$m = Z_2 m_0 \tag{4.3a}$$

$$Z_2 = 1 + \frac{3}{2\pi} \frac{\alpha}{4 - D}. \tag{4.3b}$$



The bare photon propagator is renormalised by the formal summation of vacuum polarisation diagrams  $\Pi$ , whose lowest-order contribution has been evaluated in (A1.2). Again,  $\Pi(q^2)$  can be expanded around the mass shell  $q^2 = 0$  (cf (A1.3)), yielding

$$\Pi(q^2) = P + q^2 \pi(q^2). \tag{4.4}$$

The full photon propagator can be written as  $(q_\mu q_\nu - g_{\mu\nu} q^2)\Delta$ , with

$$\Delta(q^2) = [1 - P - q^2 \pi(q^2)]^{-1}. \tag{4.5}$$

The term in brackets contributes to the renormalisation of the bare charge  $e_0$ , which relates to the renormalised charge  $e$  by

$$e^2 = e_0^2 [1 - P - q^2 \pi(q^2)]^{-1}. \tag{4.6}$$

For zero momentum transfer  $q^2 \rightarrow 0$  and for  $4 - D \ll 1$ ,  $(Z_3)^{-2} = 1 - P$  reduces to

$$(Z_3)^{-2} = 1 - \Pi(q^2 = 0) = 1 + \frac{2}{\pi} \frac{\alpha}{4 - D} \tag{4.7}$$

yielding

$$\alpha_0 = \frac{e_0^2}{4\pi} = \alpha \left( 1 + \frac{2}{\pi} \frac{\alpha}{4 - D} \right). \tag{4.8}$$

All other contributions to the renormalisation of the electric charge cancel, as can be explicitly seen by a summation of the lowest-order radiative corrections to the charge

$$e_0(1 - B + L - P). \tag{4.9}$$

As can be shown from (A1.4) and (A1.7), or derived from Ward identities,  $L$  equals  $B$  and only the  $Z_3$  factor remains for the renormalisation of the electric charge.

I shall next consider corrections to the magnetic moment due to vertex corrections as (A1.6). In particular, the term proportional to  $\sigma_{\mu\nu} q^\nu$  remains finite for Hausdorff dimensions smaller than six. It gives rise to low-order contributions to the anomalous magnetic moment as well as the  $l \neq 0$  splitting of energy levels in atoms (Lamb shift).

Utilising the expansion of the gamma function into a polynomial  $\Gamma(1 + z) = \sum_{i=0}^\infty c_i z^i$ ; with coefficients  $c_0 = 1$ ,  $c_{n+1} = (n+1)^{-1} \sum_{i=0}^n (-1)^{i+1} s_{i+1} c_{n-i}$  and  $s_1 = \mathcal{C}$ ,  $s_n = \zeta(n)$  for  $n \geq 2$ ,  $\text{Re } z > 0$ , where  $\zeta(n)$  is the Riemann zeta function (e.g.  $s_2 = \pi^2/6$ ), one obtains for small  $4 - D$

$$\begin{aligned} a_e(D=4) - a_e(D) &= \alpha [(4\pi)^{-1} - 2^{2-D} \pi^{1-D/2} \Gamma(3 - D/2)] \\ &\approx \frac{\alpha}{8\pi} (\mathcal{C} + \log(\pi))(4 - D). \end{aligned} \tag{4.10}$$

Here,  $a_e$  is the form factor of the electromagnetic current proportional to  $\sigma_{\mu\nu} q^\nu$ . Presently the difference between experimental [10] and theoretical [11] values of  $a_e$  suggests  $D \geq 4 - (5.3 \pm 2.7) \times 10^{-7}$  at the scale of the Compton wavelength of the electron.

Similarly, corrections to the  $l \neq 0$  levels for a hydrogen-like atom [11] can be derived: for  $\delta E = \Delta E(D=4) - \Delta E(D)$ ,

$$\delta E_{l \neq 0} = \frac{\alpha^3}{2\pi n^3} [\mathcal{C} + \log(\pi)](4 - D) \text{Ry}_x \begin{cases} [(l + \frac{1}{2})(l + 1)]^{-1} & \text{for } j = l + \frac{1}{2} \\ [-l(l + \frac{1}{2})]^{-1} & \text{for } j = l - \frac{1}{2}. \end{cases} \tag{4.11}$$

This correction is very small and yields

$$\delta E(2p_{\frac{1}{2}}) \approx \frac{\alpha^3}{24\pi} \text{Ry}_{\infty} [\mathcal{C} + \log(\pi)](4 - D) \leq 0.03 \pm 0.01 \text{ MHz} \quad (4.12)$$

and the lower bound  $D \geq 4 - (1.0 \pm 0.3) \times 10^{-3}$ .

The *degree of divergence*  $DIV$  of any given diagram in terms of the number of external photon lines  $B$  and external electron lines  $F$  can be calculated by power counting: let  $b$  and  $f$  be the number of internal photon and electron lines, and  $V$  the number of vertices. Then the number of loop integrals of dimension  $D$  is given by

$$L = b + f - V + 1 \quad (4.13)$$

where 1 comes from the overall  $\delta^{(D)}$  function. The degree of divergence  $DIV$  can be defined as

$$DIV = DL - 2b - f. \quad (4.14)$$

The following topological relations hold:  $2V = F + 2f$  and  $V = B + 2b$ . Substitution of  $b$  and  $f$  in (4.14) yields a cancellation of  $V$  and

$$DIV = 4 - (4 - D)L - B - \frac{3}{2}F. \quad (4.15)$$

The degree of divergence of some radiative corrections is given by

$$DIV(\Sigma) = 1 - (4 - D)L \quad (4.16a)$$

$$DIV(\Gamma) = -(4 - D)L \quad (4.16b)$$

$$DIV(\Pi) = 2 - (4 - D)L. \quad (4.16c)$$

Due to gauge invariance and symmetry considerations, the effective degree of divergence for all above graphs reduces to  $-(4 - D)L$ , indicating that for  $0 < D \leq 4$  divergencies may become weaker or disappear.

## 5. Concluding remarks

Technically, the regularisation method is almost identical to dimensional regularisation [12]. Conceptually, these notions are very different. Spacetime is embedded in a manifold of higher and integer dimension (presumably 4). The non-integer Hausdorff dimension did not emerge as an analytic continuation of a dimensional parameter to the complex plane, but rather had to be adopted to cope with an irregular spacetime structure: the assumption of fractal spacetime enforces the use of a measure with non-integer dimension and makes the application of the Lebesgue-Stieltjes integral inevitable. Advantages of this approach are that  $\gamma_5$  or the  $\varepsilon$  tensor need not be generalised to non-integer dimensions, and preservation of gauge invariance and covariance. The scheme resembles supersymmetric dimensional regularisation via dimensional reduction [13], and can be briefly formulated as follows.

(i) First perform the algebra exactly as in  $D = 4$ , with four-component Dirac matrices, tensors and vectors.

(ii) Then do the momentum integrals as in ordinary dimensional regularisation.

(iii) Apply the Fourier-Stieltjes transformation to obtain results in configuration space (instead of momentum space).

It is not evident that spacetime is a Hausdorff space, where every pair of distinct points have disjoint neighbourhoods [14]. In fact, the uncertainty relation of quantum theory suggests that there exists a maximal resolution  $\delta_{\max}$  associated to any experiment, beyond which the operational concept of distance and point separation becomes unattainable [3]. Therefore, two points separated by a distance  $\delta_{\max}$  have no (operationally realisable) disjoint neighbourhoods. Hence, in a strict (positivistic) sense, spacetime will never be proven to be Hausdorff.

It is in no way clear yet whether  $\delta_{\max}$  could, at least in principle, be reduced to arbitrary small numbers, thereby smoothly defining topologies and their associated open sets with ever smaller diameters. To encounter this question, throughout this paper, an extrinsic viewpoint has been adopted: a fractal space  $X$  embedded in  $\mathbb{R}^n$  has been considered. Alternatively, it would have been possible to adopt an intrinsic viewpoint by ‘glueing’ the ‘neighbouring’ segments of  $X$  together. This could be achieved with a sophisticated topology. Creatures living on  $X$  would perceive such a structure in the following way: they would find themselves living in  $\mathbb{R}^n$ . Its topological structure would then correspond to the graininess of  $X$ . Although both extrinsic and intrinsic viewpoints yield equivalent results, the latter would be favourable for conceptual and pragmatic reasons. Its development remains a challenge to future research.

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**Appendix 1. Lowest-order radiative corrections of QED**

In the following the electron self-energy  $\Sigma$ , the vacuum polarisation  $\Pi$  and the vertex correction  $\Lambda$  are enumerated as a function of the Hausdorff dimension  $D$  (see also [12, 13]).

The lowest-order contribution to the *vacuum polarisation*

$$\Pi_{\mu\nu}(q) = -e^2 \text{Tr} \int \frac{d^D k}{(2\pi)^D} \left( \gamma_\mu \frac{i}{\gamma k - m + i\epsilon} \gamma_\nu \frac{i}{\gamma k - \gamma q - m + i\epsilon} \right) \quad (\text{A1.1})$$

can be written as

$$\begin{aligned} \Pi_{\mu\nu}(q) &= (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2) \\ \Pi(q^2) &= \alpha 2^{4-D} \pi^{1-D/2} \Gamma(2 - D/2) m^{D-4} F(2 - D/2, 2; \frac{5}{2}; -q^2/4m^2) \end{aligned} \quad (\text{A1.2})$$

where  $\alpha$  is the fine structure constant and  $F(a, b; c; z)$  is the hypergeometric series

$$F(a, b; c; z) = 1 + \frac{ab}{c} \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{z^2}{2!} + \dots + \frac{z^{n-1}}{(n-1)!} \prod_{i=0}^{n-1} \frac{(a+i)(b+i)}{c+i} + \dots$$

The lowest-order contribution to the *electron self-energy*

$$\Sigma(p) = -ie^2 \int \frac{d^D k}{(2\pi)^D} \frac{-i}{k^2 - \lambda^2 + i\epsilon} \gamma^\mu \frac{i}{\gamma p - \gamma k - m + i\epsilon} \gamma_\mu \quad (\text{A1.3})$$

can, after some calculation, be written as

$$\Sigma(p) = A - (\gamma p - m)B + (\gamma p - m)^2 \sigma(p) \tag{A1.4}$$

where  $A$ ,  $B$  and  $\sigma$  are given by

$$A = -3\alpha 2^{2-D} \pi^{1-D/2} m^{D-3} \Gamma(2-D/2) \tag{A1.5a}$$

$$B = -Am^{-1} = 3\alpha 2^{2-D} \pi^{1-D} m^{D-4} \Gamma(2-D/2) \tag{A1.5b}$$

$$\begin{aligned} \sigma(p) = & -\alpha 2^{2-D} \pi^{1-D/2} \Gamma(2-D/2) (p^2 + m^2)^2 \\ & \times \left[ F(2-D/2, 1; D/2; -p^2/m^2) \left( -2\gamma p - \frac{4(m + \gamma p)m^2}{m^2 + p^2} \right) \right. \\ & + F(2-D/2, 1; D/2 + 1; -p^2/m^2) \left( 2m + \gamma p - \frac{2(m + \gamma p)m^2}{m^2 + p^2} \right) \\ & \left. + \gamma p - 2m + \frac{6(m + \gamma p)m^2}{m^2 + p^2} \right]. \end{aligned} \tag{A1.5c}$$

The lowest-order contribution to the *vertex term*  $\Lambda$ , with the photon momentum  $q$  and two outgoing electron momenta  $p$  and  $p'$ , is given by

$$\begin{aligned} \Lambda_\mu(q, p, p') = & -e^2 \int \frac{d^D k}{(2\pi)^D} \frac{-i}{k^2 - \lambda^2 + i\epsilon} \\ & \times \gamma_\nu \frac{i}{\gamma p - \gamma k - m + i\epsilon} \gamma_\mu \frac{i}{\gamma p' - \gamma k - m + i\epsilon} \gamma^\nu \end{aligned} \tag{A1.6}$$

can for  $q = p' - p$ , be written as

$$\Lambda_\mu(q) = [L + g(q)]\gamma_\mu + \alpha 2^{3-D} \pi^{1-D/2} \Gamma(3-D/2) \frac{i\sigma_{\mu\nu} q^\nu}{2m} \tag{A1.7}$$

with  $L = B$  defined in (A1.5b).  $g(q)$  is a function proportional to  $\Gamma(3-D/2)$  vanishing for  $q^2 \rightarrow 0$ , which will not be enumerated here. The term proportional to  $\sigma_{\mu\nu} q^\nu$  yields contributions to the anomalous magnetic moment and to the  $l \neq 0$  Lamb shift.

### Appendix 2. Measure theoretic glossary

This glossary will be of use as a quick reference to measure theoretic concepts (see also [15]). A more comprehensive treatment can be found in [6].

A *measure*  $\mu$  is a real-valued non-negative function defined on a set of sets  $E_i \subset X$ , such that

$$\mu(E_i) \geq 0 \tag{A2.1a}$$

$$\mu(\emptyset) = 0. \tag{A2.1b}$$

A measure is called *countable subadditive* if, for any sequence  $\{E_i\}$  of sets in  $X$  whose union  $\bigcup_i E_i \subset X$

$$\mu\left(\bigcup_i E_i\right) \leq \sum_i \mu(E_i). \tag{A2.2}$$

A non-empty set  $H$  of sets  $E_i \in H$  is *hereditary* if, whenever  $F \subset E_i$ , then  $F \in H$ .

An *outer measure* is a real-valued non-negative monotone and countably subadditive set function, defined on a hereditary ring  $H$ .

A metric space  $[M, d]$  is a set  $M$  and a real-valued function  $d$ , called *distance*, on  $M \times M$ , such that for all  $x, y, z \in M$ :

$$d(x, y) \geq 0 \tag{A2.3a}$$

$$d(x, y) = 0 \Rightarrow x = y \tag{A2.3b}$$

$$d(x, y) = d(y, x) \tag{A2.3c}$$

$$d(x, y) \leq d(x, z) + d(z, y). \tag{A2.3d}$$

If  $E$  and  $F$  are non-empty subsets of  $M$ , the distance between them is defined by

$$d(E, F) = \inf\{d(x, y) : x \in E, y \in F\}. \tag{A2.4}$$

A sphere with centre  $x_0$  and radius  $r$  is a subset  $E \subset [M, d]$  such that

$$E = \{x : d(x_0, x) < r\}. \tag{A2.5}$$

We are now in a position to define the Hausdorff measure. If  $X$  is a metric space,  $D$  is a positive real number and  $E$  is a subset of  $X$ , then the  $D$ -dimensional Hausdorff (outer) measure  $\mu_H(E, D)$  is defined to be the number

$$\mu_H(E, D) = \sup_{\epsilon > 0} \inf \left\{ \sum_i \omega(E_i, D) [d(E_i)]^D : E \subset \bigcup_i E_i, d(E_i) < \epsilon \forall i \right\} \tag{A2.6}$$

where  $\omega(E_i, D)$  is a geometrical factor depending on the type of covering and  $d(E)$  denotes the diameter of  $E$ , defined by  $d(E) = \sup\{d(x, y) : x, y \in E\}$ . Since  $\omega(E_i, D) [d(E_i)]^D$  is positive, the following definition, employing disjoint sets for coverings, is identical to (A2.6):

$$\mu_H(E, D) = \lim_{\epsilon \rightarrow 0} \left\{ \sum_i \omega(E_i, D) [d(E_i)]^D : E = \bigcup_i E_i, E_i \cap E_j = \emptyset, d(E_i) \leq \epsilon \forall i, j \right\}. \tag{A2.7}$$

The Hausdorff dimension  $D$  is defined by

$$d = \sup\{d \in \mathbb{R} : d > 0, \mu_H(E, d) = \infty\} = \inf\{d \in \mathbb{R} : d > 0, \mu_H(E, d) = 0\}. \tag{A2.8}$$

$D$  is invariant under variation of equivalent distances  $d_1$  and  $d_2$ , such that two positive, real numbers  $a$  and  $b$  exist with  $ad_1(E) \leq d_2(E) \leq bd_1(E)$  for all subsets  $E$  of  $X$ . Moreover, invariance under very general coordinate transformations (such as the Lorentz transformation) has been proven in [16].

If  $E$  is a subset of  $X$  (which need not necessarily be a measurable set), the characteristic function  $\chi_E$  is defined for all  $x \in X$  by the relations

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases} \tag{A2.9}$$

For instance,  $E = \{x : \chi_E(x) = 1\}$ .

A  $\sigma$  ring  $S$  is a non-empty set of sets such that

$$\text{if } E \in S \text{ and } F \in S \quad \text{then } F \setminus E \in S \tag{A2.10a}$$

$$\text{if } E_i \in S \quad \text{then } \bigcup_i E_i \in S \tag{A2.10b}$$

i.e. a  $\sigma$  ring is closed under the formation of countable unions and differences.

A measurable space  $[X, S, \mu]$  is a set  $X$  and a  $\sigma$  ring  $S$  of subsets of  $X$  with the property  $\bigcup S = X$ .

A function  $f$ , defined in a measurable space  $[X, S, \mu]$ , is called a simple function if there is a finite disjoint set of sets  $\{E_1, \dots, E_n\}$  of measurable sets and a finite set  $\{\beta_1, \dots, \beta_n\}$  of real numbers such that

$$f(x) = \begin{cases} \beta_i & \text{if } x \in E_i \\ 0 & \text{if } x \notin \bigcup_j E_j. \end{cases} \tag{A2.11}$$

To represent this function, one can also write

$$f(x) = \sum_{i=1}^n \beta_i \chi_{E_i}(x) \tag{A2.12}$$

where  $\chi_E(x)$  is the characteristic function of  $E$ . The product of two simple functions, and any finite linear combination of simple functions, are again simple functions.

A simple function  $f$  on a measurable space is *integrable* if  $\mu(E_i) < \infty$  for every index  $i$  for which  $\beta_i \neq 0$ . The Lebesgue-Stieltjes *integral* of  $f$  is defined by

$$\int f(x) \, d\mu(x) \quad \text{or} \quad \int f \, d\mu = \sum_{i=1}^n \beta_i \mu(E_i). \tag{A2.13}$$

If  $E$  is a measurable set, then the Lebesgue-Stieltjes integral of  $f$  over  $E$  is defined by

$$\int_E f \, d\mu = \int \chi_E f \, d\mu. \tag{A2.14}$$

The simplest example for  $f$  is the characteristic function  $f = \chi_E$  of a measurable set  $F$

$$\int \chi_E \, d\mu = \int_E d\mu = \mu(E). \tag{A2.15}$$

If  $f$  and  $g$  are integrable functions and  $a$  and  $b$  are real numbers, then

$$\int (af + bg) \, d\mu = a \int f \, d\mu + b \int g \, d\mu \tag{A2.16a}$$

$$\int f \, d\mu \geq \int g \, d\mu \quad \text{if } f \geq g \tag{A2.16b}$$

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu \tag{A2.16c}$$

$$\text{if } a \leq f \leq b \quad \text{then } a\mu(E) \leq \int_E f \, d\mu \leq b\mu(E). \tag{A2.16d}$$

Next, negative-valued measures are allowed and a *signed measure*  $\nu$  is a real-valued countable additive set function on the set of all measurable sets of a measurable space  $[X, S, \nu]$  such that  $\nu(\emptyset) = 0$ , and such that  $\nu$  assumes at most one of the values  $+\infty$  and  $-\infty$ .

In the following,  $\nu(x)$  will be called *finite* if  $|\nu(x)| < \infty$ .

A set function  $g$  is called *continuous* if, for every increasing or decreasing sequence  $\{E_n\}$  of sets, for which  $\lim_n E_n = E$ , we have  $\lim_n g(E_n) = g(E)$ . We proceed by establishing the *Radon-Nikodym* theorem: if  $[X, S, \mu]$  is a totally  $\sigma$ -finite measure space and if a  $\sigma$ -finite signed measure  $\nu$  on  $S$  is absolutely continuous with respect to  $\mu$ , then there exists a finite-valued measurable function on  $X$  such that

$$\nu(E) = \int_E f \, d\mu \tag{A2.17}$$

for every measurable set  $E$ . The function

$$f = d\nu/d\mu \quad (\text{A2.18})$$

is called the *Radon-Nikodym derivative*. It can be proven that if  $\lambda$  and  $\mu$  are totally  $\sigma$ -finite measures such that  $\mu \ll \lambda$  and if  $\nu$  is a totally  $\sigma$ -finite signed measure function such that  $\nu \ll \mu$ , then

$$d\nu/d\lambda = (d\nu/d\mu)(d\mu/d\lambda). \quad (\text{A2.19})$$

*Note added in proof.* A recent observation [17] concerns the induction of an *intrinsic metric*  $d'(x, y) =^{\text{def}} d(x, y)$  for any two points  $x, y \in X$ , where  $X \subset \mathbb{R}^n$  is a fractal embedded in  $\mathbb{R}^n$  and  $d(x, y)$  is the metric of  $[\mathbb{R}^n, d]$ , such as the one defined in (2.2). Hence  $[X, d']$  is a metric space, where  $d'$  defines a topology on  $X$  (see also (A2.3)). A property is called intrinsic if it can be formulated in terms of  $[X, d']$ .

Furthermore, it has been pointed out [18] that a regular fractal of Hausdorff dimension  $D = N \in \mathbb{N}$  is intrinsically perceived as  $\mathbb{R}^N$ , independent of the dimension  $n \geq N$  of the embedding space. This result can be generalised to non-integer dimensions and presents an alternative to dimensional reduction via 'compactifying' extra dimensions.

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